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Convergence of stochastic flows with jumps and Lévy processes in diffeomorphisms group

by

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ABSTRACT. — Let $\{ \xi^n_{s,t} \}$ be a sequence of stochastic flows with jumps generated by a sequence of Lévy processes $\{ X^n_t \}$ with values in the vector space of smooth mappings of R^d. We give a criterion that $\{ \xi^n_{s,t} \}$ converges weakly by means of the characteristics of the generators X^n_t, $n = 1, 2, \ldots$. As an application we study the case where a stochastic flow $\{ \xi_{s,t} \}$ takes values in diffeomorphisms group of R^d.

RÉSUMÉ. — Soit $\{ \xi^n_{s,t} \}$ une suite de flots stochastiques avec sauts engendrée par une suite de processus de Lévy dans l'espace vectoriel des applications de R^d. Nous donnons un critère pour que $\{ \xi^n_{s,t} \}$ converge faiblement, au moyen des caractéristiques des générateurs X^n_t, $n = 1, 2, \ldots$. Comme application, nous étudions le cas d'un flot stochastique $\{ \xi_{s,t} \}$ à valeurs dans le groupe des difféomorphismes de R^d.

INTRODUCTION

In the previous paper [*2*], we studied the stochastic differential equation of jump type:

$$\xi_t = x + \int_s^t dX_r(\xi_{r}-), \tag{0.1}$$

Classification AMS : 60-XX.

where $X_t = \{ X_t(x); x \in R^d \}$ is a Lévy process with values in C = the linear space of smooth vector fields over R^d. It is a natural generalization of the usual stochastic differential equation

$$(0.2)\quad \xi_t = x + \int_0^t \sigma(\xi_s) dB_s + \int_0^t b(\xi_s) ds + \int_0^t \int f(\xi_s, u)(N(dsdu) - \nu(dsdu)),$$

where B_t is a standard Brownian motion and $N(dsdu)$ is a Poisson random measure and $\nu(dsdu)$ is its intensity measure. Indeed, setting

$$X_t(x) = \sigma(x)B_t + b(x)t + \int_0^t \int f(x, u)(N(dsdu) - \nu(dsdu)),$$

it is a C-valued Lévy process and equation (0.2) can be written as (0.1).

Under certain conditions to X_t, it was shown that the equation (0.1) has a unique solution $\xi_{s,t}(x)$. Moreover, taking a suitable modification, it defines a stochastic flow with values in G_+ = the semigroup of smooth maps from R^d into itself. Namely, $\xi_{s,t}(., \omega)$ is a G_+-valued process and satisfies $\xi_{s,u} = \xi_{t,u} \circ \xi_{s,t}$ for all $s < t < u$ a. s. Moreover, $\xi_{t_i,t_{i+1}}, i=0, \ldots, n-1$ are independent for any $t_0 < t_1 < \ldots < t_n$. It is called a Lévy process with values in G_+ generated by the Lévy process X_t with values in C.

The object of this paper is two fold. The one is the study of the convergence of the sequence of G_+-valued Lévy processes $\xi^n_{s,t}$ generated by X^n_t, $n = 1, 2, \ldots$. We shall show that the weak convergence of $\{ X^n_t \}$ implies the weak convergence of $\{ \xi^n_{s,t} \}$ and *vice versa* and obtain simultaneously a criterion for the weak convergence by means of the characteristics of the Lévy processes X^n_t. Here, the characteristics (a, b, ν) of the C-valued Lévy process X_t is defined through the Lévy-Khinchin's formula:

$$E\left[\exp i \sum_{k=1}^{N} (\alpha_k, X_t(x_k))\right] = \exp t\left[i \sum_{k=1}^{N} (\alpha_k, b(x_k)) - \frac{1}{2} \sum_{k,l=1}^{N} \alpha_k a(x_k, x_l)\alpha_l \right.$$
$$\left. + \int_C \left\{ e^{i\sum_{k=1}^{N}(\alpha_k, f(x_k))} - 1 - i \sum_{k=1}^{N} (\alpha_k, f(x_k)) \right\} \nu(df) \right].$$

In the course of the discussions, the tightness criterion for $\{ X^n_t \}$ and $\{ \xi^n_{s,t} \}$ obtained in Kunita [7] play a fundamental role. We emphasize that the convergence studied in this paper is much stronger than the convergence of Markov processes with jumps, since in the latter case we usually consider the convergence of the finite dimensional projections $\xi^n_{s,t}(x)$, $n = 1, 2, \ldots$ where the state space x is fixed.

The another object of this paper is to obtain a necessary and sufficient condition that the solution $\xi_{s,t}$ should define a Lévy process in the diffeomorphisms group G, i. e., to find the conditions that the map $\xi_{s,t}(.,\omega)$; $R^d \to R^d$ becomes a diffeomorphisms for any $s < t$ a. s. A sufficient condition is given in [2]. The condition is that the characteristic measure ν is finite and is supported by the set $\{ f; f + id \in G \}$. In this paper, we shall show the diffeomorphic property of $\xi_{s,t}$ for a more general class, approximating it by a sequence of Lévy processes in G satisfying the above condition. Hence the approximation or the convergence of Lévy processes in G_+ or G will play a crucial role.

This paper is divided into three sections. In Section 1, we shall discuss the weak convergence of Lévy processes with values in C. A criterion for the weak convergence will be given in terms of the corresponding characteristics (a, b, ν). See Theorem 1.2. The strong convergence is also discussed (Theorem 1.4). In Section 2, we shall discuss the weak and the strong convergence of Lévy processes with values in G_+ which are solutions of stochastic differential equation (0.1). Theorem 2.3 is our main theorem. In Section 3, a criterion that $\xi_{s,t}$ becomes a G-valued Lévy process is obtained. The criterion is stated in terms of the characteristics of C-valued Lévy process generating $\xi_{s,t}$. See $(B, I)_{m,r}$. Then we give a criterion for the weak convergence of G-valued Lévy processes. As an example, we consider a limit theorem of stochastic flows studied by Harris [11] and Matsumoto-Shigekawa [10].

§ 1. CONVERGENCE OF LEVY PROCESSES IN C^m

1.1. C^m-valued Lévy process and its characteristics.

Let $C^m = C^m(R^d; R^d)$ be the totality of C^m-maps from d-dimensional Euclidean space R^d into itself, where m is a nonnegative integer. When $m = 0$, the space C^0 is denoted by C for simplicity. It is a Frechet space by the metric

$$\rho_m(f, g) = \sum_{|k| \leq m} \rho(D^k f, D^k g), \tag{1.1}$$

where $D^k = \left(\frac{\partial}{\partial x_1}\right)^{k_1} \dots \left(\frac{\partial}{\partial x_d}\right)^{k_d}$, $|k| = k_1 + \dots + k_d$ and ρ is the compact uniform metric

$$\rho(f, g) = \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{\sup_{|x| \leq N} |f(x) - g(x)|}{1 + \sup_{|x| \leq N} |f(x) - g(x)|}.$$

Let $X_t = X_t(\omega)$, $t \in [0, T]$ be a C^m-valued stochastic process defined on $(\Omega, \mathscr{F}, P)$, right continuous with the left hand limits a. s. It is called a *Lévy process in* C^m or C^m*-valued Lévy process* if it is continuous in probability and has the independent increments: $X_{t_{i+1}} - X_{t_i}$, $i = 0, \ldots, n-1$ are independent for any $0 \leqq t_1 < \ldots < t_n \leqq T$. In particular, if X_t is continuous in t a. s., it is called a *Brownian motion in* C^m or C^m*-valued Brownian motion.*

In the following, we always assume that X_t is stationary, i. e., the law of $X_t - X_s$ depends on $t - s$, $X_0 \equiv 0$ and $E[\,|D^k X_t(x)|^2\,] < \infty$ for any t, x and $|k| \leqq m$. Now, we define the Poisson random measure $N_p((0, t] \times A)$ over $[0, T] \times C_m$ associated to X_t by

$$(1.2)\quad N_p((0, t] \times A) = \#\{\, s \in (0, t]\,;\ \Delta X_s \in A \,\}, \qquad \Delta X_s = X_s - X_{s-},$$

where A is a Borel subset of C^m excluding 0. The intensity measure ν' is defined by $\nu'((0, t] \times A) = E[N_p((0, t] \times A)]$. Since X_t is stationary, ν' is the product measure $dt \otimes \nu(df)$. The measure ν is called the *characteristic measure* of X_t.

Let $x_1, \ldots, x_N$ be N-points in R^d and consider the N-points motion $X_t(\boldsymbol{x}) = (X_t(x_1), \ldots, X_t(x_N))$. Its characteristic function $E\left[\exp i \sum_{k=1}^{N} (a_k, X_t(x_k))\right]$ is represented by Lévy-Khinchin's formula

$$\exp t\left[i \sum_{k=1}^{N} (\alpha_k, b(x_k)) - \frac{1}{2} \sum_{k,l=1}^{N} \alpha_k a(x_k, x_l)\alpha_l + \int_{C^m} \left\{ e^{i\sum_{k=1}^{N}(\alpha_k, f(x_k))} - 1 - i \sum_{k=1}^{N} (\alpha_k, f(x_k)) \right\} \nu(df) \right],$$

where

(1.3) $b(x)$ is an m-times continuously differentiable R^d-valued function,

(1.4) $a(x, y)$ is an m-times continuously differentiable $d \times d$-matrix valued function such that $a^{kl}(x, y) = a^{lk}(y, x)$ for any $k, l = 1, \ldots, d$ and x, $y \in R^d$, and $\sum_{i,j=1}^{N} \alpha_i a(x_i, x_j)\alpha_j \geqq 0$ for any $x_i, \alpha_j \in R^d$, $i, j = 1, \ldots, N$.

(1.5) ν is a σ-finite measure on C^m such that $\nu(\{0\}) = 0$ and

$$\int_{C^m} |D^k f(x)|^2 \nu(df) < \infty \quad \text{for any } x \in R^d \quad \text{and} \quad |k| \leqq m.$$

The triple (a, b, ν) is called the *characteristics* of the C^m-valued Lévy process X_t.

Let $D^m = D([0,T]; C^m)$ be the totality of maps $X; [0,T] \to C^m(R^d; R^d)$ such that $X_t \equiv X(t)$ is right continuous with the left hand limits with respect to the metric ρ_m and $X_T = \lim_{t \uparrow T} X_t$, where T is a fixed positive number. For X, Y of D^m, we define the Skorohod metric S_m by

$$(1.6) \qquad S_m(X, Y) = \inf_{\lambda \in H} \sup_{t \in [0,T]} \{ \rho_m(X_t, Y_{\lambda(t)}) + | \lambda(t) - t | \},$$

where H is the set of homeomorphisms on [0, T]. Then D^m is a complete separable space with respect to a metric equivalent to S_m. We denote by $\mathscr{B}_{D^m}$ the topological Borel field of D^m.

Now, if $X_t(\omega)$ is a C^m-valued Lévy process, it may be considered as an element of D^m for a. e. ω. Hence it induces a law $\tilde{P}$ on $(D^m, \mathscr{B}_{D^m})$ by the relation

$$(1.7) \qquad \tilde{P}(A) = P\{ \omega ; X(\omega) \in A \}.$$

Observe that the law of the Lévy process is uniquely determined by its characteristics.

1.2. Tightness and weak convergence of Lévy processes in C^m.

Let $\{ X_t^n, n = 1, 2, \ldots \}$ be a sequence of C^m-valued processes and let P_n, $n = 1, 2, \ldots$ be the associated laws on $(D^m, \mathscr{B}_{D^m})$. The sequence $\{ X_t^n \}$ is said to be tight if for any $\varepsilon > 0$ there is a compact subset K_ε of D^m satisfying $P_n(K_\varepsilon) > 1 - \varepsilon$ for any $n = 1, 2, \ldots$. The sequence $\{ X_t^n \}$ is said to converge weakly if the sequence $\left\{ \int F(x) P_n(dx) \right\}$ converges for any bounded continuous function F over D^m. We shall obtain criteria for the tightness and the weak convergence of $\{ X_t^n \}$ in terms of their characteristics.

We shall first consider the tightness. We introduce a hypothesis for the characteristics (a_n, b_n, ν_n), $n = 1, 2, \ldots$.

Let r be a positive number greater than $d \vee 2$.

$(A, I)_{m,r}$ *There is a positive constant* L *not depending on* n *such that*

$$(1.8) \qquad | \mathrm{Tr}\, D_x^k D_y^k a_n(x, y) | \leq L(1 + |x|)(1 + |y|), \qquad x, y \in R^d$$

$$(1.9) \qquad | \mathrm{Tr} \{ D_x^k D_y^k a_n(x, y) |_{y=x} - 2 D_x^k D_y^k a_n(x, y) + D_x^k D_y^k a_n(x, y) |_{x=y} \} | \leq L |x - y|^2, \qquad x, y \in R^d.$$

Here Tr a *is the trace of the matrix* $a = (a^{ij}))$, *i. e.,* $\operatorname{Tr} a = \sum_{i=1}^{d} a^{ii}$.

$$(1.10) \qquad |D^k b_n(x)| \leqq L(1 + |x|), \qquad x \in R^d$$

$$(1.11) \qquad |D^k b_n(x) - D^k b_n(y)| \leqq L|x - y|, \qquad x, y \in R^d$$

$$(1.12) \qquad \int_{C^m} |D^k f(x)|^{r'} \nu_n(df) \leqq L(1 + |x|)^{r'}, \qquad x \in R^d, \qquad r' \in [2, r]$$

$$(1.13) \qquad \int_{C^m} |D^k f(x) - D^k f(y)|^{r'} \nu_n(df) \leqq L|x - y|^{r'}, \quad x, y \in R^d, \quad r' \in [2, r]$$

for all $n = 1, 2, \ldots$ *and* $|k| \leqq m$.

PROPOSITION 1.1. — *Let* $\{X_t^n\}$ *be a sequence of* C^m*-valued Lévy processes such that the associated characteristics satisfy* $(A, I)_{m,r}$. *Then it is tight. Furthermore, there is a positive constant* K *such that*

$$(1.14) \qquad E[|D^k X_t^n(x)|^{p'}] \leqq Kt(1 + |x|)^{p'}, \qquad x \in R^d, \qquad t \in [0, T]$$

$$(1.15) \qquad E[|D^k X_t^n(x) - D^k X_t^n(y)|^{p'}] \leqq Kt|x - y|^{p'}, \qquad x, y \in R^d, \qquad t \in [0, T].$$

for all $n = 1, 2, \ldots, |k| \leqq m$ *and* $p' \in [2, r]$.

Conversely, let $\{X_t^n\}$ *be a sequence of* C^m*-valued Lévy processes satisfying (1.14) and (1.15) for all* $n = 1, 2, \ldots, |k| \leqq m$ *and* $p' \in [2, p]$, $p > d$. *Then it is tight and the associated characteristics satisfy* $(A, I)_{m,p}$.

Proof. — We shall first consider the case $m = 0$. The first assertion is shown in Theorem 3.1 in Kunita [7]. Conversely, suppose that (1.14) and (1.15) are satisfied. Then $\{X_t^n\}$ is tight as in shown in the proof of the above cited theorem. Further, the corresponding local characteristics (a_n, b_n, ν_n) satisfies $(A, I)_{m,p}$ by Theorem 1.2 in Fujiwara-Kunita [2].

For the proof of the general case, consider $(D^k X_t^n(x), |k| \leqq m), n = 1, 2, \ldots$ as $C(R^{Nd}, R^{Nd})$-valued processes where $N = \# \{k; |k| \leqq m\}$. These are C^0-valued Lévy processes with characteristics $(D_x^k D_y^k a_n(x, y), D^k b_n(x), \tilde{\nu}_n, |k| \leqq m), n = 1, 2, \ldots$ respectively, where $\tilde{\nu}_n$ are σ-finite measures on $C(R^{Nd}; R^{Nd})$ such that

$$(1.16) \qquad \tilde{\nu}_n(A) = \nu_n\{f; (D^k f, |k| \leqq m) \in A\}.$$

Clearly, the sequence $\{X_t^n\}$ is tight in ρ_m-topology if and only if $\{(D^k X_t^n, |k| \leqq m)\}$ is tight in ρ-topology. Now the latter is tight in ρ-topology in view of hypothesis $(A, I)_{m,r}$. Hence the assertion follows.

We shall next discuss the weak convergence. The condition required will be a suitable weak convergence of the sequence of the associated characteristics (a_n, b_n, ν_n). Given $\varepsilon > 0$, we denote by $p_\varepsilon(t)$, $t \geqq 0$ a decreasing function such that $0 \leqq p_\varepsilon \leqq 1$, $p_\varepsilon(t) = 0$ if $t < \varepsilon$ and $p_\varepsilon(t) = 1$ if $t > 2\varepsilon$. Given N-points $\boldsymbol{x} = (x_1, \ldots, x_N)$, we denote by $f(\boldsymbol{x})$ the vector function $(f(x_1), \ldots, f(x_N))$ and $|f(x)|$ the norm.

(A, II) *Three conditions a), b), c) are satisfied for any* N*-points* $\boldsymbol{x} = (x_1, \ldots, x_N)$.

a) *The following limits exist for any pair* x_i, x_j *of* $\boldsymbol{x}$:

$$(1.17)\quad a(x_i, x_j) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left\{ a_n(x_i, x_j) + \int_{C^m} (1 - p_\varepsilon(|f(\boldsymbol{x})|)) f(x_i)' f(x_j) \nu_n(df) \right\} (^1).$$

b) *The following limits exist for any* x_i *of* $\boldsymbol{x}$.

$$(1.18)\qquad b(x_i) = \lim_{n \to \infty} b_n(x_i).$$

c) *There is a σ-finite measure ν on* C^m *supported by* $\{ f; f(\boldsymbol{x}) = 0 \}^c$ *such that for any bounded continuous function* F,

$$(1.19)\quad \lim_{n \to \infty} \int_{C^m} F(f(\boldsymbol{x})) p_\varepsilon(|f(\boldsymbol{x})|) \nu_n(df) = \int_{C^m} F(f(\boldsymbol{x})) p_\varepsilon(f(\boldsymbol{x})) \nu(df)$$

is satisfied.

Remark. — Suppose (1.12) and (1.13) for ν_n, $n = 1, 2, \ldots$. Then it holds $\sup_n \nu_n(U_\varepsilon^c) < \infty$ for any $U_\varepsilon^c = \{ f; |f(\boldsymbol{x})| > \varepsilon \}$. Hence (1.19) is equivalent to the weak convergence of the finite dimensional distributions of ν_n, $n = 1, 2, \ldots$ restricted to U_ε^c.

THEOREM 1.2. — *Let* $\{ X_t^n \}$ *be a sequence of* C^m*-valued Lévy processes with characteristics* (a_n, b_n, ν_n), $n = 1, 2, \ldots$ *satisfying* $(A, I)_{m,r}$. *Then it converges weakly if and only if hypothesis* (A, II) *is satisfied. Furthermore, the weak limit is a* C^m*-valued Lévy process with characteristics* (a, b, ν) *given in* (A, II).

Proof. — Assume first that the characteristics satisfy (A, II). Since $\{ X_t^n \}$

(1) $f(x)$ is a row vector and $f(x)'$ is the transpose. Hence $f(x_i)'f(x_j)$ is a $d \times d$-matrix.

is tight by Proposition 1.1, it is enough to prove that the laws of N-point processes $X_t^n(\boldsymbol{x}) = (X_t^n(x_1), \ldots, X_t^n(x_N))$, $n = 1, 2, \ldots$ converge weakly to an Nd-dimensional Lévy process. Now for each n $X_t^n(\boldsymbol{x})$ is an Nd-dimensional Lévy process with the characteristics

$$(1.20) \qquad ((a_n(x_k, x_l))_{k,l=1,\ldots,N}, (b_n(x_k))_{k=1,\ldots,N}, \tilde{\nu}_n^{\boldsymbol{x}}),$$

where

$$(1.21) \qquad \tilde{\nu}_n^{\boldsymbol{x}}(E) = \nu_n\{ f ; f(\boldsymbol{x}) \in E \}.$$

The above sequence converges to

$$(1.22) \qquad ((a(x_k, x_l))_{k,l=1,\ldots,N}, (b(x_k))_{k=1,\ldots,N}, \tilde{\nu}^{\boldsymbol{x}})$$

in the sense of Theorem (1.21) of Jacod [*4*], because of (A, II). Here the measure $\tilde{\nu}^{\boldsymbol{x}}$ is defined from ν similarly as (1.21). Therefore the sequence $\{ X_t^n(\boldsymbol{x}) \}$ converges weakly to a Lévy process with the characteristics (1.22) by the same theorem of Jacod.

Conversely if the laws of X_t^n, $n = 1, 2, \ldots$ converge weakly in (W^m, ρ_m), then for any N and $x_1, \ldots, x_N$ the N-point processes $X_t^n(\boldsymbol{x})$, $n = 1, 2, \ldots$ converge weakly. Hence condition (A, II) is satisfied by the same theorem of Jacod. The proof is completed.

In case that X_t^n, $n = 1, 2, \ldots$ are Brownian motions, the criteria for the weak convergence are simpler.

COROLLARY. — *Let* $\{ X_t^n \}$ *be a sequence of* C^m*-valued Brownian motions with characteristics* (a_n, b_n) *satisfying (1.8)-(1.11). Then it converges weakly if and only if* (A, II) *b) and the following are satisfied.*

(A, II) a') $\{ a_n(x, y) \}$ *converges as* $n \to \infty$ *for any* x, $y \in R^d$.

1.3. Strong convergence of Lévy processes on C^m.

Let $\{ X_t^n \}$ be a sequence of C^m-valued Lévy processes defined on the same probability space $(\Omega, \mathscr{F}, P)$. Let S_m be the Skorohod metric introduced in Section 1.1. We call that $\{ X_t^n \}$ is strongly convergent if $S_m(X^n, X^{n'})$ converges to 0 in probability as $n, n' \to \infty$, i. e.,

$$P(S_m(X^n, X^{n'}) > \varepsilon) \xrightarrow[n,n' \to \infty]{} 0$$

holds for any $\varepsilon > 0$.

We show that the strong convergence of C^m-valued Lévy processes X_t^n

can be reduced to the L^2-convergence of R^d-valued Lévy processes $X_t^n(x)$, $n = 1, 2, \ldots$ for each $x \in R^d$.

PROPOSITION 1.3. — *Let* $\{X_t^n\}$ *be a sequence of* C^m*-valued Lévy processes defined on the same probability space satisfying hypothesis* $(A, I)_{m,r}$ *for some* $r > d$. *Suppose that*

$$E[\,|X_t^n(x) - X_t^{n'}(x)|^2] \to 0, \qquad n, n' \to \infty$$

holds for any $x \in R^d$. *Then the sequence* $\{X_t^n\}$ *converges strongly.*

Proof. — Consider the L^2-limit of $X_t^n(x)$. Obviously it has a modification X_t of C^m-valued process since the associated characteristics (a, b, v) satisfies (1.8)-(1.13) replacing (a_n, b_n, v_n) by (a, b, v). (See [2]). We shall consider the law of the pair (X_t^n, X_t), which is defined as a probability measure $\tilde{P}_n$ on $D^m \times D^m$. Obviously, the sequence $\{\tilde{P}_n\}$ is tight. Furthermore, the N-points processes $(X_t^n(\boldsymbol{x}), X_t(\boldsymbol{x}))$, $n = 1, 2, \ldots$ converge to the N-points process $(X_t(\boldsymbol{x}), X_t(\boldsymbol{x}))$ in L^2-sense by the assumption of the proposition. Then, the sequence of laws $\tilde{P}_n$, $n = 1, 2, \ldots$ converges weakly to $\tilde{P}$, which is the law of (X_t, X_t). This means that $\tilde{P}$ is supported by the diagonal set $\Delta = \{(X, X); X \in D^m\}$ of $D^m \times D^m$. Now, since $S_m(Y, X)$, $(Y, X) \in D^m \times D^m$ is a bounded continuous function, the weak convergence of the measures $\tilde{P}_n$, $n = 1, 2, \ldots$ implies

$$\lim_{n\to\infty} E[S_m(X^n, X)] = \lim_{n\to\infty} \tilde{E}_n[S_m(Y, X)] = \tilde{E}[S_m(Y, X)] = 0.$$

This proves the strong convergence of X^n. The proof is complete.

We next characterize the strong convergence by means of the characteristics of the pairs $(X_t^n, X_t^{n'})$. It is well known that each X_t^n is decomposed as

$$(1.23)\qquad X_t^n(x) = X_t^{n,c}(x) + \int_0^t \int_{C^m} f(x)\tilde{N}_p(dsdf),$$

where $X_t^c(x)$ is a C^m-valued Brownian motion with characteristics $(a, b, 0)$ and

$$\int_0^t \int_{C^m} f(x)\tilde{N}_p^n(dsdf) = \int_0^t \int_{C^m} f(x)\{N_p^n(dsdf) - dsv^n(df)\}.$$

It is a discontinuous Lévy process with mean 0 and variance $t\int_{C^m} |f(x)|^2 v_n(df)$.

Both are independent each other. The decomposition (1.23) is often called the Lévy-Itô decomposition of X_t. Now set

$$(1.24)\qquad a_{n,n'}(x, x) = E[(X_1^{n,c}(x) - b_1^n(x))(X_1^{n',c}(x) - b_1^{n'}(x))]$$

$$(1.25)\qquad v_{n,n'}(df) = E[N^{(n,n')}((0, 1], df)]$$

where

$$N^{(n,n')}((0,t]\times A) = \#\{ s\in(0,t];\ \Delta X_s^n = \Delta X_s^{n'} \in A \}$$

where A is a Borel set of C^m excluding 0. Then we have $a_{n,n}(x,x) = a_n(x,x)$ and $\nu_{n,n}(df) = \nu_n(df)$.

THEOREM 1.4. — *Let $\{ X_t^n \}$ be a sequence of C^m-valued Lévy processes defined on the same probability space such that the associated characteristics satisfy $(A, I)_{m,r}$ for some $r > d$. Then $\{ X_t^n \}$ converges strongly if and only if the following condition is satisfied.*

(A, III). *For each $x\in R^d$, it holds*

$$\mathrm{Tr}\{ a_n(x,x) - 2a_{n,n'}(x,x) + a_{n'}(x,x) \} \to 0,$$

$$(1.26)\quad b_n(x) - b_{n'}(x) \to 0,$$

$$\int_{C^m} |f(x)|^2 \nu_n(df) - 2\int_{C^m} |f(x)|^2\nu_{n,n'}(df) + \int_{C^m} |f(x)|^2\nu_{n'}(df) \to 0$$

as $n, n' \to \infty$.

Proof. — The proof is immediate from the relation

$$E[\,|X_t^n(x) - X_t^{n'}(x)|^2] = t\Big\{ \mathrm{Tr}\{ a_n(x,x) - 2a_{n,n'}(x,x) + a_{n'}(x,x) \} + |b_n(x) - b_{n'}(x)|^2 + \int_{C^m} |f(x)|^2(\nu_n(df) - 2\nu_{n,n'}(df) + \nu_{n'}(df)).$$

§ 2. CONVERGENCE OF LEVY PROCESSES IN G_+^m

2.1. Stochastic differential equation of jump type and Lévy processes in G_+^m.

Let X_t be a C^m-valued Lévy process with the characteristics (a, b, ν) satisfying (1.8)-(1.13) for some $r > (m+1)^2 d$, replacing (a_n, b_n, ν_n) by (a, b, ν). For $s < t$, we denote by $\mathscr{F}_{s,t}$ the least sub σ-field of $\mathscr{F}$ for which $X_u - X_v;\ s \leqq u, v \leqq t$ are measurable. Then for each s and x, $Y_t(x) - Y_s(x)$, $t\in[s,T]$ where $Y_t(x) = X_t(x) - tb(x)$ is an $\mathscr{F}_{s,t}$-adapted L^2-martingale. Hence by Doob-Meyer's decomposition, there is a unique continuous process of bounded variation $\langle Y^i(x), Y^j(y)\rangle_t$ (equals 0 if $t=0$) such that

$$(2.1)\quad (Y_t^i(x) - Y_s^i(x))(Y_t^j(y) - Y_s^j(y)) - \{ \langle Y^i(x), Y^j(y)\rangle_t - \langle Y^i(x), Y^j(y)\rangle_s \}$$

is an $\mathscr{F}_{s,t}$-martingale. Since $Y_t(x)$ is a Lévy process, we have

$$\langle Y_t^i(x), Y_t^j(y)\rangle = tA^{ij}(x,y),$$

where

$$A^{ij}(x, y) = a^{ij}(x, y) + \int_{C^m} f^i(x) f^j(y) \nu(df). \tag{2.2}$$

Let $s > 0$ be a fixed number and let $\phi_t(\omega)$ be an $\mathscr{F}_{s,t}$-adapted R^d-valued process, right continuous with the left hand limits. Ito integral of ϕ_t by dY_t is defined by

$$\int_s^t dY_r^i(\phi_{r^-}) = \lim_{|\delta|\to 0} \sum_{k=0}^{n-1} \{ Y^i_{t_{k+1}\wedge t}(\phi_{t_k \wedge t}) - Y^i_{t_k \wedge t}(\phi_{t_k \wedge t}) \}, \tag{2.3}$$

where δ are partitions $\{ s = t_0 < \ldots < t_n = T \}$. The limit exists in probability and is an $\mathscr{F}_{s,t}$-adapted local martingale. Let $\psi_t(\omega)$ be an $\mathscr{F}_{s,t}$-adapted process having the same property as ϕ_t. Then it holds

$$\left\langle \int_s^t dY_r^i(\phi_{r^-}), \int_s^t dY_r^j(\psi_{r^-}) \right\rangle = \int_s^t A^{ij}(\phi_{r^-}, \psi_{r^-}) dr. \tag{2.4}$$

See Le Jan [8] and Le Jan-Watanabe [9]. Now the stochastic integral by C^m-valued Lévy process X_t is defined by

$$\int_s^t dX_r(\phi_{r^-}) = \int_s^t dY_r(\phi_{r^-}) + \int_s^t b(\phi_{r^-}). \tag{2.5}$$

We shall consider the stochastic differential equation defined by the C^m-valued Levy process X_t:

$$d\xi_t = dX_t(\xi_{t^-}). \tag{2.6}$$

The definition of the solution is as follows. Given a time s and state x, an R^d-valued $\mathscr{F}_{s,t}$-adapted process ξ_t, right continuous with the left limits, is called a solution of equation (2.6) if it satisfies for any $t > s$

$$\xi_t = x + \int_s^t dX_r(\xi_{r^-}). \tag{2.7}$$

The equation has a unique solution, which we denote by $\xi_{s,t}(x)$ if $t \geqq s$. For the convenience we set $\xi_{s,t}(x) = \xi_{s,s}(x) = x$ if $t < s$.

We summarize the properties of the process $\xi_{s,t}$. We denote by $W^m = D_l([0, T]; D^m)$ the totality of maps ξ; $[0, T] \to D^m = D([0, T]; C^m(R^d; R^d))$ such that ξ_s is *left continuous with right limits* with respect to the Skorohod topology S_m, and $\xi_{s,t}(x)$ is the projection of $\xi_s \in D^m$ to the point $(t, x) \in [0, T] \times R^d$. The following is due to Fujiwara-Kunita [2] and Kunita [7].

PROPOSITION 2.1. — *i) There is a modification of the solution such that for almost all* ω, $\xi_{s,t}(x, \omega)$ *is an element of* W^m *and satisfies* $\xi_{s,u}(x, \omega) = \xi_{t,u} \circ \xi_{s,t}(x, \omega)$ *for all* $s < t < u$ *and* x *and* $\xi_{s,t}(x, \omega) = x$ *if* $s > t$.

ii) The law of $\xi_{s,t}$ *depends on* $t - s$. *Furthermore,* $\xi_{t_i,t_{i+1}}$, $i = 0, \ldots, n-1$ *are independent for any* $0 \leqq t_1 < \ldots < t_n \leqq T$.

iii) There is a positive constant M *depending only on constants* L *and* r *appearing in (1.8)-(1.13) such that*

$$E[\,|D^k\xi_{s,t}(x) - D^k x|^{p'}] \leqq M(t-s)(1+|x|)^{p'}, \tag{2.8}$$

$$E[\,|D^k\xi_{s,t}(x) - D^k x - D^k\xi_{s,t}(y) + D^k y|^{p'}] \leqq M(t-s)\,|x-y|^{p'}, \tag{2.9}$$

hold for all $x, y \in R^d$ *and* $p' \in [2, r/(m+1)^2]$.

iv) The following limits exist

$$A^{ij}(x, y) = \lim_{h \to 0^+} \frac{1}{h} E[(\xi^i_{s,s+h}(x) - x^i)(\xi^j_{s,s+h}(y) - y^j)], \tag{2.10}$$

$$b^i(x) = \lim_{h \to 0} \frac{1}{h} E[\xi^i_{s,s+h}(x) - x^i]. \tag{2.11}$$

These are C^m*-functions of* x, y *and* x, *respectively.*

Conversely suppose that $\xi_{s,t}(x)$ *is a random field satisfying i)-iv) for any* $p' \in [2, p]$ *where* $p > (m+1)^2 d$. *Then there is a unique* C^m*-valued Lévy process* X_t *satisfying (2.7).*

Remark. — The function b of (2.11) coincides with that in the triple (a, b, ν). The function $A^{ij}(x, y)$ coincides with (2.2).

Now the space C^m may be considered as a topological semigroup if we define the product of $f, g \in C^m$ by the composition $f \circ g$ of the maps. We denote the semigroup by G^m_+. Then the solution $\xi_{s,t}$ defines a Lévy process in the semigroup G^m_+ because of properties *i*), *ii*) of the proposition. The associated C^m-valued Lévy process X_t is called the *infinitesimal generator* of the Levy process $\xi_{s,t}$.

2.2. Tightness and weak convergence of Lévy processes in G^+_m.

Let $\xi_{s,t}$ be a G^m_+-valued Lévy process. We shall define its law on the space $W^m = D_l([0, T]; D^m)$. Let S_m be the Skorohod metric on W^m such that

$$S_m(\xi, \eta) = \inf_{\lambda \in H} \sup_{s \in [0,T]} \{ S_m(\xi_s, \eta_{\lambda(s)}) + |\lambda(s) - s| \}, \tag{2.12}$$

where H is the totality of homeomorphisms of $[0, T]$ and S_m is the Skorohod topology of D^m introduced in Section 1.1. Let $\mathscr{B}_{W^m}$ be the topological Borel field of W^m. The law of G_+^m-valued Lévy process is defined by

$$(2.13) \qquad \tilde{P}(A) = P\{\omega; \xi_{.,.}(., \omega) \in A\}, \qquad A \in \mathscr{B}_{W^m}.$$

Now let $\{\xi_{s,t}^n\}$ be a sequence of G_+^m-valued Lévy processes and let $\tilde{P}_n$, $n=1, 2, \ldots$ be the associated laws over $(W^m, \mathscr{B}_{W^m})$. The tightness and the weak convergence of $\{\xi_{s,t}^n\}$ are defined by those of the associated laws $\tilde{P}_n$, $n=1, 2, \ldots$.

PROPOSITION 2.2. — *Let $\{\xi_{s,t}^n\}$ be a sequence of G_+^m-valued Lévy processes generated by C^m-valued Lévy processes X_t^n, $n = 1, 2, \ldots$ respectively. Suppose that the characteristics of $X_t, n = 1, 2, \ldots$ satisfy hypothesis $(A, I)_{m,r}$ for some $r > (m+1)^2 d$. Then, the sequence $\{\xi_{s,t}^n\}$ is tight. Furthermore, there is a positive constant M not depending on n such that*

$$(2.14) \quad E\big[\sup_{s \leq r \leq t} |D^k \xi_{s,r}^n(x) - D^k \xi_{s,r}^n(y)|^{p'}\big] \leq M(t-s)|x-y|^{p'}, \qquad x, y \in R^d,$$

$$(2.15) \quad E\big[\sup_{s \leq r \leq t} |D^k(\xi_{s,r}^n(x) - x)|^{p'}\big] \leq M(t-s)(1+|x|)^{p'}, \qquad x \in R^d,$$

hold for any k with $|k| \leq m$ and $p' \in [2, r/(m+1)^2]$.

Conversely, suppose that $\{\xi_{s,t}^n\}$ satisfies (2.14) and (2.15) for k with $|k| \leq m$ and $p' \in [2, p]$ where $p > (m+1)^2 d$. Then the associated sequence $\{X_t^n\}$ of generators is tight. Furthermore, it satisfies (1.14) and (1.15) for any k, $|k| \leq m$ and $p' \in [2, p/(m+1)^2]$.

Proof. — We only consider the case $m = 0$. The first assertion is shown in Theorem 4.1 in [7]. Conversely if (2.14) and (2.15) are satisfied for $\{\xi_{s,t}^n\}$ then $\{X_t^n\}$ satisfies (1.14) and (1.15) where the constant K depends only on M and p, by Theorem 3.2 in [2]. Therefore $\{X_t^n\}$ is tight. The proof is completed.

We shall next discuss the weak convergence of G_+^m-valued Lévy processes.

THEOREM 2.3. — *Let $\{\xi_{s,t}^n\}$ be a sequence of G_+^m-valued Lévy processes generated by C^m-valued Lévy processes $X_t^n, n = 1, 2, \ldots$ respectively. Suppose that the associated characteristics satisfy $(A, I)_{m,r}$ for some $r > (m+1)^2 d$. Then the sequence $\{\xi_{s,t}^n\}$ converges weakly if and only if $\{X_t^n\}$ converges weakly. Furthermore, the weak limit of $\{\xi_{s,t}^n\}$ is generated by the weak limit of $\{X_t^n\}$.*

COROLLARY. — *Let $\{X_t^n\}$ be a sequence of C^m-valued Brownian motions such that their characteristics satisfy (1.8)-(1.11). Then G^m-valued Brown-*

ian motions $\{\xi^n_{s,t}\}$ *generated by* $\{X^n_t\}$ *converge weakly if and only if the characteristics of* $\{X^n_t\}$ *satisfy* (A, II) *a'*), *b*).

The proof will be given at Sections 2.3 and 2.4.

2.3. **Martingale problem.**

For the proof of the theorem, we shall consider the law of the pair process $(\xi^n_{s,t}, X^n_t)$ rather than the law of $\xi^n_{s,t}$. Let $D^m = D([0,T]; C^m)$ and $W^m = D_l([0,T]; D^m)$ as before and let $\overline{W}^m = W^m \times D^m$. The law of $(\xi^n_{s,t}, X^n_t)$ is defined as a probability measure on $\overline{W}^m$, which we denote by $\overline{P}_n$. Then the sequence of laws $\overline{P}_n$, $n = 1, 2, \ldots$ is tight by Proposition 2.1 if hypothesis $(A, I)_{m,r}$ for some $r > (m+1)^2 d$ is satisfied. Let $\overline{P}_\infty$ be any limiting measure.

We shall characterize $\overline{P}_\infty$ by means of a suitable martingale problem. We shall assume that the characteristics of X^n_t, $n = 1, 2, \ldots$ satisfy (A, II). Let (a, b, v) be the triple introduced in (A, II). Given N-points $y_0 = (y^0_1, \ldots, y^0_N)$, we define an integro-differential operator over $R^{Nd} \times R^{Md}$ with the parameter y_0 by

$$L_{y^0} = L^c_{y^0} + L^d_{y^0}, \tag{2.16}$$

where

$$\begin{aligned} L^c_{y^0}F(x, y) = {} & \frac{1}{2}\sum_{k,l,i,j} a^{kl}(x_i, x_j)\frac{\partial^2}{\partial x^k_i \partial x^l_j}F(x, y) + \sum_{k,i} b^k(x_i)\frac{\partial F}{\partial x^k_i}(x, y) \\ & + \frac{1}{2}\sum_{k,l,i,j} a^{kl}(y^0_i, y^0_j)\frac{\partial^2}{\partial y^k_i \partial y^l_j}F(x, y) + \sum_{k,i} b^k(y^0_i)\frac{\partial F}{\partial y^k_i}(x, y) \\ & + \sum_{k,l,i,j} a^{kl}(x_i, y^0_j)\frac{\partial^2}{\partial x^k_i \partial y^l_j}F(x, y) \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} L^d_{y^0}F(x, y) = \int_{C^m} \Big\{ & F(x + f(x), y + f(y^0)) - F(x, y) \\ & - \sum_{i,k}\frac{\partial}{\partial x^k_i}F(x, y)f^k(x_i) - \sum_{j,l}\frac{\partial}{\partial y^l_j}F(x, y)f^l(y^0_j)\Big\}\, v(df). \end{aligned} \tag{2.18}$$

Here $x = (x_1, \ldots, x_M)$ and $y = (y_1, \ldots, y_N)$ are M and N-points of R^d, respectively and x^k_i, y^l_j, $k, l = 1, \ldots, d$ are components of $x_i = (x^1_i, \ldots, x^d_i)$ and $y_j = (y^1_j, \ldots, y^d_j)$, respectively, and $f(x) = (f^1(x), \ldots, f^d(x))$.

We want to prove the following.

PROPOSITION 2.4. — *Suppose that the characteristics of* X_t^n, $n = 1, 2, \ldots$ *satisfy* $(A, I)_{m,r}$ *for some* $r > (m+1)^2 d$ *and* (A, II). *Let*

$$(\xi_{s,t}(\boldsymbol{x}^0), X_{s,t}(\boldsymbol{y}^0)) = (\xi_{s,t}(x_1^0), \ldots, \xi_{s,t}(x_M^0), X_{s,t}(y_1^0), \ldots, X_{s,t}(y_N^0))$$

be N+M*-points motion where* $X_{s,t}(y_0) \equiv X_t(y_0) - X_s(y_0)$. *Then, for any* C^2*-function* F *over* $R^{Md} \times R^{Nd}$ *with bounded second derivatives, the process*

$$F(\xi_{s,t}(\boldsymbol{x}^0), X_{s,t}(\boldsymbol{y}^0)) - \int_s^t L_{\boldsymbol{y}^0}F(\xi_{s,r}(\boldsymbol{x}^0), X_{s,r}(\boldsymbol{y}^0))dr \tag{2.19}$$

is a martingale with respect to $\overline{P}_\gamma$.

Proof. — Taking a subsequence if necessary, we shall assume that $\overline{P}_n$, $n = 1, 2, \ldots$ converges to $\overline{P}_\infty$ weakly. Since the law of $(\xi_{s,s+t}(\boldsymbol{x}^0), X_{s,s+t}(\boldsymbol{y}^0))$ coincides with that of $(\xi_{0,t}(\boldsymbol{x}^0), X_t(\boldsymbol{y}^0))$, it is enough to prove the proposition in case $s = 0$. We write $\xi_{0,t}$ as ξ_t. We denote by $L_{\boldsymbol{y}^0}^n$ the integro-differential operator of (2.16) replacing (a, b, ν) by (a_n, b_n, ν_n). Then by Itô's formula (cf. Ikeda-Watanabe [*3*], p. 66), we find that

$$F(\xi_t(\boldsymbol{x}^0), X_t(\boldsymbol{y}^0)) - \int_0^t L_{\boldsymbol{y}^0}^n F(\xi_r(\boldsymbol{x}^0), X_r(\boldsymbol{y}^0))dr$$

is a martingale with respect to the measure $\overline{P}_n$. Now let $T_{\overline{P}_\infty}$ be those t in $[0, T]$ for which $\overline{P}_\infty(X_t \neq X_{t^-}$ or $\xi_t \neq \xi_{t^-}) = 0$. Then $[0, T] - T_{\overline{P}_\infty}$ consists at most of countable set. If $t \in T_{\overline{P}_\infty}$, then

$$\lim_{n \to \infty} \overline{E}_n[F(\xi_t(\boldsymbol{x}^0), X_t(\boldsymbol{y}^0))\Phi] = \overline{E}_\infty[F(\xi_t(\boldsymbol{x}^0), X_t(\boldsymbol{y}^0))\Phi] \tag{2.20}$$

is satisfied for any bounded continuous function Φ over $\overline{W}_m$ adapted to $\sigma(\xi_u, X_u; u \leq s)$. We shall prove

$$\lim_{n \to \infty} \overline{E}_n\left[\left\{\int_s^t L_{\boldsymbol{y}^0}^n F(\xi_r(\boldsymbol{x}^0), X_r(\boldsymbol{y}^0))dr\right\}\Phi\right] = \overline{E}_\infty\left[\left\{\int_s^t L_{\boldsymbol{y}^0} F(\xi_r(\boldsymbol{x}^0), X_r(\boldsymbol{y}^0))dr\right\}\Phi\right]. \tag{2.21}$$

Then this and (2.20) will imply the assertion of the proposition.

We shall check first that $L_{\boldsymbol{y}^0}^n F$ converges to $L_{\boldsymbol{y}^0}F$ uniformly on compact sets. Observe that the integrand of the right hand side of (2.18) is a function of $f(x)$, $f(y^0)$ with parameters x and y, which we denote by $G_{x,y}(f(x), f(y^0))$. Then $L^n_{y^0}F(x, y)$ is written as

$$(2.22)\quad L^n_{y^0}F(x, y) = L^{n,\varepsilon}_{y^0}F(x, y) + \int G_{x,y}(f(x), f(y^0))\,(1 - p_\varepsilon(|f(x)| + |f(y^0)|))\nu_n(df) + \int G_{x,y}(f(x), f(y^0)) p_\varepsilon(|f(x)| + |f(y^0)|)\nu_n(df).$$

It holds by (A, II) c).

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int G_{x,y}(f(x), f(y^0)) p_\varepsilon(|f(x)| + |f(y^0)|)\nu_n(df) = \int G_{x,y}(f(x), f(y^0))\nu(df).$$

On the other hand, the sum of the first and the second term of (2.22) is written by

$$(2.23)\quad \sum_{i,j,k,l} \tilde{a}^{kl}_{n,\varepsilon}(x_i, x_j)\frac{\partial^2 F}{\partial x_i^k \partial x_j^l} + \sum_{i,j,k,l} \tilde{a}^{kl}_{n,\varepsilon}(y_i^0, y_j^0)\frac{\partial^2 F}{\partial y_i^k \partial y_j^l} + \sum_{i,j,k,l} \tilde{a}^{kl}_{n,\varepsilon}(x_i, y_j)\frac{\partial^2 F}{\partial x_i^k \partial y_j^l} + \sum_{i,k} b^k_n(x_i)\frac{\partial F}{\partial x_i^k} + \sum_{i,k} b^k_n(y_i^0)\frac{\partial F}{\partial y_i^k} + R_{n,\varepsilon}$$

where

$$\tilde{a}^{kl}_{n,\varepsilon}(x_i, x_j) = a^{kl}_n(x_i, x_j) + \int (1 - p_\varepsilon(|f(x)| + |f(y_0)|)) f^k(x_i) f^l(x_j)\nu_n(df)$$

and the remainder term $R_{n,\varepsilon}$ is the sum of the following terms

$$\int_{C^m}\left\{\frac{\partial^2 F}{\partial x_i^k \partial x_j^l}(x + \theta f(x), y + \theta f(y^0)) - \frac{\partial^2 F}{\partial x_i^k \partial x_j^l}(x, y)\right\} \times (1 - p_\varepsilon(|f(x)| + |f(y^0)|)) f^k(x_i) f^l(x_j)\nu_n(df).$$

The sum of the first fifth terms of (2.23) converges to $L_y F$ by (A, II) a), while $\overline{\lim}_{n\to\infty} |R_{n,\varepsilon}|$ is bounded by the sum of the followings

$$\sup_{|\xi| + |\eta| < 2\varepsilon}\left|\frac{\partial^2 F}{\partial x_i^k \partial x_j^l}(x + \theta\xi, y + \theta\eta) - \frac{\partial^2 F}{\partial x_i^k \partial x_j^l}(x, y)\right| \times \overline{\lim}_{n\to\infty}\left|\int_{C^m}(1 - p_\varepsilon(|f(x)| + |f(y^0)|)) f^k(x_i) f^l(x_j)\nu_n(df)\right|,$$

which converges to 0 as $\varepsilon \to 0$. Consequently, we have $\lim_{\varepsilon\to 0} \overline{\lim}_{n\to\infty} | R_{n,\varepsilon} | = 0$. We have thus proved that $L^n_{y^0}F(x, y)$, $n = 1, 2, \ldots$ converges to $L_{y^0}F(x, y)$ for each x and y.

Now, by hypothesis $(A, I)_{m,r}$, the following families of functions

$$\left\{ \int_{C^m} G_{x,y}(f(x), f(y^0))p_\varepsilon(| f(x) | + | f(y^0) |)\nu_n(df),\ n=1, 2, \ldots, \varepsilon > 0 \right\},$$

$$\left\{ L^{n,\varepsilon}_{y^0}F(x, y) + \int_{C^m} G_{x,y}(f(x), f(y^0))(1 - p_\varepsilon(| f(x) | + | f(y^0) |))\nu_n(df),\right.$$
$$\left. n = 1, 2, \ldots, \varepsilon > 0 \right\}$$

are uniformly bounded and equicontinuous on compact sets. Therefore, the sequence $L^n_{y^0}F(x, y)$, $n = 1, 2, \ldots$ converges to $L_{y^0}F(x, y)$ uniformly on compact sets by Ascoli-Arzela's theorem.

We now proceed to the proof of (2.21). By $(A, I)_{m,r}$, there is a constant C such that $| L^n_{y^0}F(x, y) | \leqq C(1 + | x | + | y |)$ holds for all n. Then we have for any $K > 0$

$$\left| \overline{E}_n\left[\left\{ \int_s^t L^n_{y^0}F(\xi_r(x^0), X_r(y^0))dr \right\} \Phi \right] - \overline{E}_\infty\left[\left\{ \int_s^t L_{y^0}F(\xi_r(x^0), X_r(y^0))dr \right\} \Phi \right] \right|$$

$$\leqq \sup_{|\xi| + |\eta| \leqq K} | L^n_{y^0}F(\xi, \eta) - L_{y^0}F(\xi, \eta) | \, \| \Phi \| (t - s)$$

$$+ C \| \Phi \| \int_s^t \overline{E}_n[\, | \xi_r(x^0) |^2 + | X_r(y^0) |^2 ;\, | \xi_r(x^0) |^2 + | X_r(y^0) |^2 \geqq K]dr$$

$$+ \left| \overline{E}_n\left[\left\{ \int_s^t L_{y^0}F(\xi_r(x^0), X_r(y^0))dr \right\} \Phi \right] \right.$$

$$\left. - \overline{E}_\infty\left[\left\{ \int_s^t L_{y^0}F(\xi_r(x^0), X_r(y^0))dr \right\} \Phi \right] \right| .$$

For any $\varepsilon > 0$, we can choose K such that the second member of the right hand side is less than ε since we have the moment inequalities (1.14) and (2.15). Let n tend to infinity in the above. Then, the first and the third member of the above converge to 0. Therefore (2.21) is proved.

2.4. **Proof of Theorem 2.3.**

We shall first assume that the sequence $\{ X^n_t \}$ converges weakly to a C^m-valued Lévy process with characteristics (a, b, ν). Let $\overline{P}_n$ be the law of $(\xi^n_{s,t}, X^n_t)$ and let $\overline{P}_\infty$ be any limiting law of the sequence $\{ \overline{P}_n \}$. Then clearly

$(X_t, \overline{P}_\infty)$ is a C^m-valued Lévy process with characteristics (a, b, ν). We want to prove that $(\xi_{s,t}, \overline{P}_\infty)$ is generated by X_t, i. e., $d\xi_t = dX_t(\xi_{t}-)$ is satisfied. Once this is established, then the law of $(\xi_{s,t}, X_t)$ is unique, and the uniqueness of the limiting law $\overline{P}_\infty$ or the weak convergence of $\{ \xi^n_{s,t} \}$ is established.

We shall apply Proposition 2.4 to $F(x) = x^i$ and $x^i x^j$. Then we find that the both of the followings are $\overline{P}_\gamma$-martingales for any x, x_1, x_2.

$$M^i_{s,t}(x) \equiv \xi^i_{s,t}(x) - x^i - \int_s^t b^i(\xi_{s,r}(x))dr\,, \tag{2.24}$$

$$\begin{aligned} N_{s,t} \equiv\ & \xi^i_{s,t}(x_1)\xi^j_{s,t}(x_2) - \int_s^t b^i(\xi_{s,r-}(x_1))\xi^j_{s,r-}(x_2)dr \\ & - \int_s^t b^j(\xi_{s,r-}(x_2))\xi^i_{s,r-}(x_1)dr - \int_s^t a^{ij}(\xi_{s,r-}(x_1), \xi_{s,r-}(x_2))dr \\ & - \int_s^t \left\{ \int_{C^m} f^i(\xi_{s,r-}(x_1)) f^j(\xi_{s,r-}(x_2))\nu(df) \right\} dr. \end{aligned} \tag{2.25}$$

By Itô's formula, $N_{s,t}$ is written as

$$\begin{aligned} N_{s,t} =\ & \int_s^t \xi^j_{s,r-}(x_2)dM^i_{s,r}(x_1) + \int_s^t \xi^i_{s,r-}(x_1)dM^j_{s,r}(x_2) \\ & + [M^i_{s,t}(x_1), M^j_{s,t}(x_2)] - \int_s^t A^{ij}(\xi_{s,r-}(x_1), \xi_{s,r-}(x_2))dr. \end{aligned} \tag{2.26}$$

Here, the third member of the right hand side is the joint quadratic variation defined by

$$\lim_{|\delta|\to 0} \sum_{i=0}^{n-1} (M^i_{s,t_{i+1}}(x_1) - M^i_{s,t_i}(x_1))(M^j_{s,t_{i+1}}(x_2) - M^j_{s,t_i}(x_2))$$

where $\delta = \{ s = t_0 < t_1 < \ldots < t_n = t \}$. The function $A^{ij}(x, y)$ is defined by (2.2). Note that $N_{s,t}$ and the first, the second terms of the right hand side of (2.26) and

$$[M^i_{s,t}(x_1), M^j_{s,t}(x_2)] - \langle M^i_{s,t}(x_1), M^j_{s,t}(x_2) \rangle$$

are all martingales. Then we find that

$$\langle M^i_{s,t}(x_1), M^j_{s,t}(x_2) \rangle - \int_s^t A^{ij}(\xi_{s,r}(x_1), \xi_{s,r}(x_2))dr$$

is a martingale. Since it is a continuous process of bounded variation, it must be identically 0. Hence we have

$$(2.27)\qquad \langle M^i_{s,t}(x_1), M^j_{s,t}(x_2)\rangle = \int_s^t A^{ij}(\xi_{s,r^-}(x_1), \xi_{s,r^-}(x_2))dr .$$

Next apply Proposition 2.4 to $F(y) = y^i$ and y^iy^j. Then similarly as above we find that $Y^i_{s,t}(x) \equiv X^j_{s,t}(x) - (t-s)b(x)$ is a martingale and

$$(2.28)\qquad \langle Y^i_{s,t}(x_1), Y^j_{s,t}(x_2)\rangle = (t-s)A_{ij}(x_1, x_2).$$

Apply again Proposition 2.4 to $F(x, y) = x^iy^j$. The we find

$$(2.29)\qquad \langle M^i_{s,t}(x_1), Y^j_{s,t}(x_2)\rangle = \int_s^t A^{ij}(\xi_{s,r^-}(x_1), x_2)dr .$$

Now define $\tilde{M}_{s,t}(x)$ by

$$(2.30)\qquad \tilde{M}_{s,t}(x) = \int_s^t dY_r(\xi_{s,r^-}(x)),$$

where $Y_t(x) = X_t(x) - tb(x)$. Then we get from (2.28) and (2.29)

$$(2.31)\quad \langle \tilde{M}^i_{s,t}(x_1), \tilde{M}^j_{s,t}(x_2)\rangle = \langle M^i_{s,t}(x_1), \tilde{M}^j_{s,t}(x_2)\rangle = \int_s^t A^{ij}(\xi_{s,r^-}(x_1), \xi_{s,r^-}(x_2))dr .$$

Therefore, we have from (2.22), (2.31),

$$\langle M^i_{s,t}(x)-\tilde{M}^i_{s,t}(x)\rangle = \langle M^i_{s,t}(x)\rangle - 2\langle M^i_{s,t}(x), \tilde{M}^i_{s,t}(x)\rangle + \langle \tilde{M}^i_{s,t}(x)\rangle = 0 .$$

This proves $M^i_{s,t}(x) = \tilde{M}^i_{s,t}(x)$ or equivalently $\xi_{s,t}(x) - x = \int_0^t dX_r(\xi_{s,r^-}(x))$.

We shall next assume that $\{\xi^n_{s,t}\}$ converges weakly and shall prove that $\{X^n_t\}$ converges weakly. Let $\overline{P}_\infty$ any limiting measure of $\{\overline{P}_n\}$ as before, and let $\{\overline{P}_{n_k}, k = 1, 2, \dots\}$ be a subsequence converging weakly to $\overline{P}_\infty$. Then, $\{X^{n_k}_t, k = 1, 2, \dots\}$ converges weakly. The preceding argument shows that $(X_t, \overline{P}_\infty)$ is the infinitesimal generator of $(\xi_{s,t}, \overline{P}_\infty)$. Since the law of $(\xi_{s,t}, \overline{P}_\infty)$ is unique by the assumption and X_t is uniquely determined from $\xi_{s,t}$ by Proposition 2.1, the law of X_t is unique. This proves that $\{X^n_t\}$ converges weakly and the limit X_t generates the limit of $\{\xi^n_{s,t}\}$. The proof of Theorem 2.3 is completed.

2.5. **Strong convergence of Lévy processes on G^m_+.**

Let $\{X^n_t\}$ be a sequence of C^m-valued Lévy processes defined on the same probability space, and let $\xi^n_{s,t}$, $n = 1, 2, \dots$ be G^m_+-valued Lévy processes generated by X_t^n, $n = 1, 2, \ldots$, respectively. In this section we shall show that the strong convergence of $\{ X_t^n \}$ implies the strong convergence of $\{ \xi_{s,t}^n \}$ and *vice versa*. Here, the sequence $\{ \xi_{s,t}^n \}$ is said to converge strongly if $S_m(\xi^n, \xi^{n'})$ converges to 0 in probability as $n, n' \to \infty$, where S_m is the Skorohod metric on $W^m = D_l([0, T]; D^m)$.

THEOREM 2.5. — *Suppose that the associated characteristics of* C^m*-valued Lévy processes* X_t^n, $n = 1, 2, \ldots$ *satisfy* $(A, I)_{m,r}$ *for some* $r > (m+1)^2 d$. *Then the sequence of* G_+^m*-valued Lévy processes generated by* X_t^n, $n = 1, 2, \ldots$ *converges strongly if and only if* $\{ X_t^n \}$ *converges strongly. Furthermore, the strong limit* $\xi_{s,t}$ *is a* G_+^m*-valued Lévy process generated by the strong limit of* $\{ X_t^n \}$.

Proof. — We first assume that $\{ X_t^n \}$ converges strongly. In order to prove that $\{ \xi_{s,t}^n \}$ converge strongly, it is enough to show that for each s, t, x, the sequence of random variables $\xi_{s,t}^n(x)$, $n = 1, 2, \ldots$ converges strongly by Proposition 1.2. We shall show this in case $s = 0$ only, since $\xi_{s,t}^n(x)$ is time homogeneous. Set $\eta_t^{n,n'} = \xi_t^n(x) - \xi_t^{n'}(x)$. Then it satisfies the equation

$$\eta_t^{n,n'} = \left\{ \int_0^t dX_r^{n,c}(\xi_{r-}^n(x)) - \int_0^t dX_r^{n',c}(\xi_{r-}^{n'}(x)) \right\} + \left\{ \int_0^t \int_{C^m} f(\xi_{r-}^n(x))\tilde{N}_p^n(drdf) - \int_0^t \int_{C^m} f(\xi_{r-}^{n'}(x))\tilde{N}_p^{n'}(drdf) \right\},$$

where

$$X_t^n(x) = X_t^{n,c}(x) + \int_0^t f(x)\tilde{N}_p^n(drdf)$$

is the Lévy-Itô decomposition of X_t^n. We rewrite the first term of the right hand side by

$$\left\{ \int_0^t dX_r^{n,c}(\xi_{r-}^n(x)) - \int_0^t dX_r^{n,c}(\xi_{r-}^{n'}(x)) \right\} + \left\{ \int_0^t dX_r^{n,c}(\xi_{r-}^{n'}(x)) - \int_0^t dX_r^{n',c}(\xi_{r-}^{n'}(x)) \right\}$$

and the second term by

$$\int_0^t \int_{C^m} (f(\xi_{r-}^n(x)) - f(\xi_{r-}^{n'}(x)))\tilde{N}_p^n(drdf) + \int_0^t \int_{C^m} f(\xi_{r-}^{n'}(x))\tilde{N}_p^n(drdf) - \int_0^t \int_{C^m} f(\xi_{r-}^{n'}(x))\tilde{N}_p^{n'}(drdf).$$

Then $E[|\eta_t^{n,n'}|^2]$ is estimated as

$$\begin{aligned} E[|\eta_t^{n,n'}|^2] \leqq 2^6 \Big\{ & E\Big[\int_0^t \mathrm{Tr}\,\{ a_n(\xi_r^n, \xi_r^n) - 2a_n(\xi_r^n, \xi_r^{n'}) + a_n(\xi_r^{n'}, \xi_r^{n'}) \}\, dr\Big] \\ & + E\Big[\int_0^t \mathrm{Tr}\,\{ a_n(\xi_r^{n'}, \xi_r^{n'}) - 2a_{n,n'}(\xi_r^{n'}, \xi_r^{n'}) + a_{n'}(\xi_r^{n'}, \xi_r^{n'}) \}\, dr\Big] \\ & + E\Big[\Big(\int_0^t b_n(\xi_r^n) - b_n(\xi_r^{n'})dr\Big)^2\Big] \\ & + E\Big[\Big(\int_0^t b_n(\xi_r^{n'}) - b_{n'}(\xi_r^{n'})dr\Big)^2\Big] \\ & + E\Big[\int_0^t \int_{C^m} | f(\xi_r^n) - f(\xi_r^{n'}) |^2 \nu_n(df)dr\Big] \\ & + E\Big[\int_0^t \Big\{ \int_{C^m} | f(\xi_r^{n'}) |^2 \nu_n(df) - 2\int_{C^m} | f(\xi_r^{n'}) |^2 \nu_{n,n'}(df) \\ & + \int_{C^m} | f(\xi_r^{n'}) |^2 \nu_{n'}(df) \Big\} dr\Big]\Big\}. \end{aligned}$$

Using hypothesis $(A, I)_{m,r}$, we get

$$E[|\eta_t^{n,n'}|^2] \leqq C_1 \int_0^t E[|\eta_s^{n,n'}|^2]ds + K_{n,n'},$$

where

$$\begin{aligned} K_{n,n'} = 2^6 \Big\{ & E\Big[\int_0^t T_r\,\{ a_n(\xi_r^{n'}, \xi_r^{n'}) - 2a_{n,n'}(\xi_r^{n'}, \xi_r^{n'}) + a_{n'}(\xi_r^{n'}, \xi_r^{n'}) \}\, dr\Big] \\ & + tE\Big[\int_0^t | b_n(\xi_r^{n'}) - b_{n'}(\xi_r^{n'}) |^2 dr\Big] \\ & + E\Big[\int_0^t \Big\{ \int_{C^m} | f(\xi_r^{n'}) |^2 d\nu_n(f) - 2\int_{C^m} | f(\xi_r^{n'}) |^2 \nu_{n,n'}(df) \\ & + \int_{C^m} | f(\xi_r^{n'}) |^2 \nu_n(df) \Big\} dr\Big]\Big\}. \end{aligned}$$

By Gronwall's lemma, we get $E[|\eta_t^{n,n'}|^2] \leqq K_{n,n'}e^{C_1 t}$. Note that $E[|\xi_t^n|^2] \leqq C(1+|x|)^2$ and $a_n, a_{n,n'}, a_{n'}, \int | f(x) |^2 \nu_n(df)$ etc. are all bounded by $C(1 + |x|)^2$, where C does not depend on n, n'. Then we have for any N,

$$\begin{aligned} K_{n,n'} \leqq 64T \Big\{ & \sup_{|x| \leqq N} |\operatorname{Tr}\{ a_n(x,x) - 2a_{n,n'}(x,x) + a_{n'}(x,x) \}| \\ & + T \sup_{|x| \leqq N} |b_n(x) - b_{n'}(x)|^2 \\ & + \sup_{|x| \leqq N} \left| \int |f(x)|^2 \nu_n(df) - 2\int |f(x)|^2 \nu_{n,n'}(df) + \int |f(x)|^2 \nu_{n'}(df) \right| \Big\} \\ & + 64 \int_0^t P(|\xi_r^{n'}| > N) dr . \end{aligned}$$

The first member of the right hand side converges to 0 as $n, n' \to \infty$. The second member is bounded by $64tC(1+|x|)^2/N^2$, which converges to 0 as $N \to \infty$. Therefore we have $\lim_{n,n' \to \infty} K_{n,n'} = 0$. The strong convergence of $\xi_{s,t}^n$ is now established.

Conversely suppose that $\{ \xi_{s,t}^n \}$ converges strongly to $\xi_{s,t}$ as $n \to \infty$. Let X_t be the C^m-valued Lévy process generating $\xi_{s,t}$. We want to show that the sequence of generators $X_t^n, n=1,2, \ldots$ of $\xi_{s,t}^n, n=1,2, \ldots$ converges strongly to X_t. Let $\tilde{P}_n$ be the law of 4-ple $(X_t^n, \xi_{s,t}^n, X_t, \xi_{s,t})$, which is defined over $\tilde{W}^m = \overline{W}^m \times \overline{W}^m$. Then the sequence $\{ \tilde{P}_n \}$ is tight by Proposition 1.4. Therefore, a subsequence $\tilde{P}_{n_k}, k = 1, 2, \ldots$ converges weakly to a law $\tilde{P}_\infty$ over $\tilde{W}^m$. We denote by $(\tilde{X}_t, \tilde{\xi}_{s,t}, \hat{X}_t, \hat{\xi}_{s,t})$ any element of $\tilde{W}^m$. Then the law $(\hat{X}_t, \hat{\xi}_{s,t}, \tilde{P}_\infty)$ coincides with that of $(X_t, \xi_{s,t}, P)$. Theorem 2.3 indicates that $\tilde{X}_t$ generates $\tilde{\xi}_{s,t}$. Since $\{ \xi_{s,t}^n \}$ converges strongly to $\xi_{s,t}$, we have $\tilde{\xi}_{s,t} = \hat{\xi}_{s,t}$ a. s. $\tilde{P}_\infty$. Then we have $\tilde{X}_t = \hat{X}_t$ a. s. $\tilde{P}_\infty$ by the uniqueness of the generator. This proves that the limiting measure $\tilde{P}_\infty$ is unique and hence $\{ X_t^n \}$ converges to X_t strongly. The proof is complete.

§ 3. LEVY PROCESSES IN THE DIFFEOMORPHISMS GROUP

3.1. Statement of theorems.

Let G^m be the totality of C^m-diffeomorphisms of R^d. It is a complete topological group by the metric $\hat{\rho}_m(f,g) = \rho_m(f,g) + \rho_m(f^{-1}, g^{-1})$. Obviously it is a subset of G_+^m. Let $\hat{D}^m = D([0,T]; G^m)$ be the totality of maps from $[0,T]$ into G^m, right continuous with the left hand limits with respect to the metric $\hat{\rho}_m$ over G^m. The Skorohod metric $\hat{S}_m$ on $\hat{D}_m$ is defined similarly as (1.6), replacing the metric ρ_m by $\hat{\rho}_m$. We denote by $\hat{W}^m = D_l([0,T]; \hat{D}^m)$

the totality of maps ξ; $[0, T] \to \hat{D}^m$ such that ξ_s is *left continuous with the right hand limits* with respect to $\hat{S}_m$, and $\xi_{s,t}(x)$ is the projection of $\xi_s \in \hat{D}^m$ to the point $(t, x) \in [0, T] \times R^d$. Then $\hat{W}_m$ is a subset of $W^m = D_l([0, T]; D^m)$. The Skorohod metric $\hat{S}_m$ over $\hat{W}_m$ is defined similarly as (2.12).

A G^m_+-valued Lévy process is called a G^m-valued Lévy process if its law is supported by $\hat{W}^m$, i. e., the outer measure of $\hat{W}^m$ is 1. Thus the law of G^m-valued Lévy process is defined on $(\hat{W}^m, \mathscr{B}_{\hat{W}^m})$.

Our objective in Section 3 is to characterize the infinitesimal generator of G^m-valued Lévy process by means of its characteristics. In Theorem 3.1 we shall obtain some necessary conditions that the characteristics should satisfy. These will be listed as hypothesis $(B, I)_{m,r}$. It will turn out in Theorem 3.3 that the hypothesis $(B, I)_{m,r}$ is also a sufficient condition that the associated C^m-valued process generates a G^m-valued Lévy process.

THEOREM 3.1. — *Let $\xi_{s,t}$ be a G^m-valued Lévy process such that both $\xi_{s,t}$ and $\xi_{s,t}^{-1}$ satisfy (2.8), (2.9) for $p' \in [2, p]$ with $p > (m+1)^2 d$ and (2.10), (2.11) (In the case $m = 0$, we assume further that $A^{ij}(x, y)$ of (2.10) is a C^1-function of x, y). Let X_t be the infinitesimal generator. Then its characteristics (a, b, ν) satisfy the following properties* $(B, I)_{m,r}$ *for $r \in [2, p/(m+1)^2]$.*

$(B, I)_{m,r}$ *i) $a(x, y)$ is m-times continuously differentiable (In the case $m=0$, it is continuously differentiable). Further, there is a positive constant* L *such that*

$$|\operatorname{Tr}\{D_x^k D_y^k a(x, y)\}| \leqq L(1 + |x|)(1 + |y|), \tag{3.1}$$

$$|\operatorname{Tr}\{D_x^k D_y^k a(x, y)|_{y=x} - 2D_x^k D_y^k a(x, y) + D_x^k D_y^k a(x, y)|_{x=y}\}| \leqq L|x - y|^2 \tag{3.2}$$

for any $x, y \in R^d$ and $|k| \leqq m$.

ii) $b(x)$ and $c(x)$ are m-times continuously differentiable, where

$$c^j(x) = \sum_{i=1}^{d} \frac{\partial}{\partial y^i} a^{ij}(x, y)|_{y=x}. \tag{3.3}$$

Furthermore, there is a positive constant L *such that*

$$|D^k b(x)| + |D^k c(x)| \leqq L(1 + |x|), \tag{3.4}$$

$$|D^k b(x) - D^k b(y)| + |D^k c(x) - D^k c(y)| \leqq L|x - y| \tag{3.5}$$

for any $x, y \in R^d$ and $|k| \leqq m$.

iii) ν *is supported by* $\{ f ; f + id \in G^m \}$. *Further, there is a positive constant* L *such that*

$$\int | D^k f(x) |^{r'} \nu(df) \leqq L(1 + |x|)^{r'}, \tag{3.6}$$

$$\int | D^k f(x) - D^k f(y) |^{r'} \nu(df) \leqq L | x - y |^{r'} \tag{3.7}$$

hold for any $x, y \in R^d$, $| k | \leqq m$ *and* $r' \in [2, r]$. *Further, the same inequalities* (3.6) *and* (3.7) *are valid to the measure* ν^* *defined by*

$$\nu^*(A) = \nu(\{ f ; \hat{f} \in A \})$$

where

$$\hat{f} = id - (f + id)^{-1}.$$

iv) There is a sequence U_n, $n = 1, 2, \ldots$ *of Borel sets in* C^m *with* $\nu(U_n) < \infty$, $U_n \uparrow C^m$ *as* $n \uparrow \infty$ *such that functions defined by*

$$d_n(x) = \int_{U_n} \{ f(x) - \hat{f}(x) \} \nu(df), \qquad n = 1, 2, \ldots \tag{3.8}$$

and their derivatives up to m converge for any x. Furthermore, there is a positive constant L *satisfying*

$$| D^k d_n(x) | \leqq L(1 + |x|), \tag{3.9}$$

$$| D^k d_n(x) - D^k d_n(y) | \leqq L | x - y | \tag{3.10}$$

for all n and $| k | \leqq m$.

Now the inverse map $\xi_{s,t}^{-1}$ has the independent increments and the multiplicative property to the backward direction $\xi_{s,u}^{-1} = \xi_{s,t}^{-1} \circ \xi_{t,u}^{-1}$ for $s < t < u$. Hence $\xi_{s,t}^{-1}$ can be regarded as a G^m-valued Lévy process to the backward direction. Then there exists a C^m-valued Lévy process $\hat{X}_t$ such that

$$\xi_{s,t}^{-1}(x) - x = - \int_s^t \hat{d}\hat{X}_r(\xi_{r,t}^{-1}(x))$$

where the right hand side is the backward stochastic integral (The definition of the backward integral will be given at the next section). The $\hat{X}_t$ is called the *infinitesimal generator of the inverse* $\xi_{s,t}^{-1}$. It is sometimes called the *conjugate* of X_t.

We shall study the relationship between X_t and its conjugate $\hat{X}_t$. In view of Lévy-Itô decomposition we have

$$X_t(x) = X_t^c(x) + \int_0^t \int_{C^m} f(x) \tilde{N}_p(dsdf), \tag{3.11}$$

where X_t^c is a C^m-valued Brownian motion with characteristics $(a, b, 0)$ and the last term is a discontinuous Lévy process with the characteristics $(0, 0, v)$.

THEOREM 3.2. — *Under the same condition as in Theorem 3.1, the conjugate $\hat{X}_t$ is represented as*

$$\hat{X}_t(x) = X_t^c(x) + \int_0^t \int_{C^m} \hat{f}(x)\tilde{N}_p(dsdf) - (c(x) + d(x))t, \tag{3.12}$$

where $d = \lim_{n \to \infty} d_n$.

Remark. — Set $\xi^*_{t,s} = \xi^{-1}_{-s,-t}$, $-T \leqq t \leqq s \leqq 0$. Then ξ^* is a C^m-valued Lévy process (to the forward direction). Define the C^m-valued Lévy process by $X_t^* = \hat{X}_{-t}$. Then it is the infinitesimal generator of the Lévy process $\xi^*_{t,s}$.

Note that the characteristics of X_t^* is $(a, -b+c+d, v^*)$. Then $\xi_{s,t}$ and the inverse $\xi_{s,t}^{-1}$ has the same law if and only if $b = (1/2)(c + d)$ and $v = v^*$.

The hypothesis $(B, I)_{m,r}$ is almost a sufficient condition that the C^m-valued Lévy process generates a G_m-valued Lévy process. In fact, we have the following.

THEOREM 3.3. — *Suppose that the characteristics of a C^m-valued Lévy process satisfy $(B, I)_{m,r}$ for some $r > (m + 1)^2 d$. Then it generates a G^m-valued Lévy process.*

Let $\{ \xi^n_{s,t} \}$ be a sequence of G^m-valued Lévy processes. The strong and the weak convergence of the sequence $\{ \xi^n_{s,t} \}$ are defined similarly as in Section 2. However, the topology involved here is the metric $\hat{S}_m$, which is stronger than S_m obviously. We shall discuss the tightness and the convergence of the sequence of these processes.

THEOREM 3.4. — *Let $\{ \xi^n_{s,t} \}$ be a sequence of G^m-valued Lévy process such that their infinitesimal generators satisfy $(B, I)_{m,r}$ for some $r > (m+1)^2 d$. Suppose that the constant L in $(B, I)_{m,r}$ can be chosen independently of $n = 1, 2, \ldots$ Then, $\{ \xi^n_{s,t} \}$ is tight. In addition, if (A, II) is satisfied, then it converges weakly. If (A, III) is satisfied, then it converges strongly.*

The proofs of Theorems 3.1 and 3.2 will be given at the next section. The proofs of Theorems 3.3 and 3.4 will be given at Section 3.3.

3.2. **Backward stochastic differential equation for the inverse of G^m-valued Lévy process.**

Let X_t be a C^m-valued Lévy process with characteristics (a, b, v) and $\mathscr{F}_{s,t}$ be the least σ-field for which $X_u - X_v$; $s \leqq u \leqq v \leqq t$ are measurable. We fix t for a moment. Then $\mathscr{F}_{s,t}$, $s \in [0, t]$ is a decreasing family of σ-fields.

Set $Y_t = X_t - tb$. Then $Y_t - Y_s$ is a backward martingale adapted to $\mathscr{F}_{s,t}$. Now let ϕ_s, $s \in [0, t]$ be a $\mathscr{F}_{s,t}$-adapted process, right continuous with the left hand limits. The *backward Itô integral of* ϕ_s *by* X_t is defined by

$$\int_s^t \hat{d}Y_r(\phi_r) = \lim_{|\delta| \to 0} \sum_{i=0}^{n-1} Y_{t_{i+1}}(\phi_{t_{i+1}}) - Y_{t_i}(\phi_{t_{i+1}})$$

where $\delta = \{ s = t_0 < \ldots < t_n = t \}$ are partitions of $[s, t]$. It is a left continuous R^1-valued backward local martingale adapted to $\mathscr{F}_{s,t}$. Associated with the above backward local martingale, there is a unique continuous process of bounded variation ψ_s^{ij} adapted to $\mathscr{F}_{s,t}$ such that $\psi_t^{ij} = 0$ and

$$\int_s^t \hat{d}Y_r^i(\phi_r) \int_s^t \hat{d}Y_r^j(\phi_r) - \psi_s^{ij}$$

is a backward local martingale, by Meyer's decomposition. The process ψ_s^{ij} is written by $\left\langle \int_s^t \hat{d}Y_r^i(\phi_r), \int_s^t \hat{d}Y_r^j(\phi_r) \right\rangle$. Then it holds

$$\left\langle \int_s^t \hat{d}Y_r^i(\phi_r), \int_s^t \hat{d}Y_r^j(\phi_r) \right\rangle = \int_s^t A^{ij}(\phi_r, \phi_r)dr,$$

where $A^{ij}(x, y)$ is defined by (2.2). The backward integral of ϕ_t by X_t is defined by

$$\int_s^t \hat{d}X_r(\phi_r) = \int_s^t \hat{d}Y_r(\phi_r) + \int_s^t b(\phi_r)dr.$$

Now we shall prove Theorems 3.1 and 3.2 simultaneously. Let $\xi_{s,t}$ be a G^m-valued Lévy process such that both $\xi_{s,t}$ and $\xi_{s,t}^{-1}$ satisfy (2.8)-(2.11). Let X_t be the infinitesimal generator of $\xi_{s,t}$ and let (3.11) be its decomposition. Then it holds

$$(3.13)\quad \xi_{s,t}(x) - x = \int_s^t dX_r^c(\xi_{s,r^-}(x)) + \int_s^t \int_{C^m} f(\xi_{s,r^-}(x))\tilde{N}_p(drdf).$$

We want to prove the two inequalities

$$(3.14)\quad \int_s^t dX_r^c(\xi_{s,r^-}(x))\Big|_{x=\xi_{s,t}^{-1}(y)} = \int_s^t \hat{d}X_r^c(\xi_{r^+,t}^{-1}(y)) - \int_s^t c(\xi_{r^+,t}^{-1}(y))dr,$$

$$(3.15)\quad \int_s^t \int_{U_n} f(\xi_{s,r^-}(x))\tilde{N}_p(drdf)\Big|_{x=\xi_{s,t}^{-1}(y)}$$
$$= \int_s^t \int_{U_n} \hat{f}(\xi_{r^+,t}^{-1}(y))\tilde{N}_p(drdf) - \int_s^t d_n(\xi_{r^+,t}^{-1}(y))dr$$

and the convergence of each terms of (3.15) as $n \to \infty$:

$$(3.16)\quad \lim_{n\to\infty} \int_s^t \int_{U_n} f(\xi_{s,r^-}(x))\tilde{N}_p(drdf) = \int_s^t \int_{C^m} f(\xi_{s,r^-}(x))\tilde{N}_p(drdf)$$

$$(3.17)\quad \lim_{n\to\infty} \int_s^t \int_{U_n} \hat{f}(\xi_{r+,t}^{-1}(y))\tilde{N}_p(drdf) = \int_s^t \int_{C^m} \hat{f}(\xi_{r+,t}^{-1}(y))\tilde{N}_p(drdf).$$

$$(3.18)\quad \lim_{n\to\infty} \int_s^t d_n(\xi_{r+,t}^{-1}(y))dr = \int_s^t d(\xi_{r+,t}^{-1}(y))dr.$$

Indeed, substituting these relations to (3.13), we obtain

$$(3.19)\quad \xi_{s,t}^{-1}(y) - y = -\int_s^t \hat{d}X_r^c(\xi_{r+,t}^{-1}(y)) - \int_s^t \int_{C^m} f(\xi_{r+,t}^{-1}(y))\tilde{N}_p(drdf) + \int_s^t (c+d)(\xi_{r+,t}^{-1}(y))dr$$

and the assertion of Theorem 3.2 will follow.

We shall prove (3.14)-(3.18) in a series of lemmas.

LEMMA 3.5. — *Equality (3.14) holds.*

Proof. — Associated with X_t we can construct a finite or infinite number of independent standard Brownian motions B_t^k, $k = 1, 2, \ldots$ written as $B_t^k = \sum_{l,j} c_k^{lj} X_t^l(x_j)$ with suitable constants c_k^{lj} and $x_j \in R^d$ such that they span $X_t(x)$, namely,

$$X_t^c(x) = \sum_k X_k(x) B_t^k + X_0(x)t$$

where $X_k(x) = E[X_t(x)B_t^k]$ $(t = 1)$ and $X_0 = b$. See Fujiwara-Kunita [2]. Note that $X_k(x) = \sum_{l,j} c_k^{lj} a^{\cdot l}(x, x_j)$. Then these are C^m-functions by $(B, I)_{m,r}$.

We have similarly as in Kunita [6], Lemma 6.2

$$(3.20)\quad \int_s^t X_k(\xi_{s,r^-}(x))dB_r^k \Big|_{x=\xi_{s,t}^{-1}(y)} = \int_s^t X_k(\xi_{r,t}^{-1}(y))\hat{d}B_r^k - \int_s^t X_k . X_k(\xi_{r,t}^{-1}(y))dr,$$

where

$$X_k . X_k(x) = \sum_i X_k^i(x) \frac{\partial}{\partial x^i} X_k(x).$$

Taking the sum of (3.20) for $k = 1, 2, \ldots$ and noting the relation

$$\sum_{k,i} X_k^i(x) \frac{\partial}{\partial x^i} X_k^j(x) = \sum_i \frac{\partial}{\partial y^i} \Big(\sum_k X_k^i(x) X_k^j(y) \Big) \Big|_{y=x}$$
$$= \sum_i \frac{\partial}{\partial y^i} a^{ij}(x, y) \Big|_{y=x},$$

we get the formula (3.14).

LEMMA 3.6. — *Equality (3.15) holds.*

Proof. – Since

$$\xi_{s,r}(x) - \xi_{s,r^-}(x) = \int f(\xi_{s,r^-}(x)) N_p(\{r\}, df),$$

we have the relation

$$\int_s^t \int_{U_n} f(\xi_{s,r^-}(x)) \tilde{N}_p(drdf) = \sum_{s \leq r \leq t} (\xi_{s,r}(x) - \xi_{s,r^-}(x)) I_{U_n}$$
$$- \int_s^t \int_{U_n} f(\xi_{s,r^-}(x)) \nu(df) dr .$$

Substitute $x = \xi_{s,t}^{-1}(y)$ to the right hand side. We have

$$\sum_{s \leq r \leq t} (\xi_{s,r}(x) - \xi_{s,r^-}(x)) I_{U_n} \Big|_{x = \xi_{s,t}^{-1}(y)} = \sum_{s \leq r \leq t} (\xi_{r^+,t}^{-1}(y) - \xi_{r,t}^{-1}(y)) I_{U_n} .$$

Note that

$$\xi_{r,r}(x) = x + \int f(x) N_p(\{r\}, df) = \int (f + id)(x) N_p(\{r\}, df) .$$

Then we have

$$\xi_{r,r}^{-1}(y) - y = - \int \hat{f}(y) N_p(\{r\}, df),$$

where $\hat{f}(y) = y - (f + id)^{-1}(y)$. Therefore,

$$\sum_{s \leq r \leq t} (\xi_{r^+,t}^{-1}(y) - \xi_{r,t}^{-1}(y)) I_{U_n} = \sum_{s \leq r \leq t} (\xi_{r^+,t}^{-1}(y) - \xi_{r,r}^{-1}(\xi_{r^+,t}^{-1}(y))) I_{U_n}$$
$$= \int_{U_n} \hat{f}(\xi_{r^+,t}^{-1}(y)) N_p(drdf)$$

On the other hand, we have

$$\int_s^t \int_{U_n} f(\xi_{s,r^-}(x))\nu(df)dr \Big|_{x=\xi_{s,t}^{-1}(y)} = \int_s^t \int_{U_n} f(\xi_{r+,t}^{-1}(y))\nu(df)dr .$$

See [6], Lemma 6.2. The equality (3.15) follows immediately.

We next discuss the convergence of (3.16), (3.17).

LEMMA 3.7. — *The convergence (3.16) is satisfied uniformly in x on compact sets in L^p-sense for each $s < t$.*

Proof. — Set

$$Z^n_{s,t} = \int_s^t \int_{U_n} f(\xi_{s,r^-}(x))\tilde{N}_p(drdf), \qquad n = 1, 2, \ldots . \tag{3.21}$$

Note that the characteristic measure ν satisfies (3.6) and (3.7) since $\xi_{s,t}$ satisfies (2.8) and (2.9). ([2], Corollary to Theorem 3.1). Then there is a positive constant K independent of n such that

$$E[\,|Z^n_{s,t}(x)|^{p'}] \leqq K(1 + |x|)^{p'}|t - s|, \tag{3.22}$$

$$E[\,|Z^n_{s,t}(x) - Z^n_{s,t}(y)|^{p'}] \leqq K|x - y|^{p'}|t - s| \tag{3.23}$$

for any $p' \in [2, r]$. The proof can be done similarly as in the proof of Lemma 2.1 in [2]. Hence the sequence $\{Z^n_{s,t}(x)\}$ is tight. On the other hand, we can show easily

$$E[\,|Z^n_{s,t}(x) - Z^{n'}_{s,t}(x)|^2] \to 0, \qquad n, n' \to \infty .$$

Therefore $\{Z^n_{s,t}(x)\}$ converges uniformly on compact sets in $L^{p'}$-sense for each s, t.

LEMMA 3.8. — *The convergence (3.17) is satisfied uniformly in y on compact sets in L^p-sense for each $s < t$.*

Proof. — Set

$$\hat{Z}^n_{s,t}(x) = \int_s^t \int_{U_n} \hat{f}(\xi_{r+,t}^{-1}(x))\tilde{N}_p(drdf) .$$

If the characteristic measure ν satisfies

$$\int |\hat{f}(x)|^{r'}\nu(df) \leqq L(1 + |x|)^{r'}, \tag{3.24}$$

$$\int |\hat{f}(x) - \hat{f}(y)|^{r'}\nu(df) \leqq L|x - y|^{r'} \tag{3.25}$$

for $r' \in [2, r]$, then the convergence of $\{\hat{Z}^n_{s,t}\}$ follows similarly as in Lemma 3.7.

In order to prove the above inequalities, we shall consider the conjugate of X_t. Let $\hat{X}_s$ be the infinitesimal generator of the inverse flow $\xi_{r,t}^{-1}$ (backward direction) and let $\hat{\nu}$ be the associated characteristic measure. Then $\hat{\nu}$ satisfies (3.6) and (3.7) since $\xi_{s,t}^{-1}$ satisfies (2.8) and (2.9). In the following we shall prove $\hat{\nu} = \nu^*$. Note that $\xi_{r,t}^{-1}$ and $\hat{X}_t$ are related by the back ward equation,

$$\xi_{s,t}^{-1}(y) - y = - \int_s^t \hat{d}\hat{X}_r(\xi_{r+,t}^{-1}(y)) .$$

Then we have $\Delta\xi_{r,r}^{-1} = - \Delta\hat{X}_r$. Since

$$\Delta\xi_{r,r}^{-1} = (\Delta X_r + id)^{-1} - id ,$$

the Poisson random measure $\hat{N}_p((0, t] \times A)$ associated with $\hat{X}_t$ satisfies

$$\hat{N}_p((0, t] \times A) = \# \{ r \in [0, t] ; -(\Delta X_r + id)^{-1} + id \in A \} = N_p((0, t] \times \hat{A}) ,$$

where $\hat{A} = \{ \hat{f} ; f \in A \}$. Therefore the characteristic measure $\hat{\nu}$ coincides with ν^*.

Now, since we have

$$\int F(\hat{f})\nu(df) = \int F(f)\nu^*(df) = \int F(f)\,\hat{\nu}(df) ,$$

inequalities (3.6) and (3.7) for $\hat{\nu}$ for $k = 0$ is equivalent to (3.24) and (3.25). The proof is complete.

The convergence (3.18) follows from the following lemma.

LEMMA 3.9. — *The sequence d_n, $n = 1, 2, \ldots$ satisfies* $(B, I)_{m,r}$ *iv*).

Proof. — From Lemmas 3.5 and 3.6, we have

$$\xi_{s,t}^{-1}(y) - y + \int_s^t d\hat{X}_r^c(\xi_{r,t}^{-1}(y)) + \int_s^t \int_{U_n} \hat{f}(\xi_{r+,t}^{-1}(y))\tilde{N}_p(drdf) = \int_s^t d_n(\xi_{r+,t}^{-1}(y))dr .$$

The last member of the left hand side satisfies the moment inequalities like (3.22) and (3.23). Similar inequalities are valid to the first and the second term. Then we have

$$E\left[\left| \int_s^t \{ d_n(\xi_{r+,t}^{-1}(x)) - d_n(\xi_{r+,t}^{-1}(y)) \} \, dr \right|^{p'} \right] \leqq K \, | \, x - y \, |^{p'} \, | \, t - s \, | ,$$

$$E\left[\left| \int_s^t d_n(\xi_{r+,t}^{-1}(y))dr \right|^{p'} \right] \leqq K(1 + | \, x \, |)^{p'} \, | \, t - s \, | .$$

Divide the above by $|t-s|$ and let $t-s$ tend to 0. Then we get

$$|d_n(x)-d_n(y)| \leqq K|x-y|, \qquad |d_n(x)| \leqq K(1+|x|).$$

This proves that d_n, $n=1,2,\ldots$ are uniformly Lipschitz continuous and uniformly bounded. Then we can choose a subsequence $\{d_{n_k}, k=1,2,\ldots\}$ converging uniformly to d on compact sets by Ascoli-Arzela's theorem. Since $\lim_{n\to\infty}\int d_n(\xi_{r+,t}^{-1}(x))dr = \int_s^t d(\xi_{r+,t}^{-1}(x))dr$, d is uniquely determined from the sequence $\{d_n\}$. This proves that $\{d_n\}$ converges uniformly on compact sets. The uniform convergence of derivatives $\{D^k d_n(x)\}$ ($|k| \leqq m$) can be proved similarly.

Proof of Theorem 3.1. — Let (a, b, ν) be the characteristics of the infinitesimal generator. We have already shown that the measure ν satisfies $(B, I)_{m,r}$ *iii*), *iv*) in Lemmas 3.8 and 3.9. We shall prove the properties $(B, I)_{m,r}$ *i*), *ii*) in case $m=0$ only.

Let X_t be the infinitesimal generator and let $Y_t = X_t - tb$. Let Y_t^c be the continuous part of Y_t. Then it holds

$$A^{ij}(x,y) = E[Y_t^i(x)Y_t^j(y)] \qquad (t=1),$$
$$a^{ij}(x,y) = E[Y_t^{ci}(x)Y_t^{cj}(y)] \qquad (t=1).$$

Since $A^{ij}(x,y)$ is continuously differentiable, $Y_t(x)$ has the L^2-derivative $\partial_i Y_t(x) = \lim_{h\to\infty}\frac{1}{h}\{Y_t(x+he_i)-Y_t(x)\}$ for any x, where $e_i=(0,\ldots,0,1,0,\ldots 0)$ (1 is at the i-th component), and it holds

$$E[\partial_i Y_t^k(x)\partial_j Y_t^l(y)] = \frac{\partial^2}{\partial x_i \partial y_j}A^{kl}(x,y).$$

Noting that $Y_t^c(x)$ and $Y_t^d(x) \equiv Y_t(x) - Y_t^c(x)$ are independent, we see that $Y_t^c(x)$ has also the L^2-derivative $\partial_i Y_t^c(x)$ for any x. This implies the differentiability of $a(x,y)$ and the relation

$$E[\partial_i Y_t^{c,k}(x)\partial_j Y_t^{c,l}(y)] = \frac{\partial^2}{\partial x_i \partial y_j}a^{kl}(x,y)$$

follows.

The Lipschitz continuity of $a(x,y)$ follows from that of $A(x,y)$. The Lipschitz continuity of $b(x)$ is clear from the assumption. We will show the Lipschitz continuity of $c(x)$. Observe that the conjugate C^m-valued process $\hat{X}_t$ satisfies

$$|E[\hat{X}_t(x) - \hat{X}_t(y)]| \leqq L|x-y|.$$

Since $E[X_t(x)] = t(b(x)-c(x)-d(x))$ and since b and d are Lipschitz continuous, the same property holds for $c(x)$. The proof is complete.

3.3. Construction of G^m-valued Lévy process.

Let X_t be a C^m-valued Lévy process satisfying $(B, I)_{m,r}$ for some $r > (m+1)^2 d$. The purpose of this section is to prove that it generates a G^m-valued Lévy process. In [2], we have shown the fact in case where the characteristic measure ν is finite. In the following we shall prove it in case where ν is a σ-finite measure, approximating X_t by a sequence of Lévy processes having finite characteristic measures.

Let (3.1) be the decomposition of the X_t and let U_n, $n = 1, 2, \ldots$ be the sequence of Borel sets in C^m stated in $(B, III)_{m,r}$. For each n, set

$$X_t^n(x) = X_t^c(x) + \int_0^t \int_{U_n} f(x)\tilde{N}_p(drdf).$$

Let $\hat{X}_t^n$ be the conjugate of X_t^n. Then both sequences $\{ X_t^n \}$ and $\{ \hat{X}_t^n \}$ satisfy hypotheses $(A, I)_{m,r}$ and (A, III). Hence both of them converge strongly by Proposition 1.2. Let now $\xi_{s,t}^n$ be the G^m-valued Lévy process generated by X_t^n. Then the inverse $(\xi_{s,t}^n)^{-1}$ is generated by the conjugate $\hat{X}_t^n$. Then both $\{ \xi_{s,t}^n \}$ and $\{ (\xi_{s,t}^n)^{-1} \}$ satisfy the tightness criteria (2.14) and (2.15). Therefore, the sequence of the pair $\{ (\xi_{s,t}^n, (\xi_{s,t}^n)^{-1}) \}$ is tight over the product space $W^m \times W^m$ ($W^m = D_l([0, T]; D^m)$) with respect to the Skorohod metric S_m. This implies that $\{ \xi_{s,t}^n \}$ is tight over $\hat{W}^m = D_l([0, T]; \hat{D}^m)$ where $\hat{D}^m = D([0, T]; G^m)$ with respect to the Skorohod metric $\hat{S}_m$. Then $\{ \xi_{s,t}^n \}$ converges strongly with respect to $\hat{S}_m$-topology, which is proved similarly as in Theorem 2.5. Therefore the limit $\xi_{s,t}$ is a G^m-valued Lévy process. It is generated by X_t. The proof of Theorem 3.3 is thus completed.

Theorem 3.4 can be proved similarly. We omit the details.

3.4. Example.

We shall discuss a limit theorem studied by Matsumoto-Shigekawa [*10*] in the context of our approach. Let $(U, \mathscr{B}, \mu)$ be another probability space.

Let $\{ N^n(dtdu) \}$ be a sequence of Poisson random measures over the product space $[0, T] \times U$ with the intensity measures $ndt\mu(du)$, $n = 1, 2, \ldots$, respectively. Let A_u, $u \in U$ be a family of C^m-maps from R^d into itself (vector fields) with the parameter u such that $A_u(x)$ is jointly measurable. We assume that $\{ A_u, u \in U \}$ satisfies the centering condition

$$\int_U A_u(x)\mu(du) = 0, \qquad \forall x \in R^d.$$

Let $\psi^u(t, x)$, $t \in (-\infty, \infty)$ be the one parameter group of transformations generated by the vector field A_u. For each n, set

$$X_t^n(x) = \int_U \left(\psi^u\left(\frac{1}{\sqrt{n}}, x\right) - x\right) N^n((0, t], du).$$

It is a C^m-valued Lévy process. Setting $\Phi_n(u) = \psi^u\left(\frac{1}{\sqrt{n}}, .\right)$-identity, it is a measurable map from U into C^m. Define a sequence of measures ν_n over $(C^m, \mathscr{B}_{C^m})$ by

$$\nu_n(A) = n\mu(\Phi_n^{-1}(A)).$$

Then ν_n is the characteristic measure of the C^m-valued Lévy process X_t^n.

Let $\xi_{s,t}^n$ be the G_+^m-valued Lévy process generated by X_t^n. Since ν_n is supported by $\{ f = \phi - id; \phi \in G^m \}$, $\xi_{s,t}^n$ is actually a G^m-valued Lévy process. Let Q_n be the law of $\xi_{s,t}^n$ defined on $\hat{W}^m = D_l([0, T]; \hat{D}_m)$ where $\hat{D}_m = D([0, T]; G^m)$. We shall show the weak convergence of the sequence $\{ Q_n \}$ with respect to the Skorohod topology $\hat{S}_m$.

We first check the tightness of $\{ Q_n \}$. Similarly as in Lemma 3.1 in [*10*], there is a positive constant K such that

$$\int_{C^m} | D^k f(x) - D^k f(y) |^r \nu^n(df)$$
$$= \int_{C^m} \left| D^k\psi^u\left(\frac{1}{\sqrt{n}}, x\right) - D^k x - \left(D^k\psi^u\left(\frac{1}{\sqrt{n}}, y\right) - D^k y\right)\right|^r n\mu(du) \leqq \frac{K}{n^{r/2} - 1}$$

holds for any $n \geqq 1$, $r \geqq 2$ and $|k| \leqq m$. Now observe that the conjugate of X_t^n is given by

$$\hat{X}_t^n(x) = \int_U \left(\psi^u\left(-\frac{1}{\sqrt{n}}, x\right) - x\right) N^n((0, t], du).$$

Then we have similarly as the above

$$\int_{C^m} | D^k \hat{f}(x) - D^k \hat{f}(y) |^r \nu^n(df) \leqq \frac{K}{n^{r/2} - 1}.$$

Further the sequence of functions $d_n(x)$:

$$d_n(x) = \int_{C^m} (f(x) - \hat{f}(x))\nu_n(df)$$
$$= \int_{C^m} \left(\psi^u\left(\frac{1}{\sqrt{n}}, x\right) + \psi^u\left(-\frac{1}{\sqrt{n}}, x\right) - 2x\right) n\mu(d\mu)$$

satisfies

$$| D^k d_n(x) - D^k d_n(y) | \leqq K | x - y |$$
$$| D^k d_n(x) | \leqq K$$

where K does not depend on n. cf. Lemma 3.1 in [10]. Hence the hypothesis $(B, I)_{m,r}$ is satisfied with the constant $L = K$ independently of the sequence. Consequently the sequence of laws $\{ Q_n \}$ is tight by Theorem 3.4.

We shall next check the hypothesis (A, II). Let $x_1, \ldots, x_N$ be any points in R^d. For any $\varepsilon > 0$, we have

$$\int_{\bigcap_{l=1}^{N} \{|f(x_l)| < \varepsilon\}} f(x_i)' f(x_j) v^n(df)$$
$$= \int_{\bigcap_{l=1}^{N} \{|\psi^u(x_l) - x_l| < \varepsilon\}} \left(\psi^u\left(\frac{1}{\sqrt{n}}, x_i\right) - x_i\right)'\left(\psi^u\left(\frac{1}{\sqrt{n}}, x_j\right) - x_j\right) n\mu(du)$$
$$\underset{n \to \infty}{\longrightarrow} \int A_u(x_i)' A_u(x_j) \mu(du)$$

and

$$\int_{\bigcap_{l=1}^{N} \{|f(x_l)| > \varepsilon\}} f(x_i)' f(x_j) v^n(df) \underset{n \to \infty}{\longrightarrow} 0.$$

Consequently (A, II) is satisfied with

$$a(x, y) = \int_U A_u(x)' A_u(y) v(du), \qquad b = 0, \quad v = 0.$$

Then, Theorem 3.4 shows that the sequence of laws $\{ Q_n \}$ converges weakly. The limit is a G^m-valued Brownian motion.

Finally we compare the above convergence with that of Matsumoto-Shigekawa [10]. In the latter, it is shown that the sequence of laws of $\xi^n_{0,t}$ defined over $\hat{D}^m = D([0, T]; G^m)$ converges weakly with respect to the metrics S_m of (1.6). Hence our convergence theorem is a refinement of theirs.

REFERENCES

[1] P. BILLINGSLEY, *Convergence of probability measures*, John Wiley and Sons, New York, 1968.
[2] T. FUJIWARA, H. KUNITA, Stochastic differential equations of jump type and Lévy processes in diffeomorphisms group, *J. Math. Kyoto Univ.*, t. **25**, 1985, p. 71-106.
[3] N. IKEDA, S. WATANABE, *Stochastic differential equations and diffusion processes*, North-Holland/Kodansha, 1981.

[4] J. JACOD, Processus à accroissements indépendants : Une condition nécessaire et suffisante de convergence en loi, *Z. Warhscheinlichkeitstheorie verw. Gebiete*, t. **63**, 1983, p. 109-139.

[5] H. KUNITA, On the convergence of solutions of stochastic ordinary differential equations as stochastic flows of diffeomorphisms, *Osaka J. Math.*, t. **21**, 1984, p. 883-911.

[6] H. KUNITA, Stochastic differential equations and stochastic flows of diffeomorphisms, *Lecture Notes in Math.*, 1097, 1984, p. 144-303.

[7] H. KUNITA, Tightness of probability measures in D([0, T]; C) and D([0, T]; D), *J. Math. Soc. Japan*, to appear.

[8] Y. LE JAN, Flots de diffusions dans R^d, *C. R. Acad. Sci. Paris*, t. **294**, 1984, Série I, p. 697-699.

[9] Y. LE JAN, S. WATANABE, *Stochastic flows of diffeomorphisms*, *Stochastic Analysis Proc.* Taniguchi Conference, Kyoto, 1982, North-Holland/Kodansha, 1984.

[10] H. MATSUMOTO, I. SHIGEKAWA, Limit theorems for stochastic flows of diffeomorphisms of jump type, *Z. Warhscheinlichkeitstheorie verw. Gebiete*, t. **69**, 1985, p. 501-540.

[11] T. E. HARRIS, Brownian motions on the homeomorphisms of the plane, *Ann. Probab.*, t. **9**, 1981, p. 232-254.

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