ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 21, nº 1 (1985), p. 73-102
<http://www.numdam.org/item?id=AIHPB_1985__21_1_73_0>

Ann. Inst. Henri Poincaré,
Vol. 21, n° 1, 1985, p. 73-102. *Probabilités et Statistiques*

Boundary processes: the calculus of processes diffusing on the boundary

by

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ABSTRACT. — We extend the study in [*1*] of the transition probabilities of boundary processes to the case of a process which diffuses on the boundary. By means of the Malliavin Calculus, regularity is related to the degree of degeneracy at the boundary of the diffusion operators; we show interaction between the one acting within the domain and the one on the boundary.

RÉSUMÉ. — Nous étendons l'étude de [*1*] des probabilités de transition de processus frontière au cas d'un processus qui diffuse sur la frontière à l'aide du Calcul de Malliavin. La régularité est reliée au degré de dégénérescence à la frontière des opérateurs de diffusion; nous montrons l'interaction dės deux opérateurs du domaine et de la frontière.

INTRODUCTION

In [*1*], J.-M. Bismut investigates the regularity of the semi-groups of the boundary processes of a certain class of reflecting or two-sided diffusion processes. By means of the stochastic calculus of variations, this is linked to the invertibility of a certain quadratic form (related to the diffusion operators generating the process) and to the integrability of its inverse. Then sufficient conditions on the generator of the Hörmander form are given, providing for a certain degeneracy at the boundary.

Here we extend the results to the case of a process which diffuses on the boundary. We first perform the calculus in our new framework and exhibit a quadratic form taking into account the diffusion operator on the boundary. Then we extend results in [3] and give sufficient conditions for the invertibility. For the integrability, we establish some estimates and use them so as to give theorems under assumptions of a new kind. Then we give a result showing that the operator on the boundary can by itself induce regularity, and an example showing that this cannot be done purely locally; the problems arise from the non-integrability of the (multiplicative) inverse of the Brownian local time. Then we use the excursion decomposition for the two-sided process; after extending an estimate on its characteristic measure in order to have results under weaker assumptions, we show interaction between the diffusion operators on the boundary and within the domain.

I. THE CALCULUS OF VARIATIONS

We study either a reflecting or a two-sided process (x, z) in $\mathbb{R}^d \times \mathbb{R}$. We give some definitions:

$\Omega = C(\mathbb{R}_+, \mathbb{R}^m)$, $\Omega' = C(\mathbb{R}_+, \mathbb{R})$, the trajectory $w \in \Omega$ (resp. $w' \in \Omega'$) is written $w_t = (w_t^1, \ldots, w_t^m)$ (resp. z_t), $(F_t)_{t \geqslant 0}$ and $(F'_t)_{t \geqslant 0}$ are the usual filtrations regularized and completed.

P is the Brownian measure on Ω, such that $P(w_0 = 0) = 1$.

In the reflecting case, P'_{z_0} is the probability measure on Ω' such that z is a reflecting brownian motion; in the two-sided case, P'_{z_0} is the regular brownian measure. In both cases, $P(z_{(0)} = z_0) = 1$. We set $P' = P'_0$.

L is the local time at 0 of z, so normalized as to have $L_t = \sup_{0 \leqslant s \leqslant t} (-B_s)$ where $B_t = |z_t| - |z_0| - L_t$ is a Brownian motion. $A_t = \inf\{A \geqslant 0, L_A > t\}$.

$\overline{\Omega} = \Omega \otimes \Omega'$, $\overline{P} = P \otimes P'$, $\overline{F}_t = F_t \otimes F'_t$, and the K^j's are continuous $(\overline{F}_t)$-martingales with $\langle K^j \rangle = L$, $\langle K^j, M \rangle = 0$ if $M \in \{w^i, z, K^k (j \neq k)\}$. By theorem II.7.3 in [5] there exists a q-dimensional brownian motion $(C^j)_{1 \leqslant j \leqslant q}$ independent of (z, w^i) with $K^j = C^j_L$.

All other notations are as in [1]; we furthermore recall that the X_i's and D_j's are $\mathbb{R}^d$ valued vector fields, C^∞ with bounded derivatives of all orders, that d denotes the Stratonovitch differential, δ the Itô differential, and that if Y is a vector field on $\mathbb{R}^d$, h a C^∞ diffeomorphism $\mathbb{R}^d \to \mathbb{R}^d$, then $(h^{*-1}Y)(x) = \left(\frac{\partial h}{\partial x}\right)^{-1} Y(h(x))$. The equations for the processes are

$$(I.1) \quad \begin{aligned} dx_t &= X_0(x_t, z_t)dt + D_0(x_t)dL_t + \sum_{i=1}^{m} X_i(x_t, z_t)dw_t^i + \sum_{j=1}^{q} D_j(x_t)dK_t^j \\ x(0) &= x_0 \end{aligned}$$

or

$$\text{(I.2)}\quad \begin{aligned} dx_t = {} & \mathbb{1}_{z_t>0}[X_0(x_t, z_t)dt + X_i(x_t, z_t)dw_t^i] \\ & + \mathbb{1}_{z_t<0}[X_0'(x_t, z_t)dt + X_i'(x_t, z_t)dw_t^i] + D_0(x_t)dL_t + D_j(x_t)dK_t^j \\ x(0) = {} & x_0. \end{aligned}$$

Providing we replace (1.6) in [1] by

$$\text{(I.3)}\quad \begin{aligned} dx' &= D_0(x')dt + D_j(x')dC^j \\ x'_{(0)} &= x_0 \end{aligned}$$

section 1 in [1] can be easily adapted to fit into our new framework, with the possible exception of its subsection *g*) and the analytical and geometrical interpretation it gives.

Thus, we can define in a proper way the reflecting process and its associated flow, and then its boundary process. Naturally we can proceed in a similar way for the two-sided process.

The purpose of [1] is to study the regularity of the semi-group of the boundary process (A_t, x_{A_t}); the necessity to mind the component A_t appears as soon as (1.5) in [1] (which can be derived in our new framework thanks to the Burkholder-Davis-Gundy inequalities) and is exemplified in [1]-(1.37). Naturally the effect of (1.5) in [1] is felt in the calculus of variations of section 2 in [1]. This calculus brings forth the quadratic form (2.4) in [1], and by means of the calculus of section 4 in [1] and of the results in harmonic analysis recalled at the end of that section the regularity is linked to the invertibility of this form and to the integrability of its inverse. We shall now check that we can still perform the calculus of section 2 and section 4 in [1], and exhibit a quadratic form in which the D_j's, $1 \leqslant j \leqslant q$, appear. We recall that more general reflecting (or two-sided) processes may be reduced to (I.1) (or I.2)). See section 1 in [1] and IV.7, V.6 in [5].

We shall follow [1], section 2, after having replaced its definition 2.1 by

DEFINITION I.1. — On $\overline{\Omega}$, the process $\Gamma_t(\overline{w})$ with values in the linear mappings from $T^*_{x_0}(\mathbb{R}^d)$ into $T_{x_0}(\mathbb{R}^d)$ is defined by

$$\text{(I.4)}\quad p \in T^*_{x_0}(\mathbb{R}^d) \mapsto \Gamma_t(\overline{w})p = \sum_{i=1}^{m} \int_0^t \langle p, \varphi_s^{*-1} X_i(x_0) \rangle \varphi_s^{*-1} X_i(x_0) ds + \sum_{j=1}^{q} \int_0^t \langle p, \varphi_s^{*-1} D_j(x_0) \rangle \varphi_s^{*-1} D_j(x_0) dL_s.$$

It is convenient to consider Γ_t as a quadratic form: we have

$$(I.5)\quad \langle \Gamma_t p, p \rangle = \sum_{i=1}^{m} \int_0^t \langle p, \varphi_s^{*-1} X_i(x_0) \rangle^2 ds + \sum_{j=1}^{q} \int_0^t \langle p, \varphi_s^{*-1} D_j(x_0) \rangle^2 dL_s .$$

Naturally, $\varphi_s^{*-1} X_i(x_0) = \left(\frac{\partial \varphi_s}{\partial x}\right)^{-1} X_i(x_s, z_s)$ and Γ_t depends on (x_0, z_0) without this being explicitly stated.

In the two-sided case, we have

$$(I.6)\quad \langle \Gamma_t p, p \rangle = \int_0^t \mathbb{1}_{z>0} \langle p, \varphi_s^{*-1} X_i(x_0) \rangle^2 ds + \int_0^t \mathbb{1}_{z<0} \langle p, \varphi_s^{*-1} X_i'(x_0) \rangle^2 ds + \int_0^t \langle p, \varphi_s^{*-1} D_j(x_0) \rangle^2 dL_s .$$

We shall replace (2.9), (2.11), (2.15) in [1] respectively by

$$(I.7)\quad \overline{Z}_t^l = \exp\left\{ - l \int_0^T u^i \delta w^i - l \int_0^T v^k \delta K^k - \frac{l^2}{2} \int_0^T |u|^2 ds - \frac{l^2}{2} \int_0^T |v^2| dL \right\}$$

$$(I.8)\quad w_t^{l,i} = w_t^i + \int_0^t l u^i ds, \quad K_t^{l,j} = K_t^j + \int_0^t l v^j dL$$

$$(I.9)\quad d\overline{y}_t^l = (\varphi_t^{*-1} X_i)(\overline{y}^l, z) l u^i dt + (\varphi_t^{*-1} D_k)(\overline{y}^l, z) l v^k dL_t, \qquad \overline{y}_{(0)}^l = x_0 .$$

Using the properties of the Girsanov transformation [5] and the quadratic variational processes [5] we see that the law of $(w^{l,1}, \ldots, w^{l,m}, K^{l,1}, \ldots, K^{l,q}, B)$ under $\overline{R}^l = \overline{Z}_T^l d(P \otimes P'_{z_0})$ does not depend on $l \in \mathbb{R}$. We can now follow [1], section 2, that is differentiate with respect to l at $l = 0$. As L_T is in all the L_p, $p < \infty$, we may differentiate under the expectation sign, and all the rest of [1], section 2, follows with Γ_t instead of C_t, with the new choice

$$(I.10)\quad \begin{aligned} u^i &= \mathbb{1}_{s \leq T \wedge A_t} (\varphi_s^{*-1} X_i)(x_0) \\ v^k &= \mathbb{1}_{s \leq T \wedge A_t} (\varphi_s^{*-1} D_k)(x_0). \end{aligned}$$

As in section 4 the calculus of variations is performed on the component z, the presence of the K^j's does not matter. It suffices to make obvious changes, as adding in [1]-(4.9) $K_t'^j = C_{L_t'}^j$. As the C^j and z are independant, this does not affect the calculus. Also as L_T is in all the L_p there are no integrability problems. We obtain the results of the end of section 4 in [1] with Γ instead of C. We shall refer to our adapted theorems without changing their denotations. We can summarize the results from [1] we use as follows:

for $b \in C^\infty(\mathbb{R}^{d+1}, \mathbb{R})$ and $\left.\dfrac{dQ_{(x_0,z_0)}}{dP \otimes P'_{z_0}}\right|_{\overline{F}_t} = \exp\left\{ \int_0^t b(x, z)\delta B - \dfrac{1}{2}\int_0^t b^2(x, z)ds \right\}$,
we have

THEOREM I.2. — If $(x_0, z_0) \in \mathbb{R}^d \times \mathbb{R}$ and $t' > 0$ are such that $P \otimes P'_{z_0}$ a. s. $\Gamma^{x_0}_{A_t}$ is invertible, then for $t \geqslant t'$ the law under $Q_{(x_0,z_0)}$ of (A_t, x_{A_t}) is given by $p_t(a, y)dady$. If moreover for all $T \geqslant 0, p \geqslant 1$ $\mathbb{1}_{A_{t'} \leqslant T} \, |\Gamma^{-1}_{A_{t'}}| \in L_p(\overline{\Omega}, P \otimes P'_{z_0})$, then $p_t(a, y) \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$.

II. THE EXISTENCE OF A DENSITY

We shall look into the existence of a density with respect to the Lebesgue measure for the boundary semi-group. By [*1*], theorem 4.13, it suffices that Γ_t be invertible.

The main tool to prove this invertibility will be the action of φ on vector fields, as in [*1*], section 5. We shall use theorem 1.1, chapter IV, in [*2*], with the fact that if K is a vector field then $L_{X_i}K = [X_i, K]$.

To gather the most information from this action we shall use [*3*], especially theorems 2.1 and 2.3. The problem is to adapt them to our present situation; that is, to the presence of the K^j's and to the two-sided process. This will be done beforehand.

We notice here that as in [*1*] the vector fields of the half spaces $z > 0$ and $z < 0$ strongly interact; this shows nontrivial interactions of the corresponding Levy kernels.

Let $(\Omega, (F_t), P)$ be a filtered probability space, and (z, B, C) a (F_t)-Brownian motion. L denotes the local time at 0 of z.

If Y is a process, then Y^0 is the random set $Y^0 = \{ t \geqslant 0, Y_t = 0 \}$, D, H, J, K are continuous (F_t)-predictable processes, and

$$X_t = \int_0^t D_s ds + \int_0^t H_s \delta B_s + \int_0^t K_s dL_s + \int_0^t J_s \delta C_{L_s}. \tag{II.1}$$

We now have:

THEOREM II.1. — Assume that an a. s. > 0 stopping time S exists such that $z^0 \cap [0, S[\subset X^0$, a. s.

Then $z^0 \cap [0, S[\subset H^0 \cap J^0 \cap K^0$, a. s.

THEOREM II.2. — Assume that H may be written

$$H_t = H_0 + \int_0^t D'_s ds + \int_0^t K'_s dL_s + \int_0^t H'_s \delta B_s + \int_0^t J'_s \delta C_{L_s} \tag{II.2}$$

where H_0 is F_0-measurable, and D', K', H', J' are continuous predictable processes. Then if an a.s. > 0 stopping time S exists such that $z^0 \cap [0, S[\subset X^0$, then $z^0 \cap [0, S[\subset D^0 \cap K^0 \cap H^0 \cap J^0$.

Proof. — We first apply theorem 2.1 in [3] to

$$\text{(II.3)} \quad X_t^2 = 2\int_0^t X_s D_s ds + 2\int_0^t X_s K_s dL_s + 2\int_0^t X_s H_s \delta B_s + 2\int_0^t X_s J_s \delta C_{L_s} + \int_0^t H_s^2 ds + \int_0^t J_s^2 dL_s .$$

As $X_s = 0$ on $[0, S[\cap z^0$, $\int_0^t X_s J_s \delta C_{L_s} = 0$ on $[0, S[$ and $\int_0^t X_s K_s dL_s = 0$ on $[0, S[$. We may apply [3]-theorem 2.1 and we get that $J_s = 0$ on $z^0 \cap [0, S[$. This implies that $\int_0^t J_s \delta C_{L_s} = 0$ on $[0, S[$, so that we may apply [3]-theorem 2.1 to X_t itself. Our first theorem follows. For the second theorem, we subsequently apply theorem 2.1 in [3] to H_t^2 to get rid of the $\int_0^t J'_s \delta C_{L_s}$ term as well; then we may use theorem 2.3 in [3] to conclude.

Remark. — If (z, B^i, C^j) is a (F_t)-Brownian motion and if we replace in (II.1) the local martingale terms by $\sum_i \int_0^t H_s^i \delta B_s^i + \sum_j \int_0^t J_s^j \delta C_{L_s}^j$, then Theorem II.7.1' in [5] enables us to use Theorem II.1, for then we can find $\tilde{B}$, $\tilde{C}$ with $(z, \tilde{B}, \tilde{C})$ an $(\tilde{F}_t)$-Brownian motion such that the local martingale terms may be written $\int_0^t \sqrt{\sum_i (H_s^i)^2} \delta \tilde{B}_s + \int_0^t \sqrt{\sum_j (J_s^j)^2} \delta \tilde{C}_{L_s}$ (in an enlarged space $\tilde{\Omega}$).

We can now give some results on the invertibility of Γ_t:

DEFINITION II.3. — For $l \in \mathbb{N}$, E_l, E'_l, F_l, F'_l are the families of $\mathbb{R}^d$-valued vector fields defined by

$$E_1 = (X_1, \ldots, X_m) \qquad F_1 = F'_1 = (D_1, \ldots, D_q)$$
$$E_{l+1} = [(X_0, X_1, \ldots, X_m, \partial/\partial z), E_l]$$
$$F_{l+1} = [(D_0, D_1, \ldots, D_q), E_l] \cup [(D_0, D_1, \ldots, D_q, X_1, \ldots, X_m), F_l]$$
$$F'_{l+1} = [(X'_1, \ldots, X'_m, D_0, D_1, \ldots, D_q), E_l] \cup [(X_1, \ldots, X_m, D_0, D_1, \ldots, D_q), E'_l] \cup [(X_1, \ldots, X_m, X'_1, \ldots, X'_m, D_0, D_1, \ldots, D_q), F_l]$$

and E'_l is defined as E_l after replacing X_i by X'_i, $i = 0, \ldots, m$.

Naturally if A, B are two families of vector fields, [A, B] denotes the family of the vector fields $[a, b]$, $a \in \mathrm{A}$, $b \in \mathrm{B}$.

We state a theorem on the reflecting process:

THEOREM II.4. — If $x_0 \in \mathbb{R}^d$ is such that the vector space spanned by $\{ \mathrm{E}_l, \mathrm{F}_l ; 1 \leqslant l < +\infty \}$ is $\mathbb{R}^d$ in full, then $\mathrm{P} \otimes \mathrm{P}'$ a. s. for all $t > 0$, $\Gamma_t^{x_0}$ is invertible.

Moreover, if $\mathrm{X}_1, \ldots, \mathrm{X}_m$ do not depend on z, then we may enlarge F_{l+1} to $\mathrm{F}_{l+1} = [(\mathrm{D}_0, \ldots, \mathrm{D}_q), \mathrm{E}_l] \cup [(\mathrm{D}_0, \ldots, \mathrm{D}_q, \mathrm{X}_0, \ldots, \mathrm{X}_m), \mathrm{F}_l]$.

We now state a theorem on the two-sided process:

THEOREM II.5. — If x_0 is such that the vector space spanned by $\{ \mathrm{E}_l, \mathrm{E}'_l, \mathrm{F}'_l ; 1 \leqslant l < +\infty \}$ is $\mathbb{R}^d$ in full, then $\mathrm{P} \otimes \mathrm{P}'$ a. s. for all $t > 0$ $\Gamma_t^{x_0}$ is invertible.

Moreover if $\mathrm{X}_1, \ldots, \mathrm{X}_m, \mathrm{X}'_1, \ldots, \mathrm{X}'_m$ do not depend on z, F'_{l+1} may be enlarged to

$$\mathrm{F}'_{l+1} = [(\mathrm{X}'_0, \mathrm{X}'_1, \ldots, \mathrm{X}'_m, \mathrm{D}_0, \ldots, \mathrm{D}_q), \mathrm{E}_l] \cup [(\mathrm{X}_0, \mathrm{X}_1, \ldots, \mathrm{X}_m, \mathrm{D}_0, \ldots, \mathrm{D}_q), \mathrm{E}'_l] \cup [(\mathrm{X}_0, \ldots, \mathrm{X}_m, \mathrm{X}'_0, \ldots, \mathrm{X}'_m, \mathrm{D}_0, \ldots, \mathrm{D}_q), \mathrm{F}_l].$$

Proof. — As in [*1*]-Theorem 5.2, [*4*]-Theorem 2.14, we get by the 0-1 law that as soon as Γ_t is not invertible, there is a fixed $f \in \mathrm{T}^*_{x_0}(\mathbb{R}^d)$, $f \neq 0$, and an a. s. > 0 $(\overline{\mathrm{F}}_t)$-stopping time S such that $\langle f, \varphi_s^{*-1}\mathrm{X}_i(x_0) \rangle = 0$ on $[0, \mathrm{S}]$.

For the reflecting process: if $\mathrm{V}(x, z)$ is a $\mathbb{R}^d$-valued vector field, then by Theorem 1.1, Chapter IV, in [*2*] (or just by looking at [*1*], (1.4), (1.12)) we have

$$\text{(II.4)} \quad \varphi_t^{*-1}\mathrm{V} = \mathrm{V} + \int_0^t \varphi_s^{*-1}[\mathrm{X}_0, \mathrm{V}]ds + \int_0^t \varphi_s^{*-1}[\mathrm{D}_0, \mathrm{V}]d\mathrm{L}_s + \int_0^t \varphi_s^{*-1}[\mathrm{X}_i, \mathrm{V}]dw_s^i + \int_0^t \varphi_s^{*-1}[\mathrm{D}_j, \mathrm{V}]d\mathrm{K}_s^j + \int_0^t \varphi_s^{*-1}[\partial/\partial z, \mathrm{V}]dz_s$$
$$= \mathrm{V} + \int_0^t \varphi_s^{*-1}\Big([\mathrm{X}_0, \mathrm{V}] + \frac{1}{2}[\mathrm{X}_i, [\mathrm{X}_i, \mathrm{V}]] + \frac{1}{2}[\partial/\partial z, [\partial/\partial z, \mathrm{V}]]\Big)ds + \int_0^t \varphi_s^{*-1}\Big([\mathrm{D}_0, \mathrm{V}] + \frac{1}{2}[\mathrm{D}_j, [\mathrm{D}_j, \mathrm{V}]]\Big)d\mathrm{L}_s + \int_0^t \varphi_s^{*-1}[\mathrm{X}_i, \mathrm{V}]\delta w_s^i + \int_0^t \varphi_s^{*-1}[\mathrm{D}_j, \mathrm{V}]\delta\mathrm{K}_s + \int_0^t \varphi_s^{*-1}[\partial/\partial z, \mathrm{V}]\delta z.$$

By separating the local-martingale and the bounded variation terms and by using the fact the support of $d\mathrm{L}_s$ is $\{ z_s = 0 \}$ which is ds-negligible, we get that:

If $\langle f, \varphi_s^{*-1}V(x_0) \rangle = 0$ for $s \leqslant S$ then $\langle f, \varphi_s^{*-1}[X_i, V] \rangle$, $\langle f, \varphi_s^{*-1}[X_0, V] \rangle$, $\langle f, \varphi_s^{*-1}[\partial/\partial z, V] \rangle$ are equal to 0 for $s \leqslant S$ and $\langle f, \varphi_s^{*-1}[D_j, V] \rangle$, $\langle f, \varphi_s^{*-1}[D_0, V] \rangle$ are equal to 0 for $s \leqslant S$, $z_s = 0$.

Now we notice that if $W(x, z)$, $V(x, z)$ are two $\mathbb{R}^d$ valued vector fields, and if we set $\tilde{V}(x, z) \equiv V(x, 0)$, then $[W, V](x, 0) = [W, \tilde{V}](x, 0)$. Also if $\langle f, \varphi_s^{*-1}V(x_0) \rangle = 0$ for $s \leqslant S$, $z_s = 0$, then $\langle f, \varphi_s^{*-1}\tilde{V}(x_0) \rangle = 0$ for $s \leqslant S$, $z_s = 0$, and we may use theorems II.1 and II.2 now that no terms in $\int - \delta z$ are left. So we can state using theorem II.1 that:

If $\langle f, \varphi_s^{*-1}V(x_0) \rangle = 0$ for $s \leqslant S$, $z_s = 0$, then $\langle f, \varphi_s^{*-1}[D_0, V] \rangle$, $\langle f, \varphi_s^{*-1}[D_j, V] \rangle$, $\langle f, \varphi_s^{*-1}[X_i, V] \rangle = 0$ for $s \leqslant S$, $z_s = 0$.

In order to use theorem II.2 we must have that $\varphi_s^{*-1}[X_i, \tilde{V}]$ is a semi-martingale whose Itô decomposition contains no $\int - \delta z$ term. This happens as soon as for $1 \leqslant i \leqslant m$ $[\partial/\partial z, [X_i, \tilde{V}]] = 0$. The only simple assumption we can give on the X_i's to get that is that for $1 \leqslant i \leqslant m$ the X_i's do not depend on z. In which case we also get that $\langle f, \varphi_s^{*-1}[X_0, V] \rangle = 0$ for $s \leqslant S$, $z_s = 0$.

All this leads to theorem II.4 by a simple iteration.

For the two-sided process we follow the same proof after adapting the results of [*11*] instead of [*3*]. We could also adapt the techniques of [*1*]-Theorem 6.6, [*4*]-Theorem 2.19; they use the time-change $K_t = \inf \{ \gamma, C_\gamma > t \}$ where $C_t = \int_0^t \mathbb{1}_{z>0} ds$; z_{K_t} is a reflecting brownian motion, and $\int_0^{K_t} \mathbb{1}_{z>0} \delta w^i$ $(1 \leqslant i \leqslant m)$ are also independent brownian motions independent of z_{K_t}. This is followed by a quite intricate and technical proof, (see (6.37) to (6.54) in [*1*]).

Remark. — Naturally we can try to enlarge E_l, E_l', F_l, F_l' knowing that what really counts is the vector space they span. For example, it is easy to prove by using the Jacobi identity that if $B, A_1, \ldots, A_p$ $(p \geqslant 1)$ are vector fields then $[B, [A_p, [\ldots, [A_2, A_1] \ldots]]$ belongs to the vector space spanned by all the $[A_{\sigma(p)}, [A_{\sigma(p-1)}, [\ldots, [A_{\sigma(1)}, B] \ldots]]$, for σ belonging to the set of permutations of $\{ 1, \ldots, p \}$.

We would also like to get rid of the restrictions on X_0 and $\partial/\partial z$. The latter stems from the way we get back within reach of theorem II.1, and thus seems difficult to be disposed of. For the former, we used theorem II.2.

It is to no avail to use theorem 2.2 in [3] as it requires that S should be $+\infty$, or at least bounded below uniformely in w; it is difficult to further localize the theorem.

III. THE EXTENSION OF THE EXISTING ESTIMATES

We shall now extend the estimates of [1], and in particular (5.37) and (6.119). We get less interesting results, for K_t^j behaves somewhat like $t^{1/4}$ (while L_t behaves only like $t^{1/2}$) and so moves around a lot for small t's.

We use estimates on S.D.E. and also some classical estimates on Brownian motion, which we shall now recall, as we shall recall the estimate [I]-(5.29) in a slightly enhanced form.

First for $a \in \mathbb{R}_+$, if w is a brownian motion, it is classical that

$$\text{(III.1)} \qquad P\,[\sup_{0 \leqslant s \leqslant t} w_s > a] \leqslant \exp\left\{-\frac{a^2}{2t}\right\}.$$

Then for $\gamma \in \mathbb{R}_+$, following [I]-(5.42), (5.43), we have

$$\text{(III.2)} \qquad P[\sup_{0 \leqslant s \leqslant t} |w_s| \leqslant \gamma] \leqslant \sqrt{2}\exp\left\{-\frac{t}{8\gamma^2}\right\}.$$

As in [1], (5.23) to (5.29), we get from the proof of Theorem 8.31 in [10] that for

$$E'_{1,p} = (X'_1, \ldots, X'_p), \qquad E'_{l+1,p} = [(X'_0, X'_1, \ldots, X'_m), \qquad E'_{l,p}]$$

$$\langle C'^{x_0}_{t,p} f, f\rangle = \sum_{i=1}^{p} \int_0^t \langle f, \varphi_s'^{*-1} X'_i(x_0)\rangle^2 ds \qquad \text{for} \qquad f \in T^*_{x_0}(\mathbb{R}^n)$$

$$f^{l,x_0}_{t,p} = \sum_{n=1}^{l} \sum_{Y \in E'_{n,p}} \langle \varphi_t'^{*-1} Y(x_0), f\rangle^2,$$

$$\sigma = \inf\left\{ t \geqslant 0, \left|\left[\frac{\partial \varphi_s}{\partial x}(\overline{w}, x_0)\right]^{-1} - I\right| \geqslant \frac{1}{2}\right\}$$

there are $D_1 > 0$, $D_2 > 0$, $D_3 > 0$, $m_l = 20^{l-1} \times 6$, such that for any $x_0 \in \mathbb{R}^d$, $\varepsilon > 0$, we have

$$\text{(III.3)} \qquad P[\langle C'^{x_0}_{D_1\varepsilon^{3/m_l},p} f, f\rangle \leqslant \varepsilon\,; f^{l,x_0}_{t,p} \geqslant D_2 \varepsilon^{3/m_l}$$

$$\text{on} \qquad [0, D_1\varepsilon^{3/m_l}]\,; \qquad \sigma \geqslant D_1\varepsilon^{3/m_l}] \leqslant K \exp\{-D_3\varepsilon^{-\alpha/m_l}\}.$$

Let's put for $\hat{X}_0, \hat{D}_0, \hat{X}_i, \hat{D}_j$ having the same properties as X_0, D_0, X_i, D_j

$$\text{(III.4)} \quad d\hat{x}_t = \hat{X}_0(\hat{x}_t, z_t)dt + \hat{D}_0(\hat{x})dL + \sum_{j=1}^{m} \hat{X}_i(\hat{x}, z)\delta w^i + \sum_{j=1}^{q} \hat{D}_j(\hat{x})\delta K^j$$

$$\hat{x}(0) = x_0$$

$$F = (m+q) \sup_{\substack{x,z \\ 1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant q}} \{ |\hat{X}_i(x,z)|^2 \mathrm{V} |\hat{D}_j(x)|^2 \}$$

$$E = \sup_{x,z} \{ |\hat{X}_0(x,z)| \mathrm{V} |\hat{D}_0(x)| \}$$

$$T^\theta = \inf \{ t \geqslant 0, |\hat{x}_t - x_0| \geqslant \theta \}.$$

We then have

PROPOSITION III.1. — If $\theta > 0$, $T > 0$ are such that $\theta - E(T + T^{1/3}\theta^{2/3}) \geqslant 0$, then

$$\overline{P}[T^\theta \leqslant T] \leqslant \exp\left\{ -\frac{\theta^{4/3}}{2T^{1/3}} \right\} + 2d \exp\left\{ -\frac{(\theta - E(T + T^{1/3}\theta^{2/3}))^2}{2F(T + T^{1/3}\theta^{2/3})} \right\}.$$

PROPOSITION III.2. — If for $1 \leqslant j \leqslant q$ $D_j = 0$, and $\theta > 0$, $T > 0$, $\varepsilon > 0$ are such that $\theta - E(T + \varepsilon\theta) \geqslant 0$, then

$$\overline{P}[T^\theta \leqslant T] < \exp\left\{ -\frac{(\varepsilon\theta)^2}{2T} \right\} + 2d \exp\left\{ -\frac{(\theta - E(T + \varepsilon\theta))^2}{2FT} \right\}.$$

Proof. — Let

$$M_t = \int_0^t \hat{X}_i \delta w^i + \int_0^t \hat{D}_j \delta K^j; \text{ then } |\hat{x}_t - x_0| \leqslant \int_0^t |\hat{X}_0| du + \int_0^t |\hat{D}_0| dL + |M_t|.$$

$$\text{(III.5)} \quad \overline{P}[T^\theta \leqslant T] \leqslant \overline{P}[L_T > T^{1/3}\theta^{2/3}] + \overline{P}[T^\theta \leqslant T, L_T \leqslant T^{1/3}\theta^{2/3}]$$

$$\text{(III.6)} \qquad \overline{P}[L_T > T^{1/3}\theta^{2/3}] \leqslant \exp\left\{ -\frac{\theta^{4/3}}{2T^{1/3}} \right\}$$

by (III.1), knowing that $L_t = \sup_{0 \leqslant s \leqslant t} (-B_s)$. If $\theta - E(T + T^{1/3}\theta^{2/3}) \geqslant 0$, we have

$$\text{(III.7)} \quad \overline{P}[T^\theta \leqslant T, L_T \leqslant T^{1/3}\theta^{2/3}] \leqslant \overline{P}[\sup_{0 \leqslant s \leqslant T} |M_s| \geqslant \theta - E(T + T^{1/3}\theta^{2/3}), L_T \leqslant T^{1/3}\theta^{2/3}].$$

By a simple adaptation of the estimate (4.2.1) in [*9*], using that we are on $\{ L_T \leqslant T^{1/3}\theta^{2/3} \}$, we get that

$$\text{(III.8)} \quad \overline{P}[T^\theta \leqslant T, L_T \leqslant T^{1/3}\theta^{2/3}] \leqslant 2d \exp\left\{ -\frac{(\theta - E(T + T^{1/3}\theta^{2/3}))^2}{2F(T + T^{1/3}\theta^{2/3})} \right\}$$

and by using (III.5), (III.6), (III.8). Proposition III.1. is proved.

If moreover $D_1 = \ldots = D_q = 0$, we may replace (III.6) and (III.8) respectively by

(III.9) $$\overline{P}[L_T > \varepsilon\theta] \leqslant \exp\left\{ - \frac{(\varepsilon\theta)^2}{2T} \right\}$$

(III.10) $$\overline{P}[T^\theta \leqslant T, L_T \leqslant \varepsilon\theta] \leqslant 2d \exp\left\{ - \frac{(\theta - (T + \varepsilon\theta))^2}{2FT} \right\}$$

and we shall obtain Proposition III.2.

REMARK 1. — When using Prop. III.2 we shall choose $\varepsilon > 0$ such that $1 - E\varepsilon > 0$. Then when θ is large enough, $\theta - E(T + \varepsilon\theta) \geqslant 0$ and furthermore for θ going to $+\infty$ we get an estimate of the $\exp\{ - c\theta^2 \}$ kind.

REMARK 2. — In our paper, either T goes to 0 as in [*1*]-(5.37), or θ goes to infinity as in [*1*]-(6.119). There exists $T_0 > 0$ depending, only on E, d, θ, and $\theta_0 > 0$ depending only on E, d, T such that for $T \leqslant T_0$ or $\theta \geqslant \theta_0$ we have

(III.11) $$\overline{P}[T^\theta \leqslant T] \leqslant \exp\left\{ - \frac{1}{3 + 3F} \frac{\theta^{4/3}}{T^{1/3}} \right\}.$$

We shall then set

(III.12) $$G_\theta = \frac{\theta^{4/3}}{3 + 3F}, \qquad K_T = \frac{1}{(3 + 3F)T^{1/3}}.$$

REMARK 3. — Naturally to use all these estimates in our paper it suffices to write the Stratonovitch equations in Itô form. We have analogous estimates on the two-sided process, by modifying the proof in an obvious way after having changed E and F so as to take into account the vector fields in the half space $z < 0$.

IV. SMOOTHNESS: THE REFLECTING PROCESS

We shall now investigate the smoothness of the densities of the boundary semi-group. This is linked to the integrability of Γ_t^{-1} by theorems 2.4, 2.5, 4.9, 4.10, 4.11, 4.12 in [*1*].

The principle of the proofs is to estimate—by the results of the previous section—the probability that the diffusion gets away from the boundary without getting to far from the starting point, and then use the estimate (III.3) inside the half-space. We cannot thus hope to get results involving the D_j's.

As our estimates are not as good as those in [*1*], we get weaker results on local conditions. This leads us to introduce new hypotheses and new techniques in order to exploit better our estimates.

We use these new techniques to prove a result involving the D_j's.

For the reflecting process, we have

DÉFINITION IV.1.

$$E_1 = (X_1, \ldots, X_m), \qquad E_{l+1} = [(X_0, X_1, \ldots, X_m, \partial/\partial z), E_l]$$

$$k^l(x, z) = \inf_{f \in \mathbb{R}^d, \|f\|=1} \left\{ \sum_{j=1}^{l} \sum_{Y \in E_j} \langle Y(x, z), f \rangle^2 \right\}.$$

THEOREM IV.2. — Let $x_0 \in \mathbb{R}^d$ be such that for a given $l \in \mathbb{N}$, $\theta > 0$, one of the following hypotheses holds:

i) $$\lim_{\substack{z>0 \\ z \to 0}} \sqrt{z} \operatorname{Log}\left(\inf_{|x-x_0| \leqslant \theta} k^l(x, z)\right) = 0$$

ii) there is $y_0 \in \mathbb{R}^d$ such that $\lim_{\substack{z>0 \\ z \to 0}} z \operatorname{Log}\left(\inf_{|x-y_0| \leqslant \frac{\theta}{z^{1/2}}} k^l(x, z)\right) = 0.$

Then for any $t > 0$, $T > 0$, $\mathbb{1}_{A_t \leqslant T} | [\Gamma_{A_t}^{x_0}]^{-1} |$ is in all the $L_p(\overline{\Omega}, P \otimes P')$.

Proof. — We shall adapt the proof for [*1*]-Theorem 5.9. λ is a > 0 real, which will tend to $+\infty$. γ is a > 0 real number, depending on λ, which will become arbitrarily small as $\lambda \to +\infty$. We will determine γ at the end of the proof. We have

$$\text{(IV.1)} \quad \overline{P}[|\Gamma_{A_t}^{-1}| \geqslant \lambda ; A_t \leqslant T] \leqslant \overline{P}[A_t \leqslant 2t\gamma\sqrt{2}] + \overline{P}[|\Gamma_{A_t}^{-1}| \geqslant \lambda ; 2t\gamma\sqrt{2} \leqslant A_t \leqslant T]$$

we know that

$$\text{(IV.2)} \qquad \overline{P}[A_t \leqslant 2t\gamma\sqrt{2}] \leqslant 2 \exp\left\{ - \frac{t}{4\gamma\sqrt{2}} \right\}.$$

Let T_0 be the stopping time

$$\text{(IV.3)} \qquad T_0 = \inf\{ t \geqslant 0 ; |x_t - x_0| \geqslant \theta \}.$$

By Prop. III.1 we have for small enough γ

$$\text{(IV.4)} \qquad \overline{P}[T_0 \leqslant 2t\gamma\sqrt{2} \leqslant A_t] \leqslant \exp\left\{ \frac{-G_\theta}{(2t\gamma\sqrt{2})^{1/3}} \right\}$$

and thus

$$\text{(IV.5)}\quad \overline{P}[\,|\,\Gamma_{A_t}^{-1}\,| \geqslant \lambda, 2t\gamma\sqrt{2} \leqslant A_t \leqslant T] \leqslant \exp\left\{-\frac{G_\theta}{(2t\gamma\sqrt{2})^{1/3}}\right\} + \overline{P}[\,|\,\Gamma_{2t\gamma\sqrt{2}}^{-1}\,| \geqslant \lambda\,;\, A_t \wedge T_0 \geqslant 2t\gamma\sqrt{2}\,].$$

Putting $T_1^{\gamma^{2/3}} = \inf\{\, t \geqslant 0, z_t = \gamma^{2/3}\,\}$, we have the key estimate

$$\text{(IV.6)}\qquad \overline{P}[T_1^{\gamma^{2/3}} \geqslant t\gamma\sqrt{2}] \leqslant \sqrt{2}\exp\left\{-\frac{t\sqrt{2}}{8\gamma^{1/3}}\right\}.$$

Defining $T_2^{\gamma^{2/3}} = \inf\{\, t \geqslant T_1^{\gamma^{2/3}}, |\, z_t - z_{T_1^{\gamma^{2/3}}}\,| = \gamma^{2/3}/2\,\}$, we have

$$\text{(IV.7)}\qquad \overline{P}\left[T_2^{\gamma^{2/3}} - T_1^{\gamma^{2/3}} \leqslant D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\right] \leqslant 2\exp\left\{-\frac{\gamma^{4/3}\lambda^{3/2m_l}}{8D_1 2^{3/m_l}}\right\}.$$

We shall choose γ so that

$$\text{(IV.8)}\qquad t\gamma\sqrt{2} \geqslant D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}.$$

Now we have

$$\text{(IV.9)}\quad \overline{P}[\,|\,\Gamma_{2t\gamma\sqrt{2}}^{-1}\,| \geqslant \lambda\,; A_t \wedge T_0 \geqslant 2t\gamma\sqrt{2}] \leqslant \sqrt{2}\exp\left\{-\frac{t\sqrt{2}}{8\gamma^{1/3}}\right\} + 2\exp\left\{-\frac{\gamma^{4/3}\lambda^{3/2m_l}}{8D_1 2^{3/m_l}}\right\} + \overline{P}[\,|\,\Gamma_{2t\gamma\sqrt{2}}^{-1}\,| \geqslant \lambda\,;\, A_t \wedge T_0 \geqslant 2t\gamma\sqrt{2},\; T_1^{\gamma^{2/3}} \leqslant t\gamma\sqrt{2},\; T_2^{\gamma^{2/3}} - T_1^{\gamma^{2/3}} \geqslant D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l},$$

On $T_1^{\gamma^{2/3}} \leqslant t\gamma\sqrt{2}$, using (IV.8), we have ($\tilde{}$ denoting transposition)

$$\text{(IV.10)}\quad \Gamma_{2t\gamma\sqrt{2}} \geqslant \left[\frac{\partial\varphi}{\partial x}_{T_1^{\gamma^{2/3}}}\right]^{-1} \Gamma^{T_1^{\gamma^{2/3}}}_{T_1^{\gamma^{2/3}} + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}} \left[\widetilde{\frac{\partial\varphi}{\partial x}}_{T_1^{\gamma^{2/3}}}\right]^{-1}$$

where

$$\langle \Gamma_t^s f, f\rangle = \int_s^t \langle (\varphi_u \circ \varphi_s^{-1})^{*-1} X_i(x_0), f\rangle^2 du + \int_s^t \langle (\varphi_u \circ \varphi_s^{-1})^{*-1} D_j(x_0), f\rangle^2 dL_u,$$

and so

$$\text{(IV.11)}\quad \overline{P}\left[\,|\,\Gamma_{2t\gamma\sqrt{2}}^{-1}\,| \geqslant \lambda\,;\, A_t \wedge T_0 \geqslant 2t\gamma\sqrt{2},\, T_1^{\gamma^{2/3}} \leqslant t\gamma\sqrt{2},\, T_2^{\gamma^{2/3}} - T_1^{\gamma^{2/3}} \geqslant D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\right] \leqslant \overline{P}\left[\left|\frac{\partial\varphi}{\partial x}_{T_1^{\gamma^{2/3}}}\right| \geqslant \lambda^{1/4},\, T_1^{\gamma^{2/3}} \leqslant t\gamma\sqrt{2} \leqslant A_t\right] + \overline{P}\left[\,|\,[\Gamma^{T_1^{\gamma^{2/3}}}_{T_1^{\gamma^{2/3}} + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}}]^{-1}\,| \geqslant \lambda^{1/2}\,;\, T_0 \wedge T_2^{\gamma^{2/3}} \geqslant T_1^{\gamma^{2/3}} + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\right].$$

By [1]-theorem 1.1 *e*) we know that for any $p \geqslant 1$, there is $A > 0$ with

$$\text{(IV.12)} \quad \overline{P}\left[\left|\frac{\partial\varphi}{\partial x_{T_1^{\gamma^{2/3}}}}\right| \geqslant \lambda^{1/4},\ T_1^{\gamma^{2/3}} \leqslant 2t\gamma\sqrt{2} \leqslant A_t\right] \leqslant A/\lambda^p.$$

Let $T_3^{\gamma^{2/3}} = \inf \left\{ t \geqslant T_1^{\gamma^{2/3}},\ \left|\left[\frac{\partial\varphi}{\partial x_{t-T_1^{\gamma^{2/3}}}}(\theta_{T_1^{\gamma^{2/3}}}\overline{\omega}, \varphi_{T_1^{\gamma^{2/3}}}(\overline{\omega}, x_0))\right]^{-1} - I\right| \geqslant \frac{1}{2}\right\}$.

When $T_2^{\gamma^{2/3}} > T_1^{\gamma^{2/3}} + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}$, L does not increase on

$$\left[T_1^{\gamma^{2/3}},\ T_1^{\gamma^{2/3}} + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\right],$$

and so using the Markov properties of the flow

$$\text{(IV.13)} \quad \overline{P}\left[T_3^{\gamma^{2/3}} \leqslant T_1^{\gamma^{2/3}} + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l},\ T_2^{\gamma^{2/3}} \geqslant T_1^{\gamma^{2/3}} + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\right] \leqslant C \exp\{-c'\lambda^{3/2m_l}\}$$

(by the estimate (4.2.1) in [9]). Using lemma V.8.4 in [5], we get that

$$\text{(IV.14)} \quad \overline{P}\Big[\,\Big|\Big[\Gamma^{T_1^{\gamma^{2/3}}}_{T_1^{\gamma^{2/3}}+D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}}\Big]^{-1}\Big| \geqslant \lambda^{1/2}, T_0 \wedge T_2^{\gamma^{2/3}} \geqslant T_1^{\gamma^{2/3}} + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\Big]$$

$$\leqslant C \exp\{-c'\lambda^{3/2m_l}\} + \sum_{i=1}^{N} \overline{P}\Big[\Big\langle \Gamma^{T_1^{\gamma^{2/3}}}_{T_1^{\gamma^{2/3}}+D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}} f_i, f_i \Big\rangle \leqslant \frac{2}{\sqrt{\lambda}},\ T_0 \wedge T_2^{\gamma^{2/3}} \wedge T_3^{\gamma^{2/3}} \geqslant T_1^{\gamma^{2/3}} + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\Big]$$

where $f_1, \ldots, f_N$ are unit vectors of $\mathbb{R}^d$ and $N \leqslant C\lambda^{\frac{d-1}{2}}$.

We must now choose γ such that if $|x - x_0| \leqslant \theta$, $\frac{\gamma^{2/3}}{2} \leqslant z \leqslant \frac{3}{2}\gamma^{2/3}$, then $k^l(x, z) \geqslant 4D_2\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}$. Then on the set of w's

$$\left(T_0 \wedge T_2^{\gamma^{2/3}} \wedge T_3^{\gamma^{2/3}} \geqslant T_1^{\gamma^{2/3}} + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\right),$$

we see that on the interval $\left[T_1^{\gamma^{2/3}},\ T_1^{\gamma^{2/3}} + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\right]$ L does not increase, so that we may use the estimate (III.3). So for a $\beta > 0$, independent of λ,

$$\text{(IV.15)} \quad \sum_{i=1}^{N} \overline{P}\left[\langle \Gamma^{T_1^{\gamma^{2/3}}}_{T_1^{\gamma^{2/3}} + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}} f_i, f_i \rangle \leqslant \frac{2}{\sqrt{\lambda}}, T_0 \wedge T_2^{\gamma^{2/3}} \wedge T_3^{\gamma^{2/3}} \geqslant T_1^{\gamma^{2/3}} + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l} \right] \leqslant C\lambda^{\frac{d-1}{2}} \exp - \frac{D_3 \lambda^{\beta/2}}{2^{\beta}}.$$

Under the hypothesis (i), we know that for $\delta > 0$ we can find $\eta_\delta > 0$ such that if $z \leqslant \eta_\delta$ then for all $x \in \mathbb{R}^d$, $|x - x_0| \leqslant \theta$, then

$$\text{(IV.16)} \qquad \operatorname{Log} k^l(x, z) \geqslant -\frac{\delta}{\sqrt{z}}.$$

To have $k^l(x, z) \geqslant 4D_2\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}$ it is now enough that $z \leqslant \eta_\delta$ and

$$z \geqslant \left(\frac{m_l \delta}{\operatorname{Log} \lambda}\right)^2,$$

and so we take

$$\text{(IV.17)} \qquad \gamma = 2^{3/2}\left(\frac{m_l \delta}{\operatorname{Log} \lambda}\right)^3.$$

For large λ, (IV.8) will be true, and $\frac{3}{2}\gamma^{2/3} \leqslant \eta_\delta$. Checking our estimates, we see that we shall have for given $\delta > 0$ and large enough λ

$$\text{(IV.18)} \quad \overline{P}\left[|\Gamma^{x_0}_{A_t}|^{-1} \geqslant \lambda, A_t \leqslant T \right] \leqslant \frac{2}{\lambda^{\frac{t(\operatorname{Log}\lambda)^2}{16(m_l\delta)^3}}} + \frac{1}{\lambda^{\frac{G_\theta}{(2t\sqrt{2})^{1/3}\sqrt{2}m_l\delta}}}$$
$$+ \frac{\sqrt{2}}{\lambda^{\frac{t}{8m_l\delta}}} + 2\exp\{-\lambda^{1/m_l}\} + \frac{A}{\lambda^p} + C\exp\{-c'\lambda^{3/2m_l}\}$$
$$+ C\lambda^{\frac{d-1}{2}} \exp\left\{-\frac{D_3}{2^\beta}\lambda^{\beta/2}\right\}$$

and δ being arbitrarily small, the theorem follows.

Let us now suppose that hypothesis (ii) is fulfilled. We shall adapt the proof for (i). This time (as in [*1*]-Theorem 5.9) we shall take T_i^γ instead of $T_i^{\gamma^{2/3}}$, $1 \leqq i \leqq 3$, and shall change all $\gamma^{2/3}$ into γ. We shall not introduce T_0; we want to choose

$$\text{(IV.19)} \qquad \gamma = \frac{2m_l\delta}{\operatorname{Log} \lambda}$$

and the r. h. t. in (IV.4) would be to large. Under the stronger hypothesis

$\lim z \operatorname{Log} \inf_{x \in \mathbb{R}^d} k^l(x, z) = 0$ the rest of the proof would follow without any problem. Now, we need new techniques.

We first notice that if (*ii*) holds, then for all $y \in \mathbb{R}^d$ we have

$$\lim_{\substack{z>0 \\ z\to 0}} z \operatorname{Log} \Big(\inf_{|x-y| \leqslant \frac{\theta/2}{z^{1/2}}} k^l(x, z) \Big) = 0 .$$

We shall use this with $y = x_0$.

Let us now control $T_1^\gamma + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}$. If (IV.8) and $T_1^\gamma \leqslant t\gamma\sqrt{2}$ hold then

$$T_1^\gamma + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l} \leqslant 2t\gamma\sqrt{2}. \tag{IV.20}$$

For $\alpha > 0$ to be determined afterward we shall replace (IV.15) by

$$\begin{aligned} \overline{P}\Big[\langle \Gamma^{T_1^\gamma}_{T_1^\gamma + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3\, m_l}} f_i, f_i \rangle \leqslant \frac{2}{\sqrt{\lambda}}, 2t\gamma\sqrt{2} \wedge T_2^\gamma \wedge T_3^\gamma \geqslant T_1^\gamma \\ + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l} \Big] \leqslant \overline{P}\Big[\sup_{0 \leqslant s \leqslant 2t\gamma\sqrt{2}} | x_s - x_0 | > \alpha\left(\frac{\operatorname{Log} \lambda}{\delta}\right)^{1/2} \Big] \\ + \overline{P}\Big[\langle \Gamma^{T_1^\gamma}_{T_1^\gamma + D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}} f_i, f_i \rangle \leqslant \frac{2}{\sqrt{\lambda}}, 2t\gamma\sqrt{2} \wedge T_2^\gamma \wedge T_3^\gamma \geqslant T_1^\gamma + D_1\left[\frac{2}{\sqrt{2}}\right]^{3/m_l}, \\ \sup_{0 \leqslant s \leqslant 2t\gamma\sqrt{2}} | x_s - x_0 | \leqslant \alpha\left(\frac{\operatorname{Log} \lambda}{\delta}\right)^{1/2} \Big]. \end{aligned} \tag{IV.21}$$

For a given $\delta > 0$, for large λ, by taking $\gamma = \dfrac{2m_l\delta}{\operatorname{Log} \lambda}$ we have by Proposition III.1, (III.11), (III.12)

$$\begin{aligned} \overline{P}\Big[\sup_{0 \leqslant s \leqslant 2t\gamma\sqrt{2}} | x_s - x_0 | > \alpha\left(\frac{\operatorname{Log} \lambda}{\delta}\right)^{1/2} \Big] \\ \leqslant \exp\left\{ - \frac{K\left(\alpha\left(\frac{\operatorname{Log} \lambda}{\delta}\right)^{1/2}\right)^{4/3}}{t^{1/3}\left(\frac{\delta}{\operatorname{Log} \lambda}\right)^{1/3}} \right\} = \frac{1}{\lambda^{\frac{K\alpha^{4/3}}{t^{1/3}\delta}}} \end{aligned} \tag{IV.22}$$

with a K depending only on m_l and on the bounds on the vector fields and on their derivatives

For $z \leqslant \frac{3}{2}\gamma$,

$$\frac{\theta/2}{z^{1/2}} \geqslant \frac{\theta}{2}\left(\frac{\operatorname{Log} \lambda}{3m_l\delta}\right)^{1/2} \tag{IV.23}$$

which leads us to choose, so as to have $\frac{\theta/2}{z^{1/2}} \geqslant \alpha\left(\frac{\text{Log }\lambda}{\delta}\right)^{1/2}$,

$$\text{(IV.24)} \qquad \alpha = \frac{\theta}{2(3m_l)^{1/2}}$$

and then hypothesis (*ii*) enables us to estimate the second right hand term in (IV.21) as in (IV.14), (IV.15) with the difference that we take $\frac{\gamma}{2} \leqslant z \leqslant \frac{3\gamma}{2}$. Then, as $\frac{\theta/2}{z^{1/2}} \geqslant \alpha\left(\frac{\text{Log }\lambda}{\delta}\right)^{1/2}$, for (almost) all w's such that

$$\sup_{0 \leqslant s \leqslant 2t\gamma\sqrt{2}} |x_s - x_0| \leqslant \alpha\left(\frac{\text{Log }\lambda}{\delta}\right)^{1/2}, \qquad \text{for any } \delta > 0,$$

we shall have thanks to (*ii*) that for large λ's, $\text{Log } k^l(x, z) \geqslant -\frac{\delta}{z}$. Then we shall have $k^l(x, z) \geqslant 4D_2\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}$ for large λ's (as $z \geqslant \gamma/2$). We can now proceed as in (IV.14), (IV.15).

We must see what becomes of all the other estimates. They are the same as in the proof of [*1*]-theorem 5.9. More precisely if we follow the proof of theorem IV.2 (*i*) after replacing $T_i^{\gamma^{2/3}}$ by T_i^{γ}, $1 \leqslant i \leqslant 3$, the estimate (IV.2) doesn't change, (IV.4) is not to be taken into account, (IV.6) becomes

$$\text{(IV.25)} \qquad \overline{P}[T_1^{\gamma} \geqslant t\gamma\sqrt{2}] \leqslant \sqrt{2} \exp\left\{ -\frac{t\sqrt{2}}{8\gamma} \right\}$$

we replace (IV.7) by

$$\text{(IV.26)} \quad \overline{P}\left[T_2^{\gamma} - T_1^{\gamma} \leqslant D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\right] \leqslant 2 \exp\left\{ -\frac{\gamma^2 \lambda^{3/2m_l}}{8D_1 2^{3/m_l}} \right\}$$

and the only change in (IV.13) is that for $1 \leqslant i \leqslant 3$ we have T_i^{γ} instead of $T_i^{\gamma^{2/3}}$. With $\gamma = \frac{2m_l\delta}{\text{Log }\lambda}$ as in (IV.19), all the estimates above plus (IV.21), (IV.22) enable us to get (for all $p < +\infty$)

$$\text{(IV.27)} \quad \overline{P}[\,|\Gamma_{A_t}^{x_0}|^{-1} \geqslant \lambda, A_t \leqslant T] \leqslant \frac{2}{\lambda^{\frac{t\sqrt{2}}{16m_l\delta}}} + \frac{\sqrt{2}}{\lambda^{\frac{t\sqrt{2}}{16m_l\delta}}} + 2\exp\{-\lambda^{1/m_l}\}$$

$$+ \frac{A}{\lambda^p} + C\exp(-c'\lambda^{3/2m_l}) + C\lambda^{\frac{d-1}{2}} \exp\left\{ -\frac{D_3}{2^{\beta}} \lambda^{\beta/2} \right\} + \frac{1}{\lambda^{\frac{K\alpha^{4/3}}{t^{1/3}\delta}}}.$$

The theorem follows as δ is arbitrarily small.

By reasoning as for (i), we could easily adapt theorem 5.10 in [*1*], with $\lim \sqrt{z}\,\mathrm{Log}\, h(z) = 0$ in the hypotheses. We can also adapt theorem 5.11 in [*1*] and its corollary, in which $\theta = +\infty$ and so we do not need to use our new estimates to control the way the diffusion gets away from the starting point; the proof is then exactly the same as in [*1*].

For the two-sided process, this kind of estimation is not adapted, as we already saw in [*1*], section 6. Naturally if the vector fields in both half spaces $z > 0$ and $z < 0$ satisfy the assumptions of any of our theorems, then the result in the theorem holds, as seen in subsection f) of section 6 in [*1*].

We now give a result on both the reflected and the two-sided processes, which takes at last into account the D_j's.

THEOREM IV.3. — Let $x_0 \in \mathbb{R}^d$; assuming there exist $y_0 \in \mathbb{R}^d$, $\eta > 0$, $a < 4/3$, such that

$$\text{(IV.28)}\quad \forall x \in \mathbb{R}^d, \qquad \inf_{f \in \mathbb{R}^d, \| f \| = 1} \sum_{j=1}^{q} \langle D_j(x), f \rangle^2 \geqslant \eta \exp(-|x - y|^a).$$

Then $\forall T > 0$, $\forall t > 0$, $1_{A_t \leqslant T} \,|\, [\Gamma_{A_t}^{x_0}]^{-1} \,|$ is in all the $L_p(\overline{\Omega}, P \otimes P')$.

Proof. — By taking a greater $a < \dfrac{4}{3}$ and a smaller $\eta > 0$ we may take $y_0 = x_0$. For convenience we shall work only on the reflecting process. We shall end up using lemma V.8.4 in [*5*], as in [*1*]-(6.129) or (IV.14), We take $\lambda \to \infty$ and $\delta = 2/\lambda$; and we take $b > 3/4$ with $ab < 1$

$$\text{(IV.29)}\quad \overline{P}[\langle \Gamma_{A_t} f, f \rangle \leqslant \delta, A_t \leqslant T] \leqslant \overline{P}\left[\sup_{0 \leqslant s \leqslant T} \left\| \frac{\partial \varphi_s}{\partial x} \right\| > \mathrm{Log}\, 1/\delta\right]$$
$$+ \overline{P}[\sup_{0 \leqslant s \leqslant T} |x_s - x_0| > (\mathrm{Log}\, 1/\delta)^b] + \overline{P}\left[\sup_{0 \leqslant s \leqslant T} \left\| \frac{\partial \varphi_s}{\partial x} \right\| \leqslant \mathrm{Log}\, 1/\delta, \right.$$
$$\left. \sup_{0 \leqslant s \leqslant T} |x_s - x_0| \leqslant (\mathrm{Log}\, 1/\delta)^b, \langle \Gamma_{A_t} f, f \rangle \leqslant \delta, \ A_t \leqslant T\right]$$

$$\text{(IV.30)}\quad \overline{P}[\sup_{0 \leqslant s \leqslant T} |x_s - x_0| > (\mathrm{Log}\, 1/\delta)^b] \leqslant \exp\{-K(\mathrm{Log}\, 1/\delta)^{\frac{4b}{3}}\}$$

by (III.11)

$$\text{(IV.31)}\quad \overline{P}\left[\sup_{0 \leqslant s \leqslant T} \left\| \frac{\partial \varphi}{\partial x} \right\| > \mathrm{Log}\, 1/\delta\right] \leqslant \exp\{-K(\mathrm{Log}\, 1/\delta)^{4/3}\}$$

by (III.11)

If $\sup_{0 \leqslant s \leqslant T} | x_s - x_0 | \leqslant (\text{Log } 1/\delta)^b$, since $A_t \leqslant T$, for all $s \in [0, t]$ we have $\langle D_j(x_{A_s -}), f \rangle^2 \geqslant \eta \exp \{ -(\text{Log } 1/\delta)^{ab} \}$.

So if $\sup_{0 \leqslant s \leqslant T} | x_s - x_0 | \leqslant (\text{Log } 1/\delta)^b$ and $\sup_{0 \leqslant s \leqslant T} \left\| \frac{\partial \varphi}{\partial x} \right\| \leqslant \text{Log } 1/\delta$, then

$$
\text{(IV.32)} \quad \int_0^{A_t} \langle \varphi_s^{*-1} D_j(x_0), f \rangle^2 dL_s = \int_0^t \langle D_j(x_{A_s -}), \widetilde{\frac{\partial \varphi}{\partial x}}^{-1} f \rangle^2 ds
$$
$$
\geqslant t\eta \frac{\exp \{ -(\text{Log } 1/\delta)^{ab} \}}{(\text{Log } 1/\delta)^2} = t\eta \frac{\delta^{(\text{Log } 1/\delta)^{ab-1}}}{(\text{Log } 1/\delta)^2}
$$

and as $ab < 1$, for small enough δ's, $\langle \Gamma_{A_t} f, f \rangle > \delta$ and so

$$
\text{(IV.33)} \quad \overline{P}[\langle \Gamma_{A_t} f, f \rangle \leqslant \delta, A_t \leqslant T] \leqslant \exp \{ -K (\text{Log } 1/\delta)^{\frac{4b}{3}} \} + \exp \{ -K (\text{Log } 1/\delta)^{4/3} \}
$$

and we conclude as in (6.129), (6.130), (6.131) in [1] by lemma V.8.4 in [5].

We did not succeed in giving assumptions taking into account the Lie brackets of D_j's. This is probably because the set of times of sejourn at the boundary is « small ».

We now give an example similar to [1]-(6.60) to show we cannot completely localize Theorem IV.3. This is because L_t can be small for great t's, and so the diffusion can be carried away without having had enough time to be regularized on the boundary. More quantitatively, the problems arise from $E\left[\frac{1}{L_t}\right] = +\infty$.

$$
\text{(IV.34)} \quad dx_t = 1_{t \leqslant T} dC_{L_t}, \qquad x_0 = 0 .
$$

As in [1]-(6.60) we need not look into the component A_t. $x_{A_t} = C_{t \wedge L_T}$ and since z and C are independent the law of x_{A_t} is $\theta(x)dx$, where

$$
\text{(IV.35)} \quad \theta(x) = 2 \int_0^{\infty} \frac{e^{-\frac{x^2}{2(t \wedge l)}}}{\sqrt{2\pi(t \wedge l)}} e^{-\frac{l^2}{2T}} \frac{dl}{\sqrt{2\pi T}} .
$$

If we set $f(x) = \int_0^1 e^{-\frac{x^2}{2l}} e^{-\frac{l^2}{2T}} \frac{dl}{\sqrt{2\pi l}}$, f has same smoothness as θ, $f \in C^\infty(\mathbb{R}^*)$

and

$$(\text{IV}.36)\qquad f(x) = \frac{2}{\sqrt{2\pi}}\int_1^{\infty} e^{-\frac{x^2u^2}{2}} e^{-\frac{1}{2\text{T}u^4}} \frac{du}{u^2}$$

$$(\text{IV}.37)\qquad f'(x) = \frac{2}{\sqrt{2\pi}}\int_1^{\infty} (-x)e^{-\frac{x^2u^2}{2}} e^{-\frac{1}{2\text{T}u^4}} du \qquad (x \neq 0).$$

For $x > 0$, set $xu = s$

$$(\text{IV}.38)\quad f'(x) = \frac{2}{\sqrt{2\pi}}\int_x^{\infty} (-1)e^{-s^2/2} e^{-\frac{x^4}{2\text{T}s^4}} ds \quad \text{and so} \quad f'(0+) = -1.$$

Likewise $f'(0-) = +1$, and it is easy to see that

$$(\text{IV}.39)\qquad \theta'(0+) = -\frac{2}{\sqrt{2\pi\text{T}}}, \qquad \theta'(0-) = \frac{2}{\sqrt{2\pi\text{T}}}$$

and θ is not even differentiable at 0.

V. THE EXPONENTIAL MARTINGALE TECHNIQUES AND THE TWO-SIDED PROCESS

We shall now use the exponential martingale techniques proper for estimating on jump processes. This will enable us to keep track all at once of the positive excursions, the negative excursions, and of what goes on the boundary. The regularization might well be achieved by all these factors relaying one another.

We show at last interaction between the D_j and the X_i leading to C^∞-regularity of the semi-group. We fail to give assumptions taking into account the Lie brackets of the D_j.

We first investigate the dependence on η of μ in [*1*]-Theorem 6.10 (which uses the estimate (III.3) on the excursions).

We shall again use in a basic way that we may take $A_t \leqslant T$.

DEFINITION V.1. — W^+ is the set of continuous functions $e: \mathbb{R}_+ \to \mathbb{R}_+$, such that $e(0) = 0$ and that there exists $\sigma(e)$, $0 < \sigma(e) < +\infty$, for which if $0 < s < \sigma(e)$ then $e(s) > 0$ and if $s \geqslant \sigma(e)$ then $e(s) = 0$.

W^+ will be the set of positive excursions for the brownian motion z. Let n^+ the characteristic measure for the excursions of a reflecting brownian motion; let $W^- = -W^+$, n^- the image of n^+, let e_t be the process defined by $e_t(s) = z_{s+A_{t-}}$ for $0 < s < A_t - A_{t-}$ if $A_{t-} < A_t$, 0 if $s \geqslant A_t - A_{t-}$, and $e_t(s) = \delta$ if $A_t = A_{t-}$. Then e_t is a Poisson point process whose characteristic measure is $n = \frac{1}{2}(n^+ + n^-)$ on $W^+ \cup W^- \cup \{\delta\}$.

W_0 is the subset of $\Omega \times W^+$ of the elements (ε, e) such that $\varepsilon(s) = \varepsilon(s \wedge \sigma(e))$. That is, we stop w at the end of the excursion of z. For more details, see [*1*], section 3 *a*) and theorem 6.3.

For $e \in W^+$ (resp. W^-), $\Psi(\varepsilon, e, .)$ (resp. $\Psi'(\varepsilon, e, .)$) is the flow of diffeomorphisms of $\mathbb{R}^d$ associated to the stochastic differential equation on (Ω, P)

$$\text{(V.1)} \qquad dx = X_0(x, e)dt + X_i(x, e)d\varepsilon^i$$

$$(\text{resp. (V.1}')) dx = X'_0(x, e)dt + X'_i(x, e)d\varepsilon^i)$$

and $\overline{C}_t^x(\varepsilon, e)$ (resp. $\overline{C}_t'^x(\varepsilon, e)$) is the linear mapping

$$\text{(V.2)} \quad f \in T_x^*\mathbb{R}^d \to \overline{C}_t^x(\varepsilon, e)f = \int_0^t \langle \Psi_s^{*-1}X_i(x), f \rangle \Psi_s^{*-1}X_i(x)ds$$

$$\text{resp. (V.2}') \quad \overline{C}_t'^x(\varepsilon, e)f = \int_0^t \langle \Psi_s'^{*-1}X'_i(x), f \rangle \Psi_s'^{*-1}X'_i(x)ds \Big).$$

Recall that $m_l = 20^{l-1} \times 6$; E_n and E'_n have been defined in definition II.3, definition II.4, and definition IV.1.

THEOREM V.2. — There are $\eta_0 > 0, \gamma > 0$ only depending on the vector fields such that for all $x \in \mathbb{R}^d$, $f \in T_x^*\mathbb{R}^d$ with $\|f\| = 1$, $\eta \in [0, \eta_0]$, if

$$\text{(V.3)} \qquad \sum_{n=1}^{l} \sum_{Y \in E_n} \langle f, Y(x, 0) \rangle^2 \geqslant \eta$$

$$\left(\text{resp.} \sum_{n=1}^{l} \sum_{Y \in E'_n} \langle f, Y'(x, 0) \rangle^2 \geqslant \eta\right)$$

then for $0 < \rho < (\gamma\eta)^{m_l/2} = \mu(\eta)$ we have

$$\text{(V.4)} \qquad n(\langle \overline{C}_{\sigma(e)}^x(\varepsilon, e)f, f \rangle \geqslant \rho) \geqslant \frac{1/2}{\rho^{1/m_l}}$$

$$\left(\text{resp. } n(\langle \overline{C}_{\sigma(e)}'^x(\varepsilon, -e)f, f \rangle \geqslant \rho) \geqslant \frac{1/2}{\rho^{1/m_l}}\right).$$

Proof. — We shall only prove the first result. $\chi(\eta)$ is to be chosen later, with $0 < \chi(\eta) \leqslant 1/2$.

$$\mathrm{T}_0(\varepsilon, e) = \inf\{ t \geqslant 0, | \Psi_t(\varepsilon, e, x) - x | \geqslant \chi(\eta) \} \wedge \inf\left\{ t \geqslant 0, \left\| \left[\frac{\partial \Psi_t}{\partial x}(\varepsilon, e, x)\right]^{-1} - \mathrm{I} \right\| \geqslant \chi(\eta) \right\}$$

$$\mathrm{T}_1^\rho(e) = \inf\{ t \geqslant 0, e_t = \rho^{1/m_l} \}.$$

We know that

$$\text{(V.5)} \qquad \mathrm{P}[\mathrm{T}_0 < \mathrm{T}_1^\rho / e] \leqq \mathbb{1}_{\chi(\eta) < 2c'\mathrm{T}_1^\rho} + 2d.\exp\left(- \frac{\chi(\eta)^2}{4c'\mathrm{T}_1^\rho}\right)$$

by (4.2.1) in [9], for a c depending only on the vector fields. It is easy to see that

$$\text{(V.6)} \qquad \mathbb{1}_{\chi(\eta) < 2c'\mathrm{T}_1^\rho} \leqq \frac{2c'\mathrm{T}_1^\rho}{\chi(\eta)}$$

and as for positive x we have $x \leqq \exp(x)$,

$$\text{(V.7)} \qquad \exp\left(- \frac{\chi(\eta)^2}{4c'\mathrm{T}_1^\rho}\right) \leqq \frac{4c'\mathrm{T}_1^\rho}{\chi(\eta)^2}$$

and so

$$\text{(V.8)} \qquad \mathrm{P}[\mathrm{T}_0 < \mathrm{T}_1^\rho / e] \leqq \left(\frac{2c'}{\chi(\eta)} + \frac{8dc'}{\chi(\eta)^2}\right)\mathrm{T}_1^\rho$$

Also, by theorem 3.6 in [1] under n and conditionally on $(\mathrm{T}_1^\rho(e) < +\infty)$, $e_t(0 \leqslant t \leqslant \mathrm{T}_1^\rho(e))$ is a Bes(3) process starting at 0 and stopped when it hits ρ^{1/m_l}. We have $\mathrm{E}^{\mathrm{Bes}(3)}(\mathrm{T}_1^\rho) = \frac{1}{3}\rho^{2/m_l}$, so

$$\text{(V.9)} \quad n(\mathrm{T}_0(\varepsilon, e) \leqslant \mathrm{T}_1^\rho(e) / \mathrm{T}_1^\rho(e) < +\infty) \leqslant \left(\frac{2c'}{\chi(\eta)} + \frac{8dc'}{\chi(\eta)^2}\right)\frac{1}{3}\rho^{2/m_l} = \mathrm{C}_{\chi(\eta)}\rho^{2/m_l}.$$

Putting

$$g(\varepsilon, e) = \left[\frac{\partial \Psi}{\partial x}_{\mathrm{T}_1^\rho(e)}(\varepsilon, e, x \right]^{-1} f, \langle \mathrm{C}_t^s p, p \rangle = \int_s^t \langle (\varphi_u \circ \varphi_s^{-1})^{*-1}\mathrm{X}_i(x_0), f \rangle^2 du$$

$$\text{(V.10)} \quad n(\langle \overline{\mathrm{C}}_{\sigma(\;)}^x f, f \rangle \leqslant \rho / \mathrm{T}_1^\rho < +\infty) \leqslant \mathrm{C}_{\chi(\eta)}\rho^{2/m_l} + n(\langle \mathrm{C}_\sigma^{\mathrm{T}_1^\rho} g, g \rangle \leqslant \rho, \mathrm{T}_1^\rho < \mathrm{T}_0 / \mathrm{T}_1^\rho < +\infty)$$

and as soon as $\rho^{1/m_l} \leqslant \chi(\eta)$, if $\mathrm{T}_1^\rho \leqslant \mathrm{T}_0$, by the mean-value theorem

$$\text{(V.11)} \quad \sum_{n=1}^{l} \sum_{\mathrm{Y} \in \mathrm{E}_n} \langle g(\varepsilon, e), \mathrm{Y}(\Psi_{\mathrm{T}_1^\rho}(\varepsilon, e, x), \rho^{1/m_l}) \rangle^2 \geqslant \eta - c\chi(\eta)^2.$$

We shall choose η_0 such that $\sqrt{\frac{\eta_0}{2c}} \leqslant 1/2$ and such that $\chi(\eta) = \sqrt{\frac{\eta}{2c}}$ is a good choice. We now have our first restriction on ρ:

(V.12) $$\rho^{1/m_l} \leqslant \chi(\eta) = \sqrt{\frac{\eta}{2c}}.$$

Let us put $g'(\varepsilon, e) = \frac{g(\varepsilon, e)}{\| g(\varepsilon, e) \|}$. If $T_1^\rho \leqslant T_0$, as $\chi(\eta) \leqslant 1/2$, we have $\frac{1}{2} \leqslant \| g(\varepsilon, e) \| \leqslant \frac{3}{2}$, so that

(V.13) $$\sum_{n=1}^{l} \sum_{Y \in E_n} \langle g'(\varepsilon, e), Y(\Psi_{T_1^\rho}(\varepsilon, e, x), \rho^{1/m_l}) \rangle^2 \geqslant \eta/9$$

and moreover

(V.14) $$n(\langle \overline{C}_\sigma^{T_1^\rho} g, g \rangle \leqslant \rho, T_1^\rho \leqslant T_0 / T_1^\rho < +\infty) \leqslant n(\langle C_\sigma^{T_1^\rho} g', g' \rangle \leqslant 4\rho, T_1^\rho \leqslant T_0 / T_1^\rho < +\infty).$$

We know that under $n(\ / T_1^\rho < +\infty)$, if $(\mathscr{G}_t)_{t \geqslant 0}$ is the canonical filtration on $W^+ \cup W^- \cup \{\delta\}$, conditionally on $\mathscr{G}_{T_1^\rho}$, for $t \geqslant T_1^\rho$ (ε_t, e_t) is a $m+1$-dimensional brownian motion stopped when ε hits 0. On $(\overline{\Omega}, P \otimes P'_{\rho^{1/m_l}})$ let us put $S = \inf\{ t \geqslant 0, | z_t - \rho^{1/m_l} | = \rho^{1/m_l} \}$.
Let $x' \in \mathbb{R}^d$, $h \in T_{x'}^*, \mathbb{R}^d$ with $\| h \| = 1$, such that

(V.15) $$\sum_{n=1}^{l} \sum_{Y \in E_n} \langle h, Y(x', \rho^{1/m_l}) \rangle^2 \geqslant \eta/9.$$

Then we are led to estimate

(V.16) $$P \otimes P'_{\rho^{1/m_l}}(\langle \overline{C}_S^{x'}(\overline{w}) h, h \rangle \leqslant 4\rho)$$

and use this estimate in (V.14) with $x' = \Psi_{T_1^\rho}(\varepsilon, e, x)$, $h = g'(\varepsilon, e)$.

Let $\chi'(\eta)$ be such that $0 < \chi'(\eta) \leqslant 1/2$, U be the stopping time

$$U = \inf\{ t \geqslant 0, | \Psi(w, x') - x' | \geqslant \chi'(\eta) \} \wedge \inf\left\{ t \geqslant 0, \left| \left[\frac{\partial \Psi_t}{\partial x}(w, x') \right]^{-1} - I \right| \geqslant \chi'(\eta) \right\} \wedge S.$$

For $t \leqslant U$ and $2\rho^{1/m_l} \leqslant \chi'(\eta)$, by the mean-value theorem,

(V.17) $$\sum_{n=1}^{l} \sum_{Y \in E_n} \langle \Psi_t^{*-1} Y(x', z_t), h \rangle^2 \geqslant \eta/9 - c\chi'(\eta)^2.$$

We shall take $\chi'(\eta) = \chi(\eta)/3 = \sqrt{\frac{\eta}{18c}}$, and we have a stronger restriction on ρ:

(V.18) $$\rho^{1/m_l} \leqslant \frac{\chi(\eta)}{6} = \frac{1}{6}\sqrt{\frac{\eta}{2c}}.$$

We have for small ρ's, by (4.2.1) in [9], that

(V.19) $$P \otimes P'_{\rho^{1/m_l}}(U \leqslant D_1(4\rho)^{3/m_l}) \leqslant 2d \exp\left\{ -\frac{(X'(\eta) - C_2D_1(4\rho)^{3/m_l})^2}{C_2D_1(4\rho)^{3/m_l}} \right\} + 2 \exp\left\{ -\frac{(\rho^{1/m_l})^2}{2D_1(4\rho)^{3/m_l}} \right\}.$$

We shall ask for $C_2D_1(4\rho)^{3/m_l} \leqslant \rho^{1/m_l}$, which will be enough if (V.18) holds, for then $\rho^{1/m_l} \leqslant \chi'(\eta)/2$. As already $6 \leqslant m_l$, we have $4^{3/m_l} \leqslant 2$, and we only need

(V.20) $$2C_2D_1(\rho^{1/m_l})^3 \leqslant \rho^{1/m_l}$$

and there exists an $\alpha > 0$ such that (V.20) holds for $\rho^{1/m_l} \leqslant \alpha$. We shall take $\eta_0 > 0$ small enough for (V.18) to imply $\rho^{1/m_l} \leqslant \alpha$. We then have for a $k > 0$ depending only on the vector fields

(V.21) $$P \otimes P'_{\rho^{1/m_l}}(U < D_1(4\rho)^{3/m_l}) \leq 2(d+1) \exp\left\{ -\frac{k}{\rho^{1/m_l}} \right\}$$

so that

(V.22) $$P \otimes P'_{\rho^{1/m_l}}(\langle \overline{C}_S^{x'}(\overline{w})h, h \rangle \leqslant 4\rho) \leqslant 2(d+1) \exp\left\{ -\frac{k}{\rho^{1/m_l}} \right\} + P \otimes P'_{\rho^{1/m_l}}(\langle \overline{C}^x_{D_1(4\rho)^{3/m_l}}(\overline{w})h, h \rangle \leqslant 4\rho \,;\, U \geqslant D_1(4\rho)^{3/m_l}).$$

Thanks to (III.3), to estimate the second right hand term, we only need

(V.23) $$\eta/18 \geqslant D_2(4\rho)^{3/m_l}$$

or equivalently

(V.24) $$\rho^{1/m_l} \leqslant \frac{1}{4}\left(\frac{\eta}{18D_2}\right)^{1/3}$$

which will be implied by (V.18) for η_0 small enough. Then the second r.h.t. in (V.22) is dominated by $K \exp[-D_3(4\rho)^{-\alpha/m_l}]$.

By using this and (V.22) and the results following (V.14), for small ρ's

(V.25) $$n(\langle \overline{C}_\sigma^{T_1^\rho} g', g' \rangle \leqslant 4\rho, T_1^\rho \leqslant T_0/T_1^\rho < +\infty) \leqslant 2(d+1) \exp\left\{ -\frac{k}{\rho^{1/m_l}} \right\} + K \exp[-D_3(4\rho)^{-\alpha/m_l}]$$

and by (V.10), (V.14), (V.25) we get

$$(V.26)\quad n(\langle \overline{C}^x_\sigma f, f \rangle \geqslant \rho / T^\rho_1 < +\infty) \geqslant 1 - C_{\chi(\eta)}\rho^{2/m_l} - 2(d+1)\exp\left\{-\frac{k}{\rho^{1/m_l}}\right\} - K\exp\,[-D_3(4\rho)^{-\alpha/m_l}]$$

and as by theorem 3.6 in [1], $n(T^\rho_1 < +\infty) = \frac{1}{\rho^{1/m_l}}$, we shall have (V.4) as soon as

$$(V.27)\quad \left(\frac{2c'}{\chi(\eta)} + \frac{8dc'}{\chi(\eta)^2}\right)\frac{1}{3}\rho^{2/m_l} + 2(d+1)\exp\left\{-\frac{k}{\rho^{1/m_l}}\right\} + K\exp\,[-D_3(4\rho)^{-\alpha/m_l}] \leqslant \frac{1}{2}.$$

To obtain this, we first take η_0 small enough for $\chi(\eta) < \frac{1}{2}$. Then we shall have K′ such that $\frac{1}{3}\left(\frac{2c'}{\chi(\eta)} + \frac{8dc'}{\chi(\eta)^2}\right) \leqslant \frac{K'}{\chi(\eta)^2}$.

By taking again η_0 small enough, if (V.18) holds, we have

$$(V.28)\quad \frac{K'}{\chi(\eta)^2}\rho^{2/m_l} + 2(d+1)\exp\left\{-\frac{k}{\rho^{1/m_l}}\right\} + K\exp\,[-D_3(4\rho)^{-\alpha/m_l}] \leqslant \frac{2K'}{\chi(\eta)^2}\rho^{2/m_l}.$$

So, to get (V.27) it is enough to have

$$(V.29)\quad \rho^{1/m_l} \leqslant \frac{1}{2\sqrt{K'}}\chi(\eta) = \frac{1}{2}\sqrt{\frac{\eta}{2cK'}}.$$

So, if $\chi(\eta) = \sqrt{\frac{\eta}{2c}}$, taking η_0 small enough for (V.18) to imply (V.24), there will be a constant $\gamma > 0$, such that if

$$(V.30)\quad \rho^{1/m_l} \leqslant (\gamma\eta)^{1/2}$$

then (V.18), (V.29) hold. The theorem follows.

We can now state some results:

DEFINITION V.3.

$$E_1 = (X_1, \ldots, X_m), \qquad E_{l+1} = [(X_0, X_1, \ldots, X_m, \partial/\partial z), E_l]$$
$$E'_1 = (X'_1, \ldots, X'_m), \qquad E'_{l+1} = [(X'_0, X'_1, \ldots, X'_m, \partial/\partial z), E'_l]$$

$$(V.31)\quad \overline{\chi}^l(x) = \inf_{\|f\|=1}\left[\sum_{n=1}^{l}\left(\sum_{Y\in E_n}\langle f, Y(x,0)\rangle^2 + \sum_{Y'\in E'_n}\langle f, Y'(x,0)\rangle^2\right) + \sum_{j=1}^{q}\langle f, D_j(x)\rangle^2\right].$$

Theorem V.4. — Assume there exist $l \in \mathbb{N}$, $y_0 \in \mathbb{R}^d$, $\eta > 0$, $a < \frac{4}{3}$, such that for all $x \in \mathbb{R}^d$,

$$\text{(V.32)} \qquad \overline{\chi}^l(x) \geqslant \eta \exp\{-|x - y_0|^a\}.$$

Then for all $x_0 \in \mathbb{R}^d$, $T > 0$, $t > 0$, $\mathbb{1}_{A_t \leqslant T} | [\Gamma_{A_t}^{x_0}]^{-1} |$ is in all the $L_p(\Omega, P \otimes P')$. If moreover $D_1 = \ldots = D_q = 0$, we may take $a < 2$.

Proof. — Naturally, by taking perhaps a smaller $\eta > 0$ and a greater $a < \frac{8}{3m_l}$ (if $l \geqslant 1$) or $a < \frac{4}{3}$ (if $l = 0$), we may take $x_0 = y_0$. We also take $\eta \leqslant \eta_0$. We shall follow the proof of [*1*]-theorem 6.12 after writing the process [*1*]-(6.101) as follows:

$$\text{(V.33)} \quad N_t(w, \beta) = \exp\left\{ -\beta \langle \Gamma_{A_t}^{x_0} f, f \rangle + \int_0^t [\beta \langle \varphi_{A_{s-}}^{*-1} D_j(x_0), f \rangle^2 - \frac{1}{2}(\tau_s^f(\overline{w}, \beta) + \tau_s'^f(\overline{w}, \beta))] ds \right\}$$

where τ_s^f, $\tau_s'^f$ are defined by ($\tilde{}$ denoting transposition)

$$\text{(V.34)} \quad \tau_s^f(\overline{w}, \beta) = \int_{W_0} \mathbb{1}_{\sigma \leqslant 1} \{ \exp[-\beta \langle \overline{C}_{\sigma(e)}^{xA_{s-}(\overline{w})}(\varepsilon, e) \tilde{\varphi}_{A_{s-}}^{-1}(\overline{w}, x_0) f, \tilde{\varphi}_{A_{s-}}^{-1}(\overline{w}, x_0) f \rangle] - 1 \} \, dn(\varepsilon, e)$$

$$\tau_s'^f(\overline{w}, \beta) = \int_{W_0} 1_{\sigma \leqslant 1} \{ \exp[-\beta \langle \dot{C}_{\sigma(e)}^{xA_{s-}(w)}(\varepsilon, -e) \tilde{\varphi}_{A_{s-}}^{-1}(\overline{w}, x_0) f, \tilde{\varphi}_{A_{s-}}^{-1}(\overline{w}, x_0) f \rangle] - 1 \} \, dn(\varepsilon, e).$$

Take $f \in T_{x_0}^*(\mathbb{R}^d)$ with $\| f \| = 1$, and $\delta > 0$. We estimate first

$$\overline{P}[\langle \Gamma_{A_t}^{x_0} f, f \rangle \leqslant \delta, A_t \leqslant T]$$

as in (IV.31), (IV.32), (IV.33). $b > 3/4$ depends on a and is to be chosen later.

$$\text{(V.35)} \quad \overline{P}[\langle \Gamma_{A_t}^{x_0} f, f \rangle \leqslant \delta, A_t \leqslant T] \leqslant \overline{P}[\sup_{0 \leqslant s \leqslant T} |x_s - x_0| > (\text{Log } 1/\delta)^b] + \overline{P}\left[\sup_{0 \leqslant s \leqslant T} \left\| \frac{\partial \varphi_s}{\partial x} \right\| > \text{Log } 1/\delta \right] + \overline{P}\left[\langle \Gamma_{A_t}^{x_0} f, f \rangle \leqslant \delta, A_t \leqslant T, \sup_{0 \leqslant s \leqslant T} |x_s - x_0| \leqslant (\text{Log } 1/\delta)^b, \sup_{0 \leqslant s \leqslant T} \left\| \frac{\partial \varphi_s}{\partial x} \right\| \leqslant \text{Log } 1/\delta \right].$$

Let's put $T_\delta = \inf\left\{ t \geqslant 0 ; \left| \frac{\partial \varphi_t}{\partial x}(\overline{w}, x_0) \right| \vee \left| \left[\frac{\partial \varphi_t}{\partial x}(w, x_0) \right]^{-1} \right| \geqslant 1/\delta^{1/4} \right\}$. As $N_t(w, \beta)$ is exactly as in [*1*], we may reason as in [*1*], (6.95) through (6.106), that is express $\langle C_{A_t}^{x_0} f, f \rangle$ in terms of the underlying point process and notice first that for $s \leqslant L_{T_\delta}$, τ_s^f and $\tau_s'^f$ are uniformly bounded and then that $N_{t \wedge L_{T_\delta}}(\overline{w}, \beta)$ is a $(\overline{F}_{A_t})_{t \geqslant 0}$ supermartingale. So

$$\text{(V.36)} \qquad E^{\overline{P}}[N_{t \wedge L_{T_\delta}}(\overline{w}, \beta)] \leqslant 1$$

and as $\delta \to 0$, $L_{T_\delta} \to +\infty$, so that by Fatou's lemma

$$\text{(V.37)} \qquad E^{\overline{P}}[N_t] \leqslant 1 .$$

For $x' \in \mathbb{R}^d$, $g \in T^*_{x'}, \mathbb{R}^d$ with $\| g \| = 1$, $\beta > 0$, we define

$$\text{(V.38)} \quad \theta^g(x', \beta) = \int_{W_0} 1\!\!1_{\sigma \leqslant 1} \{ \exp(-\beta \langle \overline{C}^{x'}_{\sigma(e)}(\varepsilon, e)g, g \rangle) - 1 \} \, dn(\varepsilon, e)$$

$$\theta'^g(x', \beta) = \int_{W_0} 1\!\!1_{\sigma \leqslant 1} \{ \exp(-\beta \langle \overline{C}'^{x'}_{\sigma(e)}(\varepsilon, e)g, g \rangle) - 1 \} \, dn(\varepsilon, e) .$$

Now, if $| x' - x_0 | \leqslant (\text{Log } 1/\delta)^b$, at least one of the following statements is true:

$$\text{(V.39)} \qquad \sum_{n=1}^{l} \sum_{Y \in E_n} \langle g, Y(x', 0) \rangle^2 \geqslant \frac{\eta}{3} \exp\{ -(\text{Log } 1/\delta)^{ab} \}$$

$$\text{(V.40)} \qquad \sum_{n=1}^{l} \sum_{Y \in E'_n} \langle g, Y'(x', 0) \rangle^2 \geqslant \frac{\eta}{3} \exp\{ -(\text{Log } 1/\delta)^{ab} \}$$

$$\text{(V.41)} \qquad \sum_{j=1}^{g} \langle g, D_j(x_0) \rangle^2 \geqslant \frac{\eta}{3} \exp\{ -(\text{Log } 1/\delta)^{ab} \} .$$

Our aim is to choose as in [*1*]-(6.117)

$$\text{(V.42)} \qquad \beta(\delta) = \left[\frac{Dt}{2m_l \delta^{1 - \frac{1}{2m_l}}} \right]^{\frac{m_l}{m_l - 1}}$$

so as to replace (1)-(6.124), which may be written

$$\text{(V.43)} \qquad \frac{1}{2} (\tau_s^f(\overline{w}, \beta(\delta)) + \tau_s'^f(w, \beta(\delta))) \leqslant -\frac{m_l \delta \beta(\delta)}{t}$$

by

$$\text{(V.44)} \quad \frac{1}{2} (\tau_s^f(w, \beta(\delta)) + \tau_s'^f(w, \beta(\delta)) - \beta(\delta) \langle \varphi^{*-1}_{A_{s-}} D_j(x_0), f \rangle^2 \leqslant -\frac{m_l \delta \beta(\delta)}{t} .$$

If (V.39) holds, we notice that if $\rho \leqslant \mu = \mu\left(\frac{\eta}{3}\exp\{-(\text{Log }1/\delta)^{ab}\}\right)$

$$\text{(V.45)}\quad \theta^{g}(x',\beta) = -\beta\int_0^\infty e^{-\beta\rho} n(\langle \overline{C}^{x'}_{\sigma(e)}(\varepsilon, e)g, g\rangle \geqslant \rho, \sigma \leqslant 1)d\rho$$

$$\leqslant -\beta/2\int_0^\mu \frac{e^{-\beta\rho}}{\rho^{1/m_l}}d\rho = -\frac{1}{2}\beta^{1/m_l}\int_0^{\beta\mu}\frac{e^{-u}}{u^{1/m_l}}du$$

and so, as soon as

$$\text{(V.46)}\qquad \beta \geqslant \frac{1}{\mu\left(\frac{\eta}{3}\exp\{-(\text{Log }1/\delta)^{ab}\}\right)} = \frac{k_l}{\delta^{((\text{Log }1/\delta)^{ab}-1)\frac{m_l}{2}}}$$

we get

$$\text{(V.47)}\qquad \theta^{g}(x',\beta) \leqslant -D\beta^{1/m_l}.$$

We choose $b > 3/4$ with $ab < 1$ and $\beta(\delta)$ as in (V.42). Then for small enough δ (V.46) will hold. We proceed likewise when (V.40) holds.

Now if (V.41) holds, we again choose $b > 3/4$; $ab < 1$, and if

$\sup\limits_{0\leqslant s\leqslant T}\left\|\frac{\partial\varphi_s}{\partial x}\right\| \leqslant \text{Log }1/\delta$ and $A_t \leqslant T$ then

$$\text{(V.48)}\quad \langle \varphi^{*-1}_{A_{s-}}D_j(x_0), f\rangle^2$$

$$= \left\langle D_j(x_{A_{s-}}), \frac{\partial\tilde{\varphi}^{-1}}{\partial x}f\right\rangle^2 \geqslant \frac{n\delta^{(\text{Log }1/\delta)^{ab}-1}}{3(\text{Log }1/\delta)^2} \geqslant \frac{m_l\delta}{t}$$

for $t > 0$, and small enough δ's.

If we have (V.47) and (V.42), as by putting $g_s(\overline{w}) = \dfrac{\tilde{\varphi}^{-1}_{A_{s-}}(\overline{w}, x_0)f}{\|\tilde{\varphi}^{-1}_{A_{s-}}(\overline{w}, x_0)f\|}$. $h_s(\overline{w}) = \|(\tilde{\varphi}^{-1}_{A_{s-}}(\overline{w}, x_0)f\|$, we have

$$\text{(V.49)}\qquad \tau^f_s(\overline{w},\beta) = \theta^{g_s(w)}(x_{A_{s-}}(\overline{w}), \beta h^2_s(\overline{w}))$$

and as $h_s(w) \geqslant \dfrac{1}{\left\|\frac{\partial\varphi}{\partial x}A^{(\overline{w},x_0)}_{s-}\right\|} \geqslant (\text{Log }1/\delta)^{-1}$, we get for small δ's

$$\text{(V.50)}\qquad \beta(\delta)h^2_s(\overline{w}) \geqslant \frac{1}{\delta^{\frac{m_l}{2(m_l-1)}}} \times \left[\frac{Dt}{2m_l}\right]^{\frac{m_l}{m_l-1}}$$

we can conclude that

$$\text{(V.51)}\qquad \tau^f_s(\overline{w},\beta(\delta) \leqslant -D\left[\frac{Dt}{2m_l}\right]^{\frac{m_l}{m_l-1}}\frac{1}{\delta^{\frac{1}{2(m_l-1)}}} = \frac{2m_l\delta\beta(\delta)}{t}.$$

As we have either (V.39) or (V.40) or (V.41) that holds, we get (V.44) with the choice (V.42) of $B(\delta)$. Then

$$\text{(V.52)} \quad \overline{P}\Big[\langle \Gamma_{A_t}^{x_0} f, f \rangle \leqslant \delta, A_t < T, \sup_{0 \leqslant s \leqslant T} | x_s - x_0 | \leqslant (\text{Log } 1/\delta)^b, \sup_{0 \leqslant s \leqslant T} \left\| \frac{\partial \varphi}{\partial x} \right\| < \text{Log } 1/\delta \Big] \leqslant \overline{P}\Big[N_t(w, \beta(\delta)) \geqslant \exp\Big\{ - \beta(\delta)\delta + \int_0^t \Big[\beta(\delta) \langle \varphi_{A_{s-}}^{*-1} D_j(x_0), f \rangle^2 - \frac{1}{2}(\tau_s^f + \tau_s'^f) \Big] ds \Big\}, \sup_{0 \leqslant s \leqslant T} | x_s - x_0 | \leqslant (\text{Log } 1/\delta)^b, \sup_{0 \leqslant s \leqslant T} \left\| \frac{\partial \varphi}{\partial x} \right\| \leqslant \text{Log } 1/\delta \Big]$$

and so, by using (V.35), (V.52), (IV.32), (IV.33), we get

$$\text{(V.53)} \quad \overline{P}\big[\langle \Gamma_{A_t}^{x_0} f, f \rangle \leqslant \delta, A_t \leqslant T \big] \leqslant \exp\{ - K (\text{Log } 1/\delta)^{\frac{4b}{3}} \} + \exp\{ - K (\text{Log } 1/\delta)^{4/3} \} + \exp\Big\{ - (m_l - 1) \Big(\frac{D_t}{2m_l} \Big)^{\frac{m_l}{m_l - 1}} \delta^{\frac{-1}{2(m_l - 1)}} \Big\}$$

because the right hand term in (V.53) is dominated by

$$\text{(V.54)} \qquad \overline{P}[N_t(w, \beta) \geqslant \exp\{ - \beta(\delta)\delta + m_l \delta \beta(\delta) \}]$$

which by Čebyšev's inequality and (V.37) is less than $\exp\{ -(m_l - 1)\delta\beta(\delta) \}$.

We conclude by use of the lemma V.84 in [5] as in [1]-(6.129). The theorem follows.

If $D_1 = \ldots = D_q = 0$, we use theorem III.2 instead of theorem III.1 in (IV.32), so we may take $b > 1/2$ instead of $b > 3/4$.

Remark. — The statement for $l = 0$ is exactly theorem IV.3, of which we give another proof.

ACKNOWLEDGEMENT

I most heartily thank Professor Jean-Michel Bismut for having kindly led me through the intricacies of his works.

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For more relevant references, see [1], [2], [3], [4].

(Manuscrit reçu le 23 octobre 1984)