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Tightness criteria for laws of semimartingales

by

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ABSTRACT. — When the space of cadlag trajectories is endowed with the topology of convergence in measure (much weaker that Skorohod's topology), very convenient criteria of compactness are obtained for bounded sets of laws of quasimartingales and semi-martingales. Stability of various classes of processes for that type of convergence is also studied.

RÉSUMÉ. — On montre que si l'on munit l'espace des trajectoires càd-làg de la topologie de la convergence en mesure (beaucoup plus faible que la topologie de Skorohod) on obtient des critères très commodes de compacité étroite pour des ensembles bornés de lois de quasimartingales et de semimartingales. On étudie la stabilité de diverses classes de processus pour ce mode de convergence. L'étude des diffusions sera abordée dans un travail ultérieur.

INTRODUCTION

This paper was motivated by an attempt of the second author to construct the diffusions with highly singular drifts that are needed for Nelson's

« stochastic mechanics », though no mention of stochastic mechanics will be made here (the application to diffusions is the subject of another paper by the second author alone). It soon appeared that the beautiful methods of Aldous and his successors, using Skorohod's topology, weren't very well adapted to this problem. Though some substantial results could be proved in this way, the whole subject was changed by the idea of introducing a much weaker topology, of which the first author had a good experience from the works of Baxter-Chacon [*1*], Maisonneuve [*5*], Dellacherie [*3*], Dellacherie-Meyer [*4*]. Of course, weakening the topology makes it much easier to prove compactness criteria, and the whole theory finally becomes so simple that it was decided to publish it separately.

Let **D** be the space of all càdlàg functions (this is the standard abbreviation, from the French, for *right continuous with left hand limits* ([1])) from $\mathbf{R}_+$ to $\mathbf{R}$ (the trivial extension to $\mathbf{R}^d$, $d > 1$, will be left to the reader). We denote by X_t the coordinate mapping $w \mapsto w(t)$ on **D**, by $\mathscr{F}^{\circ}$, $\mathscr{F}^{\circ}_t$ the σ-fields they generate on **D**. Roughly speaking, any criterion ensuring that a deterministic function $w(t)$ belongs to **D** should give rise, when applied uniformly, to a *compactness* criterion in **D** for some suitable topology, and to a *tightness* criterion for probability measures on **D** in the corresponding weak topology (= narrow topology in the terminology of Bourbaki [*2*]). The results of Aldous can be interpreted in this way. In martingale theory, it seems more natural to use Doob's criterion on numbers of upcrossings, and it turns out that one suitable topology on **D** is that of convergence in measure (in Lebesgue's sense). Contrary to Skorohod's topology, **D** will no longer be a Polish space, but this isn't a serious drawback, since the only result in weak convergence which requires the space to be Polish in an essential way is the converse to Prohorov's theorem, which is seldom used.

We apply this compactness/tightness criterion to probability laws on **D**, i. e. to laws of stochastic processes. Our main result is the following: any set of laws of *quasimartingales* on **D** which is bounded in variation is tight, and its closure consists of laws of quasimartingales. This is the probabilistic analogue of the elementary tightness criterion for functions of bounded variation on $\mathbf{R}_+$, but apparently the result wasn't known even for martingales. On the other hand, for laws of decreasing processes it is just another form of the Baxter-Chacon theorem on compactness of the set of randomized stopping times.

([1]) Including a limit at $+\infty$.

The last part of the paper is devoted to the problem of stability of given classes of processes under weak convergence. We first give a general result, implying that finite dimensional distributions converge at least within a set of full Lebesgue measure (this is weaker than the corresponding result for Skorohod's topology, but still sufficient to give a lot of information about the limiting measure). Then we give more detailed stability results, concerning processes with finite variation paths, continuous local martingales, etc. This section owes much to discussions with C. Stricker, which led to great improvements in results and proofs.

PSEUDO-PATHS

Our account of pseudo-paths is reduced here to the strictly necessary facts. For more details, see the book [*4*], chapter IV, n^{os} 40-46.

Let $\lambda(dt)$ the measure $e^{-t}dt$ on $\mathbf{R}_+$. Let $w(t)$ be a real valued Borel function on $\mathbf{R}_+$ (Lebesgue measurable functions would do as well, but this extension isn't really useful). By definition, the *pseudo-path* of w is a probability law on $[0, \infty] \times \overline{\mathbf{R}}$: the image measure of λ under the mapping $t \mapsto (t, w(t))$. We denote by ψ the mapping which associates to a path w its pseudo-path: it is clear that ψ identifies two paths if and only if they are equal a. e. in Lebesgue's sense. In particular, ψ is $1-1$ on $\mathbf{D}$, and provides us with an imbedding (²) of $\mathbf{D}$ into the compact space $\overline{\mathscr{P}}$ of all probability laws on the compact space $[0, \infty] \times \overline{\mathbf{R}}$. For a moment, we shall give to the induced topology on $\mathbf{D}$ the name of *pseudo-path topology.*

The definition of pseudo-paths in [*4*] is slightly different: Lebesgue measure is used there instead of the bounded measure λ.

Let us introduce some intermediate sets between $\mathbf{D}$ and $\overline{\mathscr{P}}$: $\mathscr{P}$ will be the set of all measures $\mu \in \overline{\mathscr{P}}$ which are carried by $[0, \infty[\times \mathbf{R}$; Λ will be the set of all measures $\mu \in \mathscr{P}$ whose projection on $[0, \infty]$ is $\lambda(dt)$. Finally, Ψ will be the set of all pseudo-paths. We have obvious inclusions

$$\mathbf{D} \subset \Psi \subset \Lambda \subset \mathscr{P} \subset \overline{\mathscr{P}}.$$

For the reader's information (we'll not use these results) let us mention that $\mathscr{P}$, Λ, Ψ are Polish spaces: the only delicate point is to show that Ψ is a $\mathscr{G}_\delta$ in Λ, which is done in [*4*], IV.43. On the other hand, it has been shown by B. V. Rao (see appendix) that $\mathbf{D}$ isn't a $\mathscr{G}_\delta$ in Ψ, and hence can't

(²) Note that the σ-field induced on $\mathbb{D}$ by $\overline{\mathscr{P}}$ is the usual one.

be Polish. Our characterization of $\mathbf{D}$ in theorem 2 will show that $\mathbf{D}$ is Borel in $\overline{\mathscr{P}}$, hence a Lusin space.

The following lemma is adapted from Dellacherie [*3*].

LEMMA 1. — *The pseudopath topology on* Ψ *is just convergence in measure.*

Proof. — Both topologies being metrizable, we may restrict ourselves to sequences. If w_n is a sequence of paths which converges in measure to w, we have for any bounded continuous function $f(s, x)$ on $[0, \infty] \times \overline{\mathbf{R}}$

$$\lim_n \int f(s, w_n(s))\lambda(ds) = \int f(s, w(s))\lambda(ds) \tag{1}$$

Otherwise stated, w_n tends to w in the pseudo-path topology. Conversely, assume (1). Taking first $f(s, x) = a(s) \operatorname{Arctg}(x)$, with $a(s)$ continuous bounded on $[0, \infty]$, we deduce from (1) that the paths $u_n(s) = \operatorname{Arctg}(w_n(s))$ converge to $u = \operatorname{Arctg}(w)$ in the weak topology of $L^2(\lambda)$. Then taking $f(s, x) = (\operatorname{Arctg}(x))^2$ we see that $\| u_n \|^2 \to \| u \|^2$. Hence u_n converges *strongly* in $L^2(\lambda)$ to u, and w_n converges in measure to w.

Of course, lemma 1 is also valid on $\mathbf{D} \subset \Psi$.

CHARACTERIZATION OF $\mathbb{D}$ USING NUMBERS OF UPCROSSINGS

We are going to extend to all of $\overline{\mathscr{P}}$ some familiar functionals of stochastic processes.

First of all, let $\mu \in \overline{\mathscr{P}}$. We define $S(\mu) = \mu^* \in [0, \infty]$ by

$$S(\mu) = \mu^* = \inf \{ c : \mu \text{ is carried by } [0, \infty] \times [-c, c] \}. \tag{2}$$

For a pseudo-path $\mu = \psi(w)$, $S(\mu)$ is simply $w^* = \operatorname{ess\,sup}_t | w(t) |$, whence the notation μ^*. It is clear that the set $\{ \mu : S(\mu) \leqq c \}$ is closed in $\overline{\mathscr{P}}$, i. e., S is a l. s. c. function on $\overline{\mathscr{P}}$.

Let $\mathscr{R}$ be the set of all rational pairs (u, v) with $u < v$. Let τ be a finite subdivision on $[0, \infty]$

$$\tau : 0 = t_0 < t_1 \ldots < t_n = +\infty$$

We define for $\mu \in \overline{\mathscr{P}}$ a positive integer $N_\tau^{uv}(\mu)$ by the following condition: $N_\tau^{uv}(\mu) \geqq k$ if and only if there exist elements of τ denoted as follows

$$0 \leqq t_{i_1} < t_{i'_1} < t_{i_2} < t_{i'_2} \ldots < t_{i_k} < t_{i'_k} < \infty$$

such that μ charges (i. e. gives strictly positive measure to) each one of the open sets in $[0, \infty] \times \overline{\mathbf{R}}$

$$]t_{i_i}, t_{i_1+1}[\times[-\infty, u[,]t_{i'_1}, t_{i'_1+1}[\times]v, \infty],]t_{i_2}, t_{i_2+1}[\times[-\infty, u[\dots$$

The sets $\{\mu : \mathrm{N}_\tau^{\mu v}(\mu) \geqq k\} = \{\mu : \mathrm{N}_\tau^{uv}(\mu) > k-1\}$ are open in $\overline{\mathscr{P}}$, so that N_τ^{uv} is a l. s. c. function, and the same is true for the function

$$\mathrm{N}^{uv} = \sup_\tau \mathrm{N}_\tau^{uv}.$$

If μ is the pseudo-path of a càdlàg function w, $\mathrm{N}^{uv}(\mu)$ is the classical number of upcrossings of $[u, v]$ by the path w, as defined in martingale theory.

THEOREM 2. — *A measure $\mu \in \overline{\mathscr{P}}$ belongs to* **D** *if and only if it belongs to* Λ *and satisfies the conditions*

(3) $$\mu^* < \infty, \qquad \forall (u, v) \in \mathscr{R}, \mathrm{N}^{uv}(\mu) < \infty.$$

Proof. — These conditions are clearly necessary. Let us prove they are sufficient.

1) We disintegrate the measure $\mu \in \Lambda$ as $\int_0^\infty \varepsilon_s \otimes \rho_s \, \lambda(ds)$, and show first that ρ_s has (for a. e. s) a support reduced to one single point $w(s)$. Indeed, assume the contrary, we may then find a set $\mathrm{A} \subset \mathbb{R}_+$ of strictly positive Lebesgue measure, a pair $(u, v) \in \mathscr{R}$, such that for every $s \in \mathrm{A}$ the measure ρ_s charges both $[-\infty, u[$ and $]v, +\infty]$. From this, it is easy to deduce that $\mathrm{N}^{uv}(\mu) \geqq k$ for every integer k, a contradiction.

It isn't difficult to choose a Borel version of the mapping w, and to verify that $\mu = \psi(w)$.

2) Assume there is some point $t \in [0, \infty[$ such that

$$\lim_{\varepsilon \downarrow 0} \operatorname{ess} \inf_{t<s<t+\varepsilon} w(s) < \lim_{\varepsilon \downarrow 0} \operatorname{ess} \sup_{t<s<t+\varepsilon} w(s)$$

then we may insert between these numbers some rationals $u < v$, and it is again very easy to see that $\mathrm{N}^{uv}(w) \geqq k$ for every k, a contradiction to our assumptions. Otherwise stated, we have proved that our function w has essential limits from the right (in $\overline{\mathbf{R}}$) at any point $t \in [0, \infty[$.

According to a result of Chung, Doob and Walsh (see [*4*], IV.37), w is a. e. equal to a *right continuous function* (in the ordinary sense). This is the only delicate point of the proof, and it is fortunate that other people did the hard work for us.

3) Then the generalized numbers of upcrossings we have been using turn out to be ordinary numbers of upcrossings for this right continuous

function, and our assumptions imply that it has also left limits in $\overline{\mathbf{R}}$. Finally, the assumption on μ^* implies that this càdlàg function is bounded in $\mathbf{R}$, and the theorem is proved.

COROLLARY. — *Any subset* A *of* **D** *such that*

$$\sup_{\mu \in A} \mu^* < \infty, \qquad \sup_{\mu \in A} N^{uv}(\mu) < \infty \qquad \textit{for} \qquad (u, v) \in \mathscr{R}$$

is relatively compact in **D** *for the pseudo-path topology.*

Proof. — According to the lower semi-continuity of the functions μ^*, $N^{uv}(\mu)$, the closure of A in $\overline{\mathscr{P}}$ (which is compact) is contained in **D**. It is likely that the above properties characterize the relatively compact subsets of **D**, but we didn't try to prove this point.

QUASIMARTINGALES

In this section we recall classical results on quasimartingales and on the space H^1, which will be used in the next section. Quasimartingales have been studied by Fisk, Orey, but the definitive results concerning them are due to Murali Rao.

Let Ω be a probability space with a filtration $(\mathscr{F}_t)_{t \geq 0}$. Let (X_t) be a càdlàg adapted process such that $X_t \in L^1$ for every t. Given a subdivision τ : $0 = t_0 < t_1 \ldots < t_n = \infty$, we define X_∞ to be 0 and set

$$V_\tau(X) = \Sigma_{0 \leq i < n} \quad \mathbf{E}[\,|\,\mathbf{E}[X_{t_{i+1}} - X_{t_i} \,|\, \mathscr{F}_{t_i}]\,|\,] \tag{4}$$

and $V(X) = \sup_\tau V_\tau(X)$. If this number (the *conditional variation* of X) is finite, then X is said to be a quasimartingale. For a martingale X, V(X) is $\sup \mathbf{E}[\,|X_t|\,]$; for a positive supermartingale, $V(X) = \mathbf{E}[X_0]$.

We may interpret (4) as

$$V_\tau(X) = \sup_\varphi \mathbf{E}[\Sigma_{i<n} \varphi_{t_i}(X_{t_{i+1}} - X_{t_i})] \tag{5}$$

where the random variables φ_{t_i} are assumed to be $\mathscr{F}_{t_i}$-measurable and bounded by 1 in absolute value, the sup being attained for

$$\varphi_{t_i} = \operatorname{sgn}(\mathbf{E}[X_{t_{i+1}} - X_{t_i} \,|\, \mathscr{F}_{t_i}])$$

The right hand side of (5) can be interpreted as a stochastic integral $\mathbf{E}\left[\int \varphi_s dX_s\right]$ of a predictable elementary process φ, bounded by 1 in absolute value.

Rao's main result is the unique decomposition of X as a difference of two positive supermartingales Y, Z such that $V(X)=E[Y_0+Z_0]$ (see [*4*], chapter VI, 40; the uniqueness part is due to Stricker). It follows from this that we don't change V(X) by allowing random subdivisions (T_i) using stopping times instead of deterministic subdivisions (t_i). We shall need the following extension to quasimartingales of the classical Doob inequalities. It is well known [3], but we reprove it for the reader's convenience.

LEMMA 3. — *Let X be a quasimartingale. Then*

$$(6)\qquad c\mathbf{P}\{X^* \geqq c\} \leqq V(X) \qquad \mathbf{E}[N^{uv}(X)] \leqq \frac{|u| + V(X)}{v-u}.$$

Proof. — Using a deterministic change of time, we may assume $X_t=0$ for t large. We then apply the fact that $V(X) \geqq V_\tau(X)$, where τ is the random subdivision consisting of the stopping times 0, T, ∞ with

$$T = \inf\{t : |X_t| > c\}$$

Then since $|\mathbf{E}[X_\infty - X_T | \mathscr{F}_T]| \geqq c$ on $\{T < \infty\}$ we get the first inequality. For the second one, we follow the usual proof of Doob's inequality, defining

$$T_0 = 0,\quad T_1 = \inf\{t > 0 : X_t < u\},$$
$$T_1' = \inf\{t > T_1, X_t > v\},\quad T_2 = \inf\{t > T_1', X_t < u\}\ \ldots$$

Then we have

$$\Sigma(X_{T_i'} - X_{T_i}) + u^+ \geqq (v-u)N^{uv}(X)$$

(the u^+ term is there to compensate for a last uncompleted upcrossing, which may contribute a negative value $(X_\infty - X_{T_i})$ with $X_\infty=0$, $X_{T_i} \in [0,u]$). Integrating, we get $V(X) + u^+ \geqq (v-u)\mathbf{E}[N^{uv}]$.

A useful result due to Yoeurp is the following (a simple proof due to Stricker is given in [*6*]): if f is a convex Lipschitz function, such that $f(0)=0$, and X is a quasimartingale, then $Y = f(X)$ is a quasimartingale with $V(Y) \leqq 2KV(X)$ (K being the Lipschitz constant of f). We shall apply this to the truncation functional $T_n(x) = (x \vee (-n)) \wedge n$. This function isn't convex, but is the difference of the two convex Lipschitz functions $x \vee (-n)$ and $(x-n)^+$, so $V(T_n(X)) \leqq 4V(X)$.

Finally, a few words about H^1. The most convenient definition of the

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[3] See for instance Métivier, Reelle und vektorwertige Quasimartingale, p. 148-153. Lecture Notes in M. 607, Springer 1977.

norm $N(X) = \| X \|_{H^1}$ will be, in analogy with (5), to define $N(X) = \sup_\tau N_\tau(X)$ where

$$(7) \qquad N_\tau(X) = \sup_\varphi \mathbf{E}[\,|\,\Sigma \quad \varphi_{t_i}(X_{t_{i+1}} - X_{t_i})\,|\,].$$

(cf. [4], VIII.104). On the other hand, to say that X belongs to H^1 amounts to saying that X has a canonical decomposition $X = A + M$ (A being a previsible process of finite variation with $A_0 = 0$, and M a local martingale) such that $\mathbf{E}\left[\int |\,dA_s\,| + M^*\right] < \infty$, and this expectation defines a norm equivalent to N(X). So if X belongs to H^1, X is a quasimartingale and X^* belongs to L^1. Conversely, assume that X is a quasimartingale and X^* belongs to L^1. Decompose X as $Y - Z$, a difference of two positive supermartingales, with $V(X) = \mathbf{E}[Y_0 + Z_0]$, and apply the Doob decomposition theorem ([4], VII.13) to Y and Z; it follows that $X = A + M$ with V(X) controlling $\mathbf{E}\left[\int |\,dA_s\,|\right]$, and hence $\mathbf{E}[A^*]$. Then it is clear that $\mathbf{E}[X^*]$ controls $\mathbf{E}[M^*]$, and we see that the norm

$$(8) \qquad N'(X) = V(X) + \mathbf{E}[X^*]$$

is also equivalent to the H^1 norm N(X).

From Yoeurp's result and the preceding remarks follows at once the fact that, if X is a quasimartingale, the truncated process $T_n \circ X$ belongs to H^1. On the other hand, H^1 has better stability properties than quasimartingales: if $\mathbf{Q}$ is a probability law, absolutely continuous w. r. to $\mathbf{P}$ with a density ρ which is bounded by some constant C, it is clear that the norm of X in $H^1(\mathbf{Q})$ satisfies $N_{\mathbf{Q}}(X) \leqq CN_{\mathbf{P}}(X)$, while we don't know whether such a relation holds for the conditional variations.

It is sometimes convenient to extend the definition of quasimartingales and H^1 from probability laws to general positive bounded measures: if η is such a measure, we decide that $V_\eta(X) = N_\eta(X) = 0$ if $\eta = 0$, and if $\eta \neq 0$, $V_\eta(X) = \eta(1)V_{\mathbf{P}}(X)$, $N_\eta(X) = \eta(1)N_{\mathbf{P}}(X)$ where $\mathbf{P}$ is the probability law $\eta/\eta(1)$. This extension is quite easy, and most results will be left to the reader.

TIGHTNESS FOR LAWS OF QUASIMARTINGALES

We give now the main result of this paper.

THEOREM 4. — *Let $\mathbf{P}_n$ be a sequence of probability laws on* $\mathbf{D}$, *such that under $\mathbf{P}_n$ the coordinate process (X_t) is a quasimartingale with conditional* *variation* $V_n(X)$ *uniformly bounded in n. Then there exists a subsequence* (P_{n_k}) *which converges weakly on* **D** *to a law* **P**, *and* (X_t) *is a quasimartingale under* **P**.

Proof. — There exists a subsequence $(\mathbf{P}_{n_k})$ which converges *weakly on* $\overline{\mathscr{P}}$ to some law **P**, and we are going to prove that **P** is carried by **D**. This will imply ([*4*], III.58) that $\mathbf{P}_{n_k}$ converges to **P** *weakly on* **D**. For the sake of simplicity of notation, we assume that indexes have been renamed so that the whole sequence $\mathbf{P}_n$ converges to **P**. We denote by $\mathbf{E}_n$, **E** expectations relative to $\mathbf{P}_n$, **P**.

By lower semicontinuity, we have

$$c\mathbf{P}\{ S(.)>c \} \leqq \liminf_n c\mathbf{P}_n \{ S(.) > c \}, \qquad E[N^{uv}] \leqq \liminf_n \mathbf{E}_n[N^{uv}]$$

and according to lemma 3 the right hand sides are uniformly bounded in n. So the random variables S, N^{uv} are **P**-a. s. finite, and theorem 2 implies that **P** is carried by **D**. A slightly different version of this reasoning would prove that the measures $\mathbf{P}_n$ are carried uniformly by compact subsets of **D**, i. e. tightness of the set $\{ \mathbf{P}_n, n \geqq 0 \}$, but here it was easier to work directly, without using Prohorov's theorem. It remains to show that **P** is a quasimartingale law.

Let $\tau = ({}_i)_{i \leqq n}$ be a finite subdivision ($t_n = +\infty$, $X_\infty = 0$ by convention). Let φ_{t_i} for $i < n$ be *continuous* $\mathscr{F}^\circ_{t_i}$-measurable functions on **D**, bounded by 1 in absolute value. Let X^c be the process X truncated at c. According to Yoeurp's result we have for $s \geqq 0$

$$\mathbf{E}_n[\Sigma_{i<n}\varphi_{t_i}(X^c_{s+t_{i+1}} - X^c_{s+t_i})] \leqq V_n(X^c) \leqq 4V_n(X)$$

We integrate over $s \in [0, \varepsilon[$ and apply Fubini's theorem

$$\mathbf{E}_n\left[\frac{1}{\varepsilon}\int_0^\varepsilon (\Sigma_i\varphi_{t_i}(X^c_{s+t_{i+1}} - X^c_{s+t_i}))ds\right] \leqq 4V_n(X)$$

The functional in the brackets is bounded and continuous. Let us define $v = \liminf_n V_n(X)$; we deduce from this

$$\mathbf{E}\left[\frac{1}{\varepsilon}\int_0^\varepsilon (\Sigma_i \ldots)ds\right] \leqq 4v$$

and letting $\varepsilon \to 0$, by right continuity

$$\mathbf{E}[\Sigma_i\varphi_{t_i}(X^c_{t_{i+1}} - X^c_{t_i})] \leqq 4v \tag{9}$$

On the other hand, we have for fixed $t\mathbf{E}_n[|X_t|] \leqq V_n(X)$. Using the same kind of argument we deduce that $\mathbf{E}\left[\frac{1}{\varepsilon}\int_0^\varepsilon |X_{t+s}|\,ds\right] \leqq v$, and by Fatou's lemma $\mathbf{E}[|X_t|] \leqq v$ as $\varepsilon \to 0$. Letting then $c \to \infty$ in (9), we get the same relation without truncation. Finally, we let φ_{t_i} tend in $L^1(\mathbf{P})$ to the random variable sgn $(\mathbf{E}[X_{t_{i+1}} - X_{t_i} \mid \mathscr{F}^\circ_{t_i}])$ and get that under $\mathbf{P}$ $V_\tau(X) \leqq 4v$. Since τ is arbitrary, $\mathbf{P}$ is a quasimartingale law. Note that we have used the filtration $\mathscr{F}^\circ_t$ without any enlargement: this is permitted, see [*4*], Appendix 2, n° 2, or Stricker in Sém. Prob. XV, Lect. Notes in M. 850, p. 495.

Remarks. — 1) Usually, $\mathbf{P}_n$ will be the law on the canonical space $\mathbf{D}$ of some càdlàg quasimartingale X^n defined on some other filtered probability space Ω^n, $\mathscr{F}^n_.$, possibly dependent on n. Then $V_{\mathbf{P}_n}(X)$ is *smaller* than the original conditional variation $V(X^n)$: this amounts to restricting the filtration to be the natural filtration of X^n on Ω^n.

2) We are going to sketch another proof, which will show that the constant 4 may be reduced to 1. We keep the notation suggested in the remark 1) above.

First, assume that X^n is a *positive supermartingale* (X then is a positive supermartingale under $\mathbf{P}_n$, and $V_n(X) = \mathbf{E}_n[X_0] = \mathbf{E}^n[X_0]$). Then we show that X is a positive supermartingale under the limit law P, and that $\mathbf{E}[X_0] \leqq \liminf_n E_n[X_0]$. To prove the first point: a.s. positivity under $\mathbf{P}$ is obvious (the set of positive paths in $\mathbf{D}$ is closed, and carries each law $\mathbf{P}_n$, hence their limit $\mathbf{P}$). Then, to prove that X is a supermartingale under $\mathbf{P}$ it suffices to prove that for any pair $s < t$, any positive bounded continuous $\mathscr{F}^\circ_s$-measurable function φ, any $c > 0$, we have

$$\mathbf{E}[\varphi X^c_t] \leqq \mathbf{E}[\varphi X^c_s] \qquad (X^c = X \wedge c)$$

We leave it to the reader to pass from $(\mathscr{F}^\circ_{t-})$ to $(\mathscr{F}^\circ_{t+})$. In turn, this can be reduced to

$$\mathbf{E}\left[\frac{1}{\varepsilon}\int_0^\varepsilon \varphi X^c_{t+r}\,dr\right] \leqq \mathbf{E}\left[\frac{1}{\varepsilon}\int_0^\varepsilon \varphi X^c_{s+r}\,dr\right]$$

which is now stable under weak convergence. Finally, for a positive supermartingale we have $V(X) = \sup_{c,\varepsilon} \mathbf{E}\left[\frac{1}{\varepsilon}\int_0^\varepsilon X^c_r\,dr\right]$ from which the relation

$$V_{\mathbf{P}}(X) \leqq \limsup_n V_{\mathbf{P}_n}(X)$$

follows at once. But we may also extract a subsequent so that $V_{P_{n_k}}(X)$ converges to $\liminf_n V_{P_n}(X)$ and apply the relation to this subsequence, thus replacing lim sup by lim inf.

Now we use the notation of remark 1), to deal with the general case. Let $X^n = Y^n - Z^n$ be the Rao decomposition of X^n, and let $\mathbf{Q}_n$ be the law of the pair (Y^n, Z^n) on $\mathbf{D} \times \mathbf{D}$—we denote by (Y, Z) the canonical process on this space. Our tightness criteria apply as well to vector valued processes so that we may assume, by extracting a subsequence if necessary, that $\mathbf{Q}_n$ converges weakly on $\mathbf{D} \times \mathbf{D}$ to some law $\mathbf{Q}$, and it is clear that $\mathbf{P}$ is the law of the càdlàg process $Y - Z$ under $\mathbf{Q}$. On the other hand, the same reasoning as above will show that Y, Z are positive supermartingales under $\mathbf{Q}$, relative to their joint natural filtration, so that

$$\mathbf{E}_Q[Y_0 + Z_0] \leqq \liminf_n (V_{Q_n}(Y) + V_{Q_n}(Z)) \leqq \liminf_n V(X^n)$$

and on the other hand $V_{\mathbf{P}}(X) \leqq \mathbf{E}_Q[Y_0 + Z_0]$.

This proof is interesting also as an illustration of the power of simple arguments using vector valued processes: in some cases, one isn't interested only in the weak convergence of the law of a semimartingale X, but in the weak convergence of the joint law of (X, A, M), its canonical decomposition, for instance, or of the joint law of $(X, A, \langle X, X \rangle)$. We'll leave that for the second part of the paper.

Note also that the norm $N'(X) = V(X) + \mathbf{E}[X^*]$ defining H^1 also behaves like a l. s. c. function under weak convergence.

3) We have defined above the quasimartingale and H^1 norms of X relative to a measure on $\mathbf{D}$ which isn't a probability measure, but is just bounded and positive. It is clear that theorem 4 remains valid in this setup, provided the total mass of $\mathbf{P}_n$ is uniformly bounded in n.

4) The deterministic analogue of theorem 4 is the following theorem, which is well known, but is rarely stated in this way: the set H_C of all càdlàg functions $h(t)$ on $\mathbb{R}_+$, such that $\int_{0-}^{\infty} |dh_s| \leqq C$, is compact under the topology of convergence in measure. More precisely, every sequence (h_n) on elements of H_C contains a subsequence which converges to some $h \in H_C$ outside some *countable* set (however, since we are dealing with signed measures dh_n, this exceptional set may be larger than the set of discontinuities of h). Our aim in the next section will be to find a weaker analogue of this last result for quasimartingales.

CONVERGENCE OF FINITE DIMENSIONAL DISTRIBUTIONS

Let $\mathbf{P}_n$ be a sequence of measures of uniformly bounded mass on $\mathbf{D}$, such that $V_n(X) = V_{\mathbf{P}_n}(X)$ is uniformly bounded by some constant K. According to theorem 4, we may assume that $\mathbf{P}_n$ converges weakly on $\mathbf{D}$ to some measure $\mathbf{P}$. Let I be any countable set in $\mathbf{R}_+$, and let $\mathbf{P}_n^{\mathrm{I}}$, $\mathbf{P}^{\mathrm{I}}$ be the law of the process $(X_t)_{t\in \mathrm{I}}$ on the Polish space $\mathbf{R}^{\mathrm{I}}$, under $\mathbf{P}_n$, $\mathbf{P}$. The property $c\mathbf{P}_n\{X^* > c\} \leqq K$ implies that the sequence $(\mathbf{P}_n^{\mathrm{I}})$ is tight in $\mathbf{R}^{\mathrm{I}}$. Our problem will be in this section: can we find a *dense* countable I such that, at least after extracting a subsequence, $\mathbf{P}_n^{\mathrm{I}}$ converges weakly on $\mathbf{R}^{\mathrm{I}}$ to $\mathbf{P}^{\mathrm{I}}$ in the usual weak topology? Since knowing the finite distributions of a càdlàg process on a countable dense set completely determines the law, this result will be sufficient to determine the limit law in the case of semi-martingales.

It turns out that we have a better result: I can be taken to be a set of full Lebesgue measure (then $\mathbf{R}^{\mathrm{I}}$ will not be a Polish space, but this is unimportant). The proof (which replaces a more complicated one in former versions) is a simple application of the celebrated Skorohod theorem on weak convergence, and doesn't require any regularity assumption on the processes.

We consider real valued processes for simplicity, but everything extends to $\mathbf{R}^d$ valued (hence to manifold valued) processes.

Let X^n, X be measurable processes, with pseudo-laws μ_n, μ, such that μ_n tends weakly to μ on Ψ. Since Ψ is a Polish space, Skorohod's theorem implies that one may find on some probability space $(\Omega, \mathscr{F}, \mathbf{P})$ random variables Y^n, Y with laws μ_n, μ such that $Y^n(\omega) \to Y(\omega)$ for a. e. ω. Using a mapping from $]-\infty, +\infty[$ to $]-1, 1[$, we may assume that X^n, X, Y^n, Y are bounded.

We may consider Y^n, Y as *processes* on Ω: this amounts to the remark that a *path* can be canonically associated with a given pseudo-path w: assuming w to be bounded, one just defines $w(t) = \liminf\limits_{h\downarrow 0} \frac{1}{h}\int_t^{t+h} w(s)ds$.

Then for a. e. ω the bounded functions $Y^n_{\cdot}(\omega)$ converge to $Y_{\cdot}(\omega)$ in measure. So $\int |Y_t^n(\omega) - Y_t(\omega)| e^{-t}dt$ tends boundedly to 0. Integrating in ω, we get L^1 convergence. We extract a subsequence so that we have a. e. convergence w. r. to the measure $dP \times dt$ (we do not change notation to denote this subsequence). Applying Fubini's theorem, we find that for every t in some set A of full Lebesgue measure, $Y_t^n \to Y_t$ a. s. Hence the finite distributions of the processes $(Y_t^n)_{t\in A}$ converge in the usual sense to those of $(Y_t)_{t\in A}$.

Coming back to the original processes, there is a set B of full Lebesgue measure such that, for every $t \in B$ and every n, we have

$$X_t^n = \liminf_{h\downarrow 0} \frac{1}{h} \int_t^{t+h} X_s^n \, ds \qquad \text{a. s.}$$

and the fact that X^n and Y^n have the same pseudo-law implies that the processes $(X_t^n)_{t\in B}$ and $(Y_t^n)_{t\in B}$ have the same law on $\mathbf{R}^B$. So finally, the finite dimensional distributions of $(X_t^n)_{t\in A\cap B}$ converge weakly to those of $(X_t)_{t\in A\cap B}$.

Let us give a formal statement:

THEOREM 5. — *Let (X_t^n), (X_t) be a measurable processes, such that the pseudo-law of X^n converges to that of X. Then there exists a subsequence $(X_t^{n_k})$ and a set I of full Lebesgue measure, such that the finite dimensional distributions of $(X_t^{n_k})_{t\in I}$ converge to those of $(X_t)_{t\in I}$.*

The proof also yields the following corollary:

THEOREM 6. — *Let (X_t^n), (X_t) be as above. Let f be a continuous bounded function on $\mathbf{R}_+^k$. Then the functions*

$$(t_1, \ldots, t_k) \mapsto \mathbf{E}[f(X_{t_1}^n, \ldots, X_{t_k}^n)]$$

converge in measure to the corresponding function relative to (X_t), as $n \to \infty$.

Proof. — Just apply the following remark: to check that a sequence (h_n) of measurable functions on $\mathbf{R}_+^k$ (or any measure space) converges in measure to h, it is sufficient to check that any subsequence itself contains a subsequence which converges a. s. to h. This is the case according to theorem 5.

LOCALIZATION

Up to now, we have been working with quasimartingales. Let us extend the results to *semi*martingales. We are going to work on semimartingales *up to* ∞ (otherwise, their sample functions might not belong to $\mathbf{D}$ as we have defined it). The extension to ordinary semimartingales is left to the reader.

THEOREM 7. — *For each n, let* $\mathbf{P}_n$ *be the law on* $\mathbf{D}$ *of a semimartingale* X^n *defined on some filtered probability space* $(\Omega^n, \mathscr{F}^n, \mathbf{P}^n, \mathscr{F}^n_t)$. *Assume that for every* $\varepsilon > 0$ *there is a quasimartingale* Y^n *on* Ω^n *such that*

(10) $\mathscr{P}^n\{(X^n - Y^n)^* > 0\} \leqq \varepsilon$ *and* $\mathrm{Var}(Y^n)$ *is bounded uniformly in n.*

Then the sequence $(\mathbf{P}_n)$ *contains a subsequence which converges weakly on* $\mathbf{D}$ *to a law* $\mathbf{P}$, *and under* $\mathbf{P}$ X *is a semimartingale up to* ∞.

Usually Y^n is X^n stopped at T or at T- for some stopping time T on Ω^n.

Proof. — One first checks the tightness of the laws $\mathbf{P}_n$, using theorem 2 and its corollary. For instance, with the same notations as in (10)

$$\mathbf{P}_n\{N^{uv} > c\} \leqq \frac{1}{c}\mathbf{E}^n[N^{uv}(Y^n)] + \mathbf{P}^n\{(X^n - Y^n)^* > 0\}$$

is uniformly small in n for c large. So extracting a subsequence if necessary we may assume that $\mathbf{P}_n$ converges to $\mathbf{P}$, and it remains to prove that X is a semimartingale under $\mathbf{P}$.

Call $\overline{\mathbf{P}}_n$ the law of the pair (X^n, Y^n) on $\mathbf{D} \times \mathbf{D}$, and call (X_t, Y_t) the coordinate process on $\mathbf{D} \times \mathbf{D}$. Since Y^n has uniformly bounded variation, we may find a subsequence such that $\overline{\mathbf{P}}_{n_k}$ converges to a law $\overline{\mathbf{P}}$ on $\mathbf{D} \times \mathbf{D}$ such that:

— The law of X under $\overline{\mathbf{P}}$ is $\mathbf{P}$,

— Y is a quasimartingale under $\overline{\mathbf{P}}$ in the joint natural filtration of (X, Y) (th. 4),

— $\overline{\mathbf{P}}\{(X - Y)^* > 0\} \leqq \varepsilon$ (lower-semicontinuity !).

This implies X is a semimartingale up to ∞. The simplest (if not the most elementary) way to prove it is to use the Dellacherie-Mokobodzki (theorem [*4*], VIII.80): if (j_n) is a sequence of previsible elementary processes in the natural filtration of X, which converges uniformly to 0, we must prove that $(j_n \,.\, X)_\infty$ (these stochastic integrals are really finite sums) converges to 0 in probability. Now (j_n) can also be interpreted by projection as a previsible process on $\mathbf{D} \times \mathbf{D}$ w. r. to the filtration of (X, Y), and since Y is a quasimartingale $(j_n \,.\, Y)_\infty$ tends to 0 in probability. On the other hand, $j_n \,.\, X$ and $j_n \,.\, Y$ differ only on a set of probability smaller than ε, and the conclusion follows.

Remark. — C. Stricker has proved the following version of theorem 7, which doesn't use at all approximation by quasimartingales. We assume that for every finite t, for every sequence of previsible elementary processes j_k on $\mathbf{D}$ which converges uniformly to 0 the stochastic integrals $\int_0^t j_{ks} dX_s$ w. r. to the coordinate process on $\mathbf{D}$ converge to 0 in probability under $\mathbf{P}_n$, *uniformly in n*. Then the sequence $(\mathbf{P}_n)$ is tight, and its limit laws are laws of semimartingales.

CLOSURE PROPERTIES: FINITE VARIATION PROCESSES

We are going now to investigate some stability properties, which are necessary to apply weak convergence in concrete cases.

The first trivial case is that of processes of finite variation. For simplicity, we deal only with processes of *integrable* variation, though the localization procedure in theorem 7 will extend this to processes whose total variation remains bounded in probability. The case of processes with integrable variation can be reduced in a pedantic way to that of quasimartingales if one makes the remark (Meyer-Yan, Sem. Prob. IX, p. 466, Lecture Notes in M. 465) that one may find an increasing, even continuous and bounded, process A such that knowing the r. v. A_t for any $t > 0$ gives all the information in $\mathscr{F}_\infty$, and therefore X is a process of integrable variation if and only if and only if the pair (X, A) is a quasimartingale. Here is a less sophisticated proof: what is really involved is the following easy result (essentially the same as remark 4 after theorem 4). We denote by c a finite number.

LEMMA 8. — *The function* $w \mapsto \int_0^\infty |dX_s(w)| = J(w)$ (including the mass $|X_0(w)|$ at 0) *is lower semi-continuous on* $\mathbf{D}$, *and the sets* $\{ J \leqq c \}$ *are compact.*

Proof. — Lower semi-continuity means that $\{ J \leqq c \}$ is *closed*, so it suffices to prove the last statement. Now if (w_n) is a sequence such that $J(w_n) \leqq c$, it contains a subsequence which converges to a path $w \in \mathbf{D}$ with $J(w) \leqq c$ outside a countable set, and hence in measure.

Now it is clear that the condition $\mathbf{E}[J] \leqq K$ is preserved under weak convergence, so we have from lemma 8 and Prohorov's theorem:

COROLLARY 9. — *Any sequence* (X_t^n) *of processes* (on variable spaces Ω^n) *of integrable variation, with integral* $\mathbf{E}^n\left[\int |dX_s^n|\right] \leq K$, *contains a subsequence which converges in law on* $\mathbf{D}$, *and the limit law satisfies*

$$\mathbf{E}\left[\int |dX_t|\right] \leqslant K.$$

A much more interesting problem is to study the stability of *absolute continuity* under convergence in law. We give only one result in this direction, which is useful in many applications. It is clear that the exponent 2 plays no particular role.

THEOREM 10. — *In the preceding result, assume that* $X_t^n = \int_0^t H_s^n ds$, *with* $E^n\left[\int_0^\infty |H_s^n|^2 ds\right] \leq K$ (independent of n). *Then under the limit law* **P** *the canonical process* (X_t) *is a. s. absolutely continuous, and its density process* (H_t) *satisfies* $E\left[\int_0^\infty |H_s|^2 ds\right] \leq K$.

It is more natural for such a statement to work on $[0, t]$ for t finite rather than on $[0, \infty]$, since on $[0, t]$ the condition of corollary 10 follows from Schwarz's inequality. But we'll prove th. 11 as stated.

Proof. — We already know that X has integrable variation. To check it is absolutely continuous with density H in $L^2(dP \times dt)$ we must check that

$$\left|E\left[\int_0^\infty f_s dX_s\right]\right| \leq KE\left[\int_0^\infty |f_s|^2 ds\right]^{1/2} \tag{11}$$

for any measurable process $f(s, w)$ on **D**. Now it is sufficient to check it when

$$f(s, w) = \Sigma_i f_i(w) I_{[t_i, t_{i+1}[}(s)$$

(t_i) being a finite subdivision of $[0, \infty[$, with its points taken in some fixed countable dense I, and the functions f_i being continuous on **D** (no filtration appears here) bounded by 1 in absolute value. We may take for I some countable set given by theorem 5. Then we have an inequality like (11) on Ω^n for X^n, which carries over to the image law P_n on **D**, and finally passes to the limit.

CLOSURE PROPERTIES: MARTINGALES AND CONTINUOUS LOCAL MARTINGALES

The following result is almost obvious:

THEOREM 11. — *Let* (X_t^n) *a sequence of martingales, uniformly bounded in* L^1 (on variable probability spaces Ω^n), *whose laws* $\mathbf{P}_n$ *converge weakly to* **P** *in the space* **D**. *Assume that for every t the r. v.* X_t^n *are uniformly integrable w. r. to the* (variable) *laws* $\mathbf{P}^n$. *Then* X *is a* L^1*-bounded martingale under* **P**.

Proof. — First of all, for martingales the quasimartingale variation is just the L^1 norm. So theorem 5 applies, and extracting a subsequence if necessary we may find a countable dense set I such that $(X_t^n)_{t\in I}$ converges in law to $(X_t)_{t\in I}$. The martingale property for X^n

$$E^n[f_1 \circ X_{s_1}^n \dots f_j \circ X_{s_j}^n X_s] = E^n[f_1 \circ X_{s_1}^n \dots f_j \circ X_{s_j}^n X_t]$$

for $s_1, \dots, s_j, s, t \in I$, $s_1 \leqq s_2 \dots \leqq s_j \leqq s \leqq t$, $f_1, \dots, f_j$ bounded continuous on $\mathbf{R}$, will pass to the limit as $n \to \infty$, because of the uniform integrability we have assumed:

$$E^n[\,|X_t^n|\, I_{\{|X_t^n|>c\}}] \to 0 \quad \text{uniformly in } n \quad \text{as } c \to \infty$$

Therefore $(X_t)_{t\in I}$ is a martingale, and it is well known that the property can be then extended to $\mathbf{R}_+$.

Of course, the uniform integrability mentioned in the preceding statement will usually follow from some boundedness property in L^p, $p > 1$. In contrast to such an essentially trivial result, the theory of weak convergence for *local* martingales seems to be very delicate. So we restrict ourselves to the case of *continuous* local martingales.

The idea which underlies the following theorem is the main idea of Rebolledo, Métivier, Joffe...: to get continuity results on a martingale defined as a weak limit, one tries to get continuity results on its associated increasing process (and these in turn will follow from *absolute* continuity estimates as in theorem 10).

We denote by (X_t^n) a sequence of *continuous local martingales* [1] with $X_0^n = 0$, and by (A_t^n) the corresponding increasing processes. As usual, each pair (X^n, A^n) is defined on its own space and filtration. We assume that *the r. v. 's* A_∞^n *are uniformly bounded in probability.* Then setting

$$T^n = \inf\{\, t : A_t^n > K \,\}$$

we have $P^n\{\,T^n < \infty\,\} < \varepsilon$ for K large, uniformly in n. On the other hand, the processes X^n stopped at T^n are quasimartingales with uniformly bounded variation (since they are martingales uniformly bounded in L^2...) and theorem 7 applies to X^n, theorem 9 (localized) to A^n, implying together the existence of a subsequence of (X^n, A^n) which converges weakly on $\mathbf{D} \times \mathbf{D}$. For the coherence of notation, we denote by (X_t, A_t) the coordinate mappings in this case. Here is a reasonably satisfactory result.

THEOREM 12. — *Let* (X^n, A^n) *converge weakly to* (X, A) *as described above. Assume that* $A_0 = 0$ *and* A *is continuous. Then* X *is a continuous local martingale* [4], *and* $A = \langle X, X \rangle$ *a. s.*

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[4] On the interval $[0, \infty]$.

To prove this we need a lemma due to Yor (see Sém. Prob. X, Lecture Notes in M. 511, p. 497). The version we give is due to Stricker, and is more convenient for our purposes. The simplified proof (no longer using Lévy measures) is due to him too.

LEMMA. — *Let* X *be càdlàg adapted with* $X_0 = 0$, A *be a continuous increasing process such that* $A_0 = 0$. *Assume that for* $\lambda \in \mathbf{R}$ *the process*

$$Z_t^\lambda = \exp\left(\lambda X_t - \frac{\lambda^2}{2} A_t\right)$$

is a local martingale (resp. a supermartingale). Then X *is a continuous local martingale, and* $A = \langle X, X \rangle$ *(resp.* $A - \langle X, X \rangle$ *is increasing).*

Proof of the lemma. — Set $V_t = A_t + t$, which is continuous strictly increasing, and tends to infinity with t, and introduce the change of time $\tau_t = \inf\{ s : V_s > t \}$. Then we have for $s < t$

$$A_{\tau_t} - A_{\tau_s} = (t - s) - (\tau_t - \tau_s) \leqq t - s \tag{12}$$

The supermartingale property of Z_t^λ implies $\mathbf{E}[Z_t^\lambda / Z_s^\lambda] \leqq 1$. Writing this explicitly and using (12) we deduce that

$$\mathbf{E}\,[\exp(\lambda(X_{\tau_t} - X_{\tau_s}))] \leqq \exp\left(\frac{\lambda^2}{2}(t - s)\right)$$

Since the result is also valid for $-\lambda$, we have

$$\mathbf{E}\,[\exp(\lambda\,|\,X_{\tau_t} - X_{\tau_s}\,|)] \leqq 2\exp\left(\frac{\lambda^2}{2}(t - s)\right)$$

from which a Kolmogorov lemma type argument deduces that X_{τ_t} is continuous in t. Inverting the change of time we get the continuity of X itself.

To prove that X is a local martingale and that $A = \langle X, X \rangle$, we may now reduce by stopping to the case where X and A are *bounded*. The Ito formula tells us that

$$Z_t^\lambda - 1 = \lambda \int_0^t Z_s^\lambda dX_s - \frac{\lambda^2}{2} \int_0^t Z_s^\lambda \quad d(A_s - \langle X, X \rangle_s)$$

This is a supermartingale, and remains so after dividing by λ if $\lambda > 0$. Letting $\lambda \to 0$ (which is easy to justify since X and A are bounded) we find that X is a supermartingale. Similarly, taking $\lambda < 0$ and dividing by $-\lambda$ we find that $-$ X is a supermartingale, hence X is a *martingale.* Then if Z^λ is a local martingale (supermartingale) the integral on the right must be 0 (decreasing), whence the conclusion on $A - \langle X, X \rangle$.

Proof of theorem 12. — For every n, the processes $k \wedge \exp\left(\lambda X_t^n - \frac{\lambda^2}{2} A_t^n\right)$ are bounded positive supermartingales. A passage to the limit like in theorem 12 will show that $k \wedge \exp\left(\lambda X_t - \frac{\lambda^2}{2} A_t\right)$ is a supermartingale under $\mathbf{P}$, of expectation $\leqq 1$. Letting $k \to \infty$ and applying the lemma to $X - X_0$ and A, we see that X is a continuous local martingale, and $A - \langle X - X_0, X - X_0 \rangle$ is increasing. From $A_0 = 0$ we deduce that $X_0 = 0$ a. s. We know that for every real λ we have $\mathbf{E}[\exp(\lambda X_0)] \leqq 1$. It first follows that the law of X_0 has exponential moments of all orders, which allows us to expand

$$\mathbf{E}[e^{\lambda X_0} - 1] = \lambda \mathbf{E}[X_0] + \frac{\lambda^2}{2}\mathbf{E}[X_0^2] + o(\lambda^2)$$

and to deduce that $\mathbf{E}[X_0] = 0$, $\mathbf{E}[X_0^2] = 0$, hence $X_0 = 0$ a. s. •

It remains to show that $A = \langle X, X \rangle$. We return to the constant K and stopping time T^n before the statement of th. 12 and denote by B^n, Y^n the processes A^n, X^n stopped at T^n. Extracting a subsequence if necessary, we may assume that (X^n, A^n, Y^n, B^n) converge weakly to processes (X, A, Y, B), and $\mathbf{P}\{(X - Y)^* \neq 0\}$, $\mathbf{P}\{(A - B)^* \neq 0\}$ is small as in the proof of theorem 7, if K has been chosen large enough. So it is sufficient to prove that $B = \langle Y, Y \rangle$ a. s. On the other hand, the processes B^n are uniformly bounded by K, the processes Y^n are bounded in every L^p, and therefore the relation $\mathbf{E}[B_t^n] = \mathbf{E}[Y_t^{n2}]$ will pass to the limit for t in a set of full measure, implying that $\mathbf{E}[B_t] = \mathbf{E}[\langle Y, Y \rangle_t]$. Since $B \geqq \langle Y, Y \rangle$, the equality follows.

An example. — The following trivial example will show that the continuity of A and the condition $A_0 = 0$ are essential, and at which place. Take for X^n a gaussian centered martingale with increasing process $A_t^n = 0$ for $t \leqq a$, $A_t^n = 1$ for $t \geqq a + 1/n$, and linear in between. Then $A_t = 1_{\{t \geqq a\}}$, $X_t = g 1_{\{t \geqq a\}}$,where g is a r. v. with distribution N(0, 1). So neither A nor X is continuous (if $a = 0$, the property that $X_0 = 0$ a. s. gets lost). On the other hand it is true that $\exp\left(\lambda X_t - \frac{\lambda^2}{2} A_t\right)$ is a martingale for every λ, so the failure comes from Yor's lemma itself.

APPENDIX

We give here the nice proof, by B. V. Rao, that **D** isn't a Polish space. Since it is known that Ψ is a Polish space, and that a subspace of a Polish space is Polish if and only if it is a $\mathscr{G}_\delta$ (Bourbaki, Top. Gén. IX, § 6, n° 1, th. 1) it is sufficient to show that **D** isn't a $\mathscr{G}_\delta$. Now **D** is dense in Ψ and, according to Baire's theorem, the intersection of two dense $\mathscr{G}_\delta$ sets is dense, and hence non empty. So to prove that **D** isn't a $\mathscr{G}_\delta$ it is sufficient to find a *dense* $\mathscr{G}_\delta$ which is disjoint from **D**.

Let u, v be two rationals such that $u < v$, and let A be the set $\{ N^{uv} = +\infty \}$ in Ψ, which is disjoint from **D**, and is a $\mathscr{G}_\delta$ ($N^{uv}(.)$ is l. s. c.). To see that we may approximate any path w by elements w_n of A, we simply define w_n to be equal to w on $[1/n, \infty[$, and to have an infinite number of upcrossings of $[u, v]$ on the interval $[0, 1/n[$. It is clear that w_n then tends to w in measure.

REFERENCES

[*1*] J. R. Baxter, R. V. Chacon, Compactness of stopping times. *ZW*, t. **40**, 1977, p. 169-181.

[*2*] N. Bourbaki, *Intégration*, Chap. IX, A. S. I. 1343, Hermann, Paris, 1969.

[*3*] C. Dellacherie, *Convergence en probabilité et topologie de Baxter-Chacon*. Sém. Prob. XII, LN **649**, p. 424, Springer, 1978.

[*4*] C. Dellacherie, P. A. Meyer, *Probabilités et Potentiel*, volumes A et B. Hermann, Paris, North Holland, Amsterdam.

[*5*] B. Maisonneuve, Ensembles régénératifs. *Astérisque*, t. **15**, 1974, p. 31.

[*6*] C. Stricker, *Semimartingales et valeur absolue*. Sém. Prob. XIII, LN **721**, p. 473, Springer, 1979.

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