ANNALES DE L'I. H. P., SECTION B

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*Annales de l'I. H. P., section B*, tome 22, nº 3 (1986), p. 263-285

<http://www.numdam.org/item?id=AIHPB_1986__22_3_263_0>

*Ann. Inst. Henri Poincaré,*
Vol. 22, n° 3, 1986, p. 263-285. *Probabilités et Statistiques*



# On the Skorokhod topology

by

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ABSTRACT. — Let E be a completely regular topological space. Mitoma [9], extending the classical case $E = R^1$, has recently introduced the Skorokhod topology on the space D( [0, 1 ]: E). This topology is investigated in detail. We find families of continuous functions which generate the topology, examine the structure of the Borel and Baire $\sigma$-algebras of D( [0, 1 ] : E) and prove tightness criteria for E-valued stochastic processes. Extensions to $D(R^+ : E)$ are also given.

RÉSUMÉ. — Soit E un espace topologique complètement régulier. Étendant le cas classique, Mitoma [9] vient d'introduire récemment la topologie de Skorokhod sur l'espace D( [0, 1 ]: E). Nous examinons en détail cette topologie, donnons des familles d'applications continues engendrant la topologie : nous étudions la structure des tribus de Borel et de Baire de D( [0, 1 ]: E) et démontrons des critères de tension pour les processus stochastiques. Nous terminons par des extensions à $D(R^+ : E)$.

## INTRODUCTION

The space D( [0, 1 ]: E), where E is a separable metric space, being a model of various physical phenomena, has been a point of interest of Probability Theory since the fundamental paper by Skorokhod [*12*].

*Annales de l'Institut Henri Poincaré* - Probabilités et Statistiques - 0246-0203
Vol. 22/86/03/263/23/$ 4,30/© Gauthier-Villars

Recently Mitoma [9] has introduced the Skorokhod topology on the space $D([0,1]:E)$, where E is *a completely regular topological space.* This paper is devoted to the study of the Skorokhod topology just in this case. In particular, it is proved in Sec. 1 (Theorem 1.7) that the Skorokhod topology is the coarsest topology with respect to which all mappings of the form

$$\tilde{f}: D([0,1]:E) \to D([0,1]:R^1)$$
$$[\tilde{f}(x)](t) = f(x(t)), \qquad t \in [0,1],$$

are continuous, where $f: E \to R^1$ is continuous.

In Sec. 2 we consider the problem when Borel subsets of $D([0,1]:E)$ are equal to cylindrical subsets of $D([0,1]:E)$. A simple « permanence » theorem (Theorem 2.1) and some sufficient conditions for the cylindrical subsets to be Baire subsets of $D([0,1]:E)$ (Theorem 2.5) as well as several corollaries are given.

Section 3 contains « weak » tightness criteria for families of Borel measures on $D([0,1]:E)$ (Theorem 3.1). Here « weak » means that the problem of tightness of a sequence $\{X_n\}_{n\in\mathbb{N}}$ of E-valued stochastic processes with trajectories in $D([0,1]:E)$ can be reduced to tightness of *real* processes $\{f(X_n)\}_{n\in\mathbb{N}}$, where $f: E \to R^1$ is continuous, *plus* uniform concentration in probability of trajectories of processes $\{X_n\}_{n\in\mathbb{N}}$ on subspaces $D([0,1]:K) \subset D([0,1]:E)$, where K is a compact subset of E.

In Sec. 4 the results for $D([0,1]:E)$ are extended over the space $D(R^+:E^\perp)$.

Measurability properties of the Skorokhod spaces and, especially, weak tightness criteria, are examined in Sec. 5 in a few examples.

## 1. THE SKOROKHOD TOPOLOGY ON D([0,1] : E)

Let $(E, \tau)$ be a topological Hausdorff space. Denote by $D_1(E,\tau) = D([0,1]:(E,\tau))$ the space of mappings $x: [0,1] \to E$ which are right-continuous and admit left-hand limits for every $t > 0$ (in topology $\tau$ !). One can prove

1.1. PROPOSITION. — *Let* $x \in D_1(E,\tau)$. *Then the closure of the set* $\{x(t) \mid t \in [0,1]\}$ *is compact in* $(E,\tau)$ *and coincides with*

$$\{x(t) \mid t \in [0,1]\} \cup \{x(t-) \mid t \in [0,1]\} \qquad \blacksquare$$

If $(E,\tau)$ is a metrisable space, each element $x \in D_1(E,\tau)$ has only countably many discontinuities. This is not so in the general case:

1.2. EXAMPLE. — *Let* $E = (R^1)^{[0,1]}$ *with the product topology* $\tau_\pi$. *Consider the mapping*

$$[0,1] \ni t \mapsto (x(t,.) : [0,1] \to R^1) \in E$$
$$x(t,u) = I_{[u,1]}(t).$$

Clearly $x \in D_1(E, \tau_\pi)$ and it is easy to see that the set of jumps of the element $x$ is uncountable:

$$\{ t' \in [0,1] \mid x(t) \neq x(t-) \} = (0,1]$$

(by definition $x(0-) = x(0)$). ■

Now assume that $(E, \tau)$ is a completely regular topological space. Then there exists a family $\{ d_i \}_{i \in \mathbb{I}}$ of pseudometrics on E such that

$$(1.1) \qquad \forall_{a,b \in E} \exists_{i \in \mathbb{I}} \; d_i(a,b) > 0.$$

$$(1.2) \qquad \forall_{i,j \in \mathbb{I}} \exists_{k \in \mathbb{I}} \; \max(d_i, d_j) \leqslant d_k$$

and open balls in pseudometrics $d_i$, $i \in \mathbb{I}$, form a basis for the topology $\tau$ (see [*4*]).

Following Mitoma [*9*], one can define on $D_1(E, \tau)$ a completely regular topology by considering pseudometrics

$$(1.3) \quad \tilde{d}_i(x,y) = \inf_{\lambda \in \Lambda} \max\big( \sup_{t \in [0,1]} |\lambda(t) - t|, \sup_{t \in [0,1]} d_i(x(\lambda(t)), y(t))\big), \; i \in \mathbb{I},$$

where $\Lambda$ is the set of strictly increasing continuous functions $\lambda : [0,1] \to [0,1]$, $\lambda(0) = 0$, $\lambda(1) = 1$.

It is easy to see that the family $\{ \tilde{d}_i \}_{i \in \mathbb{I}}$ satisfies the conditions (1.1) and (1.2), hence open balls in pseudometrics $\{ \tilde{d}_i \}_{i \in \mathbb{I}}$ form a basis for a completely regular topology on $D_1(E, \tau)$, called the Skorokhod topology on $D_1(E, \tau)$. We shall see that this topology does not depend on the particular choice of pseudometrics $\{ d_i \}_{i \in \mathbb{I}}$.

1.3. THEOREM. — *Let the family* $\{ d_i \}_{i \in \mathbb{I}}$ *of pseudometrics on* E *satisfies the assumptions (1.1) and (1.2). Let* $\tau$ *be the topology on* E *generated by the family* $\{ d_i \}_{i \in \mathbb{I}}$.

*Then the Skorokhod topology on* $D_1(E, \tau)$ *defined by pseudometrics* $\{ \tilde{d}_i \}_{i \in \mathbb{I}}$ *depends only on the topology* $\tau$ *on* E.

We will divide the proof into two lemmas.

1.4. LEMMA. — *Let* $p_n = \{ 0 = t_{n0} < t_{n1} < \ldots < t_{nk_n} = 1 \}$, $n \in \mathbb{N}$, *be a sequence of partitions of the interval* $[0,1]$. *Suppose that* $\{ p_n \}_{n \in \mathbb{N}}$ *is normal, i.e.*

$$(1.4) \qquad |p_n| = \max_{1 \leqslant k \leqslant k_n} |t_{nk} - t_{n,k-1}| \to 0.$$

*and define the mappings* $T_n = T(p_n): D_1(E, \tau) \to D_1(E, \tau)$ *by the formula*

$$(1.5)\quad T_n(x)(t) = \begin{cases} x(t_{nk}) & \text{if} \quad t \in [t_{nk}, t_{n,k+1}), \quad k=0, 1, \ldots, k_n - 1 \\ x(1) & \text{if} \quad t = 1. \end{cases}$$

*Then*

i) *for every* $x \in D_1(E, \tau)$ *and* $i \in \mathbb{I}$, $\tilde{d}_i(T_n(x), x) \to 0$, *i. e.* $T_n(x) \to x$ *in the completely regular topological space* $(D_1(E, \tau), \{ \tilde{d}_i \}_{i \in \mathbb{I}})$.

ii) *for each n, the set* $T_n(D_1(E, \tau)) \subset D_1(E, \tau)$ *is homeomorphic with* $E^{k_n+1}$ *equipped with the product topology.*

iii) *The set* $\bigcup_{n \in \mathbb{N}} T_n(D_1(E, \tau))$ *is sequentially dense in* $D_1(E, \tau)$.

*Proof.* — *i*) Follows by suitable adjusted Lemma 1, p. 110, [3], and implies *iii*), easily. *ii*) is an immediate consequence of the definition (1.3) of pseudometrics $\tilde{d}_i$, $i \in \mathbb{I}$. ■

1.5. LEMMA. — *Let* $\{ d_i \}_{i \in \mathbb{I}}$ *and* $\{ \zeta_j \}_{j \in \mathbb{J}}$ *be two families of pseudometrics on* E *satisfying (1.1) and (1.2). Let the topology* $\tau$ *generated by* $\{ d_i \}_{i \in \mathbb{I}}$ *be coarser then the topology* $\sigma$ *generated by* $\{ \zeta_j \}_{j \in \mathbb{J}}$.

*Then obviously*

$$D_1(E, \tau) \supset D_1(E, \sigma)$$

*and the topology on* $D_1(E, \sigma)$ *generated by pseudometrics* $\{ \tilde{\zeta}_j \}_{j \in \mathbb{J}}$ *is finer than the topology induced from* $(D_1(E, \tau), \{ \tilde{d}_i \}_{i \in \mathbb{I}})$.

*Proof.* — Observe that for every partition $p_n$,

$$T(p_n)(D_1(E, \tau)) = T(p_n)(D_1(E, \sigma)) \subset D_1(E, \tau) \cap D_1(E, \sigma).$$

By Lemma 1.4 *iii*) and the property (1.2) of pseudometrics $(\tilde{d}_i)_{i \in \mathbb{I}}$, it suffices to check that for each $x_0$ of the form

$$x_0(t) = \begin{cases} a_{t_k} & \text{for} \quad t \in [t_k, t_{k+1}), \qquad k = 0, 1, 2, \ldots, N - 1, \\ a_1 & \text{for} \quad t = 1. \end{cases}$$

where $0 = t_0 < t_1 < \ldots < t_N = 1$, and for each $i \in \mathbb{I}$ and $\varepsilon > 0$ the $\tilde{d}_i$-ball

$$S_\tau(x_0, \varepsilon) = \{ y \in D_1(E, \tau) \mid \tilde{d}_i(x_0, y) < \varepsilon \}$$

contains a certain $\sigma$-ball.

Let for $k = 0, 1, \ldots, N$, $j_k \in \mathbb{J}$ be such that for some $\eta_k > 0$, $\eta_k < \varepsilon$,

$$\{ a \in E \mid d_i(a_{t_k}, a) < \varepsilon \} \supset \{ a \in E \mid \zeta_{j_k}(a_{t_k}, a) < \eta_k \}.$$

Let $\eta = \min_{1 \leqslant k \leqslant N} \eta_k$ and $j$ be such that $\zeta_j \geqslant \max_{0 \leqslant k \leqslant N} \zeta_{j_k}$. If

$$z \in S_\sigma(x_0, \eta) = \{ z \in D_1(E, \sigma) \mid \tilde{\zeta}_j(x_0, z) < \eta \},$$

then there exists $\lambda \in \Lambda$ such that $\sup_{t\in[0,1]} |\lambda(t) - t| < \eta < \varepsilon$ and

$$\sup_{t\in[0,1]} \zeta_j(z(\lambda(t)), x_0(t)) = \max\big(\max_{0\leq k\leq N-1}\big(\sup_{t\in[t_k,t_{k+1})} \zeta_j(z(\lambda(t)), a_{t_k})\big), \zeta_j(z(1), a_1)\big) < \eta .$$

In particular, $\sup_{t\in[t_k,t_{k+1})} d_i(z(\lambda(t)), a_{t_k}) < \varepsilon, k = 0, \ldots, N-1$ and $d_i(z(1), a_1) < \varepsilon$. Hence $S_\sigma(x_0, \eta) \subset S_\tau(x_0, \varepsilon)$. ∎

Let us note some immediate consequences of Theorem 1.3 and Lemma 1.4.

1.6. PROPOSITION. — i) *The continuous mappings* $f : [0,1] \to E$ *form a closed subset* $C([0,1]:(E,\tau))$ *of* $D_1(E,\tau)$ *and the Skorokhod topology on* $C([0,1]:(E,\tau))$ *coincides with the usual compact-open (or « uniform ») topology on* $C([0,1]:(E,\tau))$.

ii) $D_1(E,\tau)$ *is separable iff* $(E,\tau)$ *is separable.*

iii) $D_1(E,\tau)$ *is metrisable iff* $(E,\tau)$ *is metrisable.*

iv) *If* $(F, \tau \cap F)$ *denotes the subspace* $F \subset E$ *with the topology* $\tau \cap F$ *induced from* $(E,\tau)$, *then the Skorokhod topology on* $D_1(F, \tau\cap F)$ *coincides with the topology induced on* $D_1(F,\tau\cap F)$ *by the Skorokhod topology on* $D_1(E,\tau)$.

v) *If* $G$ *is an open subset of* $(E,\tau)$, *then* $D_1(G,\tau\cap G)$ *is an open subset of* $D_1(E,\tau)$. *Similarly, if* $F$ *is a closed subset of* $(E,\tau)$ *then* $D_1(F,\tau\cap F)$ *is a closed subset of* $D_1(E,\tau)$.

vi) *If* $\mathcal{K} \subset D_1(E,\tau)$ *is compact, then there exists a compact subset* $K \subset (E,\tau)$ *such that* $\mathcal{K} \subset D_1(K, \tau\cap K)$.

vii) *All compact subsets of* $D_1(E,\tau)$ *are metrisable iff* $(E,\tau)$ *has this property.*

*Proof.* — *i*) The compact-open topology on $C([0,1]:(E,\tau))$ is simply the complete regular topology induced by uniform pseudometrics

$$\tilde{d}_i(x,y) = \sup_{t\in[0,1]} d_i(x(t), y(t)), \quad i \in \mathbb{I}, \tag{1.6}$$

where $\{d_i\}_{i\in\mathbb{I}}$ is an arbitrary family of pseudometrics on $(E,\tau)$ generating the topology $\tau$. The arguments quite similar as in the metric case (see [*3*], p. 112) show that the pseudometrics $\tilde{d}_i$ and $\tilde{\tilde{d}}_i$ are equivalent when restricted to $C([0,1]:(E,\tau))$.

*ii*) Follows from Lemma 1.4 *iii*). For the proof of *iii*) and *iv*) see Theorem 1.3.

*v*) Let $G$ be open in $(E,\tau)$ and let $x_0 \in D_1(G, \tau\cap G)$.

By Proposition 1.1 the closure $\bar{x}_0$ of the set $\{x_0(t) \mid t \in [0,1]\}$ is compact in $(G, \tau \cap G)$. Since $G$ is open one can find a pseudometric $d_i$ and $\varepsilon > 0$ such that $\{ a \in E \mid d_i(a, \bar{x}_0) < \varepsilon \} \subset G$. So necessarily $S_{\tilde{d}_i}(x_0, \varepsilon) \subset D_1(G, \tau \cap G)$.

Observe that arguing in the same way one can prove that the sets $\tilde{G}_{t_0} = \{ x \mid x(t) \in G$ in some neighbourhood of $t_0 \}$, $t_0 \in [0, 1)$ and $\tilde{G}_1 = \{ x \mid x(1) \in G \}$ are open in $D_1(E, \tau)$. Let $F$ be closed in $(E, \tau)$. Then $[D_1(F, \tau \cap F)]^c = \bigcup_{t_0 \in [0,1]} (\tilde{F^c})_{t_0}$ is open.

*vi*) Let $\mathscr{K}$ be compact in $D_1(E, \tau)$. Let $K = \bigcup_{x \in \mathscr{K}} \bar{x}$. We have to show that $K$ is compact. Take any covering of $K$ by open sets : $K \subset \bigcup_{\alpha \in A} G_\alpha$. By Proposition 1.1 we know that for each $x \in \mathscr{K}$, there exists a finite number of the sets $G_\alpha$: $G_{\alpha_1(x)}, G_{\alpha_2(x)}, \ldots, G_{\alpha_m(x)}$, such that $\bar{x} \subset \bigcup_{1 \leqslant i \leqslant m} G_{\alpha_i(x)}$. Consider open sets of the form $G_{A_0} = \bigcup_{\alpha \in A_0} G_\alpha$, where $A_0$ is a finite subset of $A$. We have

$$\mathscr{K} \subset \bigcup_{\substack{A_0 \subset A \\ A_0\text{-finite}}} D_1(G_{A_0}, \tau \cap G_{A_0}).$$

By *v*) and the compactness of $\mathscr{K}$ one can find a finite sequence $A_1, A_2, \ldots, A_n$ of finite subsets of $A$ such that $\mathscr{K}$ is covered by the sum of $D_1(G_{A_j}, \tau \cap G_{A_j})$. But this implies that $K = \bigcup_{x \in \mathscr{K}} \bar{x} \subset \bigcup_{1 \leqslant j \leqslant n} G_{A_j} = \bigcup_{1 \leqslant j \leqslant n} \bigcup_{\alpha \in A_j} G_\alpha$. The statement *vii*) follows from *i*), *iii*), *iv*) and *vi*). ∎

The next theorem is based on the essential property of completely regular topological spaces and seems to be the most interesting application of Theorem 1.3.

1.7. THEOREM. — *Let* $(E, \tau)$ *be a completely regular topological space. Suppose that the family* $\mathbb{F} = \{ f : E \to R^1 \}$ *of continuous functions on* $(E, \tau)$ *has the following two properties*:

$$\text{(1.7)} \qquad \mathbb{F} \text{ generates the topology } \tau \text{ on } E.$$

$$\text{(1.8)} \qquad \text{If } f, g \in \mathbb{F}, \text{ then } f + g \in \mathbb{F}.$$

*Then the Skorokhod topology on* $D_1(E, \tau)$ *is generated by the family* $\tilde{\mathbb{F}} = \{ \tilde{f} \mid f \in \mathbb{F} \}$ *of mappings* $\tilde{f} : D_1(E, \tau) \to D_1(R^1)$ *of the form*

$$[\tilde{f}(x)](t) = f(x(t)), \tag{1.9}$$

*where* $f$ *belongs to* $\mathbb{F}$.

1.8. LEMMA. — *Let* $(E_2, \tau_2)$ *be a completely regular topological space. Consider a family* $\{ f_i : E_1 \to (E_2, \tau_2) \}_{i \in \mathbb{I}}$ *which separates points in* $E_1$. *Denote by* $\tau_1$ *the topology on* $E_1$ *generated by the family* $\{ f_i \}_{i \in \mathbb{I}}$.

*Then the Skorokhod topology on* $D_1(E_1, \tau_1)$ *is generated by maps*

$$\{ \tilde{f}_{I_0} : D_1(E_1, \tau_1) \to D_1((E_2)^{I_0}, (\tau_2)^{I_0}) \}_{I_0 \subset \mathbb{I},\, I_0\text{-finite}}$$

*of the form*

$$[\tilde{f}_{I_0}(x)](t) = (f_i(x(t)))_{i \in I_0} \in (E_2)^{I_0}. \tag{1.10}$$

*Proof.* — The product topology on $(E_2)^{I_0}$ can be defined by the pseudo-metrics

$$\zeta_{I_0,j}((a_i)_{i \in I_0}, (b_i)_{i \in I_0}) = \max_{i \in I_0} \zeta_j(a_i, b_i).$$

where the family $\{ \zeta_j \}_{j \in \mathbb{J}}$ of pseudometrics on $E_2$ determines the topology $\tau_2$. So the topology $\tau_1$ on $E_1$ can be defined by the basis consisting of all open balls in the pseudometrics $d_{I_0,j}(a, b) = \zeta_{I_0,j}(f_{I_0}(a), f_{I_0}(b))$.

Now, it is sufficient to see that

$$\{ y \in D_1(E, \tau) \mid \tilde{d}_{I_0,j}(x, y) < \varepsilon \} \\ = (\tilde{f}_{I_0})^{-1} \{ z \in D_1((E_2)^{I_0}, (\tau_2)^{I_0}) \mid \tilde{\zeta}_{I_0,j}(z, \tilde{f}_{I_0}(x)) < \varepsilon \}. \qquad \blacksquare$$

By the above lemma, if the family $\mathbb{F} = \{ f : E \to R^1 \}$ generates the topology $\tau$ on E, the Skorokhod topology on $D_1(E_1, \tau)$ is generated by all « vectors »

$$D_1(E, \tau) \ni x \mapsto [(f_1(x(.)), \dots, f_m(x(.))) : [0, 1] \to R^m] \in D_1(R^m),$$

where $f_1, f_2, \dots, f_m \in \mathbb{F}$. The final reduction follows now from the fact that the Skorokhod topology on $D_1(R^m)$ is generated by the mappings

$$\tilde{l}_k : D_1(R^m) \to D_1(R^1), \qquad 1 \leqslant k \leqslant m,$$
$$\widetilde{l_j + l_k} : D_1(R^m) \to D_1(R^1), \qquad 1 \leqslant j < k \leqslant m,$$

where $l_k : R^m \to R^1$, $l_k(a_1, a_2, \dots, a_m) = a_k$. Indeed, if $x_n \to x_0$ in $D_1(R^m)$ then

$$\tilde{l}_k(x_n) \to l_k(x_0), \qquad 1 \leqslant k \leqslant m, \tag{1.11}$$
$$\widetilde{l_j + l_k}(x_n) \to \widetilde{l_j + l_k}(x_0), \qquad 1 \leqslant j < k \leqslant m,$$

since $l_k$, $1 \leqslant k \leqslant m$, are continuous.

Conversely, suppose that (1.11) holds. If $x_n \not\to x_0$ then $\{ x \}_{n \in \mathbb{N}}$ *cannot be relatively compact.*

1.9. LEMMA (A version of Lemma 27.3 [1], see also Lemma 5.2 [6]). — *Let* K *be a compact metric space. A subset* $\mathscr{K} \subset D_1(K)$ *is not relatively compact if and only if at least one of the following three conditions is satisfied.*

(1.12) *There exist*

a) *a sequence* $\{ x_n \} \subset \mathscr{K}$
b) $t \in [0, 1]$
c) *three sequences* $t \leftarrow s_n < t_n < u_n \rightarrow t$
d) *elements* $a \neq b \neq c$ *of* K

*such that* $x_n(s_n) \rightarrow a$, $x_n(t_n) \rightarrow b$, $x_n(u_n) \rightarrow c$.

(1.13) *There exist*

a) *a sequence* $\{ x_n \} \subset \mathscr{K}$
b) *two sequences* $0 \leqslant s_n < t_n \rightarrow 0$
c) *elements* $a \neq b$ *of* K.

*such that* $x_n(s_n) \rightarrow a$, $x_n(t_n) \rightarrow b$.

(1.14) *There exist*

a) *a sequence* $\{ x_n \} \subset \mathscr{K}$
b) *two sequences* $1 > t_n \geqslant s_n \rightarrow 1$
c) *elements* $a \neq b$ *of* K

*such that* $x_n(s_n) \rightarrow a$, $x_n(t_n) \rightarrow b$. ■

Observe that all $x_n$ take values in some compact subset $K \subset R^m$, provided (1.11) is fulfilled. Hence we can apply Lemma 1.9.

Suppose that for some subsequence $\{ x_{n'} \}$ of $\{ x_n \}$ the condition (1.12) is satisfied. In particular, for all $k$, $1 \leqslant k \leqslant m$,

$$(1.15) \quad \tilde{l}_k(x_{n'})(s_{n'}) \rightarrow l_k(a), \quad \tilde{l}_k(x_{n'})(t_{n'}) \rightarrow l_k(b) \quad \tilde{l}_k(x_{n'})(u_{n'}) \rightarrow l_k(c).$$

If for some $k$, $l_k(a) \neq l_k(b) \neq l_k(c)$, then we get a contradiction with the relative compactness of $\{ \tilde{l}_k(x_{n'}) \}$. Hence for all $k$, $l_k(a) = l_k(b)$ or $l_k(b) = l_k(c)$. Since $a \neq b \neq c$, we can find $j$ and $k$, such that

$$l_j(a) \neq l_j(b)(= l_j(c)) \quad \text{and} \quad (l_k(a) = )l_k(b) \neq l_k(c).$$

Hence $(l_j + l_k)(a) \neq (l_j + l_k)(b) \neq (l_j + l_k)(c)$ and writing down the convergence (1.15) for $\widetilde{l_j + l_k}$ instead of $\tilde{l}_k$, we get a contradiction with relative compactness of $\widetilde{l_j + l_k}(x_{n'})$. Similarly one can eliminate the conditions (1.13) and (1.14). Hence $x_n \rightarrow x_0$. ■

## 2. MEASURABILITY PROBLEMS IN D([0,1] : E)

Let $(E, \tau)$ be a completely regular topological space and let $\mathbb{X} = (X_t)_{t\in[0,1]}$ be a stochastic process with values in E, i. e. a family of measurable mappings

$$X_t : (\Omega, \mathscr{F}, P) \to (E, \mathscr{B}_E), \qquad t \in [0,1].$$

where $(\Omega, \mathscr{F}, P)$ is a probability space and $\mathscr{B}_E$ denotes the $\sigma$-algebra of Borel subsets of E.

Suppose that all trajectories of $\mathbb{X}$ belong to the space $D_1(E)$ (here and in the sequel we drop the symbol $\tau$ of topology, for $\tau$ is fixed). Then $\mathbb{X}$ considered as a map

$$\mathbb{X} : (\Omega, \mathscr{F}, P) \to D_1(E)$$

is measurable, if $D_1(E)$ is equipped with the $\sigma$-algebra $\mathscr{C}_{D_1(E)}$ generated by simple cylindrical subsets of $D_1(E)$. Recall that

$$(2.1)\quad \mathscr{C}_{D_1(E)} = \sigma(\pi_t^{-1}(\mathscr{B}_E) \mid t \in [0,1]) = \sigma((\pi_{t_1,\dots,t_m})^{-1}(\mathscr{B}_E)^{\otimes m} \mid t_1, \dots, t_m \in [0,1], m \in \mathbb{N})$$

where for $T_0 = \{t_1, t_2, \dots, t_m\} \subset [0,1]$, the projection $\pi_{T_0} = \pi_{t_1,t_2,\dots,t_m} : D_1(E) \to E^m$ is defined by

$$(2.2)\qquad \pi_{T_0}(x) = (x(t_1), x(t_2), \dots, x(t_m)).$$

In the case $E = R^1$, a remarkable equality $\mathscr{C}_{D_1(R^1)} = \mathscr{B}_{D_1(R^1)}$ holds (see [3], Th. 14.5) and it is reasonable to ask which spaces E behave similarly to $R^1$, i. e. which of them preserve the equality

$$(2.3)\qquad \mathscr{C}_{D_1(E)} = \mathscr{B}_{D_1(E)}.$$

We shall prove several results in this direction.

2.1. THEOREM. — i) *Suppose that* $(E, \tau)$ *has the property (2.3). Then every subspace* $(E_1, \tau \cap E_1)$ *of* $(E, \tau)$ *has the property (2.3).*

ii) *If every finite product space* $\prod_{1\leqslant i\leqslant m} (E_i, \tau_i)$ *composed of the elements of a sequence* $\{(E_i, \tau_i)\}_{i\in\mathbb{N}}$ *has the property (2.3), then this property is fulfilled also for the infinite product* $\prod_{i\in\mathbb{N}} (E_i, \tau_i)$.

*Proof.* — Let $E_1 \subset E$. By the property *iv*), Proposition 1.6, of the Skorokhod topology, $\mathscr{B}_{D_1(E_1)} = \mathscr{B}_{D_1(E)} \cap D_1(E_1)$, and by the definition of $\sigma$-algebras $\mathscr{C}_{D_1(E)}$ and $\mathscr{C}_{D_1(E_1)}$, $\mathscr{C}_{D_1(E_1)} = \mathscr{C}_{D_1(E)} \cap D_1(E_1)$. Those two equalities prove the part *i*) of Theorem 2.1.

In order to make the proof of *ii*) more concise, consider the following lemma.

2.2. LEMMA. — *Let* $(E, \tau)$ *be a Hausdorff topological space and let the topology* $\tau$ *be generated by a family of mappings* $\{ f_i : (E, \tau) \to (E_i, \tau_i) \}_{i \in \mathbb{I}}$.

i) *For every subset* $A \subset E$,

$$\overline{A} = \bigcap_{\substack{I_0 \subset \mathbb{I} \\ I_0\text{-finite}}} f_{I_0}^{-1}(\overline{f_{I_0}(A)}) \tag{2.4}$$

*where* $\overline{A}$ *denotes the closure of* A *and for a finite subset* $I_0 \subset \mathbb{I}$ *the map* $f_{I_0} : E \to \prod_{i \in I_0} E_i$ *is defined by*

$$f_{I_0}(a) = (f_i(a))_{i \in I_0}. \tag{2.5}$$

ii) *If the family* $\{ f_i \}_{i \in \mathbb{I}}$ *is countable, then the* $\sigma$*-algebra of Borel subsets of* E *is generated by « finite dimensional » Borel subsets*:

$$\mathscr{B}_E = \sigma((f_{I_0})^{-1}(\mathscr{B}_{\prod_{i \in I_0} E_i}) \mid I_0 \subset \mathbb{I}, I_0\text{-finite}) \tag{2.6}$$

iii) *If the family* $\{ f_i \}_{i \in \mathbb{I}}$ *is countable and, in addition, for each two elements* $f_j$ *and* $f_k$ *of the generating family, there exists an element* $f_m \in \{ f_i \}_{i \in \mathbb{I}}$ *such that* $f_j$ *and* $f_k$ *can be factorized by continuous mappings and* $f_m$:

$$f_j = g_{jm} \circ f_m, \qquad f_k = g_{km} \circ f_m,$$

*where* $g_{jm} : E_m \to E_j$ *and* $g_{km} : E_m \to E_k$ *are continuous, then the structure of Borel subsets of* E *can be described in especially simple way*:

$$\mathscr{B}_E = \sigma((f_i)^{-1}(\mathscr{B}_{E_i}) \mid i \in \mathbb{I}). \quad \blacksquare \tag{2.7}$$

Now, since the topology of the infinite product is generated by projections $f_j : \prod_{i \in \mathbb{N}} E_i \to R_j$, $j \in \mathbb{N}$, the Skorokhod topology on $D_1\left(\prod_{i \in \mathbb{N}} E_i\right)$ is generated by mappings (see Lemma 1.8).

$$F_m = f_{\{1,2,\ldots,m\}} : D_1\left(\prod_{i \in \mathbb{N}} E_i\right) \to D_1\left(\prod_{1 \leq i \leq m} E_i\right), \quad m \in \mathbb{N}.$$

By Lemma 2.2 *iii*), the Borel $\sigma$-algebra in $D_1\left(\prod_{i\in\mathbb{N}} E_i\right)$ is generated by the $\sigma$-algebras

$$F_m^{-1}(\mathscr{B}_{D_1(\prod_{1\le i\le m} E_i)}) = F_m^{-1}(\mathscr{C}_{D_1(\prod_{1\le i\le m} E_i)}), \qquad m \in \mathbb{N}.$$

On the other hand, a similar consideration applied to the $\sigma$-algebra $\mathscr{C}_{D_1(\prod E_i)}$ yields

$$\begin{aligned}
\mathscr{C}_{D_1(\prod_{i\in\mathbb{N}} E_i)} &= \sigma((\pi_t)^{-1}(\mathscr{B}_{\prod_{i\in\mathbb{N}} E_i}) \mid t \in [0,1]) \\
&= \sigma((\pi_t)^{-1}(f^{-1}_{\{1,2,\dots,m\}}(\mathscr{B}_{\prod_{1\le i\le m} E_i})) \mid t \in [0,1], m \in \mathbb{N}) \\
&= \sigma(F_m^{-1}(\mathscr{C}_{D_1(\prod_{1\le i\le m} E_i)}) \mid m \in \mathbb{N}).
\end{aligned}$$

Hence the generators of $\mathscr{B}_{D_1(\prod_{i\in\mathbb{N}} E_i)}$ and $\mathscr{C}_{D_1(\prod_{i\in\mathbb{N}} E_i)}$ are the same. ■

2.3. REMARK. — *It follows from Lemma 1.4 iii), that the property (2.3) implies*

$$(2.8) \qquad \mathscr{B}_{E^n} = (\mathscr{B}_E)^{\otimes n} \quad \text{for every} \quad n \in \mathbb{N}.$$

By Lemma 2.2 *ii*), (2.8) is equivalent to $\mathscr{B}_{E^\infty} = (\mathscr{B}_E)^{\otimes\infty}$.

Let $E = R^1$ with *discrete* topology. Then $\mathscr{B}_E \otimes \mathscr{B}_E$ is strictly contained in $\mathscr{B}_{E\times E}$. Hence (2.3) may fail even in case of metric spaces.

If E is a separable metric space, then it is homeomorphic to a subset of $R^\infty$. From Theorem 2.1 immediately follows

2.4. COROLLARY. — *The property (2.3) is fulfilled for separable metric spaces.*

It is well known that for a metric space E, the Borel $\sigma$-algebra in E coincides with the Baire $\sigma$-algebra generated by all continuous functions. In the case when such a coincidence can be verified, one can prove at least one inclusion in (2.3).

2.5. THEOREM. — i) *Suppose that in* E *the Borel and the Baire $\sigma$-algebras coincide*:

$$(2.9) \qquad \mathscr{B}_E = \sigma(C(E : R^1)).$$

*Then the simple cylindrical subsets of* $D_1(E)$ *are Baire subsets of* $D_1(E)$, *in particular*,

$$(2.10) \qquad \mathscr{C}_{D_1(E)} = \sigma(f \circ \pi_t \mid f \in C(E : R^1), t \in [0,1]) \subset \mathscr{B}_{D_1(E)}.$$

ii) *Suppose, in addition, that* $\mathscr{B}_{E^n} = (\mathscr{B}_E)^{\otimes n}$, $n \in \mathbb{N}$. *Then every continuous function on* $D_1(E)$ *is* $\mathscr{C}_{D_1(E)}$*-measurable, i. e.* $\mathscr{C}_{D_1(E)}$ *coincides with the Baire* $\sigma$*-algebra of subsets of* $D_1(E)$:

$$(2.11) \qquad \mathscr{C}_{D_1(E)} = \sigma(C(D_1(E) : R^1)).$$

*Proof.* — Let $f \in C(E : R^1)$ and $t \in [0, 1]$. The function $f \circ \pi_t$ is a superposition of the countinuous mapping $\tilde{f}: D_1(E) \to D_1(R^1)$ (see (1.9)) and the $\{ t \}$-projection $\tilde{\pi}_t : D_1(R^1) \to R^1$ which is measurable. Hence $f \circ \pi_t = \tilde{\pi}_t \circ \tilde{f}$ is Baire-measurable on $D_1(E)$. This proves i).

Now, take any continuous function $f: D_1(E) \to R^1$. In notations of Lemma 1.4, for every normal sequence $\{ p_n \}_{n \in \mathbb{N}}$ of partitions of $[0, 1]$,

$$f(x) = \lim_{n \to \infty} [f \circ T(p_n)](x), \qquad x \in D_1(E).$$

Hence it suffices to verify, that $f \circ T(p_n)$ is measurable with respect to $\mathscr{C}_{D_1(E)}$. Let $p_n = \{ 0 = t_{n0} < t_{n1} < \ldots < t_{nk_n} = 1 \}$ and let

$$h : T(p_n)(D_1(E)) \to E^{(k_n + 1)}$$

be the natural homeomorphism between these two spaces (see Lemma 1.4 *ii*)). Then

$$f \circ T(p_n) = f \circ h^{-1} \circ \pi_{\{p_n\}}$$

where $f \circ h^{-1} : E^{(k_n+1)} \to R^1$ is continuous and $\pi_{\{p_n\}} = \pi_{t_{n0}, t_{n1}, \ldots, t_{nk_n}}$. Hence

$$\begin{aligned}[][f \circ T(p_n)]^{-1}(\mathscr{B}^1) &= \pi_{\{p_n\}}^{-1} \circ (f \circ h^{-1})^{-1}(\mathscr{B}^1) \\ &\subset \pi_{\{p_n\}}^{-1}(\mathscr{B}_E^{(k_n+1)}) = \pi_{\{p_n\}}^{-1}((\mathscr{B}_E)^{\otimes (k_n+1)}) \subset \mathscr{C}_{D_1(E)}\end{aligned}$$

provided $\mathscr{B}_{E^n} = (\mathscr{B}_E)^{\otimes n}$ for all $n \in \mathbb{N}$. ■

By assuming a more special structure of E one can use Theorem 2.5 *ii*) to derive the stronger property (2.3). Here two examples are presented.

2.6. COROLLARY. — *Suppose that the space* $(E, \tau)$ *has the following two properties*:

(2.12) *Compact subsets of* E *are metrisable.*

(2.13) *There exists a sequence* $\{ K_n \}_{n \in \mathbb{N}}$ *of compact subsets of* E *such that for every* $x \in D_1(E)$ *one can find* $K_n$ *containing the set* $\bar{x} = \{ x(t) \mid t \in [0, 1] \}$.

Then E has the property (2.3).

2.7. LEMMA. — *If* E *is* $\sigma$*-compact and every compact subset of* E *is metrisable, then every compact* $K \subset E$ *is a Baire subset of* E.

*Proof.* — Let $E = \bigcup_{n \in \mathbb{N}} K_n$, where $K_n$ are compact in E. For each $n \in \mathbb{N}$, let $\{ f_{nk} \}_{k \in \mathbb{N}}$ be a sequence of continuous functions on E, separating points in $K_n$:

$$\forall_{\substack{a,b \in K_n \\ a \neq b}} \exists_{k \in \mathbb{N}} f_{nk}(a) \neq f_{nk}(b) .$$

If the sequence $\{ g_m \}_{m \in \mathbb{N}}$ exhausts the sum $\bigcup_{n \in \mathbb{N}} \{ f_{nk} \mid k \in \mathbb{N} \}$, then it separates points in E. Now, consider a Hausdorff topology $\tau'$ on E generated by the sequence $\{ g_m \}_{m \in \mathbb{N}}$. Clearly, compact set K in $(E, \tau)$ is also compact in $(E, \tau')$, hence is measurable with respect to the $\sigma$-algebra generated by the sequence $\{ g_n \} \subset C(E : R^1)$. Thus each compact subset of $(E, \tau)$ is a Baire subset of $(E, \tau)$. ■

In order to prove Corollary 2.6 we will see that

(2.14) the Baire and Borel $\sigma$-algebras in E coincide,

(2.15) $\mathscr{B}_{E^n} = (\mathscr{B}_E)^{\otimes n}$ for every $n \in \mathbb{N}$,

Then we will apply Theorem 2.5 *ii*) and show that

(2.16) in $D_1(E)$ the Baire and Borel $\sigma$-algebras coincide.

Let A be a Borel subset of E. By (2.13) $A = \bigcup_{n \in \mathbb{N}} A \cap K_n$, where for each $n \in \mathbb{N}$, $A \cap K_n$ is a Borel set in $K_n$, hence by (2.12) a Baire subset of $K_n$. Now $A \cap K_n$ is also a Baire subset of E by Lemma 2.7.

The statement (2.15) follows immediately from $\sigma$-compactness of E and the equality

$$(\mathscr{B}_{E^n}) \cap K^n = (\mathscr{B}_E \cap K)^{\otimes n}$$

which is fulfilled by (2.12) for every compact set $K \subset E$.

The statement (2.16) can be proved quite similarly as (2.14) provided the decomposition $D_1(E) = \bigcup_{n \in \mathbb{N}} D_1(K_n)$ (given by (2.13)) is known and $D_1(K)$ is a Baire subset of $D_1(E)$ for every compact $K \subset E$. But

$$D_1(K) = \bigcap_{t \in Q \cap [0,1]} \pi_t^{-1}(K)$$

where Q is the set of rationals, and in Theorem 2.5 *i*) we have already verified that under (2.14) the cylindrical set $\pi_t^{-1}(K)$ is a Baire subset of $D_1(E)$. ■

2.8. COROLLARY. — *For each $j \in \mathbb{N}$, let the space $E_j$ has the properties (2.12) and (2.13) from Corollary 2.6. Then following the line of the proof of the preceeding corollary, one can check, that every finite product $E_1 \times E_2 \times \ldots \times E_m$ has the property (2.3). By Theorem 2.1 ii), the* infinite *product $\prod_{j \in \mathbb{N}} E_j$ has the property (2.3), too.*

## 3. WEAK TIGHTNESS CRITERIA

The aim of this section is to prove tightness criteria for probability measures on $D_1(E)$.

Recall that a family $\{ \mu_i \}_{i \in \mathbb{I}}$ of probability measures on a topological space $(E, \tau)$ is tight, iff for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset E$ such that

$$\mu_i(K_\varepsilon) > 1 - \varepsilon, \qquad i \in \mathbb{I}.$$

3.1. THEOREM. — *Let $(E, \tau)$ be a completely regular topological space with* metrisable *compacts.*

*Let $\mathbb{F}$ be a family of continuous functions on* E. *Suppose that:*

(3.1) *$\mathbb{F}$ separates points in* E

(3.2) *$\mathbb{F}$ is closed under addition, i. e. if $f, g \in \mathbb{F}$, then $f + g \in \mathbb{F}$.*

i) *A family $\{ \mu_i \}_{i \in \mathbb{I}}$ of probability measures on $\mathscr{B}_{D_1(E)}$ is tight iff the following two conditions hold:*

(3.3) *For each $\varepsilon > 0$ there is a compact $K_\varepsilon \subset E$ such that*

$$\mu_i(D_1(K_\varepsilon)) > 1 - \varepsilon, \qquad i \in \mathbb{I}.$$

(3.4) *The family $\{ \mu_i \}_{i \in \mathbb{I}}$ is $\mathbb{F}$-weakly tight, i. e. for each $f \in \mathbb{F}$ the family $\{ \mu_i \circ (\tilde{f})^{-1} \}_{i \in \mathbb{I}}$ of probability measures on $D_1(R^1)$ is tight.*

ii) *If the family $\{ \mu_i \}_{i \in \mathbb{I}}$ is tight, then it is relatively compact in the weak topology.*

*Proof.* — The part *ii*) is true in general case: for families of measures on completely regular topological space with metrisable compacts (we know by Proposition 1.6 *vii*) that compact subsets of $D_1(E)$ are metrisable provided E has this property), see [*13*], Th. 2, § 5, or [*7*], Th. 6, § 4.

Let us consider the problem of necessity in part *i*). The condition (3.3) follows from Proposition 1.6 *vi*). Tightness implies also (3.4) due to the two following facts:

— for every $f \in C(E : R^1)$, $\tilde{f}$ is a continuous mapping of $D_1(E)$ into $D_1(R^1)$ (see Theorem 1.7),

— the image under a continuous map of a tight family of measures is tight again.

In order to prove the sufficiency of conditions (3.3) and (3.4), let us establish two simple topological lemmas.

3.2. LEMMA. — *Assume that* E *and* $\mathbb{F}$ *are as in Theorem 3.1. Then for every compact* $K \subset E$ *there exists a* countable *family* $\mathbb{F}_K \subset \mathbb{F}$ *satisfying (3.1) and (3.2) when restricted to* K.

*Proof.* — If $\mathbb{F}'_K$ satisfies (3.1), then finite sums of elements of $\mathbb{F}'_K$ form a class $\mathbb{F}_K$ satisfying (3.1) and (3.2). Hence we have to prove only that for every K one can find a countable separating subfamily of $\mathbb{F}$.

Consider the compact space $K \times K$. Let $(a, b) \in K \times K$, $a \neq b$. By (3.1) there exists $f \in \mathbb{F}$ such that $f(a) \neq f(b)$.

Let $U(a, b, f)$ be a neighbourhood of $(a, b)$ of the form $V \times W$, where $a \in V, b \in W: f(V) \cap f(W) = \phi$. Then $\Delta \cap V \times W = \phi$, if $\Delta = \{ (a, a) \mid a \in K \}$. $K \times K \backslash \Delta$ is a separable metric space, and the family $\{ U(a, b, f) \mid (a, b) \in K \times K \}$ covers $K \times K \backslash \Delta$. Hence there exists countable subcovering $\{ U(a_k, b_k, f_k) \}_{k \in \mathbb{N}}$. The family $\{ f_k \}_{k \in \mathbb{N}}$ separates points in K. ■

3.3. LEMMA. — *Let* K *be a metrisable compact. Suppose that a countable family* $\mathbb{F}$ *of continuous functions on* K *satisfies (3.1) and (3.2). Then a closed subset* $\mathcal{K} \subset D_1(K)$ *is compact if and only if the set* $f(\mathcal{K})$ *is compact in* $D_1(R^1)$ *for each* $f \in \mathbb{F}$.

*Proof.* — Only relative compactness of $\mathcal{K}$ remains to be proved. For this we apply Lemma 1.9 in the same way as in the proof of Theorem 1.7. ■

Now, the proof of the sufficiency of (3.3) and (3.4) is immediate. Let $\varepsilon > 0$. Let K be such that

$$\mu_i(D_1(K)) > 1 - \varepsilon/2, \qquad i \in \mathbb{I}.$$

Given K, let $\mathbb{F}_K$ be a countable subfamily of $\mathbb{F}$ satisfying on K the conditions (3.3) and (3.4) (Lemma 3.2). Denumerating the elements of $\mathbb{F}_K$ we obtain a sequence $\{ f_k \}_{k \in \mathbb{N}}$. For every $k \in \mathbb{N}$, let $\mathcal{K}_k$ be such a compact in $D_1(R^1)$ that

$$\mu_i \circ (\tilde{f}_k)^{-1}(\mathcal{K}_k) > 1 - \varepsilon/2^{k+1}, \qquad i \in \mathbb{I}.$$

Then the set $\mathcal{K} = D_1(K) \cap \bigcap_{k \in \mathbb{N}} (\tilde{f}_k)^{-1}(\mathcal{K}_k)$ is compact by Lemma 3.3 and has the property

$$\mu_i(\mathcal{K}) > 1 - \varepsilon, \qquad i \in \mathbb{I}.$$

Hence the family $\{ \mu_i \}_{i \in \mathbb{I}}$ is tight. ■

## 4. THE SKOROKHOD TOPOLOGY ON D($R^+$ : E)

According to our convention D($R^+$: E) (or simply D(E)) will denote the space of mappings $x: R^+ \to E$, which are right-continuous and have left limits in every $t > 0$.

In the case $E = R^1$, Lindvall [8] has defined the Skorokhod topology on D($R^1$) as the topology generated by mappings

$$r_n : D(R^+ : R^1) \to D([0, n+1] : R^1), \qquad n \in \mathbb{N},$$

where

$$r_n(x)(t) = g_n(t) . x(t), \qquad t \in [0, n+1] \tag{4.1}$$

and

$$g_n(t) = \begin{cases} 1 & \text{if} \quad t \in [0, n] \\ -t+n+1 & \text{if} \quad t \in [n, n+1] . \end{cases}$$

Since the generating family is countable, D($R^1$) is metric and separable (and even topologically complete).

When E is a linear topological space, Lindvall's ideas are applicable without any change. It is not so in the general case and we suggest to proceed in the following way.

Let $d$ be any pseudometric on E. By $\tilde{d}_s$ we will denote the pseudometric on D([0, $s$]: E) defined by the formula (1.3) (where the interval [0, 1] is replaced by [0, $s$]).

Let the map $q_s$: D($R^+$ : E) $\to$ D([0, $s+1$]: E) be defined as:

$$q_s(x)(t) = \begin{cases} x(t) & \text{if} \quad 0 \leqslant t \leqslant s \\ x(s) & \text{if} \quad s \leqslant t \leqslant s+1 . \end{cases} \tag{4.2}$$

Fix $x$ and $y$ in D(E) and consider the function

$$R^+ \ni s \mapsto \widetilde{d_{s+1}}(q_s(x), q_s(y)) = \zeta_s^d(x, y) . \tag{4.3}$$

Clearly, it is an element of D($R^+$ : $R^+$), hence is Borel measurable. Define

$$\zeta^d(x, y) = \int_0^{\infty} e^{-s} \min (1, \zeta_s^d(x, y)) ds . \tag{4.4}$$

$\zeta^d$ is a pseudometric on D(E) and Proposition 4.1 below shows that the convergence of sequences in $\zeta^d$ is just the « Skorokhod convergence ».

4.1. PROPOSITION. — *Let* $\{ x_n \}_{n\in\mathbb{N}}$ *be a sequence of elements of* D(E). *Let* $\zeta^d$ *be defined by (4.4). Then for some* $x \in$ D(E) *the convergence* $x_n \underset{\zeta^d}{\to} x$ *holds if and only if there exists a sequence* $\{ \lambda_n \}_{n\in\mathbb{N}} \subset \Lambda_\infty = \{ \lambda : \mathrm{R}^+ \to \mathrm{R}^+ \mid \lambda$ *is increasing, continuous,* $\lambda(0)=0$, $\lambda(t) \to +\infty$ *if* $t \to +\infty \}$, *such that for each* $t \in \mathrm{R}^+$

$$\sup_{0\leqslant s\leqslant t} |\lambda_n(s) - s| \to 0 \tag{4.5}$$

$$\sup_{0\leqslant s\leqslant t} d(x_n(\lambda_n(s)), x(s)) \to 0. \tag{4.6}$$

*Proof.* — The conditions (4.5) and (4.6) imply that for every $s \in \mathrm{R}^+$ with the property $d(x(s), x(s-)) = 0$, we have the convergence $q_s(x_n) \to q_s(x)$ in $(\mathrm{D}([0, s+1]: \mathrm{E}), \tilde{d}_{s+1})$. The distance $d(x(s), x(s-))$ may differ from 0 at most in countably many $s$, hence by the Lebesgue dominated theorem

$$\zeta^d(x_n, x) \to 0.$$

Conversely, suppose that $\zeta^d(x_n, x) \to 0$. It suffices to find an increasing sequence $\{ s_m \}_{m\in\mathbb{N}} \subset \mathrm{R}^+$, $s_m \nearrow +\infty$ such that $\zeta^d_{s_m}(x_n, x) = \tilde{d}_{s_m+1}(q_{s_m}(x_n), q_{s_m}(x)) \underset{n\to\infty}{\to} 0$, $m \in \mathbb{N}$.

In fact, we shall prove much more, namely that $\zeta^d_s(x_n, x) \to 0$ for each point of continuity of $x$ with respect to $d$.

Suppose that for some $s$ with the property $d(x(s), x(s-)) = 0$ we have $\zeta^d_s(x_n, x) \nrightarrow 0$. Then we can find a subsequence $\{ n' \} \subset \{ n \}$ such that

$$\zeta^d_s(x_n, x) \leqslant \eta \qquad \text{for some } \eta > 0. \tag{4.7}$$

But $\zeta^d_{(.)}(x_n, x) \to 0$ in measure $\mu$ (where $\mu$ is the standard exponential distribution on $\mathrm{R}^+$) so one can choose a subsequence $\{ n'' \} \subset \{ n' \}$ such that $\zeta^d_{(.)}(x_{n''}, x) \to 0$ $\mu$-a. s. Hence there exists $s' \geqslant s$ such that $\zeta^d_{s'}(x_{n''}, x) \to 0$. Consequently $\zeta^d_s(x_{n''}, x) \to 0$ which contradicts (4.7). ■

4.2. REMARK. — During the preparation of this paper the author has been aware of the paper [*10*] by G. Pages, in which another way of metrisation of $\mathrm{D}(\mathrm{R}^+ : \mathrm{E})$ (E-Polish space) has been introduced. One may easily adopt the technique of [*10*] to get new « Skorokhod » pseudometrics on $\mathrm{D}(\mathbb{R}^+ : \mathrm{E})$. It is evident that both the approaches are equivalent.

Similarly as in Sec. 1., one can define the Skorokhod topology on $\mathrm{D}(\mathrm{R}^+ : \mathrm{E})$ as the completely regular topology induced on $\mathrm{D}(\mathrm{R}^+ : \mathrm{E})$ by a family of pseudometrics $\{ \zeta^{d_i} \}_{i\in\mathbb{I}}$, where $\{ d_i \}_{i\in\mathbb{I}}$ determine the topology on E and satisfy (1.1) and (1.2).

Just the same way as Theorem 1.7 one can prove

4.3. THEOREM. — i) *The Skorokhod topology on* $D(R^+ : E)$ *does not depend on the particular choice of the family* $\{ d_i \}_{i \in \mathbb{I}}$.

ii) *If* $\mathbb{F}$ *is any family of continuous functions on* E, *generating the topology on* E *and closed under addition, then the Skorokhod topology on* D(E) *is generated by the family* $\{ \tilde{f} \mid f \in \mathbb{F} \}$, *where, as previously,*

$$\tilde{f} : D(E) \to D(R^1),\ [f(x)](t) = f(x(t)). \qquad \blacksquare$$

Repeating the proofs from Sec. 2, one can obtain Theorems 2.1 and 2.5 for $D(R^+ : E)$.

Although it seems to be evident, we cannot prove that D(E) has the property $\mathscr{B}_{D(E)} = \mathscr{C}_{D(E)}$ if and only if $\mathscr{B}_{D_1(E)} = \mathscr{C}_{D_1(E)}$. Here is a partial result in this direction.

4.4. PROPOSITION. — *Suppose that* E *is a linear topological space. If* $\mathscr{B}_{D_1(E)} = \mathscr{C}_{D_1(E)}$ *then the same is true for the space* D(E).

*Proof.* — It is based on Lemma 2.2 *iii*) and uses the fact that the topology of D(E) is generated by a countable family $\{ r_n \}_{n \in \mathbb{N}}$. $\blacksquare$

4.5. REMARK. — Let $F \subset E$ be any subset of E. Then $D_t(F) = D([0, t] : F)$ is a subset of $D_t(E)$. But $D_t(F)$ can also be interpreted as a subset of $D(E) = D(R^+ : E)$, namely the set of those $x \in D(E)$ which take values in F when restricted to $[0, t]$: $\forall_{0 \leqslant s \leqslant t} x(s) \in F$. The latter case will be used below and in Sec. 5.

We end this section with tightness criteria in D(E).

4.6. THEOREM. — i) *For* E *and* $\mathbb{F}$ *such as in Theorem 3.1, a family* $\{ \mu_i \}_{i \in \mathbb{I}}$ *of probability measures on* $\mathscr{B}_{D(E)}$ *is tight iff the following two conditions hold*:

(4.8) *For each* $t > 0$ *and* $\varepsilon > 0$ *there is a compact subset* $K_{t,\varepsilon} \subset E$ *such that*

$$\mu_i(D_t(K_{t,\varepsilon})) > 1 - \varepsilon, \qquad i \in \mathbb{I}.$$

(4.9) *The family* $\{ \mu_i \}_{i \in \mathbb{I}}$ *is* $\mathbb{F}$*-weakly tight.*

ii) *If the family* $\{ \mu_i \}_{i \in \mathbb{I}}$ *is tight, then it is relatively compact in the weak topology.* $\blacksquare$

## 5. THREE PARTICULAR CASES

Let E be a linear topological space. Denote by E′ the topological dual of E.

For the sake of brevity we introduce a special type of E-valued stochastic processes.

A weakly measurable stochastic process is a family $\mathbb{X} = (X_t)_{t\in R^+}$ of mappings $X_t : (\Omega, \mathcal{F}, P) \to E$ such that for each $y \in E'$, the family $\{ \langle X_t, y \rangle \}_{t\in R^+}$ is a real stochastic process.

If a weakly measurable stochastic process $\mathbb{X}$ satisfies additionally:

*a*) for almost all $\omega \in \Omega$ the trajectory

$$\mathbb{X}(\omega) = (R^+ \ni t \mapsto X_t(\omega) \in E)$$

belongs to $D(R^+ : E)$,

*b*) the « weak distribution » of $\mathbb{X}$, i. e. the probability measure induced by $\mathbb{X}$ on $(D(E), \sigma(\tilde{y} \mid y \in E'))$, admits a unique extension to a probability measure on $\mathcal{B}_{D(E)}$,

then we say that $\mathbb{X}$ has a distribution on D(E).

A family $\{ \mathbb{X}^i \}_{i\in\mathbb{I}}$ of weakly measurable stochastic processes with distributions on D(E) is weakly tight, if for each $y \in E'$, the family of real stochastic processes $\{ \langle \mathbb{X}^i, y \rangle \}_{i\in\mathbb{I}}$ is tight on $D(R^1)$. This definition of weak tightness is just the E′-weak tightness of distributions of processes introduced in Sec. 3.

## I. E is a real separable Banach space.

In such a case the weak measurability of $X_t : (\Omega, \mathcal{F}) \to E$ implies the Borel-measurability (see [*2*]) and a weakly measurable stochastic process is a Borel-measurable mapping into D(E)-see Corollary 2.4. Hence its « weak distribution » is simply its distribution.

In limit theorems for Banach space-valued random variables the « linearized » notion of tightness, called flat concentration, is more useful than the usual tightness. We shall adopt this notion to Skorokhod spaces in order to get weak tightness criteria for stochastic processes in Banach spaces.

Let $\| \cdot \|$ be a norm on E. For any subset $F \subset E$, $F^\varepsilon (F^{\varepsilon]})$ denotes the (closed) $\varepsilon$-neighbourhood of F:

$$F^\varepsilon = \{ b \in E \mid \| a - b \| < \varepsilon \quad \text{for some } a \in F \}$$
$$F^{\varepsilon]} = \overline{F^\varepsilon}.$$

5.1. THEOREM. — *The family* $\{ \mathbb{X}^i \}_{i\in\mathbb{I}}$ *of stochastic processes with distributions on* D(E) *is tight if and only if it is weakly tight and flatly concentrated, i. e. for every* $t > 0$ *and* $\varepsilon > 0$ *there exists a* finite-dimensional subspace $F \subset E$ *such that*

$$P(\mathbb{X}^i \in D_t(F^\varepsilon)) > 1 - \varepsilon, \qquad i \in \mathbb{I}.$$

*Proof.* — Given Theorem 4.6, the proof of Theorem 5.1 follows completely along the line of the proof of Theorem 4.5 on p. 24 of [2]. Indeed, it suffices to check only that weak tightness and flat concentration imply (4.8), i. e.

$$P(\mathbb{X}^i \in D_t(K_{t,\varepsilon})) > 1 - \varepsilon, \qquad i \in \mathbb{I},$$

for some compact $K_{t,\varepsilon} \subset E$.

Let $F_n$, $n \in \mathbb{N}$ be finite-dimensional subspaces of E such that

$$P(\mathbb{X}^i \in D_t((F_n)^{\varepsilon/2^{n+1}})) > 1 - \varepsilon/2^{n+1}, \qquad i \in \mathbb{I}, \quad n \in \mathbb{N}.$$

Moreover, let $r > 0$ be chosen so great, that (by weak tightness)

$$P(\tilde{y}_k(\mathbb{X}^i) \in D_t([-r, r])) = P(\mathbb{X}^i \in D_t(y_k^{-1}([-r, r]))) > 1 - \varepsilon/2m, \qquad i \in \mathbb{I}, \quad 1 \leqslant k \leqslant m,$$

where $\{ y_1, y_2, \ldots, y_m \} \subset E'$ satisfy

$$E' = \{ y \in E' \mid y|_{F_1} = 0 \} \oplus \operatorname{span} \{ y_1, y_2, \ldots, y_m \}.$$

Set

$$K_{t,\varepsilon} = \bigcap_{n \in \mathbb{N}} (F_n)^{(\varepsilon/2^{n+1})]} \cap \bigcap_{1 \leqslant k \leqslant m} y_k^{-1}([-r, r])$$

and observe that by Lemmas 4.4 and 4.3, p. 23-24 of [2], the set $K_{t,\varepsilon}$ is compact. Moreover,

$$P(\mathbb{X}^i \in D_t(K_{t,\varepsilon})) > 1 - \varepsilon, \qquad i \in \mathbb{I},$$

since $D_t\left(\bigcap_{j \in \mathbb{J}} A_j\right) = \bigcap_{j \in \mathbb{J}} D_t(A_j)$ for arbitrary family $\{ A_j \}_{j \in \mathbb{J}}$. ■

From the proof it is clear that in the sufficiency part of Theorem 5.1, the assumption on weak (or, more precisely, E'-weak) tightness may be replaced by D-weak tightness, where D is an arbitrary total and *closed under addition* subset of E'.

5.2. COROLLARY. — *Let* H *be a separable Hilbert space with the inner product* $\langle ., . \rangle$. *For an orthonormal basis* $\{ e_k \}_{k \in \mathbb{N}}$ *in* H, *define the function* $r_N^2$: $H \to R^+$ *by*

$$r_N^2(x) = \sum_{k \geqslant N+1} \langle x, e_k \rangle^2, \qquad N \in \mathbb{N}.$$

*Let* D *be a total and closed under addition subset of* H.

*Then the sequence* $\{ \mathbb{X}^n \}_{n \in \mathbb{N}}$ *of stochastic processes with trajectories in* D(H) *is tight iff it is* D-*weakly tight and for every* $\varepsilon > 0$ *and* $t > 0$

$$\lim_{N \to \infty} \limsup_{n \to \infty} P(r_N^2(X^n(s)) > \varepsilon \text{ for some } s,\ 0 \leqslant s \leqslant t) \to 0. \qquad ■$$

## II. E is the topological dual of a Frechét nuclear space.

Here we are going to look at Mitoma's result [*9*] from our general point of view.

Let $\Phi$ be a Frechét nuclear space (see [*11*]).

Let $\|\,.\,\|_1 \leqslant \|\,.\,\|_2 \leqslant \ldots$ be an increasing sequence of Hilbertian seminorms defining the topology on $\Phi$. Denote by $(\Phi_p, \|\,.\,\|_p)$ the Hilbert space arising by completion of the quotient space $\Phi/\|\,.\,\|_p$ and by $(\Phi'_{-p}, \|\,.\,\|_{-p})$ the topological dual of $(\Phi_p, \|\,.\,\|_p)$. After obvious identification, $\Phi'_{-p}$ is a subset of $\Phi'$ and $\Phi' = \bigcup_{p\in\mathbb{N}} \Phi'_{-p}$.

$\Phi'$ will be always equipped with the strong topology $\beta$ (hence $(\Phi'_\beta)' \sim \Phi$ by reflexibility of $\Phi$). Note that the topology induced from $\Phi'$ onto $\Phi'_{-p}$ is *strictly* weaker than the Hilbert topology $(\Phi'_{-p}, \|\,.\,\|_{-p})$. But on compact subsets of $(\Phi'_{-p}, \|\,.\,\|_{-p})$ both topologies coincide.

Moreover, for every $m \in \mathbb{N}$ there exists $n > m$ such that the canonical mapping $i_{n,m}: \Phi_n \to \Phi_m$ is nuclear. Hence the conjugate imbedding

$$i^*_{n,m} : \Phi'_{-m} \to \Phi'_{-n}$$

is also nuclear. We will use this fact in the proof of

5.3. PROPOSITION. — i) *For each* $t \in \mathrm{R}^+$ *there exists a decomposition*

$$\mathrm{D}_t(\Phi') = \bigcup_{n\in\mathbb{N}} \mathrm{D}_t(\mathrm{K}_n)$$

*where for each* $n$, $\mathrm{K}_n$ *is compact* both *in* $\Phi'_\beta$ *and in* $(\Phi'_{-p}, \|\,.\,\|_{-p})$ *for some* $p = p(n)$ (*i. e. an element* $x \in \mathrm{D}(\Phi')$ *is « locally Hilbertian »*).

ii) $\Phi'$ *has the property (2.3), i. e.* $\mathscr{B}_{\mathrm{D}_1(\Phi')} = \mathscr{C}_{\mathrm{D}_1(\Phi')}$.

*By Proposition 4.4 also* $\mathscr{B}_{\mathrm{D}(\Phi')} = \mathscr{C}_{\mathrm{D}(\Phi')}$.

*Proof.* — Let $x \in \mathrm{D}_t(\Phi')$. For each $s \in [0, t]$, $x(s)$ is a continuous linear functional on the complete metric space $\Phi$. The family $\{ x(s) \mid s \in [0, t] \}$ is pointwisely bounded

$$\sup_{0\leqslant s\leqslant t} |\langle x(s), \varphi \rangle| < +\infty, \varphi \in \Phi,$$

hence by the Banach-Steinhaus Theorem, it is equicontinuous on $\Phi$, i. e. there exists $m \in \mathbb{N}$ and $\mathrm{R} > 0$ such that

$$\sup_{0\leqslant s\leqslant t} \sup_{\|\varphi\|_m\leqslant \mathrm{R}} |\langle x(s), \varphi \rangle| \leqslant 1$$

or

$$\sup_{0 \leqslant s \leqslant t} \| x(s) \|_{-m} \leqslant 1/R = C < +\infty .$$

Hence each $x \in D_t(\Phi')$ belongs to a set of the form

$$K_{m,N} = \{ a \in \Phi' \mid \| a \|_{-m} \leqslant N \}, m \in \mathbb{N}, \qquad N \in \mathbb{N} ,$$

and

$$D_t(\Phi') = \bigcup_{m \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} D_t(\overline{K_{m,N}}) .$$

Let $m$ be fixed. There axists $n > m$ such that the identity imbedding $i^*_{m,n}$: $(\Phi'_{-m}, \| \cdot \|_{-m}) \to (\Phi'_{-n}, \| \cdot \|_{-n})$ is nuclear, hence $K_{m,N}$ as an image of the bounded set under compact operator is pre-compact in $(\Phi'_{-n}, \| \cdot \|_{-n})$. The part *ii*) follows then by Corollary 2.6 (all compact subsets of $\Phi'$ are metrisable, since the space $\Phi$ separates points in $\Phi'$ and is separable itself). ■

Note that by similar arguments one can prove Borel-measurability of weakly measurable stochastic process with values in $\Phi'$.

5.4. Remark. — In the proof of Proposition 5.3 we have used the compactness of the operators $i_{m,n} : \Phi_n \to \Phi_m$ only. In the sequel the nuclearity of $i_{mn}$ is unavoidable.

5.5. Theorem (Mitoma [9]). — *A family $\{ \mathbb{X}^i \}_{i \in \mathbb{I}}$ of stochastic processes in $D(\Phi')$ is tight if and only if it is weakly (i. e. $\Phi$-weakly) tight.*

*Proof.* — By Theorem 4.6 and the compactness of $i^*_{mn} : \Phi'_{-m} \to \Phi'_{-n}$ it suffices to prove only that for each $\varepsilon > 0$ and $t > 0$ there exists $m \in \mathbb{N}$ and $N \in \mathbb{N}$ such that

$$P(\mathbb{X}^i \in D_t(K_{m,N})) = P(\sup_{0 \leqslant s \leqslant t} \| X^i(s) \|_{-m} \leqslant N) > 1 - \varepsilon , \qquad i \in \mathbb{I} ,$$

provided the family is weakly tight. But we cannot make it better than Mitoma [9] did (see also Fouque [5]).

## III. E is the topological dual of the strict inductive limit of a sequence of Frechét nuclear spaces.

Let $\Phi$ be the strict inductive limit of a sequence $\{ \Phi_n \}_{n \in \mathbb{N}}$ of Frechét nuclear space. Then $\Phi'$ is isomorphic with a subspace of the product space $\prod_{n \in \mathbb{N}} \Phi'_n$ (see [11]). From this representation of $\Phi'$, Corollary 2.8 and Theorem 2.1 *i*) one can derive the equality $\mathscr{B}_{D(\Phi')} = \mathscr{C}_{D(\Phi')}$.

Moreover, it follows from the previous case, that any $\Phi'$-valued weakly measurable stochastic process is a Borel-measurable stochastic process.

Suppose that a family $\{ \mathbb{X}^i \}_{i\in\mathbb{I}}$ of stochastic processes with distributions on $D(\Phi')$ is weakly tight.

Let $\pi_j : \prod_{n\in\mathbb{N}} \Phi'_n \to \Phi'_j$ be the natural projection, $j \in \mathbb{N}$. By Theorem 5.5, the family $\{ \tilde{\pi}_j \circ \mathbb{X}^i \}_{i\in\mathbb{I}}$ is tight in $D(\Phi'_j)$ for each $j \in \mathbb{N}$, i. e. for each $\varepsilon > 0$, $t > 0$ and $j \in \mathbb{N}$ there is a compact $K_j \subset \Phi'_j$ such that

$$P(\tilde{\pi}_j \circ \mathbb{X}^i \in D_t(K_j)) > 1 - \varepsilon/2^j, \qquad i \in \mathbb{I}.$$

Hence we have for every $i \in \mathbb{I}$

$$P\left( \mathbb{X}^i \in \bigcap_{j\in\mathbb{N}} (\tilde{\pi}_j)^{-1}(D_t(K_j) \;= D_t\left( \prod_{j\in\mathbb{N}} K_j \right) \right) > 1 - \varepsilon .$$

Now, applying Theorem 4.6 we get

5.6. THEOREM (Fouque [5]). — Tightness in $D(\Phi')$ is equivalent to $\Phi$-weak tightness. ■

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*(Manuscrit reçu le 4 juillet 1985)*