ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 21, nº 4 (1985), p. 323-361
<http://www.numdam.org/item?id=AIHPB_1985__21_4_323_0>

Ann. Inst. Henri Poincaré,
Vol. 21, n° 4, 1985, p. 323-361. *Probabilités et Statistiques*

Analytic and sequential Feynman integrals on abstract Wiener and Hilbert spaces, and a Cameron-Martin formula

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ABSTRACT. — Sequential Feynman integrals are defined for classes of functions on a Hilbert space and on an abstract Wiener space. A Cameron-Martin formula is proved for analytic and sequential Feynman integrals for the classes $\mathscr{G}^q(B)$ and $\mathscr{G}^q(H)$.

Key-words: Analytic Feynman and sequential Feynman integrals, abstract Wiener space, m-lifting, Cameron-Martin formula, Maslov index.

RÉSUMÉ. — Les intégrales de Feynman séquentielles sont définies pour des classes de fonctions sur un espace de Hilbert et sur un espace de Wiener abstrait. Une formule de Cameron-Martin est démontrée pour des intégrales de Feynman analytiques et séquentielles, pour les classes $\mathscr{G}^q(B)$ et $\mathscr{G}^q(H)$.

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This research has been supported by Air Force office of Scientific Research grant no AFSOR F49620 82 C 0009.

1. INTRODUCTION AND SUMMARY

The present paper is a continuation of the study of Feynman integrals undertaken in Kallianpur and Bromley [*11*]. A major theme of that paper, not taken up here, was the idea of using analytic continuation in several complex variables to define Feynman integrals for different classes of integrals. The purpose of the present work is two-fold: (1) To define *sequential* Feynman integrals by means of finite dimensional approximations and (2) to establish the existence of both analytic and sequential Feynman integrals for a wider class of integrals than the Fresnel class considered in Kallianpur and Bromley or in Albeverio and Høegh-Krohn [*11*] [*1*]. The latter results will be collectively referred to as Cameron-Martin formulas because of their formal similarity to problems of equivalence of Gaussian measures. A special case of the Cameron-Martin formula was given in [*12*]. Both the analytic and sequential Feynman integrals and the Cameron-Martin formula will be investigated in this paper at the level of generality adopted in [*11*], namely for classes of functionals on abstract Wiener and abstract Hilbert spaces.

A special feature of the paper is the definition of the analytic Feynman integral for classes of functions on a Hilbert space H. This is done by the use of finitely additive Gauss measure on H and the introduction of the « *m*-lifting » map. These ideas, together with preliminaries on abstract Wiener spaces are discussed in Section S. Section 3 is devoted to theorems on analytic Feynman integrals. All the results pertaining to sequential Feynman integrals are given in Section 2. These include the Cameron-Martin formula for integrands in $\mathcal{G}^q$(H) and for the class $\mathcal{G}^q$(B) of functionals on abstract Wiener spaces.

In Section 5, we specialize the theory to Feynman path integrals and briefly indicate how the solution of the Schrödinger equation can be represented as a Feynman integral either on a Hilbert space of paths or on the space of paths of the Wiener process. The results of Section 5 (except possibly for subsections (*d*) and (*e*) and the remarks in (*e*)) are not new and are included as an application of the theory of the earlier sections and also to enable the reader to appreciate the physical background that initially led Feynman to his integral [*6*]. Moreover, while making Feynman's arguments rigorous in this section, we have tried to adhere as closely as possible to his original approach as described in his book with Hibbs ([*7*], Chapter 3, especially Sections 3.5, 3.6 and 3.11).

The relationship of our paper to other work in this area is discussed in Section 6. Sequential definitions of the Feynman integral have appeared in a very recent Memoir by Cameron and Storvick [*3*] and in papers by Truman and Elworthy and Truman [*15*] [*5a*] [*5b*]. These papers deal with a Hilbert space of paths, the RKHS of the Wiener process and define a sequential Feynman integral based on polygonal path approximations. Cameron and Storvick confine their investigation essentially to what we call the Fresnel class over C [0, t] and hence their results cannot be applied to any problem involving unbounded potentials. Elworthy and Truman, on the other hand, give a version of the Cameron-Martin formula for their sequential integral on $\mathscr{H}_t$ in their paper [*5 a*]. Since this paper was written, we have seen a copy of a recent paper by Elworthy and Truman, « Feynman maps, Cameron-Martin formulas and anharmonic oscillators », [*5 b*] kindly sent to us by the authors. In it, a Cameron-Martin formula is established for Feynman path integrals. Theorems 3.2 and 4.2 of our paper may be regarded as generalizations of this result. Our definition of the sequential Feynman integral is also connected in some respects with Tarski's [*14*] and is alluded to in Section 6.

As a final comment it may be useful to summarize some of the special features of this paper:

(1) In the literature on Feynman integrals—and we refer here not to the work of physicists but to theoretical investigations (e. g., in much of the work of Cameron and his co-workers)—Wiener measure and Wiener space provide a basic setting for the analytic continuation procedure. The present paper sheds some light on the role of the RKHS of Wiener space in the various definitions of the Feynman integral. In fact, our work shows that the basic definition is that of the integral on a Hilbert space (any separable, real Hilbert space H) and that the part played by Wiener space and Wiener measure is secondary. As a consequence, the use of a (finitely additive) Gauss measure on Hilbert space enters the problem in a natural way and provides a probabilistic setting for Albeverio and Høegh-Krohn's theory. The results of Sections 3 and 4 extend the latter to a larger class of integrands, viz. to $\mathscr{G}^q$(H). Definitions of analytic Feynman integrals for H are given directly (via m-lifting maps). Section 4 provides a *sequential* Feynman integral theory in the set-up of [*1*].

(2) The definition of the sequential integral is given in terms of arbitrary sequences $\{ P_n \}$ of finite dimensional orthogonal projections converging strongly to the identity in H. This generality makes the proofs of the main theorems somewhat harder but has wider applicability even for the case

$H = \mathscr{H}_t$, the RKHS of the Wiener process over $C[0, t]$. For example, in the latter case, it makes it possible for us to rigorously establish the Feynman integral also via approximation by finite Fourier sums. To the best of our knowledge this approach, already known to Feynman ([7], p. 71-73), has not received as much attention in the mathematical literature as his other idea, viz. analytic continuation.

(3) The work of the present paper shows that the Feynman integral can be obtained no matter how it is defined, by means of a very general, single limit finite dimensional approximation procedure, set forth in Theorem 4.3.

2. PRELIMINARIES: ABSTRACT WIENER SPACES AND *m*-LIFTING MAPS

The basic notions of abstract Wiener space, measurable norm and « *m*-lifting » map are due to L. Gross (see [8] and the references given there). We briefly summarize them below for the reader's convenience.

Let H be a real separable infinite dimensional Hilbert space with inner product $(.,.)$ and norm $|\,.\,|$. Let $\mathscr{P}$ be the set of all orthogonal projections on H with finite dimensional range. For $P \in \mathscr{P}$, let

$$\mathscr{C}_P = \{ P^{-1}B : B \text{ a Borel set in Range } P \}$$

and

$$\mathscr{C} = \bigcup_P \mathscr{C}_P .$$

A cylinder measure is a finitely additive nonnegative measure on $(H, \mathscr{C})$ such that its restriction to $\mathscr{C}_P$ is countably additive for all $P \in \mathscr{P}$. The canonical Gauss measure m on H is the cylinder measure on $(H, \mathscr{C})$ characterized by

$$\int e^{i(h,h_1)} dm(h) = e^{-1/2|h_1|^2} . \tag{2.1}$$

Let $\|\,.\,\|$ be a *measurable* norm on H, i. e. for every $\varepsilon > 0$, there exists $P_0 \in \mathscr{P}$ such that for all $P \perp P_0$, $P \in \mathscr{P}$, we have

$$m\{ h \in H : \| Ph \| > \varepsilon \} < \varepsilon .$$

It can be shown that H is not complete with respect to $\|\,.\,\|$. (See [13]). Let B denote the completion of H under $\|\,.\,\|$ and let i denote the natural injection. The adjoint operator i^* maps the strong dual B^* continuously, one-to-one, onto a dense subspace of H* (which is identified with H). By a well known result of Gross, the induced measure mi^{-1} on the cylinder sets in B is indeed countably additive and hence extends to a countably additive measure ν on $\mathscr{B}$-the Borel σ-field on B. The pair (H, B) is called an abstract Wiener space and ν is called the abstract Wiener measure. If $H = \left\{ f \in C[0,1] : f \text{ absolutely continuous, } f(0) = 0, \frac{df}{dt} \in L^2[0,1] \right\}$ with the inner product

$$(f_1, f_2) = \int_0^1 \left(\frac{df_1}{dt}\right) \cdot \left(\frac{df_2}{dt}\right) . dt ,$$

then the uniform norm on H is measurable and in this case B is $C_0[0,1]$ and ν is the classical Wiener measure on $C_0[0,1]$. The concept of measurable norm and abstract Wiener space is due to Gross. See Kuo [13] for forther details.

We will briefly describe the integration theory on $(H, \mathscr{C}, m)$. We now fix a CONS $\{ e_j \}$ of H, such that $e_j \in B^*$, for all j. For $h \in H$, $x \in B$, let

$$(2.2) \qquad (h, x)^{\sim} = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} (h, e_j)e_j(x), & \text{if the limit exists}, \\ 0 & \text{otherwise}. \end{cases}$$

Let $L : H \to \mathscr{L}(B, \mathscr{B}, \nu)$ be defined by

$$L(h)(x) = (h, x)^{\sim} .$$

Then L is a representative of the weak distribution corresponding to m i. e., for $h_1, h_2, \ldots, h_k \in H$ and A a Borel set in $\mathbb{R}^k$,

$$(2.3) \quad m\{ h : ((h_1, h), \ldots, (h_k, h)) \in A \} = \nu\{ (L(h_1), \ldots, L(h_k)) \in A \} .$$

For a proof that (2.3) holds, see [11]. For a cylinder function f on H given by

$$(2.4) \qquad f(h) = \phi((h_1, h), \ldots, (h_k, h)) ,$$

where $h_i \in H$ and ϕ is a complex valued Borel function on $\mathbb{R}^k$, we denote by $R(f)$ the random variable $\phi((h_1, x)^{\sim}, \ldots, (h_k, x)^{\sim})$ on $(B, \mathscr{B}, \nu)$. We extend this mapping as follows:

DEFINITION. — Let $\mathscr{L}(H, \mathscr{C}, m)$ be the class of complex valued continuous functions f on H such that the net $\{ R(f \circ P) : P \in \mathscr{P} \}$ ($P_1 < P_2$ if

Range $P_1 \subseteq$ Range P_2) is Cauchy in ν-probability. Further, for $f \in \mathscr{L}(H, \mathscr{C}, m)$, let

$$\text{(2.5)} \qquad R(f) = \lim_{P \in \mathscr{P}} \text{ in } \nu\text{-probability } R(f \circ P).$$

The mapping R will be called an « m-lifting ».

DEFINITION. — Let

$$\mathscr{L}^1(H, \mathscr{C}, m) = \left\{ f \in \mathscr{L}(H, \mathscr{C}, m) : \int | R(f) | d\nu < \infty \right\}$$

and for $f \in \mathscr{L}^1(H, \mathscr{C}, m)$ and $C \in \mathscr{C}$, define

$$\text{(2.6)} \qquad \int_C f dm = \int R(1_C) \cdot R(f) d\nu .$$

REMARK 1. — In the above definition we have taken $(B, \mathscr{B}, \nu)$ as the « representation space » for the weak distribution L and for the m-lifting. Other linear probability spaces can also be chosen leading to Feynman integrals of different classes of functionals. This point will be taken up in Section 5.

We will now introduce the Fresnel class $\mathscr{F}(H)$ of functions on H. This class plays an important role in the later sections. Let $\mathscr{M}(H)$ be the class of all countably additive complex measures on Borel subsets of H with finite absolute variation. Let $\mathscr{F}(H)$ be the class of all functions f of the form

$$\text{(2.7)} \qquad f(h_1) = \int_H e^{i(h,h_1)} d\mu(h)$$

for some $\mu \in \mathscr{M}(H)$. $\mathscr{F}(H)$ is the Fresnel class of Albeverio and Høegh-Krohn [*1*] and has been discussed also in [*11*].

The next result shows that $\mathscr{F}(H) \subseteq \mathscr{L}^1(H, \mathscr{C}, m)$ and gives a representation for $R(f)$ for $f \in \mathscr{F}(H)$. For convenience we will use the following notation throughout this paper. Let E be a vector space and let $\theta : E \to \mathbb{C}$. For $\lambda > 0$, we denote by θ^λ the function $\theta^\lambda(e) = \theta(\lambda^{-1/2}e)$, $e \in E$.

LEMMA 2.1. — Let $f \in \mathscr{F}(H)$ be as in (2.7). Then $f \in \mathscr{L}^1(H, \mathscr{C}, m)$ and $R(f) = F$, where F is given by

$$\text{(2.8)} \qquad F(x) = \int_H e^{i(h,x)^\sim} d\mu(h), \; x \in B .$$

Further, for $\lambda > 0$, we have

$$\text{(2.9)} \qquad R(f^{\lambda}) = F^{\lambda} \qquad \text{for all } \lambda > 0 .$$

Proof. — Fix $\mu \in \mathcal{M}(H)$ and let f be given by (2.7). Continuity of f follows from the Dominated Convergence Theorem for μ. For $P \in \mathcal{P}$,

$$(f \circ P)(h_1) = f(Ph_1) = \int e^{i(h,Ph_1)} d\mu(h) = \int e^{i(Ph,h_1)} d\mu(h) .$$

From the definition of m-lifting for cylinder functions, it follows that

$$\text{(2.10)} \qquad R(f \circ P)(x) = \int e^{i(Ph,x)^{\sim}} d\mu(h) .$$

Using Fubini's theorem

$$\text{(2.11)} \qquad \int_B |F(x) - R(f \circ P)(x)| dv(x) \leq \int_H \int_B |e^{i(Ph,x)^{\sim}} - e^{i(h,x)^{\sim}}| dv(x) d|\mu|(h)$$
$$\leq \int_H \int_B |1 - e^{i(h-Ph,x)^{\sim}}| dv(x) d|\mu|(h),$$

where $|\mu|$ denotes the total variation measure for μ. For $\sigma > 0$, let

$$\text{(2.12)} \qquad u(\sigma) = \int_{\mathbb{R}} |1 - e^{i\sigma y}| 1/\sqrt{2\pi} e^{-1/2 y^2} dy .$$

Since the distribution of $L(h - Ph)$ under v is normal with mean 0 and variance $|h - Ph|^2$, we have from (2.11) and (2.12),

$$\text{(2.13)} \qquad \int_B |F(x) - R(f \circ P)(x)| dv(x) \leq \int_H u(|h - Ph|) d|\mu|(h) .$$

Since $u(\sigma) \to 0$ as $\sigma \to 0$ and u is bounded, the Dominated Convergence Theorem for $|\mu|$ implies that

$$\left\{ \int_B |F(x) - R(f \circ P)(x)| dv(x) \right\}_{P \in \mathcal{P}} \to 0 .$$

Hence

$$\text{(2.14)} \qquad R(f) = F .$$

From the definition of $(h, x)^{\sim}$, it follows that for $\lambda > 0$,

$$(h, \lambda x)^{\sim} = \lambda(h, x)^{\sim} = (\lambda h, x)^{\sim} .$$

Hence

$$f^{\lambda}(h_1) = \int e^{i(h,h_1)} d\mu_{\lambda}(h)$$

and

$$F^{\lambda}(x) = \int e^{i(h,x)^{\sim}} d\mu_{\lambda}(h)$$

where $\mu_{\lambda} \in \mathcal{M}(H)$ is defined by

$$\mu_{\lambda}(B) = \mu(\lambda^{1/2}B).$$

Hence invoking (2.14) for f^{λ}, μ^{λ}, F^{λ}, we get

$$R(f^{\lambda}) = F^{\lambda}. \qquad \square$$

REMARK 2. — The same calculations as above also give us the following result. If $P_n \xrightarrow{s} I$, $P_n \in \mathcal{P}$, then

$$(2.15) \qquad R(f^{\lambda} \circ P_n) \to R(f^{\lambda}) = F^{\lambda} \text{ in } \mathcal{L}^1(B, \mathcal{B}, \nu).$$

(Here $P_n \xrightarrow{s} I$ means P_n converges strongly to I).

We now wish to evaluate the m-lifting for a wider class of integrands on H which correspond, in physical problems, to certain unbounded potentials such as the anharmonic potential. In the latter context and for the RKHS of the Wiener process the class was introduced by Elworthy and Truman and also by Ph. Combe et al. [*5*] [*4*].

Our immediate aim is to establish Proposition 2.4.

LEMMA 2.2. — Let A be a trace class operator on H and let $P_n \in \mathcal{P}$ be such that $P_n \xrightarrow{s} I$. Then, $\| \cdot \|_1$ denoting a trace norm, we have

$$(2.16) \qquad \| P_n A P_n - A \|_1 \to 0.$$

This is a well known result. For a proof see Gross ([*8*], Corollary 3.2).

LEMMA 2.3. — Let A be a self adjoint trace class operator with eigenvalues $\{ \alpha_k \}$ and corresponding eigenfunctions $\{ e_k \}$. Let $u(h) = (h, Ah)$, $h \in H$. Then, for all $\lambda > 0$, $u^{\lambda}(H, \mathcal{C}, m)$ and

$$(2.17) \qquad R(u^{\lambda}) = v^{\lambda}$$

where v is given by

$$(2.18) \qquad \begin{aligned} v(x) &= \lim_n \sum_{j=1}^{n} \alpha_k [(e_k, x)^{\sim}]^2 && \text{if the limit exists,} \\ &= 0 && \text{otherwise.} \end{aligned}$$

Proof. — Since we can write $A = A_+ - A_-$ where $\dot{A}_+$ and A_- are self adjoint, positive trace class operators, we have $u(h) = u_+(h) - u_-(h)$ where $u_\pm(h) = (h, A_\pm h)$. It suffices therefore, to prove the result for a self adjoint, positive, trace class operator A with eigenvalues $\{ \alpha_j \}$ and corresponding eigenfunctions $\{ e_j \}$. Accordingly set $u(h) = (h, Ah) = \| Bh \|^2$ where the self adjoint, Hilbert-Schmidt operator B is the square root of A. Now Theorem 2 of Gross [8] can be applied to u and it follows that $R(u)$ exists. Furthermore, Corollary 5.3 of [8] implies that $R(u) = \lim_{n\to\infty}$ in ν-probability $R(u \circ P_n)$ where $P_n \in \mathscr{P}$ is any sequence converging strongly to the identity. Choosing P_n to be the orthogonal projection onto span $\{ e_1, \ldots, e_n \}$ it is easy to see that $R(u) = v$ is given by (2.18). Note that the above limit is finite ν-a. s. since the series $\sum_{j=1}^{\infty} \alpha_j$ converges. If we now fix $\lambda > 0$, we have $u^\lambda(h) = \frac{1}{\lambda} u(h)$ and so $R(u^\lambda) = \frac{1}{\lambda} R(u) = \frac{1}{\lambda} v$. From the definition of $(e_j, x)^\sim$ it follows that $\frac{1}{\lambda} v(x) = v^\lambda(x)$ and we have $R(u^\lambda) = v^\lambda$. □

We will henceforth use the more suggestive notation $(x, Ax)^\sim$ for $v(x)$.

PROPOSITION 2.4. — Let $\mu \in \mathscr{M}(H)$ and A be a self adjoint trace class operator on H. Let g, G be defined by

$$(2.19) \qquad g(h) = e^{i/2(h,Ah)} \int_H e^{i(h_1,h)} d\mu(h_1)$$

and

$$(2.20) \qquad G(x) = e^{i/2(x,Ax)^\sim} \int_H e^{i(h,x)^\sim} d\mu(h) = e^{i/2(x,Ax)^\sim} F(x), \quad \text{say}.$$

Then for $\lambda > 0$, we have

$$(2.21) \qquad R(g^\lambda) = G^\lambda$$

and further if $P_n \xrightarrow{s} I$, then

$$(2.22) \qquad R(g^\lambda \circ P_n) \to G^\lambda \quad \text{in } \mathscr{L}^1(B, \mathscr{B}, \nu).$$

Proof of Proposition 2.4. — If $\lambda > 0$, $g^\lambda(h) = e^{\frac{i}{2}u^\lambda(h)} f^\lambda(h)$ and (2.21) follows from the multiplicative property of R. Next for $P \in \mathscr{P}$,

$$R(g^\lambda \circ P) - G^\lambda = e^{\frac{i}{2}R(u^\lambda \circ P)} R(f^\lambda \circ P) - e^{\frac{i}{2}v^\lambda} F^\lambda,$$

and

$$\int_B |R(g^\lambda \circ P) - G^\lambda| dv \leq |\mu_\lambda| \int_B |e^{\frac{i}{2}R(u^\lambda \circ P)} - e^{\frac{i}{2}v^\lambda}| dv, + \int_B |R(f^\lambda \circ P) - F^\lambda| dv, \tag{2.23}$$

where $|\mu_\lambda|$ is the total variation of the complex-valued measure μ_λ introduced in Lemma 2.1. The integrals on the R. H. S. of (2.23) converge to zero as $P \to I$ along $\mathscr{P}$ by the dominated convergence theorem. We can, in fact, replace P by P_n in (2.23) and take the limit as $P_n \xrightarrow{s} I$. This proves (2.22).

3. ANALYTIC WIENER AND FEYNMAN INTEGRALS

a) Integrals on abstract Wiener space.

Here we recall the definition of analytic Wiener and Feynman integrals given in [*11*] and obtain a « Cameron-Martin » type formula for the analytic Feynman integrals. (For a special case, see [*12*]).

DEFINITION. — Let F be a measurable complex-valued function on B such that

(*i*) $J_F(\lambda) = \int F(\lambda^{-1/2}x)dv(x)$ exists for all real $\lambda > 0$.

(*ii*) There is an analytic function J_F^* on $\Omega = \{ x \in \mathbb{C} : \operatorname{Re} z > 0 \}$ such that $J_F^*(\lambda) = J_F(\lambda)$ for real $\lambda > 0$.

Then we will define $\mathscr{I}_a^z(F) = J_F^*(z)$ and call $\mathscr{I}_a^z$ the *analytic Wiener integral* of F over B with parameter z.

If $\lim_{\substack{z \to -iq \\ z \in \Omega}} \mathscr{I}_a^z(F)$ exists for some q real, we will denote the value of this limit by $I_a^q(F)$ and define it to be the *analytic Feynman integral* of F over B with parameter q.

If F and G are functions on B such that F = G a. s. v, then it does not imply that $J_F(\lambda) = J_G(\lambda)$ for all $\lambda > 0$ and thus F = G a. s. v does not imply that $\mathscr{I}_a^z(F) = \mathscr{I}_a^z(G)$. (See [*11*] for a discussion on this point). These considerations lead us to the definition of *s-equivalence* of functions on B. Given two complex valued functions F and G on B, we say that F = G *s-almost surely* if for each $\alpha > 0$,

$$v\{ x \in B : F(\alpha x) = G(\alpha x) \} = 0.$$

It is easy to see that $J_F(\lambda)$ and $J_G(\lambda)$ exist simultaneously and coincide if $F = G$ s-a. s. For a function F on B, we will denote by [F] the equivalence class of functions G which are equal to F s-a. s.

We will now introduce the Fresnel class $\mathscr{F}(B)$ of functions on B.

$$\mathscr{F}(B) = \left\{ [F] : F(x) = \int e^{i(h,x)^{\sim}} d\mu(h),\ \mu \in \mathscr{M}(H) \right\}.$$

As is customary, we will identify a function with its s-equivalence class and think of $\mathscr{F}(B)$ as a class of functions on B rather than as a class of equivalence classes.

For $\mu_1, \mu_2 \in \mathscr{M}(H)$, let $\mu_1 * \mu_2$ denote the convolution of μ_1 and μ_2. Also let $\| \mu \|$ denote the total variation of $\mu \in \mathscr{M}(H)$. Then $\mathscr{M}(H)$ is a Banach algebra. If for $f \in \mathscr{F}(H)$ given by (2.7), we define $\| f \|_0 = \| \mu \|$, then it can be easily seen that $\mathscr{F}(H)$ is a Banach algebra and that the mapping $\mu \to f$ (μ, f related by (2.7)) is a Banach algebra isomorphism between $\mathscr{M}(H)$ and $\mathscr{F}(H)$.

It is shown in [*11*] [*3*] that $\mathscr{F}(B)$ is also a Banach algebra with the norm $\| F \|_0 = \| \mu \|$ and the mapping $\mu \to F$ (μ, F related by (2.8)) is a Banach algebra isomorphism.

The following result gives an evaluation of the analytic Wiener and Feynman integrals for $F \in \mathscr{F}(B)$. This result is taken from [*11*] and the short proof is included here for the sake of completeness.

THEOREM 3.1. — Let $F \in \mathscr{F}(B)$ be given by

$$F(x) = \int_H e^{i(h,x)^{\sim}} d\mu(h), \quad \mu \in \mathscr{M}(H). \tag{3.1}$$

Then for all $z \in \Omega$, the analytic Wiener integral $\mathscr{I}_a^z(F)$ exists and

$$\mathscr{I}_a^z(F) = \int_H e^{-\frac{1}{2z}|h|^2} d\mu(h). \tag{3.2}$$

The analytic Feynman integral $I_a^q(F)$ exists for all $q \in \mathbb{R}$, $q \neq 0$ and

$$I_a^q(F) = \int_H e^{-\frac{i}{2q}|h|^2} d\mu(h). \tag{3.3}$$

Proof. — By Fubini's theorem, we have

$$\begin{aligned} \text{(3.4)} \qquad \mathrm{J_F}(\lambda) &= \int_{\mathrm{B}}\int_{\mathrm{H}} e^{i(h,\lambda^{-1/2}x)^{\sim}} d\mu(h)d\nu(x) \\ &= \int_{\mathrm{H}}\int_{\mathrm{B}} e^{i\lambda^{-1/2}(h,x)^{\sim}} d\nu(x)d\mu(h) \\ &= \int_{\mathrm{H}} e^{-\frac{1}{2\lambda}|h|^2} d\mu(h). \end{aligned}$$

Let

$$\text{(3.5)} \quad \mathrm{J}_{\mathrm{F}}^{*}(z) = \int_{\mathrm{H}} e^{-\frac{1}{2z}|h|^2} d\mu(h), \qquad z \in \overline{\Omega} - \{0\}, \quad \overline{\Omega} = \{ z \in \mathbb{C} : \operatorname{Re} z \geq 0 \}.$$

Then $\mathrm{J}_{\mathrm{F}}^{*}(\lambda) = \mathrm{J_F}(\lambda)$ for real $\lambda > 0$ and by the dominated convergence theorem, $\mathrm{J}_{\mathrm{F}}^{*}(z)$ is continuous in $\overline{\Omega} - \{0\}$. For each $h \in \mathrm{H}$, $e^{-\frac{1}{2z}|h|^2}$ is analytic in Ω so that $\int_{\mathrm{C}} e^{-\frac{1}{2z}|h|^2} dz = 0$ for every rectifiable closed curve C in Ω. Since $| e^{-\frac{1}{2z}|h|^2} | \leq 1$ for $z \in \Omega$, a simple application of Fubini's theorem and Morera's theorem give the analyticity of $\mathrm{J}_{\mathrm{F}}^{*}(z)$. The proof of (3.3) is immediate. □

The classes $\Lambda^q(\mathrm{H})$ *and* $\Lambda^q(\mathrm{B})$. For a real number q, $q \neq 0$, we denote by $\Lambda^q(\mathrm{H})$ [resp. $\Lambda^q(\mathrm{B})$] the class of functions g [resp. G] given by (2.22) [resp. (2.23)] for some $\mu \in \mathcal{M}(\mathrm{H})$ and some self adjoint, trace class operator A on H such that the bounded inverse $(\mathrm{I} + 1/q\mathrm{A})^{-1}$ exists.

Recall that for a self adjoint trace class operator A with eigenvalues $\{\alpha_j\}$, the Fredholm determinant of (I + A) (denoted by det (I + A)) is defined by

$$\text{(3.8)} \qquad \det(\mathrm{I} + \mathrm{A}) = \prod_{j=1}^{\infty} (1 + \alpha_j)$$

and the Maslov index of (I + A) (denoted by ind (I + A)) is the number of negative eigenvalues of (I + A), i. e.

$$\text{(3.9)} \qquad \operatorname{ind}(\mathrm{I} + \mathrm{A}) = \# \{ j : 1 + \alpha_j < 0 \}.$$

With this notation, we have the following result on the analytic Feynman integrals for the class $\mathcal{G}^q(\mathrm{B})$. (See also [*12*]).

THEOREM 3.2. (Cameron-Martin formula for analytic Feynman integrals). — Let A be a self adjoint trace class operator on H such that $(I + 1/qA)$ is invertible ($q \in \mathbb{R}$, $q \neq 0$) and let $F \in \mathscr{F}(B)$. Let

$$G(x) = \exp\{ i/2(x, Ax)^{\sim} \} F(x). \tag{3.10}$$

Then the analytic Feynman integral $I_a^q(G)$ exists and

$$I_a^q(G) = |\det(I + 1/qA)|^{-1/2} e^{-\frac{i\pi}{2}\operatorname{ind}(I + 1/qA)} \cdot \int_H e^{\frac{i}{2q}((I + 1/qA)^{-1}h, h)} d\mu(h), \tag{3.11}$$

where μ is related to F by (3.1). We will give a proof for $q = 1$. The proof in the general case is similar.

Proof. — Let $\bar{e}_j$ be the eigenfunctions and α_j the eigenvalues of A. Let $\xi_j = (\bar{e}_j, x)^{\sim}$ and $h_j = (\bar{e}_j, h)$. Clearly, $\Sigma_{j=1}^{\infty} \alpha_j \xi_j^2 < \infty$ a. s. and hence we have

$$(x, Ax)^{\sim} = \sum_{j=1}^{\infty} \alpha_j \xi_j^2 < \infty \qquad s\text{-a. s.}$$

Now,

$$\begin{aligned} J_G(\lambda) &= \int_B G(\lambda^{-1/2}x) d\nu(x) \\ &= \int_H \left[\int_B e^{\frac{i}{2\lambda}\Sigma_1^\infty \alpha_j \xi_j^2 + i/\sqrt{\lambda}\Sigma_1^\infty h_j \xi_j} d\nu \right] d\mu(h) \\ &= \int_H \prod_{j=1}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2\left(1 - \frac{i\alpha_j}{\lambda}\right)y^2 + \frac{ih_j}{\sqrt{\lambda}} y} dy \right] d\mu(h) \end{aligned} \tag{3.12}$$

since (ξ_j) is a sequence of independent standard normal random variables on (B, ν). To evaluate the infinite product, use the fact if $\operatorname{Re} b > 0$ and c is real, then

$$1/\sqrt{2\pi} \int_{-\infty}^{\infty} e^{-1/2by^2 + icy} dy = 1/b^{1/2} e^{-\frac{c^2}{2b}}. \tag{3.13}$$

In this formula and in the sequel, for a complex number $z = re^{i\theta}$, r real positive $-\pi < \theta < \pi$, $z^{1/2}$ will denote the number $\sqrt{r}e^{i\theta/2}$, where $\sqrt{r}$ is positive square root of r. Using (3.13) in (3.12), we get

$$J_G(\lambda) = \left\{ \prod_{j=1}^{\infty} \left(1 - \frac{i\alpha_j}{\lambda}\right) \right\}^{-1/2} \int_H e^{-\frac{1}{2\lambda}\Sigma_{j=1}^{\infty} \frac{h_j^2}{\left(1 - \frac{i\alpha_j}{\lambda}\right)}} d\mu(h). \tag{3.14}$$

Observe that since A is trace class, $\Sigma |\alpha_j| < \infty$ and hence the infinite product and series appearing in (3.14) converge absolutely.

By renumbering the α_j if necessary, assume $1+\alpha_j<0$ for $j=1, 2, \ldots$, $m = \mathrm{Ind}(\mathrm{I} + \mathrm{A})$ and $(1 + \alpha_j) > 0$ for $j \geq m + 1$. Choose $\delta > 0$ such that $\alpha_j \notin [-1-\delta, -1+\delta]$ for all j. This can be done because $\alpha_j \neq -1$ for all j and $|\alpha_j| \to 0$ as $j \to \infty$. Let

$$\Omega' = \Omega \cup \{ z \in \mathbb{C} : \mathrm{Re}\, z = 0, |1 + \mathrm{I}_m z| \leq \delta \}.$$

For $z \in \Omega'$, let

$$\mathrm{A}_1(z) = \prod_{j=1}^{m} (z)^{1/2}, (z - i\alpha_j)^{-1/2}, \tag{3.15}$$

$$\mathrm{A}_2(z) = \prod_{j=m+1}^{\infty} \left(1 - \frac{i\alpha_j}{z}\right)^{-1/2} \tag{3.16}$$

and

$$\mathrm{A}_3(z) = \int_{\mathrm{H}} e^{-1/2\Sigma_{j=1}^{\infty} \frac{h_j^2}{z - i\alpha_j}} d\mu(h). \tag{3.17}$$

We will first show that A_1, A_2, A_3 are continuous functions on Ω' and analytic in Ω.

Since $w \to (w)^{1/2}$ is an analytic function on $\mathbb{C}' = \{ re^{i\theta} : r>0, -\pi<\theta<\pi \}$ and for $z \in \Omega'$, z, $z - i\alpha_j \in \mathbb{C}'$, $\mathrm{A}_1(z)$ is continuous on Ω' and analytic on Ω.

Now, let $1 + u_j(z) = \left(1 - \frac{i\alpha_j}{z}\right)^{-1/2}$ for $z \in \Omega'$, $j \geq m$ and

$$\Omega'_{r_0} = \{ z \in \Omega' : 0 < 1/r_0 \leq |z| \leq r_0 \}.$$

It is easy to see that for some constant K_0, we have

$$|(1 + w)^{-1/2} - 1| \leq \mathrm{K}_0 |w| \qquad \text{for} \quad w \in \mathbb{C}, \quad |w| \leq 1/2.$$

Since $\Sigma_{j=1}^{\infty} |\alpha_j| < \infty$, $|\alpha_j| \to 0$ and hence for fixed r_0, there exists a j_0 such that for $j \geq j_0$, $z \in \Omega'_{r_0}$, we have $\left|\frac{i\alpha_j}{z}\right| \leq 1/2$ and hence

$$|u_j(z)| \leq \mathrm{K}_0 r_0 |\alpha_j| \tag{3.18}$$

for all $j \geq j_0$, $z \in \Omega'_{r_0}$. Thus, $\Sigma_{j=m+1}^{\infty} |u_j(z)|$ converges uniformly on Ω'_{r_0}. Since $\Omega' = \cup_{r=2}^{\infty} \Omega'_r$, this implies that

$$\mathrm{A}_2(z) = \prod_{j=m+1}^{\infty} (1 + \alpha_j(z))$$

is continuous on Ω' and analytic in Ω.

As for $A_3(z)$, first observe that for $h \in H$ fixed, the series

$$- 1/2 \sum_{j=1}^{\infty} \frac{h_j^2}{z - i\alpha_j} = \phi(h, z) \tag{3.19}$$

converges uniformly on Ω'_{r_0} for all $r_0 \geq 1$ and hence for all $h \in H$, $\phi(h, z)$ is continuous on Ω' and analytic in Ω. Also, it is easy to see that $\mathrm{Re}(\phi(h, z)) \leq 0$. The dominated convergence theorem now implies that $A_3(z)$ is continuous on Ω' and Morera's theorem along with Fubini's theorem shows that $A_3(z)$ is analytic in Ω.

Thus, $J^*_G(z) = A_1(z). A_2(z). A_3(z)$ is continuous on Ω' and analytic in Ω. It is easy to see that $J^*_G(\lambda) = J_G(\lambda)$ for real $\lambda > 0$ and hence (by definition) $I_a(G)$ exists and

$$I_a^1(G) = J^*_G(- i)\,. \tag{3.20}$$

Now,

$$A_1(- i) = \prod_{j=1}^{m} \frac{(- i)^{1/2}}{(- i - i\alpha_j)^{1/2}} \tag{3.21}$$

$$= \prod_{j=1}^{m} \frac{e^{- i\pi/4}}{e^{i\pi/4} \mid 1 + \alpha_j \mid^{1/2}}, \quad \text{since } 1 + \alpha_j < 0$$

$$= \left(\prod_{j=1}^{m} \mid (1 + \alpha_j) \mid^{- 1/2} e^{- i\pi/2 \, \mathrm{ind}(I + L)} \right),$$

$$A_2(- i) = \prod_{j=m+1}^{\infty} (1 + \alpha_j)^{- 1/2} \tag{3.22}$$

$$= \prod_{j=m+1}^{\infty} \mid (1 + \alpha_j) \mid^{- 1/2}, \quad \text{as } 1 + \alpha_j \geq 0\,,$$

and

$$A_3(- i) = \int e^{- 1/2 \sum_{j=1}^{\infty} \frac{h_j^2}{- i - i\alpha_j}} d\mu(h) \tag{3.23}$$

$$= \int e^{- i/2 \sum_{j=1}^{\infty} h_j^2/(1 + \alpha_j)} d\mu(h)$$

$$= \int e^{- i/2 (h, (I + A)^{- 1} h)} d\mu(h)\,.$$

Thus,

$$(3.24)\quad I_a^1(G) = \left|\prod_{j=1}^{\infty}(1+\alpha_j)\right|^{-1/2} e^{-\frac{i\pi}{2}\operatorname{ind}(I+A)} \int_H e^{-\frac{i}{2}(h,(I+A)^{-1}h)} d\mu(h). \qquad \square$$

b) Integrals on Hilbert space.

We now define analytic Feynman integrals for functions on H. Suppose $f : H \to \mathbb{C}$ is such that for all real $\lambda > 0$, $f^\lambda \in \mathscr{L}^1(H, \mathscr{C}, m)$. For real $\lambda > 0$, let

$$(3.25)\qquad K_f(\lambda) = \int_H f^\lambda dm .$$

DEFINITION. — Let f be such that there exists an analytic function $K_f^*(z)$ on Ω such that $K_f^*(\lambda) = K_f(\lambda)$ for real $\lambda > 0$. Then we define $K_f^*(z)$ to be the analytic Gauss integral of f over H with parameter z and denote it by $\mathscr{I}_a^z(f)$.

Further, if for q real, the limit

$$I_a^q(f) = \lim_{\substack{z \to -iq \\ z \in \Omega}} \mathscr{I}_a^z(f)$$

exists, we define $I_a^q(f)$ to be the analytic Feynman integral of f over H with parameter q.

REMARK 3. — Suppose f is such that there exists an $F : B \to \mathbb{C}$ with the property

$$R(f^\lambda) = F^\lambda$$

for all real $\lambda > 0$. Then, it is easy to see that for all $\lambda > 0$,

$$J_F(\lambda) = K_f(\lambda)$$

and hence $I_a^q(F)$ exists if and only if $I_a^q(f)$ exists and in that case both are equal.

Now Lemma 2.1, Theorem 3.1 and Remark 3 give us the following results.

THEOREM 3.3. — Let $f \in \mathscr{F}(H)$ be given by (2.7). Then for $z \in \Omega$, $\mathscr{I}_a^z(f)$ exists and

$$\mathscr{I}_a^z(f) = \int_H e^{-\frac{1}{2z}|h|^2} d\mu(h) .$$

Further, for all q real, $q \neq 0$, $I_a^q(f)$ exists and

$$(3.26) \qquad \mathcal{I}_a^q(f) = \int_H e^{-\frac{i}{2q}|h|^2} d\mu(h).$$

Also, Remark 3, Proposition 2.4 and Theorem 2.2 yield the Cameron-Martin formula for $g \in \mathcal{G}^q(H)$.

THEOREM 3.4. (Cameron-Martin formula). — Let $\mu \in \mathcal{M}(H)$ and let A be a self adjoint trace class operator such that $\left(I + \frac{1}{q}A\right)^{-1}$ exists, ($q \in \mathbb{R}$, $q \neq 0$). Let

$$(3.27) \qquad g(h) = e^{\frac{i}{2}(h, Ah)} \int_H e^{i(h_1, h)} d\mu(h_1).$$

Then $I_a^q(g)$ exists and is given by

$$(3.28) \quad I_a^q(g) = \left|\det\left(I + \frac{1}{q}A\right)\right|^{-1/2} e^{-\frac{i\pi}{2}\operatorname{ind}\left(I + \frac{1}{q}A\right)} \int_H e^{-\frac{i}{2q}\left(h, \left(I + \frac{1}{q}A\right)^{-1} h\right)} d\mu(h).$$

4. SEQUENTIAL FEYNMAN INTEGRAL

a) On Hilbert space.

In this section, we define the sequential Feynman integral and prove an analogue of Theorem 3.4 (Cameron-Martin formula) for the same.

Let $f : H \to \mathbb{C}$ be such that for all $P \in \mathcal{P}$, for all real $\lambda > 0$,

$$(4.1) \qquad \int_{\mathbb{R}^m} \left| f\left(\sum_{j=1}^m \xi_j e'_j\right)\right| e^{-\frac{\lambda}{2}\Sigma_{j=1}^m \xi_j^2} d\xi < \infty$$

where $m = \dim PH$ and $(e'_1, \ldots, e'_m)$ is an orthonormal basis for PH and then for $z \in \Omega$ define

$$(4.2) \qquad J_f(z, P) = \left[\left(\frac{z}{2\pi}\right)^{1/2}\right]^m \int_{\mathbb{R}^m} f\left(\sum_{j=1}^m \xi_j e'_j\right) e^{-\frac{z}{2}\Sigma_{j=1}^m \xi_j^2} d\xi.$$

Observe that (4.1) implies that the integral appearing in (4.2) is a proper integral.

DEFINITION. — Let f satisfy (4.1) for all $\lambda > 0$ and $P \in \mathscr{P}$. Let $q \neq 0$ be a real number. Suppose that

$$\lim_{n \to \infty} J_f(z_n, P_n)$$

exists for all $z_n \to -iq$, $z_n \in \Omega$ and for all $P_n \xrightarrow{s} I$, $P_n \in \mathscr{P}$. Then we define the limit, easily seen to be independent of $\{ z_n \}$, $\{ P_n \}$, to be the sequential Feynman integral of f with parameter q and denote it by $I_s^q(f)$.

REMARK 4. — It is easy to see that (4.1) is equivalent to

$$(4.1)' \qquad f^\lambda \circ P \in \mathscr{L}^1(H, \mathscr{C}, m)$$

and further that $J_f(\lambda, P) = K_{f \circ P}(\lambda)$. Also, Morera's theorem and Fubini's theorem imply that if (4.1) holds for all $\lambda > 0$, $J_f(z, P)$ is analytic on Ω and thus

$$(4.3) \qquad \mathscr{I}_a^z(f \circ P) = K_{f \circ P}^*(z) = J_f(z, P).$$

So, the sequential Feynman integral can equivalently be defined as

$$(4.4) \qquad I_s^q(f) = \lim_n \mathscr{I}_a^{z_n}(f \circ P_n)$$

for $z_n \to -iq$, $z_n \in \Omega$, $P_n \xrightarrow{s} I$, if the limit in (4.4) exists for all such $\{ z_n \}$, $\{ P_n \}$.

REMARK 5. — The sequential Feynman integral $I_s^q(f)$ can be regarded as an integral of the function $f(h)e^{iq|h|^2}$ with respect to a « uniform » (complex valued) measure $\mathscr{D}$, normalized such that the integral of $e^{iq|h|^2}$ is 1. Of course, such a measure does not exist and hence this indirect definition. Many authors prefer the notation $\int e^{iq|h|^2} f(h) \mathscr{D}(h)$ (or some variant of this) for $I_s^q(f)$. In physical problems, it is useful to think of $I_s^q(f)$ as « $\int e^{iq|h|^2} f(h) \mathscr{D}(h)$. »

We now show the existence of I_s^q for the classes $\mathscr{F}(H)$ and $\mathscr{G}^q(H)$ and obtain the Cameron-Martin formula for the sequential integral.

THEOREM 4.1. — Let $f \in \mathscr{F}(H)$. Then for $q \neq 0$, $I_s^q(f)$ exists and is equal to $I_a^q(f)$,

$$(4.5) \qquad I_s^q(f) = I_a^q(f) = \int_H e^{-\frac{i}{2q}|h|^2} d\mu(h),$$

where f is given by (2.7).

Proof. — Let $P_n \xrightarrow{s} P$. Let $\mu_n = \mu \circ P_n^{-1}$. Then

$$(4.6) \qquad f \circ P_n(h) = \int e^{i(h,h_1)} d\mu_n(h_1)$$

and hence by Remark 4 above and Theorem 3.3, for $z \in \Omega$

$$(4.7) \qquad J_f(z, P_n) = \mathscr{I}_a^z(f \circ P_n) = \int e^{-\frac{1}{2z}|h|^2} d\mu_n(h) = \int e^{-\frac{1}{2z}|P_n h|^2} d\mu(h).$$

Now, if $z_n \to -iq$, $z_n \in \Omega$; then $\mathrm{Re}\, z_n \geq 0$, and hence by the dominated convergence theorem,

$$(4.8) \qquad \lim_n J_f(z_n, P_n) = \int e^{-\frac{i}{2q}|h|^2} d\mu(h).$$

Thus $I_a^q(f)$ exists and (4.5) holds. □

THEOREM 4.2 (Cameron-Martin formula). — Let $g \in \mathscr{G}^q(H)$ be given by

$$(4.9) \qquad g(h) = e^{\frac{i}{2}(h, Ah)} f(h)$$

where A is a self adjoint, trace class operator on H such that $\left(I + \frac{1}{q}A\right)$ is invertible and $f \in \mathscr{F}(H)$ is of the form

$$(4.10) \qquad f(h) = \int_H e^{i(h,h_1)} d\mu(h_1), \quad \mu \in \mathscr{M}(H).$$

Then $I_s^q(g)$ exists, equals $I_a^q(g)$ and has the value

$$(4.11) \qquad I_s^q(g) = \left|\det\left(I + \frac{1}{q}A\right)\right|^{-1/2} e^{-\frac{i\pi}{2}\mathrm{ind}\left(I + \frac{1}{q}A\right)} \int_H e^{-\frac{i}{2q}\left(h, \left(I + \frac{1}{q}A\right)^{-1} h\right)} d\mu(h).$$

Proof. — We will prove the case $q = 1$. The proof for general q is similar. Let $P_n \in \mathscr{P}$, $P_n \xrightarrow{s} I$. Let $A_n = P_n A P_n$, $\mu_n = \mu \circ P_n^{-1}$. Then

$$(4.12) \qquad g \circ P_n(h) = e^{\frac{i}{2}(h, A_n h)} \int e^{i(h,h_1)} d\mu_n(h_1).$$

nOw if

$$(4.13) \qquad A_n h = \sum_{j=1}^{k_n} (h, e_j^n)\alpha_j^n$$

where α_j^n are the eigenvalues and e_j^n the eigenfunctions of A_n, then proceeding as in the proof of Theorem 3.2, it is easily seen

$$J_g(z, P_n) = \prod_{j=1}^{k_n} \frac{(z)^{1/2}}{(z - i\alpha_j)^{1/2}} \int e^{-1/2\Sigma_{j=1}^{k_n} \frac{(h, e_j^n)^2}{(z - i\alpha_j^n)}} d\mu_n(h).$$

Fix $z_n \to -i$, $z_n \in \Omega$. Then

$$J_f(z_n, P_n) = a_n . b_n \tag{4.14}$$

where

$$a_n = \sum_{j=1}^{k_n} \frac{(z_n)^{1/2}}{(z_n - i\alpha_j^n)^{1/2}} \tag{4.15}$$

and

$$b_n = \int e^{-1/2\Sigma_{j=1}^{k_n} \frac{(P_n h, e_j^n)^2}{(z_n - i\alpha_j^n)}} d\mu_n(h) \tag{4.16}$$

$$= \int e^{-1/2\Sigma_{j=1}^{k_n} \frac{(h, e_j^n)^2}{(z_n - i\alpha_j^n)}} d\mu(h).$$

We will now show that a_n, b_n have limits and evaluate them.

Let α_j be the eigenvalues of A, enumerated such that $1 + \alpha_j < 0$ for $j = 1, \ldots, m$; $1 + \alpha_j > 0$ for $j \geq m + 1$ and $m = \text{ind}(I + A)$. (Recall that $1 + \alpha_j \neq 0$ for all j, as $I + A$ is invertible.) Since $A_n \to A$ is trace norm, we can rearrange $\{ \alpha_j^n \}$, $j = 1, \ldots, k_n$ such that

$$\lim_n \alpha_j^n = \alpha_j, \qquad \text{uniformly in } j. \tag{4.17}$$

Also, $A_n \to A$ in trace norm implies that $\| A_n \|_1 \to \| A \|_1$ i. e.

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} | \alpha_j^n | = \sum_{j=1}^{\infty} | \alpha_j | < \infty . \tag{4.18}$$

We claim that (4.17) and (4.18) imply that

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} \| \alpha_j^n | - | \alpha_j \| = 0. \tag{4.19}$$

For this write

$$| | \alpha_j^n | - | \alpha_j | | = | \alpha_j^n | + | \alpha_j | - 2 | \alpha_j^n | \wedge | \alpha_j | \tag{4.20}$$

and use dominated convergence theorem and (4.17) to get

$$\lim_{n\to\infty}\sum_{j=1}^{\infty}|\alpha_j^n| \wedge |\alpha_j| = \sum_{j=1}^{\infty}|\alpha_j|. \tag{4.21}$$

Now, (4.18), (4.20) and (4.21) imply (4.19). Using usual arguments it can be shown that (4.19) implies

$$\lim_{k\to\infty}\left[\sup_{n\geq 1}\sum_{j=k}^{\infty}|\alpha_j^n|\right] = 0. \tag{4.22}$$

Now write

$$a_n = \prod_{j=1}^{\infty}(1 + u_j^n) \tag{4.23}$$

where

$$u_j^n = \frac{(z_n)^{1/2}}{(z_n - i\alpha_j^n)^{1/2}} - 1 = \left(1 - \frac{i\alpha_j^n}{z_n}\right)^{-1/2} - 1.$$

(Here we have used that if $\operatorname{Re} z_1 > 0$, $\operatorname{Re} z_2 > 0$, then $(z_1 z_2)^{1/2} = z_1^{1/2}.z_2^{1/2}$.) Now, (4.17), $z_n \to -i$ and $|\alpha_j| \to 0$ implies that there exist n_0, j_0 such that for $n \geq n_0$, $j \geq j_0$,

$$\left|\frac{-i\alpha_j^n}{z_n}\right| \leq 1/2,\ |\alpha_j^n| \leq 1/2 \tag{4.24}$$

and hence for $n \geq n_0$, $j \geq j_0$,

$$|u_j^n| \leq K_1 . \left|\frac{-i\alpha_j^n}{z_n}\right| \leq K_1.K_2.|\alpha_j^n| \tag{4.25}$$

where K_1, K_2 are constants such that

$$||1 + w|^{-1/2} - 1| \leq K_1|w| \qquad \text{for } |w| \leq 1/2 \tag{4.26}$$

and

$$\left|\frac{-i}{z_n}\right| \leq K_2 \qquad \text{for all } n. \tag{4.27}$$

Now (4.25) and (4.22) give

$$\lim_{k\to\infty}\left[\sup_{n}\sum_{j=k}^{\infty}|u_j^n|\right] = 0. \tag{4.28}$$

Also, (4.28) and the inequality

$$\left|\prod_{j=k}^{\infty}(1 + u_j^n) - 1\right| \leq e^{\Sigma_{j=k}^{\infty} u_j^n}$$

imply

(4.29)
$$\lim_{k\to\infty} \sup_n \left[\prod_{j=k}^{\infty}(1 + u_j^n) - 1\right] = 0.$$

Now, (4.29) and the usual arguments give

(4.30)
$$\lim_{n\to\infty} \prod_{j=1}^{\infty}(1 + u_j^n) = \prod_{j=1}^{\infty}(1 + u_j)$$

where

$$u_j = \lim_{n\to\infty} u_j^n = \frac{(-i)^{1/2}}{(-i - i\alpha_j)^{1/2}}.$$

For $j \leq m$, $(1 + \alpha_j) < 0$ and thus

(4.31)
$$u_j = |1 + \alpha_j|^{-1/2} \cdot \frac{e^{-i\pi/4}}{e^{i\pi/4}} = |1 + \alpha_j|^{-1/2} . e^{-i\pi/2}$$

and for $j > m$, $(1 + \alpha_j) > 0$ and thus

(4.32)
$$u_j = |1 + \alpha_j|^{-1/2}.$$

From (4.23), (4.30), (4.31) and (4.32) we have

(4.33)
$$\lim_n a_n = e^{-i\pi/2.m}\left[\left|\prod_{j=1}^{\infty}(1+\alpha_j)\right|\right]^{-1/2} = |\det(I+A)|^{-1/2} e^{-i\pi/2.\mathrm{ind}(I+L)}.$$

For $h \in H$ and $n \geq 1$, let

$$\phi_n(h) = -1/2 \sum_{j=1}^{k_n} \frac{(P_n h, e_j^n)^2}{(z_n - i\alpha_j^n)},$$

so that

$$b_n = \int e^{-\phi_n(h)} d\mu(h).$$

We claim that for all $h \in H$,

(4.34)
$$\phi_n(h) \to \phi(h) \stackrel{\text{def}}{=} -\frac{i}{2}(h, (I + A)^{-1}h).$$

To see this, let

$$\phi_n'(h) = -1/2 \sum_{j=1}^{k_n} \frac{(P_n h, e_j^n)^2}{(-i - i\alpha_j^n)} = -\frac{i}{2}(h, (I + A_n)^{-1}h).$$

Now,

$$(4.35)\qquad \left| \frac{1}{z_n - i\alpha_j^n} - \frac{1}{-i - i\alpha_j^n} \right| \leq \frac{|z_n + i|}{|z_n| \left| \left(1 - \frac{i\alpha_j^n}{z_n}\right)(1 + \alpha_j^n) \right|} \to 0$$

as $n \to \infty$, since $z_n \to -i$. In fact in view of (4.24), this limit is uniform in j. Let LHS of (4.35) be less than ε_n for all j, where $\varepsilon_n \to 0$. Then

$$(4.36)\qquad |\phi_n(h) - \phi_n'(h)| \leq \varepsilon_n \sum_{j=1}^{k_n} (P_n h, e_j^n)^2 \leq \varepsilon_n \| P_n h \|^2 \to 0 \quad \text{as } n \to \infty.$$

Also, $A_n \to A$ in trace class implies that

$$(4.37)\qquad (I + A_n)^{-1} - I \to (I + A)^{-1} - I$$

in trace class (see Lemma XI.9.15, Dunford and Schwartz [*16*]) and hence $\phi_n'(h) \to \phi(h)$. This and (4.36) imply (4.34). Since $z_n \in \Omega$, it is easily seen that

$$Re(\phi_n(h)) \leq 0$$

and hence

$$|e^{\phi_n(h)}| \leq 1.$$

Thus, from the dominated convergence theorem and (4.34),

$$(4.38)\qquad b_n \to \int e^{\phi(h)} d\mu(h) = \int e^{-\frac{i}{2}(h, (I + A)^{-1}h)} d\mu(h).$$

Now, (4.33) and (4.38) imply that $J_f(z_n, P_n)$ converges to the RHS in (4.11). Thus, $I_s^1(g)$ exists and (4.11) holds. Also, in view of (3.28), $I_s^1(g) = I_a^1(g)$. □

We now consider sequential Feynman integrals.

b) On abstract Wiener space.

Suppose $F : B \to \mathbb{C}$ is such that

$$(4.39)\qquad R(f^\lambda) = F^\lambda \quad \text{for all } \lambda > 0$$

for some $f \in \mathscr{L}(\mathrm{H}, \mathscr{C}, m)$. For $\mathrm{P} \in \mathscr{P}$, define $\mathrm{F_P}$ by

$$\mathrm{F_P} = \mathrm{R}(f \circ \mathrm{P}). \tag{4.40}$$

By the definition of the R-mapping for cylindrical functions, it can be checked that

$$\mathrm{R}(f^\lambda \circ \mathrm{P}) = \mathrm{F_P^\lambda} \quad \text{for all } \lambda > 0. \tag{4.41}$$

Now, $\mathrm{F_P^\lambda}$ converges in v-probability to F^λ and thus for each λ, $\{ \mathrm{F_P^\lambda} \}$ is a finite dimensional approximation to F^λ. Suppose that the analytic Wiener integral $\mathscr{I}_a^z(\mathrm{F_P})$ exists for all $\mathrm{P} \in \mathscr{P}$ and further assume that for all $z_n \in \Omega$, $z_n \to -iq$, $(q \neq 0)$ and for all $\mathrm{P}_n \xrightarrow{s} \mathrm{I}$, $\mathrm{P}_n \in \mathscr{P}$, the limit

$$\lim_{n \to \infty} \mathscr{I}_a^{z_n}(\mathrm{F}_{\mathrm{P}_n}) = \mathrm{I}_s^q(\mathrm{F}) \tag{4.42}$$

exists. Then we define $\mathrm{I}_s^q(\mathrm{F})$ to be the sequential Feynman integral of F with parameter q.

It is easy to see in view of Remark 4 that

$$\mathscr{I}_a^z(\mathrm{F_P}) = \mathrm{J}_f(z, \mathrm{P})$$

and hence $\mathrm{I}_s^q(\mathrm{F})$ exists if and only if $\mathrm{I}_s^q(f)$ exists (where F, f are related by (4.39)) and then both are equal.

Thus, Lemma 2.1 and Theorem 4.1 imply that for $\mathrm{F} \in \mathscr{F}(\mathrm{B})$, $\mathrm{I}_s^q(\mathrm{F})$ exists and is equal to $\mathrm{I}_a^q(\mathrm{F})$. Also Proposition 2.4 and Theorem 4.2 imply that for $\mathrm{G} \in \mathscr{G}^q(\mathrm{B})$, $\mathrm{I}_s^q(\mathrm{G})$ exists and is equal to $\mathrm{I}_a^q(\mathrm{G})$.

REMARK 6. — The equality of the sequential and analytic Feynman integrals can be viewed as an approximation result for the analytic Feynman integral in the following sense: Let $g \in \mathscr{G}^q(\mathrm{H})$, $z_n \in \Omega$, $z_n \to -iq$ and $\mathrm{P}_n \xrightarrow{s} \mathrm{I}$. $\mathrm{P}_n \in \mathscr{P}$. Then

$$\mathrm{I}_a^q(g) = \lim_n \mathscr{I}_a^{z_n}(g \circ \mathrm{P}_n). \tag{4.43}$$

Similarly, for $\mathrm{G} \in \mathscr{G}(\mathrm{B})$ given by (2.21) and P_n as above let G_n be defined by

$$\mathrm{G}_n(x) = e^{i(x, \mathrm{P}_n \mathrm{A} \mathrm{P}_n x)^\sim} \int_{\mathrm{H}} e^{i(\mathrm{P}_n h x)^\sim} d\mu(h). \tag{4.44}$$

Then, for $z_n \to -iq$, $z_n \in \Omega$ we have

$$\mathrm{I}_a^q(\mathrm{G}) = \lim_n \mathscr{I}_a^{z_n}(\mathrm{G}_n). \tag{4.45}$$

In (4.43), $\mathscr{I}_a^{z_n}(g \circ \mathrm{P}_n)$ can be evaluated explicitly as we have seen already.

Using these observations, we can give a formula for analytic Feynman integrals for the $\mathscr{G}$ class involving a single limit and proper integrals. We state this result as a theorem for future reference.

THEOREM 4.3. — Let $g \in \mathscr{G}^q(\mathrm{H})$ and $\mathrm{G} \in \mathscr{G}^q(\mathrm{B})$ be given by (2.22) and (2.23) respectively. Let $\{ e'_n \}$ be any complete orthonormal basis in H and let $z_n \in \Omega$ be such that $z_n \to -iq$ (say $z_n = -iq + 1/n$). Then

$$(4.46) \qquad \mathrm{I}_a^q(g) = \mathrm{I}_s^q(g) = \mathrm{I}_a^q(\mathrm{G}) = \mathrm{I}_s^q(\mathrm{G})$$

$$= \lim_{n\to\infty} \left[\left(\frac{z_n}{2\pi} \right)^{1/2} \right]^n \int_{\mathbb{R}^n} g\left(\sum_{j=1}^{n} \xi_j e'_j \right) e^{-\frac{z_n}{2}\sum_{j=1}^n \xi_j^2} d\xi .$$

From now on, we will drop the suffix « a » and « s » from I_a^q and I_s^q when the integrands belong to the class $\mathscr{F}$ or $\mathscr{G}^q$.

5. APPLICATIONS TO FEYNMAN PATH INTEGRALS

a) Feynman path integrals and the Schrödinger equation.

Feynman's fundamental idea was to show that the solution of the Schrödinger equation of Quantum Mechanics (for a single particle of mass m)

$$(5.1) \qquad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \mathrm{V}\psi$$

$$\psi(0, x) = \phi(x)$$

can be expressed as

$$(5.2) \qquad \psi(t, x) = \int_{\gamma(t)=x} e^{\frac{i}{2}\frac{m}{\hbar}\int_0^t (\dot\gamma(s))^2 ds - \frac{i}{\hbar}\int_0^t \mathrm{V}(\gamma(s))ds} \phi(\gamma(0))\mathscr{D}(\gamma)$$

where the integral is carried over a suitable space of paths and $\mathscr{D}(\gamma)$ is a uniform « measure » on the space of paths normalized so that

$$\int_{\gamma(t)=x} e^{\frac{i}{2}\frac{m}{\hbar}\int_0^t (\dot\gamma(s))^2 ds} \mathscr{D}(\gamma) = 1 .$$

In (5.1) above, Δ is the Laplacian, $\hbar = \dfrac{h}{2\pi}$ where h is Planck's constant and V is a suitable potential.

For simplicity, let us consider the one dimensional case. In (5.2), let us write $X(s) = \gamma(t - s) - x$, so that $X(0) = 0$ and $\gamma(s) = X(t - s) + \dot{x})$, to get

$$(5.2)' \quad \psi(t, x) = \int_{X(0)=0} e^{\frac{i}{2}\frac{m}{\hbar}\int_0^t (\dot{X}(s))^2 ds - \frac{i}{\hbar}\int_0^t V(X(t-s)+x)ds} . \phi(X(t) + x)\mathscr{D}(X).$$

Assuming that the « paths » γ have finite kinetic energy, we get that $X \in \mathscr{H}_t$, where $\mathscr{H}_t$ is the real separable Hilbert space of functions $X : [0, t] \to \mathbb{R}$ with $X(0) = 0$, $\dot{X} = \frac{dX}{ds} \in L^2[0, t]$ with the inner product

$$(5.3) \qquad (X_1, X_2) = \int_0^t \dot{X}_1(s)\dot{X}_2(s)ds.$$

Then (5.2)′ can be rewritten as

$$(5.2)'' \quad \psi(t, x) = \int_{X \in \mathscr{H}_t} e^{\frac{i}{2}\cdot\frac{m}{\hbar}|X|^2 - \frac{i}{\hbar}\int_0^t V(X(t-s)+x)ds} \phi(X(t - s) + x)\mathscr{D}(X)$$

and in view of Remark 4, the integral appearing in (5.2)′ can be regarded as a Feynman integral with parameter $q = m/\hbar$ of the function $g_{t,x}$ defined by

$$(5.4) \qquad g_{t,x}(X) = e^{-\frac{i}{\hbar}\int_0^t V(X(t-s)+x)ds} \phi(X(t) + x)$$

over the Hilbert space $\mathscr{H}_t$. Thus, Feynman's idea can be expressed as follows: The solution to the equation (5.1) can be represented by

$$(5.5) \qquad \psi(t, x) = I^{\left(\frac{m}{\hbar}\right)}(g_{t,x}),$$

where $g_{t,x}$ is given by (5.4).

Let the potential V be given by

$$(5.6) \qquad V(x) = ax^2 + bx + c + \int_{\mathbb{R}} e^{ixy}d\mu_1(y)$$

and let ϕ be given by

$$(5.7) \qquad \phi(x) = \int_{\mathbb{R}} e^{ixy}d\mu_2(y)$$

where μ_1, μ_2 are complex Borel measures on $\mathbb{R}$ with bounded variation. Assume that $\phi \in L^2(\mathbb{R})$. Then, it can be shown that $g_{t,x} \in \mathscr{G}^q(\mathscr{H}_t)$ (with $q = m/\hbar$) and ψ defined by (5.5) is a (weak) solution of the Schrödinger equation (5.1) (see [*1*] [*15*]). We will not give a proof of this assertion. We just remark that ψ defined by (5.5) can be computed using our Cameron-

Martin formula (Theorem 3.3 and 4.2) and then we can proceed as in [1] or [15] to show that Ψ is a solution to (5.1).

In (5.7) above, the solution to the Schrödinger equation was represented as an integral over the path space $\mathcal{H}_t$-which happens to be the RKHS of the Wiener measure. We now show that instead of $\mathcal{H}_t$, we can take the path space to be $C_0[0, t]$-the space of real valued continuous functions X on $[0, t]$ with $X(0) = 0$.

For $X \in \mathcal{H}_t$, let $\| X \|_0 = \sup_{0 \leq s \leq t} | X(t) |$. Then $\| \, . \, \|_0$ is a measurable norm on $\mathcal{H}_t$ and the completion of $\mathcal{H}_t$ under $\| \, . \, \|_0$ is $C_0[0, t]$. (See Kuo [13]).

For V, ϕ satisfying (5.6) and (5.7), let

$$(5.8)\quad G_{t,x}(X) = \exp\left(- \frac{i}{\hbar} \int_0^t V(X(t-s) + x) ds \right) \phi(X(t)+x), \quad X \in C_0[0, t].$$

Then, it is easy to see that $G^{\lambda}_{t,x}$ is a continuous function on $C_0[0, t]$ for all $\lambda > 0$ and its restriction to $\mathcal{H}_t$ is $g^{\lambda}_{t,x}$. Thus we have by Theorem 6.3 in [13])

$$(5.9)\qquad R(g^{\lambda}_{t,x}) = G^{\lambda}_{t,x} \quad \text{for all } \lambda > 0,$$

where R is the « m-lifting » (see Section 2). Hence, by Remark 3,

$$(5.10)\qquad I^q(g_{t,x}) = I^q(G_{t,x})$$

and thus the solution ψ to the Schrödinger equation can also be represented as

$$(5.11)\qquad \psi(t, x) = I^{(m/\hbar)}(G_{t,x}).$$

In other words, the solution to the Schrödinger equation can be represented as a Feynman integral of a functional over either $\mathcal{H}_t$ or $C_0[0, t]$.

In most of the physical literature on Feynman integrals, H is taken to be $\overline{\mathcal{H}}_t$—the RKHS of the standard Wiener process with paths $X \in C[0, t]$ with $X(t) = 0$. We have used $\mathcal{H}_t$ instead of $\overline{\mathcal{H}}_t$, so that the representation (5.6)-(5.7) is similar to the Feynman-Kac formula.

b) Feynman integrals on the RKHS of the pinned Wiener process and the Green's function for the Schrödinger equation.

According to Feynman, the Green's function G (or the fundamental solution) for the Schrödinger equation (5.1) is given by

$$(5.13)\qquad G(t + s, b, s, a) = G(t, b, 0, a)$$

and

$$G(t, b, 0, a) = \int_{\substack{X \in \Gamma, X(0)=a \\ X(t)=b}} e^{\frac{i}{\hbar} S(X)} \mathscr{D}(x) \tag{5.14}$$

where the action functional

$$S(X) = \int_0^t L(X_s) ds = \int_0^t \left[\frac{m}{2} (\dot{X}(s))^2 - V(X(s)) \right] ds,$$

L being the Lagrangian and Γ is the ensemble of all possible quantum mechanical paths with finite kinetic energy and $\mathscr{D}(X)$ is a « uniform measure » on Γ with the normalization

$$\int_{X \in \Gamma, X(0)=X(t)=0} e^{\frac{i}{2} \frac{m}{\hbar} \int_0^t \dot{X}_s^2 ds} \mathscr{D}(X) = \left(2\pi i t \frac{\hbar}{m} \right)^{-1/2}. \tag{5.15}$$

Here $\Gamma' = \{ X \in \Gamma : X(0) = a, X(t) = b \}$ is not a linear space (unless $a = b = 0$) and thus we cannot use our definition of Feynman integrals directly for the integral appearing in (5.14). Ito [9] has given a proof of (5.14) (for a class of potentials V) by directly defining Feynman integrals over Γ' via an isometric mapping between Γ' and a Hilbert space.

We shall proceed somewhat differently and follow Feynman more closely. If one regards the path X of a quantum mechanical particle which is at position at a time $\tau = 0$ and at position b at time $\tau = t$ as a random path which deviates in a random fashion from the classical path $\bar{X}$ (i. e. the path it would follow under the laws of classical mechanics), then it is natural to write $X = \bar{X} + Y$, Y being the random deviate and write the integral in (5.14) as an integral over Y.

In his book [7] with Hibbs, Feynman has included a brief discussion of this point of view with special emphasis on the case of the harmonic potential.

Let $V(x) = \frac{m}{2} \omega^2 x^2$ so that the action S(X) is given by

$$S(X) = \frac{m}{2} \int_0^t (\dot{X}_s^2 - \omega^2 X_s^2) ds.$$

Since the classical path $\bar{X}$ satisfies

$$\ddot{\bar{X}}_s + \omega^2 \bar{X}_s = 0,$$

it can be easily checked, using integration by parts, that

$$S(X) = S(\bar{X}) + S(Y). \tag{5.16}$$

Since $\mathscr{D}(X)$ is a « uniform » measure, $\mathscr{D}(\overline{X} + Y) = \mathscr{D}(Y)$, Feynman argues that

$$(5.17) \qquad \int_{X \in \Gamma'} e^{\frac{i}{\hbar}S(X)} \mathscr{D}(X) = \int_{Y \in \Gamma_0} e^{\frac{i}{\hbar}S(\overline{X}+Y)} \mathscr{D}(Y) = e^{\frac{i}{\hbar}S(\overline{X})} \int_{Y \in \Gamma_0} e^{\frac{i}{\hbar}S(Y)} \mathscr{D}(Y)$$

in view of (5.16), where $\Gamma_0 = \{ Y \in \Gamma : Y(0) = Y(t) = 0 \}$. It is a simple exercise to show that the first factor on the RHS in (5.17) equals

$$(5.18) \qquad \exp\left[\frac{im\omega}{2\hbar \sin \omega t}((a^2 + b^2)\cos \omega t - 2ab)\right].$$

We now show that the second factor can be expressed as a Feynman integral as defined in Sections 3 and 4, and can be evaluated using the Cameron-Martin formula (Theorem 3.4 and 4.2) and that G given by (5.13), (5.14) is the Green's function for the harmonic oscillator.

Let $\mathscr{H}_{0,t}$ be the RKHS of the pinned Wiener process on $[0, t]$ i. e. $\mathscr{H}_{0,t} = \{ Y \in \mathscr{H}_t : Y(t) = 0 \}$. Then, as sets $\mathscr{H}_{0,t} = \Gamma_0$. Let

$$(5.19) \qquad g(Y) = e^{-\frac{i}{2}\frac{m}{\hbar}\omega^2 \int_0^t (Y(s))^2 ds}$$

Then

$$e^{\frac{i}{\hbar}S(Y)} = e^{\frac{i}{2}\cdot\frac{m}{\hbar}|Y|^2} \cdot g(Y).$$

We shall regard Feynman's heuristic integral

$$(5.20) \qquad \int_{Y \in \Gamma_0} e^{\frac{i}{\hbar}S(Y)} \mathscr{D}(Y) = \int_{Y \in \mathscr{H}_{0,t}} e^{\frac{i}{2}\cdot\frac{m}{\hbar}|Y|^2} g(Y) \mathscr{D}(Y)$$

as the integral given by $\left(2\pi i t \frac{\hbar}{m}\right)^{-1/2} I^{\left(\frac{m}{\hbar}\right)}(g)$ in view of Remark 5 and the normalization (5.15)

We now show that $g \in \mathscr{G}^q(\mathscr{H}_{0,t})$ and evaluate $I^q(g)$ (with $q = m/\hbar$) using Theorem 4.2. We adopt a method used by Ph. Combe et al. in [*4*] for the case of $\mathscr{H}_t$.

Define a bilinear form $\mathscr{A}$ on $\mathscr{H}_{0,t}$ by

$$(5.21) \qquad \mathscr{A}(Y_1, Y_2) = \omega^2 \int_0^t Y_1(s) Y_2(s) ds.$$

It is easy to see that $\mathscr{A}$ is continuous and symmetric and hence

$$\mathscr{A}(Y_1, Y_2) = (Y_1, AY_2)$$

for a symmetric bounded operator A. It can be checked that the eigenvalues of A are $\frac{\omega^2 t^2}{n^2\pi^2}$ for $n = 1, 2, \ldots$ and hence A is a trace class operator. Let $A_1 = -qA$. Then

$$(Y, A_1 Y) = -q\omega^2 \int_0^t (Y(s))^2 ds$$

and hence

$$g(Y) = e^{i(Y, A_1 Y)}.$$

For $t \neq \frac{n\pi}{\omega}$, $\left(I + \frac{1}{q} A_1\right) = (I - A)$ is invertible and hence $g \in \mathcal{G}^q(\mathcal{H}_{0,t})$ and by Theorem (4.2)

$$\text{(5.22)} \qquad I^q(g) = \left| \det\left(I + \frac{1}{q} A_1\right)\right|^{-1/2} e^{-\frac{i\pi}{2}\operatorname{ind}\left(I + \frac{1}{q}A_1\right)}$$

$$= |\det(I - A)|^{-1/2} e^{-\frac{i\pi}{2}\operatorname{ind}(I - A)}$$

$$= \left|\prod_{n=1}^{\infty} 1 - \frac{\omega^2 t^2}{n^2\pi^2}\right|^{-1/2} e^{-\frac{i\pi}{2}\left[\frac{\omega t}{\pi}\right]}$$

$$= \left|\frac{\sin \omega t}{\omega t}\right|^{-1/2} e^{-\frac{i\pi}{2}\left[\frac{\omega t}{\pi}\right]}$$

where $\left[\frac{\omega t}{\pi}\right]$ is the largest integer less than $\frac{\omega t}{\pi}$. Finally combining (5.17)-(5.22), we have

$$\text{(5.23)} \quad G(t, b, 0, a)$$

$$= \left(\frac{m}{2\pi i t \hbar}\right)^{1/2} \cdot \left(\left|\frac{\omega t}{\sin \omega t}\right|\right)^{1/2} . e^{-\frac{i\pi}{2}\left[\frac{\omega t}{\pi}\right]} . e^{\frac{i}{\hbar} S(\bar{X})}$$

$$= \left|\frac{m\omega}{2\pi i \hbar \sin \omega t}\right|^{1/2} . e^{-\frac{i\pi}{2}\left[\frac{\omega t}{\pi}\right]} . \exp\left[\frac{i m\omega}{2\hbar \sin \omega t}\{(a^2 + b^2)\cos \omega t - 2ab\}\right]$$

which is the Green's function for the Schrödinger equation when $V(x) = \frac{\omega^2 x^2}{2}$. (See [9]). The Maslov index does not appear in the expression (111) obtained by Ito in [9]. This is because he considers values of $t < \frac{\pi}{\omega}$. The Maslov index is also missing in the expression given in Feynman-Hibbs [7, Chapter 3].

c) **Polygonal approximations on path space.**

As pointed out earlier, one important consequence of the equality of analytic and sequential integrals in $\mathcal{G}$ is that the analytic Feynman integral can be written as a limit of finite dimensional integrals. When the underlying Hilbert space is $\mathcal{H}_t$, these approximating integrals can be taken to be integrals over « polygonal paths ».

Let us fix a partition $\Pi_k = \{ 0 \leq t_0^k < t_1^k < \ldots < t_{m_k}^k = t \}$ of $[0, t]$ and let

$$(5.24) \qquad \phi_j^k(s) = \frac{1}{\sqrt{t_j^k - t_{j-1}^k}} \int_0^s 1(u)_{[t_{j-1}^k, t_j^k]} du .$$

Then it is easily seen that for $i \neq j$, ϕ_i^k and ϕ_j^k are orthogonal and $| \phi_i^k | = 1$, $i = 1, \ldots, m_k$. Let P^k denote the orthogonal projection onto span $(\phi_1^k, \ldots, \phi_{m_k}^k)$. Then it is easily seen that for $X \in \mathcal{H}_t$

$$(5.25) \quad (P^k X)(s) = \int_0^s \sum_{j=1}^{m} \left(\frac{X(t_j^k) - X(t_{j-1}^k)}{t_j^k - t_{j-1}^k} \right) 1(u)_{[t_{j-1}^k, t_j^k]} du$$

$$= X(t_{j-1}^k) + \frac{X(t_j^k) - X(t_{j-1}^k)}{t_j^k - t_{j-1}^k}(s - t_{j-1}^k) \quad \text{for} \quad t_{j-1}^k \leq s \leq t_j^k$$

which is the usual polygonal approximation of X for the partition $\{ t_j^k \}$.

It can be checked that if $\delta(\pi_k) = \sup_j | t_j^k - t_{j-1}^k |$ tends to zero as $k \to \infty$, then $P^k \xrightarrow{s} I$. Let us fix such a double sequence $\{ t_j^k \}$.

Then for $g \in \mathcal{G}^q(\mathcal{H}_t)$, we have by the definition of the sequential integral,

$$(5.26) \quad I^q(g) = \lim_{k \to \infty} \left[\left(\frac{z_k}{2\pi} \right)^{1/2} \right]^{m_k} \int_{\mathbb{R}^{m_k}} \left(\sum_{j=1}^{m_k} \xi_j \phi_j^k \right) \exp\left(- \frac{z_k}{2} \sum_{j=1}^{m_k} \xi_j^2 \right) d\xi$$

where $z_k \in \Omega$, $z_k \to -iq$.

Here, $\Sigma_{j=1}^{m_k} \xi_j \phi_j^k$ is a polygonal path which is equal to $\Sigma_{j=1}^{l} \xi_j (t_j^k - t_{j1}^k)^{1/2}$ at $t = t_l^k$ and linear elsewhere. Thus, for $g \in \mathcal{G}^q(\mathcal{H}_t)$, the Feynman integral of g can be obtained as the limit of « proper » integrals over polygonal paths. The same is true for $g \in \mathcal{G}^q(\mathcal{H}_{0,t})$ as P^k given by (5.25) is also an orthogonal projection on $\mathscr{H}_{0,t}$. Of course, in this case $\phi_i^k \notin \mathscr{H}_{0,t}$ and we have to choose a different basis for $P^k(\mathscr{H}_{0,t})$.

d) Fourier series approximation of the Feynman path integrals.

In the sequential path integrals defined by Cameron and Storvick and by Elworthy and Truman [3] [5] only the sequence of projections on polygonal paths on $\mathscr{H}_t$ or $\overline{\mathscr{H}}_t$ are considered (see (*b*)). Theorem 4.3 enables us to choose other sequences $\{ P_n \}$ that lead to interesting approximations. For example, fix the complete orthonormal system $\{ e_n \}$, where $e_n(\tau) = \frac{\sqrt{2t}}{n\pi} \sin \frac{n\pi\tau}{t}$, in $\mathscr{H}_{0,t}$ and define P_n to be the orthogonal projection with range, span $\{ e_1, \ldots, e_n \}$. In view of Theorem 4.3, for any $g \in \mathscr{G}^q(\mathscr{H}_{0,t})$ $I^q(g)$ can be calculated using $\{ P_n \}$. With this choice of $\{ P_n \}$, Theorem 4.3 makes rigorous Feynman's ideas of an alternative method of evaluating path integrals using « Fourier series » (see [7], page 71 where the integrand g of (*b*) is considered).

e) Remarks on the *m*-lifting approach.

In Section 2, the *m*-lifting $R(f)$ has been defined as a random variable on the abstract Wiener space B associated with a measurable norm $\| \cdot \|$ on H. It will be recalled that R has been defined in terms of $(h, x)^\sim$ which itself has been defined to be a Gaussian random variable on (B, ν) with zero mean and variance $| h |^2$.

Besides the examples of (H, B, ν) already considered one could take (*i*) $B = C[0, t]$ and ν to be a general Gaussian Markov process and (*ii*) B to be the space of continuous functions $x(\tau_1, \tau_2)$, $(\tau_1, \tau_2) \in [0, t_1] \times [0, t_2]$ and ν to be the 2-parameter Wiener measure (sometimes also called the Yeh-Wiener process). It would appear that the former problem would lead to Feynman integral representations of solutions of other types of Schrödinger-like equations. At present, we do not know of a physical motivation for studying problem (*ii*) in detail.

We have defined *m*-lifting in such a way as to facilitate comparison of our results on Feynman integrals for integrands on Hilbert space with analogous results for classes of integrands on abstract Wiener space. Our interest in abstract Wiener spaces as a « ground space » for functionals for which Feynman integrals can be defined is due principally to the following reasons: (1) A great deal of effort has been devoted to the analytic continuation approach based on Wiener integrals; (2) The formal connection between Feynman's representation of the solution to Schrödinger's equation and the so-called Feynman-Kac formula; (3) The observation, apparently due to Feynman (which might have inspired most of the Wiener space approach) that the quantum mechanical paths may be compared to the irregular paths of a particle performing Brownian motion.

The m-lifting approach seems to make clear, however, that the Hilbert space is the basic path space rather than Wiener space of any sort. It is possible, indeed to extend our definition of m-lifting to enable us to define Feynman integrals for functionals of Gaussian white noise which may be conceded to be even more « irregular » than the paths of Brownian motion. We outline this procedure without going into details.

Take $H = L^2(\mathbb{R}^1)$. The canonical Gauss measure on H is then called Gaussian white noise. Let us write $(\ , \ )_0$ for the inner product in H. Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ be the space of rapidly decreasing functions regarded as a countably Hilbertian nuclear space with semi-norms $\| y \|_p$ such that $\| y \|_p^2 = \Sigma_{j=0}^p \int (1 + u^2)^p | y^{(j)}(u) |^2 du$, $(p = 0, 1, \ldots)$, $y^{(j)}$ being the jth order derivative of y.

Then $\mathcal{S}'$ the strong dual of $\mathcal{S}$ is also a nuclear space (but no longer metrizable). For convenience let us write $Q(y, y') = (y, y')_0$ for $y, y' \in \mathcal{S}$. By the Minlos-Bochner theorem, there is a unique (countably additive) Gaussian probability measure ν with covariance kernel Q (the « white noise » measure) on $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$ where $\mathcal{B}(\mathcal{S}')$ is the σ-field generated by the cylinder sets $\{ x \in \mathcal{S}' : [\langle y_1, x \rangle, \ldots, \langle y_k, x \rangle] \in B \}$, B being a Borel set in $\mathbb{R}^k$, $y_1, \ldots, y_k \in \mathcal{S}$. It is well known that $\mathcal{S}$ is $| \ . \ |_0$-dense in $L^2(\mathbb{R})$ and we have the following continuous imbedding $\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}'$. For any $h \in L^2(\mathbb{R})$ we now define $(h, x)^\sim$ exactly as was done in Section 2, taking $\{ e_j \} \subset \mathcal{S}$ to be an orthonormal basis in $L^2(\mathbb{R})$ and $\langle e_j, x \rangle$ to the evaluation of the functional $x \in \mathcal{S}'$ at e_j. For any continuous, complex-valued function f on $L^2(\mathbb{R})$ the definition of $R(f)$ is now obvious with (B, ν) replaced by $(\mathcal{S}', \nu)$. Thus the entire theory of analytic Feynman and sequential Feynman integrals can be transferred to functionals defined on $(\mathcal{S}', \nu)$. Theorems 3.1 and 3.2 immediately apply to classes $\mathcal{F}(\mathcal{S}')$ and $\mathcal{G}^q(\mathcal{S}')$ whose definition is analogous to that of $\mathcal{F}(B)$ and $\mathcal{G}^q(B)$ and the solution to the Schrödinger equation can be represented as a Feynman integral over $\mathcal{S}'$.

6. CONNECTIONS WITH SOME RELATED WORK

1) Relationship to Cameron and Storvick's papers.

We now turn to some recent work of Cameron and Storvick. In [2], they have given a definition of the analytic Feynman integral on $C_0^\nu[a, b]$, the space of continuous functions $x : [a, b] \to \mathbb{R}^\nu$ vanishing at a. In their latest paper [3], Cameron and Storvick have introduced a sequential definition of the Feynman integral in apparently a more general setting than [15]. The main result of [3] establishes the existence of this integral for integrands belonging to two classes $\hat{S}$ and S^* which are closely related to the class S of [2]. We shall now discuss in some detail the relationship of Cameron and Storvick's work with the approach and results of the present paper. Before we do so, however, two general comments seem to be in order: (*i*) The integrands in [3] are functions on domains contained in $C_0^\nu[a, b]$ or on the RKHS of ν-dimensional Wiener process. Even for this case, while the sequential integral defined by Cameron and Storvick deals only with polygonal approximations, the application of our results permits other kinds of finite dimensional approximations. One typical and important special case of the latter, as pointed out in the preceding section, leads to a rigorous justification of the « Fourier series » method of approximation alluded to in Feynman and Hibbs. (*ii*) All the integrands considered in [3] belong to the Fresnel class (as will be seen later)—except for an example of a cylinder function (see Sec. 2) which can be treated easily in our set up. The Fresnel class cannot be used when unbounded potentials are to be considered (see example of the harmonic potential of the previous section). For this purpose, we need to consider the classes $\mathcal{G}^q(H)$ and $\mathcal{G}^q(B)$ for which a Cameron-Martin formula has been proved.

To relate their work with our present work, let us assume, for simplicity, that $\nu = 1$ and $[a, b] = [0, 1]$. Let $\mathcal{H}$ be the RKHS of the Wiener measure on $C_0[0, 1]$. Then, as observed in Section 5, $(i, \mathcal{H}, C_0[0, 1])$ is an abstract Wiener space (there, $\mathcal{H}$ was denoted by $\mathcal{H}_1$). The analytic Feynman integral, $\int^{anf_q} F(x)dx$ in the notation of [2] coincides with $I_a^q(F)$ in our notation for the choice $(H, B) = (\mathcal{H}, C_0[0, 1])$. In fact, the definitions themselves coincide. See Kallianpur and Bromley, and Johnson [11] [10] for the relationship of this definition to that of Albeverio and Høegh-Krohn [1] when F belongs to the Fresnel class.

Let us begin by recalling Cameron and Storvick's definition of sequential integral, again for the case $v = 1$, $[a, b] = [0, 1]$ for simplicity.

The sequential Feynman integral of a functional F on $C_0[0, 1]$, denoted by $\int^{sf_q} F(x)dx$, was defined in [3] as

$$(6.11)\qquad \int^{sf_q} F(x)dx = \lim_n C_n \int_{\mathbb{R}^{m_n}} \exp\left(-\frac{z_n}{2}\int_0^1 |X(t, \Pi_n, \xi)|^2 dt\right) F(X(., \Pi_n, \xi))d\xi$$

where $\Pi_n = \{ 0 = t_0^n < t_1^n < \ldots < t_{m_n}^n = 1 \}$ is a sequence of partitions of $[0, 1]$ such that $\delta(\Pi_n) = \max_{1 \le j \le m_n} | t_j^n - t_{j-1}^n | \to 0$; $z_n \to -i$, $c_n = \left(\frac{z_n}{2\pi}\right)^{\frac{m_n}{2}} \Pi_{j=1}^{m_n}(t_j - t_{j-1})^{-1/2}$ and for $\xi \in \mathbb{R}^{m_n}$, $X(., \Pi_n, \xi)$ is the polygonal path given by $X(0, \Pi_n, \xi) = 0$, $X(t_j^n, \Pi_n \xi) = \xi_j$ and linear in each of the intervals $[t_{j-1}^n, t_j^n]$; $1 \le j \le m_n$. (It is required that the lim in 6.11 exists for all $z_n \to -i$, $\delta(\Pi_n) \to 0$ and is independent of the sequences $\{ z_n \}$, $\{ \Pi_n \}$).

It is easy to see that for all Π_n, ξ $X(., \Pi_n, \xi)$ belongs to $\mathscr{H}$ and thus in (6.1) above, we can replace F by f, its restriction to $\mathscr{H}$. Indeed, denoting by P_{Π_n} the orthogonal projection on $\mathscr{H}$ given by (5.25), we have

$$X(., \Pi_n, \xi) \in P_{\Pi_n}(\mathscr{H}).$$

Also, if $(\phi_1^n, \ldots, \phi_{m_n}^n)$ is the orthogonal basis for $P_{\Pi_n}(\mathscr{H})$ given by (5.24), then it is easy to see that

$$(6.2)\qquad X(., \Pi_n, \xi) = \sum_{j=1}^{m_n} \frac{(\xi_j - \xi_{j-1})}{(t_j^n - t_{j-1}^n)^{1/2}} . \phi_j^n, \ (\xi_0 \equiv 0)$$

and

$$(6.3)\qquad \int_0^1 | \dot{X}(t, \Pi_n, \xi) |^2 dt = \| X(., \Pi_n, \xi) \|_{\mathscr{H}}^2 = \sum_{j=1}^{m_n} \frac{(\xi_j - \xi_{j-1})^2}{t_j^n - t_{j-1}^n}.$$

Thus, substituting $\xi_j' = (\xi_j - \xi_{j-1}).(t_j^n - t_{j-1}^n)^{-1/2}$ in (6.1) (with $\xi_0 \equiv 0$) and recalling that $f = F|_{\mathscr{H}}$, $X(., \Pi_n, \xi) \in \mathscr{H}$, we get (using (6.2), (6.3)), that the integral appearing on the RHS of (6.1) is $J_f(z_n, P_{\Pi_n})$ (see (4.1)).

Thus their definition can be rewritten in our notations as

$$(6.4)\qquad \int^{sf_q} F(x)dx = \lim_n J_f(z_n, P_{\Pi_n})$$

if the limit exists for all $z_n \to -i$, $\{\Pi_n\}$ such that $\delta(\Pi_n) \to 0$ and $f = F|_{\mathscr{H}}$. Thus it is clear that in the definition of the sequential integral of F, the values of F outside $\mathscr{H}$ do not play any role. Also it is easy to see that

$$(6.5) \qquad \int^{sf_q} F(\xi)d\xi = I_s^q(f)$$

if the latter exists, where $f = F|_{\mathscr{H}}$. These remarks show that the sequential integral of Cameron-Storvick is really an integral over $\mathscr{H}$ and not over $C_0[0, 1]$. (The authors of [3] themselves seem to have realized this. See note at the end of Section 3 and the counterexample in [*3*]).

In [*3*], Cameron and Storvick have shown the existence of and evaluated the integral $\int^{anf_q} F(\xi)d\xi$ for $F \in S$, where (in our notation)

$$S = \left\{ F : F(\xi) = \int \exp(i(\eta, \xi)^{\sim})d\mu(\eta) s\text{-a. s.}, \mu \in \mathscr{M}(\mathscr{H}) \right\}.$$

As has been pointed out in Johnson [*10*] and Kallianpur-Bromley [*11*], the evaluation of this integral follows immediately upon observing that S is, in fact, the Fresnel class

$$S = \mathscr{F}(C_0[0, 1]).$$

As regards the sequential integral, they have shown its existence for $F \in \hat{S}$, where

$$\hat{S} = \{ F : \mathscr{D}(F) \to \mathbb{C};\ \mathscr{D}(F) \supseteq \mathscr{H} \quad \text{and for some } \mu \in \mathscr{M}(\mathscr{H})$$
$$F(\eta) = \int \exp(i(\eta, \eta'))d\mu(\eta') \quad \text{for all } \eta \in \mathscr{H} \}.$$

It is clear that $\hat{S}$ essentially coincides with $\mathscr{F}(\mathscr{H})$ and the existence of the sequential integral for $F \in \hat{S}$ follows from our result (Theorem 4.1), the proof of the latter being quite elementary and short.

In addition to S and $\hat{S}$, Cameron and Storvick introduce yet another class of integrands S*, the motivation for which seems to be to get a class of functions F on C[0, 1] such that the restriction f of F to $\mathscr{H}$ contains all the « information » about F (or in other words uniquely determines F) so that even though their definition of the sequential integral involves only f, the integral can be called « the integral of F ». The class S* is defined as

$$S^* = \left\{ F : \exists \mu \in \mathscr{M}(H) \text{ such that } F(\xi) = \int_{\mathscr{H}} e^{i(\eta, \xi)^{\sim}} d\mu(\eta),\ s\text{-a. e. } \xi \right.$$

and

$$\mathrm{F}(\eta) = \int_{\mathscr{H}} e^{i(\eta,\eta')} d\mu(\eta') \qquad \text{for all } \eta \in \mathscr{H} \Big\}.$$

For $\mathrm{F} \in \mathrm{S}^*$, we have for all $\lambda > 0$,

$$\mathrm{R}(f^{\lambda}) = \mathrm{F}^{\lambda} \tag{6.6}$$

where $f = \mathrm{F}|_{\mathscr{H}}$ and thus for $\mathrm{F} \in \mathrm{S}^*$, we have

$$\int_{\mathrm{C}_0[0,1]}^{sf_q} \mathrm{F}(\xi) d\xi = \mathrm{I}_s^q(\mathrm{F}). \tag{6.7}$$

Thus, for $\mathrm{F} \in \mathrm{S}^*$, the Cameron-Storvick definition does give the « right » answer, but that is because (6.7) holds and the fact remains that their definition of the sequential Feynman integral of F over $\mathrm{C}_0[0, 1]$ really defines the sequential Feynman integral of $f = \mathrm{F}|_{\mathscr{H}}$ over $\mathscr{H}$.

We feel that though the analytic Feynman integral on C[0, 1] can be defined without any reference to the RKHS $\mathscr{H}$ of the Wiener measure, the abstract Wiener space structure $(i, \mathscr{H}, \mathrm{C}[0, 1])$ and the m-lifting R are crucial for the definition of a sequential Feynman integral over C[0, 1], as indeed they are for the general theory developed in this paper.

2) **Elworthy and Truman** [5].

These authors have given a sequential definition of the Feynman integral, not for an abstract Hilbert space, but for the Hilbert space $\overline{\mathscr{H}}_t$ of paths. They use polygonal projections P_π of the form (5.25). For a function f on H, they first define the Feynman integral of the finite dimensional functional $f \circ \mathrm{P}_\pi$, denoted by $\mathscr{F}_\pi(f)$ as an appropriate improper integral in [15] and oscillatory integral in [5]. Then the Feynman integral of f, $\mathscr{F}(f)$ is defined as the limit (if it exists) of $\mathscr{F}_\pi(f)$ as $\delta(\mathrm{P}_\pi) \to 0$.

Their definition for the class $\mathscr{G}^1(\mathrm{H})$ can be stated in our notation as follows:

$$\mathscr{F}_\pi(f) = \lim_{\substack{z \to -i \\ z \in \Omega}} \mathrm{J}_f(z, \mathrm{P}_\pi)$$

and

$$\mathscr{F}(f) = \lim_{\delta(\mathrm{P}_\pi) \to 0} \mathscr{F}_\pi(f).$$

The above definition involves a repeated limit and as a consequence does not yield a formula like (4.46) that evaluates the Feynman integral as a single limit of proper Lebesgue integrals.

3) In [*14*]. Tarski has given a sequential definition for an abstract Hilbert space which is closer in spirit to our sequential definition. His definition can be roughly described in our notation as follows: If

$$\hat{J}_f(z) = \lim_{P \in \mathscr{P}} J_f(z, P) \tag{6.8}$$

and

$$I^q(f) = \lim_{z \to -iq} \hat{J}_f(z) \tag{6.9}$$

exist, then define $I^q(f)$ to be the Feynman integral. He states (6.8) in a different form:

$$\hat{J}_f(z) = \lim_{n} J_f(z, P_n) \tag{6.8$'$}$$

where $\{ P_n \}$ belongs to a « determining » class. (6.8)′ can easily be seen to be equivalent to (6.8). However, this is not his precise definition. It is more complicated, and it is such that the formula

$$I^q(f) = e^{1/2iq(a,a)} I^q(e^{-iq(.,a)} f(.))$$

is built into the definition.

Observe that the limit in (6.8)′ is taken along increasing sequences $\{ P_n \}$ in a « determining class », and thus in the case when the underlying Hilbert space is the space of paths $\mathscr{H}_t$ or $\overline{\mathscr{H}}_t$, the polygonal approximation to the Feynman integral may not be valid. Even if the determining class consists of *all* increasing sequences, the polygonal approximation will be valid for only successively finer partitions.

Since Tarski's definition also uses a repeated limit (see (6.8) and (6.9) above) like that of Elworthy and Truman—though in the reverse order—the same remarks apply to Tarski's definition as well.

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(Manuscrit reçu le 27 juin 1984)

(révisé le 3 septembre 1984)