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Stochastic analysis and local times for (N, d)-Wiener process

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ABSTRACT. — Let N, $d \in \mathbb{N}$, $k \in \mathbb{N}_0^d$ be such that $2|k| + d < 2\mathrm{N}$. It is well-known that in this case the (N, d)-Wiener process W has local times possessing (jointly in (t, x), the time resp. space variables being t resp. x) continuous derivatives in x of order k. In the framework of an appropriated stochastic calculus for (N, d)-Wiener process which generalizes Wong's and Zakai's calculus for the Wiener sheet, we derive Tanaka-like formulas for versions $\mathrm{L}^{(k)}$ of these derivatives. Using a method provided by the underlying calculus, we prove (t, x)-continuity for $\mathrm{L}^{(k)}$: with the help of Burkholder's inequalities for the stochastic integral processes occuring in Tanaka's formula we establish Kolmogorov's continuity criterion. More generally, for $\emptyset \neq \mathrm{V} \subset \{1, \ldots, \mathrm{N}\}$, $d \in \mathbb{N}$, $k \in \mathbb{N}_0^d$ such that $2|k| + d < 2|\mathrm{V}|$, the local time of (N, d)-Wiener process can be obtained by integrating the local times of the $(|\mathrm{V}|, d)$-processes $\mathrm{W}_{(.,t_{\overline{\mathrm{V}}})}$ over $t_{\overline{\mathrm{V}}}$. Using this observation, we get Tanaka-like formulas for the joint (t, x)-derivatives of the local time of W (k^{th} partial in x, w. r. to t_i, $i \in \overline{\mathrm{V}}$, in t) for which the above mentioned method yields continuity results, too.

RÉSUMÉ. — Soient N, $d \in \mathbb{N}$, $k \in \mathbb{N}_0^d$ tels que $2|k| + d < 2\mathrm{N}$. Il est bien connu qu'en ce cas le (N, d)-processus de Wiener possède un temps local ayant des dérivées en x d'ordre k continues (en (t, x), t étant la variable du temps, x celle de l'espace). Dans le cadre d'un calcul stochastique approprié pour le (N, d)-processus de Wiener qui généralise le calcul de Wong et Zakai pour le drap Brownien, on obtient des formules à la Tanaka pour

des versions $L^{(k)}$ de ces dérivées. Utilisant une méthode provenant de ce calcul stochastique même, on démontre la continuité en (t, x) pour $L^{(k)}$: à l'aide des inégalités de Burkholder pour les processus intégraux stochastiques figurant dans la formule de Tanaka on vérifie le critère de Kolmogorov pour continuité. Plus généralement, étant donné $\emptyset \neq V \subset \{1, \ldots, N\}$, $d \in \mathbb{N}$, $k \in \mathbb{N}_0^d$ tels que $2|k| + d < 2|V|$, le temps local du (N, d)-processus de Wiener s'obtient en intégrant les temps locaux des $(|V|, d)$-processus $W_{(.,t_{\overline{V}})}$ en $t_{\overline{V}}$. On utilise cette observation pour obtenir des formules à la Tanaka pour les dérivées en (t, x) du temps local de W (la k^{ieme} en x, par rapport à t_i, $i \in \overline{V}$, en t) pour lesquelles la méthode déjà mentionnée fournit des résultats de continuité aussi.

INTRODUCTION

It is well-known that N-parameter Wiener process with values in $\mathbb{R}^d$ for $d < 2N$ has local times whose k^{th} partial derivatives in the space variables exist and are jointly continuous (in space and time) up to $k \in \mathbb{N}_0^d$ such that $2|k| + d < 2N$. Roughly speaking, local times become smoother if N increases. The opposite is true if d increases. To prove this result, Ehm [*8*] has generalized Berman's [*3*] method of Fourier-analyzing occupation times (in fact, Ehm considered a large class of « Lévy-processes »): take the Fourier-transform of occupation time and study its integrability and differentiability properties. This yields very sharp results on moduli of continuity of local time (for this method, see also Tran [*17*] and Adler [*1*]).

But local times are quite generally accessible from stochastic analysis, too. This fact is well-known from one-parameter semimartingale theory (see Meyer [*14*], p. 361-371, Azema, Yor [*2*], Bichteler [*5*]). There are a few results for multi-parameter processes, too: Cairoli, Walsh [*6*] give a representation by a Tanaka-like formula for a local time of the Wiener sheet; Walsh [*19*] investigates smoothness properties of a local time of the Wiener sheet by means of Tanaka's formula. Local times for N-parameter « semimartingales » have been studied in [*11*]. See Geman, Horowitz [*9*] for a survey on local times.

This paper's aim is two-fold: firstly, to describe the local time of (N, d)-Wiener process with $d < 2N$ and its (space-and-time) partial derivatives by Tanaka-like formulae in the framework of an appropriate stochastic calculus; secondly, to prove the smoothness properties of these functions by means of the underlying stochastic calculus. Hereby, no attempt is made to cope the sharpness of the Fourier-analytic method's results.

An « appropriate stochastic calculus » has been presented in a previous paper (see Imkeller [*10*]). As direct generalizations of Wong, Zakai's [*20*], the stochastic integrals necessary for a complete calculus are constructed in the following way: for each partition $\mathscr{C}$ of $\{1, \ldots, N\}$ we take a function $\phi : \mathscr{C} \to \{0, 1, \ldots, d\}$ to note whether in T-direction the measure with respect to which we integrate is Lebesgue measure ($\phi(T) = 0$, i. e. « $T \in \mathscr{C}^0$ ») or is the stochastic measure associated with W^j ($\phi(T)=j$, i. e. « $T \in \mathscr{C}^j$ »), $1 \leqq j \leqq d$, $T \in \mathscr{C}$. We obtain a set of integrals $I^{(\mathscr{C},\phi)}$, such that for $f \in C^{2N}(\mathbb{R}^d)$, with $D^{(\mathscr{C},\phi)} f(W)$ square integrable w. r. t. $P \times \lambda^N$ for all $(\mathscr{C}, \phi)$, we have the (« Ito's ») formula

$$f(W_t) - f(0) = \sum_{(\mathscr{C},\phi)} \frac{1}{2^{|\mathscr{C}^0|}} I^{(\mathscr{C},\phi)}([1_{\Omega \times]0,t]} D^{(\mathscr{C},\phi)} f(W)]^{\mathscr{C}}), \quad t \in [0, 1]^N .$$

Here $D^{(\mathscr{C},\phi)}$ is a differential operator obtained by applying $|\mathscr{C}^0|$ times the Laplacian $\mathbb{D}$ and $|\mathscr{C}^j|$ times partial differentiation in direction j, $1 \leqq j \leqq d$; for any process Y, $Y^{\mathscr{C}}$ is the « $\mathscr{C}$-corner function » of Y: $(s^T)_{T \in \mathscr{C}} \to Y(\sup_{T \in \mathscr{C}} s^T)$. To derive a Tanaka-like formula, we take the term of highest differentiation order

$$\frac{1}{2^N} \int_{]0,t]} \mathbb{D}^N f(W_u) \prod_{1 \leq i \leq N} u_i^{N-1} du$$

in Ito's formula for formally describing a local time of W over $]0, t]$ at $x \in \mathbb{R}^d$ by

$$\int_{]0,t]} \delta_{W_u - x} \prod_{1 \leq i \leq N} u_i^{N-1} du ,$$

δ_y being Dirac's δ-distribution at $y \in \mathbb{R}^d$, which is « natural » for our calculus. Therefore, a representation of local time is obtained by generalizing Ito's formula to the solutions $F^{N,d}(x,.)$ of the partial differential equations $\mathbb{D}^N F^{N,d}(x,.) = \delta_{.-x}$, $x \in \mathbb{R}^d$. This, however, requires allowing the integrals $I^{(\mathscr{C},\phi)}$ to be distribution-valued (for N = 1, see Ustunel [*18*]). It turns out that there is, yet, another possibility which requires starting with a modification of Ito's formula (but keeps the values of the representing integrals in $\mathbb{R}$): by « partial stochastic integration » like in the classical Gauss' integral theorem we replace integrals over intervals by integrals

$I^{(\mathscr{T},\phi,t_{\bar{U}})}$ of the processes $W_{(.,t_{\bar{U}})}$, i. e. integrals over « affine submanifolds » of $[0,1]^N$, $U \subset \{1,\ldots,N\}$, $(\mathscr{T},\phi)$ being related to $|U|$-parameter space. This procedure essentially reduces the orders of occuring differential operators to at most N; the sum in the resulting formula extends over $(\mathscr{T},\phi)\in\Lambda$, i. e. each $T\in\mathscr{T}^0$ has at least two elements:

$$f(W_t^v),-f(0)=\sum_{(\mathscr{T},\phi)\in\Lambda,\ \bigcup_{T\in\mathscr{T}}T=U}\frac{1}{2^{|\mathscr{T}^0|}}\alpha_{(\mathscr{T},\phi)}I^{(\mathscr{T},\phi,t_{\bar{U}})}([1_{\Omega\times]0,t_U]}D^{(\mathscr{T},\phi)}f(W_{(.,t_{\bar{U}})})]^{\mathscr{T}})$$
$$+\frac{1}{2^N}\int_{]0,t]}\mathbb{D}^N f(W_u)\prod_{1\leq i\leq N}u_i^{N-1}du,\qquad t\in[0,1]^N,$$

with suitable constants $\alpha_{(\mathscr{T},\phi)}$ (theorem 4 of [*10*]).

By showing that the corresponding integrals for $D^{(\mathscr{T},\phi)}F^{N,d}(x,W)$ exist, we prove that this formula makes sense for $F^{N,d}$. Indeed, it even makes sense for $D^{(k)}F^{N,d}$, if $k\in\mathbb{N}_0^d$ is such that $2|k|+d<2N$, a fact which leads us directly to a Tanaka-like formula for the k^{th} partial derivative of local time:

$$M^{(k)}(.,t,x)=2^N[D^{(k)}F^{N,d}(x,W_t)-D^{(k)}F^{N,d}(x,0)$$
$$-\sum_{(\mathscr{T},\phi)\in\Lambda,\ \bigcup_{T\in\mathscr{T}}T=U}\frac{1}{2^{|\mathscr{T}^0|}}\alpha_{(\mathscr{T},\phi)}I^{(\mathscr{T},\phi,t_{\bar{U}})}([1_{\Omega\times]0,t_U]}D^{(k)}D^{(\mathscr{T},\phi)}F^{N,d}x,W(.,t_{\bar{U}}))]^{\mathscr{T}})].$$

In theorem 1 we show that $M^{(0)}$ is in fact a good candidate for local time, whereas $M^{(k)}$ is the k^{th} distributional derivative of $M^{(0)}$ in the space variables. The remainder of this paper is devoted to establishing the smoothness of $M^{(k)}$ in space and time by means of the stochastic calculus presented in [*10*]. To do this, a method proposed by Walsh [*19*] is employed. In order to establish Kolmogorov's criterion for continuity of $M^{(k)}$ in space and time, the moments of each one of the terms figuring in Tanaka's formula are estimated with the help of Burkholder's martingale inequalities for the « martingales » $I^{(\mathscr{T},\phi,t_{\bar{U}})}$. The latter are developed in proposition 1, generalizing Metraux's [*13*] inequalities for discrete martingales and using ideas of Cairoli, Walsh [*7*] for the continuous parameter case. Thus, in theorem 2 we obtain functions $L^{(k)}$ such that $L^{(k)}(.,s,t,.)$ is a version of the usual k^{th} partial derivative of a local time of W over the interval $]s,t]$ which is jointly continuous in (s,t,x) as long as s is not on $\partial\mathbb{R}_+^N$. Of course, $L^{(k)}(.,0,.,.)$ is a version of $M^{(k)}$. If $\emptyset\neq V\subset\{1,\ldots,N\}$, $d\in\mathbb{N}$ is such that $d < 2|V|$, the local times of the $(|V|, d)$-processes $W_{(.,t_{\overline{V}})}$ can be integrated over $t_{\overline{V}}$ such as to give a local time of W. This observation is used to treat the joint differentiability in (t, x) of local time. It first yields another Tanaka-like formula, defining, in a similar manner as above, functions $M^{(k,\overline{V})}$ ($k \in \mathbb{N}_0^d$ with $2|k| + d < 2|V|$), which turn out to be good candidates for joint distributional derivatives of local time: $D^{(k)}$ in space and w. r. to t_i, $i \in \overline{V}$, in time. Finally, the methods indicated above yield corresponding continuity results for $M^{(k,\overline{V})}$ (theorem 3).

0. NOTATIONS, PRELIMINARIES AND DEFINITIONS

This article is based upon an application of the main theorem of the stochastic calculus developed in *[10]* to local times. Consequently, it largely depends not only on the results proved there. It is convenient to take the same notation, too. Therefore, the reader is referred to *[10]* for general notations concerning processes, filtrations, parameter space, etc. as well as for special notations necessary for a neater treatment of the technical aspects of the stochastic calculus used here. By W we always denote Wiener process with parameter space $\mathbb{I} = [0, 1]^N$, taking its values in $\mathbb{R}^d$, N, $d \in \mathbb{N}$ (occasionally, W is called (N, d)-Wiener process). The symbol $\hat{\mathbb{I}}_0^2$ is used for the set of all pairs $(s, t) \in \mathbb{I}^2$ with $0 < s \leqq t$.

The following concept of occupation time is natural for the representation of local time of W by means of a Tanaka-like formula: for $J \in \mathscr{I}$, a function $\nu(., J, .): \Omega \times \mathscr{B}(\mathbb{R}^d) \to \mathbb{R}$ is called « *occupation time of* W *over* » J, if

$$\nu(\omega, J, B) = \int_J 1_B(W_u) \prod_{1 \leqq i \leqq N} u_i^{N-1} du, \quad \omega \in \Omega, \quad B \in \mathscr{B}(\mathbb{R}^d).$$

(In terms of *[10]*, $\nu(\omega, J, B)$ measures the « $\mu^{(\mathscr{S},\psi)}$-amount of time » spent by $W(\omega,.)$ in B during the time interval J, where $\mathscr{S} = \{\{i\}: 1 \lneqq i \leqq N\}$, $\psi = 0$; cf. corollary 1 of theorem 3 in *[10]*).
A function $L(., J,.) \in \mathscr{M}(\mathscr{F} \times \mathscr{B}(\mathbb{R}^d), \mathscr{B}(\mathbb{R}))$ is called « *local time of* W *over* » J, if for P-a. e. $\omega \in \Omega$

$$\int_B L(\omega, J, x) dx = \nu(\omega, J, B), \quad B \in \mathscr{B}(\mathbb{R}^d). \tag{0.1}$$

Finally, a function $L \in \mathscr{M}(\mathscr{F} \times \mathscr{B}(\mathbb{I}) \times \mathscr{B}(\mathbb{R}^d), \mathscr{B}(\mathbb{R}))$ is called « *local time of* W », if for P-a. e. $\omega \in \Omega$

$$\int_B L(\omega, t, x) dx = \nu(\omega, R_t, B), \quad t \in \mathbb{I}, \quad B \in \mathscr{B}(\mathbb{R}^d). \tag{0.2}$$

1. TANAKA'S FORMULA FOR W

We now show how to generalize Ito's formula (theorem 4 of (*10*)) in order to obtain a representation of local time of W by stochastic integrals (Tanaka's formula). By formally differentiating occupation time over J, we conclude that local time over J at $x \in \mathbb{R}^d$ should be given by the « integral » $\int_J \delta_{W_u - x} \prod_{1 \leq i \leq N} u_i^{N-1} du$, where δ_y is Dirac's δ-distribution at $y \in \mathbb{R}^d$. Let us briefly recall Ito's formula (theorem 4 of [*10*]). For $f \in C^{2N}(\mathbb{R}^d)$ such that $D^{(\mathscr{E},\phi)} f(W) \in L^2(\Omega \times \mathbb{I}, \mathscr{P}, P \times \lambda^N)$, $(\mathscr{E}, \phi) \in \Psi_N$ and $D^{(\mathscr{E},\phi)} f(W_{(.,t_{\bar{\mathscr{E}}})})^{\mathscr{E}} \in L^{t_{\bar{\mathscr{E}}}}_{(\bar{\mathscr{E}},\phi)}$, $(\mathscr{E}, \phi) \in \Psi$, $t_{\bar{\mathscr{E}}} \in \mathbb{I}_{\bar{\mathscr{E}}}$, we have, putting

$$\alpha_{(\mathscr{E},\phi)} := \prod_{T \in \mathscr{E}^0} (|T|-1)(-1)^{|\underline{\mathscr{E}}|-1} \sum_{0 \leq i \leq |\mathscr{E}|} \sum_{i \leq k \leq |\mathscr{E}|} (-1)^{|\mathscr{E}|-i} \binom{k}{i} i^{|\mathscr{E}|}, \quad (\mathscr{E}, \phi) \in \Lambda,$$

$$(1.1) \quad \Delta_J f(W) = \sum_{(\mathscr{E},\phi) \in \Lambda} \frac{1}{2^{|\mathscr{E}^0|}} \alpha_{(\mathscr{E},\phi)} \Delta_{J_{\bar{\mathscr{E}}}} I^{(\mathscr{E},\phi,.)}([1_{\Omega \times J_{\underline{\mathscr{E}}}} D^{(\mathscr{E},\phi)} f(W_{(.,.)})]^{\mathscr{E}})$$
$$+ \frac{1}{2^N} \int_J \mathbb{D}^N f(W_u) \prod_{1 \leq i \leq N} u_i^{N-1} du, \qquad J \in \mathscr{I}.$$

Moreover, for each product ρ of finite measures ρ_i, $1 \leq i \leq N$, on $\mathscr{B}(\mathbb{I})$, the existence of (in (ω, s, t)) measurable versions of the integrals occuring in (1.1) can be assured, such that (1.1) is valid for ρ^2-a. e. $(s, t) \in \hat{\mathbb{I}}^2$, with $J =]s, t]$. Comparing the last term of (1.1) to the above « integral », we find that local time should be given by an extension of Ito's formula to a family of functions $F^{N,d}(x,.): \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ which satisfies the partial differential equations

$$(1.2) \qquad \mathbb{D}^N F^{N,d}(x,.) = \delta_{x-.}, \qquad x \in \mathbb{R}^d.$$

It is well-known (cf. Schwartz [*16*], p. 44-47) that

$$F^{N,d}: \mathbb{R}^{2d} \to \mathbb{R} \cup \{\infty\}, (x, y) \to$$
$$\begin{cases} \left(\prod_{1 \leq j \leq N-1} 2j \prod_{1 \leq j \leq N} (2j-d)\gamma_d \right)^{-1} |y-x|^{2N-d}, & \text{if } d \text{ is odd}, \\ \left(\prod_{1 \leq j \leq N-1} 2j \prod_{\substack{1 \leq j \leq N \\ 2j \neq d}} (2j-d)\gamma_d \right)^{-1} |y-x|^{2N-d} \log|y-x|, & \text{if } d \text{ is even}, \end{cases}$$

is a solution of (1.2), γ_d being the measure of the surface of the d-dimensional unit sphere ($\gamma_1 := 2$). We will now establish that in case $d < 2N$, an extension of Ito's formula to $F^{N,d}$ exists. But it turns out, that we can do better: if $k \in \mathbb{N}_0^d$ is such that $2|k| + d < 2N$, we can even show that Ito's formula makes sense for $D^{(k)}F^{N,d}$. We thus obtain not only Tanaka's formula for W in case $d < 2N$, but a candidate for the k^{th} partial derivative (in the space variables) of the local time of W, if $2|k| + d < 2N$. Considering the terms of (1.1), our task can be put in the following words: establish that

$$D^{(k)}D^{(\mathscr{C},\phi)}F^{N,d}(x, W_{(.,t_{\overline{\mathscr{C}}})})^{\mathscr{C}} \in L^{t_{\overline{\mathscr{C}}}}_{(\mathscr{C},\phi)}$$

for $(\mathscr{C}, \phi) \in \Lambda$, $t_{\overline{\mathscr{C}}} \in \mathbb{I}_{\overline{\mathscr{C}}}$, $x \in \mathbb{R}^d$, $|k| < [(2N - d)/2]$.

For doing so, we have to estimate the partial derivatives of $F^{N,d}(x,.)$. Use induction on the order $|q|$ of the differential operator and observe that for each $\delta > 0$ the function $u \to u^\delta \log u$ is bounded on $[1, \infty[$ to conclude that for $q \in \mathbb{N}_0^d$, $\delta > 0$ there is a constant $c \in \mathbb{R}$ such that

(1.3) $|D^{(q)}F^{N,d}(x, y)| \leqq c[|x - y|^{2N-d-|q|+\delta} + |x - y|^{2N-d-|q|-\delta}]$.

Consequently,

(1.4) for $(\mathscr{C}, \phi) \in \Lambda$ with order m, $k \in \mathbb{N}_0^d$, $\delta > 0$ there exists $c \in \mathbb{R}$ such that

$$|D^{(k)}D^{(\mathscr{C},\phi)}F^{N,d}(x, y)| \leqq c[|y-x|^{2N-d-m-|k|+\delta}+|y-x|^{2N-d-m-|k|-\delta}].$$

With the help of (1.4) we can prove

LEMMA 1. — Let $d \in \mathbb{N}$, $k \in \mathbb{N}_0^d$ such that $2|k|+d < 2N$, $(\mathscr{C}, \phi) \in \Lambda$ with order m be given. Then $(t_{\overline{\mathscr{C}}}, x) \to \| D^{(k)}D^{(\mathscr{C},\phi)}F^{N,d}(x, W_{(.,t_{\overline{\mathscr{C}}})})^{\mathscr{C}} \|^{t_{\overline{\mathscr{C}}}}_{(\mathscr{C},\phi)}$ is locally bounded on $\mathbb{I}_{\overline{\mathscr{C}}} \times \mathbb{R}^d$.

Proof. — Since $(\mathscr{C}, \phi) \in \Lambda$, we have $m \leqq |\underline{\mathscr{C}}| \leqq N$ and thus

$$2N - d - m - |k| < - d/2 \vee (1/2 - m).$$

Taking (1.4) into account, it is enough to show for $l > - d/2 \vee (1/2 - m)$

$(t_{\overline{\mathscr{C}}}, x) \to \| [|x - W_{(.,t_{\overline{\mathscr{C}}})}|^l]^{\mathscr{C}} \|^{t_{\overline{\mathscr{C}}}}_{(\mathscr{C},\phi)}$ is locally bounded on $\mathbb{I}_{\overline{\mathscr{C}}} \times \mathbb{R}^d$.

In case $l \geqq 0$ this is a simple consequence of the integrability of $|W_{\underline{1}}|^l$. In case $l < 0$ the fact that $x \to E(|\xi - x|^l)$ has its global maximum at $x = 0$ for any Gaussian unit vector ξ and scaling imply

$$\| [|x - W_{(.,t_{\overline{\mathscr{C}}})}|^l]^{\mathscr{C}} \|^{t_{\overline{\mathscr{C}}}}_{(\mathscr{C},\phi)} \leqq \| [|W_{(.,t_{\overline{\mathscr{C}}})}|^l]^{\mathscr{C}} \|^{t_{\overline{\mathscr{C}}}}_{(\mathscr{C},\phi)}$$

$$= \prod_{i \in \overline{\mathscr{C}}} t_i^{l/2+m/2} \| [|W_{(.,\underline{1}_{\overline{\mathscr{C}}})}|^l]^{\mathscr{C}} \|^{\underline{1}_{\overline{\mathscr{C}}}}_{(\mathscr{C},\phi)}.$$

Since $l > 1/2 - m$ we are left with the assertion

$$(1.5) \qquad \| [| W |^l]^{\mathscr{C}} \|_{(\mathscr{C},\phi)} < \infty, \quad \text{if} \quad (\mathscr{C}, \phi) \in \Lambda_N .$$

Let $\beta_{2l} := E(| W_{\underline{1}} |^{2l})$. We have

$$\| [| W |^l]^{\mathscr{C}} \|^2_{(\mathscr{C},\phi)} = E\left(\int_{\mathbb{I}^{\mathscr{C}^1}} \left(\int_{\mathbb{I}^{\mathscr{C}^0}} [| W |^l]^{\mathscr{C}}(., \jmath) d\jmath_{\mathscr{C}^0} \right)^2 d\jmath_{\mathscr{C}^1} \right)$$

$$\leq \int_{\mathbb{I}^{\mathscr{C}^1}} \int_{\mathbb{I}^{\mathscr{C}^0}} \int_{\mathbb{I}^{\mathscr{C}^0}} [E([| W |^{2l}]^{\mathscr{C}}(., \jmath_{\mathscr{C}^1}, u_{\mathscr{C}^0}))]^{1/2} \qquad \text{(Hölder)}$$
$$[E([| W |^{2l}]^{\mathscr{C}}(., \jmath_{\mathscr{C}^1}, v_{\mathscr{C}^0}))]^{1/2} du_{\mathscr{C}^0} dv_{\mathscr{C}^0} d\jmath_{\mathscr{C}^1}$$

$$= \beta_{2l} \int_{\mathbb{I}^{\mathscr{C}^1}} \int_{\mathbb{I}^{\mathscr{C}^0}} \int_{\mathbb{I}^{\mathscr{C}^0}} 1_{\mathbb{I}_{\mathscr{C}}}(\jmath_{\mathscr{C}^1}, u_{\mathscr{C}^0}) 1_{\mathbb{I}_{\mathscr{C}}}(\jmath_{\mathscr{C}^1}, v_{\mathscr{C}^0}) \quad \text{(definition of } [\cdot]^{\mathscr{C}})$$
$$\prod_{T \in \mathscr{C}^1} \prod_{i \in T} (s_i^T)^l \prod_{T \in \mathscr{C}^0} \prod_{i \in T} (u_i^T v_i^T)^{1/2} du_{\mathscr{C}^0} dv_{\mathscr{C}^0} d\jmath_{\mathscr{C}^1}$$

$$= \beta_{2l} c_1 \left(\int_0^1 r^{m+l-1} dr \right)^N$$

with a suitable constant $c_1 \in \mathbb{R}$. Since $l > -d/2$, β_{2l} is finite and since $l > 1/2 - m$, $r \to r^{m+l-1}$ is integrable over $[0, 1]$. This gives (1.5). □

Remark. — The assertion of lemma 1 is not necessarily true for $(\mathscr{C}, \phi) \in \Psi \backslash \Lambda$. This shows that the formula of theorem 3 of *[10]* cannot be generalized in the same way as (1.1) to give a representation of local time.

By what has been said above we are motivated and by lemma 1 we are allowed to give

DEFINITION 1. — Let $d \in \mathbb{N}$, $k \in \mathbb{N}_0^d$ be such that $2 | k | + d < 2N$. For $x \in \mathbb{R}^d$, $(s, t) \in \hat{\mathbb{I}}^2$ let, setting $J =]s, t]$,

$$M^{(k)}(., s, t, x) := 2^N [\Delta_J D^{(k)} F^{N,d}(x, W)$$
$$- \sum_{(\mathscr{C},\phi) \in \Lambda} \frac{1}{2^{|\mathscr{C}^0|}} \alpha_{(\mathscr{C},\phi)} \Delta_{J_{\mathscr{C}}} I^{(\mathscr{C},\phi,.)}([1_{\Omega \times J_{\mathscr{C}}} D^{(k)} D^{(\mathscr{C},\phi)} F^{N,d}(x, W_{(.,.)})]^{\mathscr{C}})] .$$

We will show now that $M^{(k)}$ is, in fact, a good candidate for the k^{th} partial derivative of local time of W. For this purpose, we need a measurable version of $M^{(k)}$ and some knowledge about the exchangeability of « $I^{(\mathscr{C},\phi,t)}$ » and « dx ».

LEMMA 2. — Let $(\mathscr{C}, \phi) \in \Lambda$, ρ on $\mathscr{B}(\mathbb{I})$ be a product of finite measures ρ_i,

$1 \leqq i \leqq \mathrm{N}$. Further, suppose that a function $g: \mathbb{R}^{2d} \to \mathrm{R} \cup \{\infty\}$ satisfies

i) $(t_{\overline{\mathscr{C}}}, x) \to \| g(x, \mathrm{W}_{(.,t_{\overline{\mathscr{C}}})})^{\mathscr{C}} \|_{(\mathscr{C},\phi)}^{t_{\overline{\mathscr{C}}}}$ is locally bounded on $\mathbb{I}_{\overline{\mathscr{C}}} \times \mathbb{R}^d$,

ii) $x \to g(x, y)$ is continuous on $\mathbb{R}^d \backslash \{ y \}$, $y \in \mathbb{R}^d$.

Then there exists $\mathrm{G} \in \mathscr{M}(\mathscr{F} \times \mathscr{B}(\hat{\mathbb{I}}^2) \times \mathscr{B}(\mathbb{R}^d), \mathscr{B}(\mathbb{R}))$ such that

iii) $\mathrm{G}(., s, t, x) = \Delta_{\mathrm{J}_{\overline{\mathscr{C}}}} \mathrm{I}^{(\mathscr{C},\phi,.)}([1_{\Omega \times \mathrm{J}_{\overline{\mathscr{C}}}} g(x, \mathrm{W}_{(..,.)})]^{\mathscr{C}})$ for $\rho^2 \times \lambda^d$ – a. e. $(s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$, putting $\mathrm{J} =]s, t]$,

iv) $\mathrm{G}(\omega, s, t, .)$ is locally square integrable w. r. t. λ^d, for all $(\omega, s, t) \in \Omega \times \hat{\mathbb{I}}^2$,

v) $$\int_{\mathbb{R}^d} \mathrm{G}(., s, t, x)h(x)dx = \Delta_{\mathrm{J}_{\overline{\mathscr{C}}}} \mathrm{I}^{(\mathscr{C},\phi,.)}\left(\left[1_{\Omega \times \mathrm{J}_{\overline{\mathscr{C}}}} \int_{\mathbb{R}^d} h(x)g(x, \mathrm{W}_{(.,.)})dx\right]^{\mathscr{C}}\right)$$

for ρ^2 – a. e. $(s, t) \in \hat{\mathbb{I}}^2$, putting $\mathrm{J} =]s, t]$, all $h \in \mathrm{C}_c^0(\mathbb{R}^d)$.

Proof. — Since ρ is a product measure and *iv)* and *v)* are « local » properties, it is enough to show, that for $m \in \mathbb{Z}^d$ there exists

$$\mathrm{G}_m \in \mathscr{M}(\mathscr{F} \times \mathscr{B}(\mathbb{I}) \times \mathscr{B}(]m - \underline{1}, m]), \mathscr{B}(\mathbb{R}))$$

such that

iii') $\mathrm{G}_m(., t, x) = \mathrm{I}^{(\mathscr{C},\phi,t_{\overline{\mathscr{C}}})}([1_{\Omega \times (\mathrm{R}_t)_{\overline{\mathscr{C}}}} g(x, \mathrm{W}_{(.,t_{\overline{\mathscr{C}}})})]^{\mathscr{C}})$ for $\rho \times \lambda^d$ – *a. e.* $(t, x) \in \mathbb{I} \times]m - \underline{1}, m]$,

iv') $\mathrm{G}_m(\omega, t, .)$ is square integrable w. r. t. λ^d for all $(\omega, t) \in \Omega \times \mathbb{I}$,

v') $$\int_{]m-\underline{1},m]} h(x)\mathrm{G}_m(., t, x)dx = \mathrm{I}^{(\mathscr{C},\phi,t_{\overline{\mathscr{C}}})}([1_{\Omega \times (\mathrm{R}_t)_{\overline{\mathscr{C}}}} \int_{]m-\underline{1},m]} h(x)g(x, \mathrm{W}_{(.,t_{\overline{\mathscr{C}}})})dx]^{\mathscr{C}})$$

for ρ – a. e. $t \in \hat{\mathbb{I}}$, all $h \in \mathrm{C}_b^0(]m - \underline{1}, m])$.

For simplicity, take $m = \underline{1}$. Put $\Lambda =:]\underline{0}, \underline{1}] \times \mathbb{I}$, $\mathscr{G} =: \mathscr{B}(]\underline{0}, \underline{1}]) \times \mathscr{B}(\mathbb{I})$, $\nu := \lambda^d \mid_{\mathscr{B}(]\underline{0},\underline{1}])} \times \rho$ (instead of $]0, 1]$ resp. $\mathscr{B}(]0, 1])$ resp. $\lambda \mid_{\mathscr{B}(]0,1])}$) in the proof of lemma 2 of [*11*]. To make this proof work, we further must replace $\| \cdot \|_q$ by $\| \cdot \|_{(\mathscr{C},\phi)}^{t_{\overline{\mathscr{C}}}}$ and resort to « lemma 5 and its corollary » of [*10*] instead of « lemma 1 » of [*11*]. □

THEOREM 1. — Let $d < 2\mathrm{N}$, ρ on $\mathscr{B}(\mathbb{I})$ be a product of finite measures ρ_i, $1 \leqq i \leqq \mathrm{N}$. Then for each $k \in \mathbb{N}_0^d$ such that $2 | k | + d < 2\mathrm{N}$ there exists $\mathrm{K}^{(k)} \in \mathscr{M}(\mathscr{F} \times \mathscr{B}(\hat{\mathbb{I}}^2) \times \mathscr{B}(\mathbb{R}^d), \mathscr{B}(\mathbb{R})$ which satisfies

i) $\mathrm{K}^{(k)}(., s, t, x) = \mathrm{M}^{(k)}(., s, t, x)$ for $\rho^2 \times \lambda^2$ – a. e. $(s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$,

ii) $\mathrm{K}^{(k)}(\omega, s, t, .)$ is locally square integrable w. r. t. λ^d for all $(\omega, s, t) \in \Omega \times \hat{\mathbb{I}}^2$,

iii) $$\int_{\mathbb{R}^d} h(x)\mathrm{K}^{(k)}(., s, t, x)dx = (-1)^{|k|} \int_{\mathbb{R}^d} \mathrm{D}^{(k)}h(x)\mathrm{K}^{(0)}(., s, t, x)dx$$

for ρ^2 – a. e. $(s, t) \in \hat{\mathbb{I}}^2$, all $h \in \mathrm{C}_c^\infty(\mathbb{R}^d)$.

Moreover, $\mathrm{K}^{(0)}(., s, t, .)$ is a local time of W over $]s, t]$ for ρ^2 – a. e. $(s, t) \in \hat{\mathbb{I}}^2$. W has a local time.

Proof. — Let $k \in \mathbb{N}_0^d$ satisfy $2 \mid k \mid + d < 2N$. For $(\mathscr{C}, \phi) \in \Lambda$ set

$$g^k_{(\mathscr{C},\phi)} := D^{(k)} D^{(\mathscr{C},\phi)} F^{N,d}$$

According to lemma 1, $g^k_{(\mathscr{C},\phi)}$ fulfils *i*) and *ii*) of lemma 2. Therefore we can choose $G^k_{(\mathscr{C},\phi)}$ such that *iii*)-*v*) of lemma 2 are valid for the pair $(g^k_{(\mathscr{C},\phi)}, G^k_{(\mathscr{C},\phi)})$. Define

$$K^{(k)}(.,s,t,x) := 2^N \left[\Delta_{]s,t]} D^{(k)} F^{N,d}(x, W) - \sum_{(\mathscr{C},\phi)\in\Lambda} \frac{1}{2^{|\mathscr{C}^0|}} \alpha_{(\mathscr{C},\phi)} G^k_{(\mathscr{C},\phi)}(.,s,t,x) \right],$$

$(s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$.

Of course, $K^{(k)} \in \mathscr{M}(\mathscr{F} \times \mathscr{B}(\hat{\mathbb{I}}^2) \times \mathscr{B}(\mathbb{R}^d), \mathscr{B}(\mathbb{R}))$. *i*) is a consequence of lemma 2, *iii*) and definition 1; *ii*) follows from lemma 2, *iv*); lemma 2, *v*) and the equality

$$\int_{\mathbb{R}^d} h(x) D^{(k)} D^{(\mathscr{C},\phi)} F^{N,d}(x, W) dx$$
$$= (-1)^{|k|} \int_{\mathbb{R}^d} D^{(k)} h(x) D^{(\mathscr{C},\phi)} F^{N,d}(x, W) dx, \quad h \in C_c^\infty(\mathbb{R}^d), (\mathscr{C}, \phi) \in \Lambda$$

together impli *iii*).

Now fix $t \in \mathrm{II}$ and take $\rho := \bigtimes_{1 \leq i \leq N} (\varepsilon_{\{0\}} + \varepsilon_{\{t_i\}})$, ε_s being the point mass in $s \in \mathbb{I}$.

For $h \in C_c^\infty(\mathbb{R}^d)$ set $f := \int_{\mathbb{R}^d} h(x) F^{N,d}(x,.) dx$. We have $f \in C^\infty(\mathbb{R}^d)$,

$$D^{(q)} f = (-1)^{|q|} \int_{\mathbb{R}^d} D^{(q)} h(x) F^{N,d}(x,.) dx, \quad q \in \mathbb{N}_0^d,$$

and particularly

$$\mathbb{D}^N f = h.$$

By (1.4) and since W possesses moments of all orders, the hypotheses of theorem 4 of [*10*] are fulfilled. Therefore,

$$(1.6) \quad \int_{\mathbb{R}^d} K^{(0)}(.,0,t,x) h(x) dx = 2^N [\Delta_{R_t} f(W)$$
$$- \sum_{(\mathscr{C},\phi)\in\Lambda} \frac{1}{2^{|\mathscr{C}^0|}} \alpha_{(\mathscr{C},\phi)} I^{(\mathscr{C},\phi,t_{\bar{\mathscr{C}}})}([1_{\Omega \times (R_t)_{\bar{\mathscr{C}}}} D^{(\mathscr{C},\phi)} f(W_{(.,t_{\bar{\mathscr{C}}})})]^{\mathscr{C}})]$$
$$= \int_{R_t} \mathbb{D}^N f(W_u) \prod_{1 \leq i \leq N} u_i^{N-1} du = \int_{R_t} h(W_u) \prod_{1 \leq i \leq N} u_i^{N-1} du.$$

It is clear how (1.6) has to be generalized so as to give (0.1) for $J = R_t$.

To obtain (0.2) with a suitable $L \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\mathbb{I}) \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$, first define $L(., t,.)$ by $K^{(0)}(., 0, t,.)$ for rational $t \in \mathrm{II}$. Then use monotonicity in t of the occupation time of W over R_t.

For arbitrary ρ, the argument which proves that the corresponding $K^{(0)}(., s, t,.)$ is a local time over $]s, t]$ for ρ^2 – a. e. $(s, t) \in \hat{\mathbb{I}}^2$, is contained in the one which was given. □

Theorem 1 particularly says that $K^{(k)}(\omega, s, t,.)$ is the k^{th} distributional derivative of a local time of $W(\omega,.)$ over $]s, t]$ for $P \times \lambda^2$ – a. e. $(\omega, s, t) \in \Omega \times \hat{\mathbb{I}}^2$. Our next aim is to improve this statement by means of the stochastic calculus we dispose of: we will show that there exists a version $L^{(k)}$ of $K^{(k)}$ which is continuous in (s, t, x). $L^{(k)}$ proves to be the « classical » k^{th} partial derivative of a local time of W. Hereby, the following technique will be used (cf. Walsh [*19*]): Kolmogorov's well-known continuity criterion for stochastic processes is verified for each term of $M^{(k)}$ separately. This makes it necessary for example to estimate the moments of

$$I^{(\mathscr{C}, \phi, t_{\underline{\mathscr{C}}})}([1_{\Omega \times (R_t)_{\underline{\mathscr{C}}}} D^{(k)} D^{(\mathscr{C}, \phi)} [F^{N,d}(x, W_{(.,t_{\underline{\mathscr{C}}})}) - F^{N,d}(y, W_{(.,t_{\underline{\mathscr{C}}})})]]^{\mathscr{C}}),$$
$$t \in \mathbb{I}, \quad x, y \in \mathbb{R}^d, \quad (\mathscr{C}, \phi) \in \Lambda.$$

Since for $(\mathscr{C}, \phi) \in \Psi$ the integral process in the $\underline{\mathscr{C}}$-variables of $I^{(\mathscr{C}, \phi)}$ can be seen to be a $\underline{\mathscr{C}}^1$-martingale (cf. remark after lemma 4 of [*10*]), this is a job for Burkholder's martingale inequalities for $I^{(\mathscr{C}, \phi)}$.

2. BURKHOLDER'S INEQUALITIES FOR $I^{(\mathscr{C}, \phi)}$

Burkholder's inequalities for martingales with a discrete parameter set are well-known (cf. Metraux [*13*] and Merzbach [*12*], p. 43). They imply

(2.1) for $1 < p < \infty$ there are constants $A_p, B_p > 0$ such that for all martingales M and all partitions $(J^k: \underline{1} \leqq k \leqq r)$ of $\mathbb{I}$ in $\mathcal{I}$

$$A_p E\left(\left[\sum_{\underline{1} \leqq k \leqq r} (\Delta_{J^k} M)^2\right]^{p/2}\right) \leqq E(|M_{\underline{1}}|^p) \leqq B_p E\left(\left[\sum_{\underline{1} \leqq k \leqq r} (\Delta_{J^k} M)^2\right]^{p/2}\right).$$

In particular, (2.1) can be applied to the $\underline{\mathscr{C}}^1$-parameter martingales

$$I^{(\mathscr{C}; \phi)}_{0,(.,\underline{1}_{\underline{\mathscr{C}}^{1}})}(Y_0), \quad Y_0 \in \mathscr{E}_{\mathscr{C}}, \quad (\mathscr{C}, \phi) \in \Psi \qquad \text{(cf. lemma 4 of [10])}$$

to yield inequalities which, however, still depend on the partition chosen. Now suppose that a sequence of partitions of $\mathbb{I}$ in $\mathcal{I}$ is given, whose mesh goes to zero. If we can establish convergence of the corresponding quadratic sums on the left and right sides of (2.1) to a suitable limit, we obtain inequalities depending only on this limit (« quadratic variation »). The following lemma shows that this can be done for $Y_0 \in \mathscr{E}_{\mathscr{T}}$, $(\mathscr{T}, \phi) \in \Psi$. Finally, an appeal to density of $\mathscr{E}_{\mathscr{T}}$ in $L_{(\mathscr{T},\phi)}$ will yield Burkholder's inequalities for all $Y \in L_{(\mathscr{T},\phi)}$.

LEMMA 3. — For $\emptyset \neq U \in \Pi_N$ let $(\mathscr{T}, \phi) \in \Psi_U$, $\mathscr{T}^1 = \mathscr{T}$. Suppose that $Y_0 \in \mathscr{E}_{\mathscr{T}}$ has an $\mathbb{I}_{\mathscr{T}}$-representation $Y_0 = \sum_{1 \leq k^T \leq q} \alpha_{\boldsymbol{k}} \prod_{T \in \mathscr{T}} 1_{K^{k^T}}$, where $K^k =]u^k, v^k]$, $1 \leq k \leq q$. For $n \in \mathbb{N}$ let $(J^{j,n} : 1 \leq j \leq r(n))$ be the partition which is generated by $\{ u^k, v^k : 1 \leq k \leq q \} \cup \left\{ \frac{i}{n} : 0 \leq i \leq n \right\}$, $(J_U^{j_U,n} : 1_U \leq j_U \leq r(n)_U)$ the partition of $\mathbb{I}_U$ defined by the projections of $J^{j,n}$ on $\mathbb{I}_U$, $1 \leq j \leq r(n)$. Then

$$(L^2-) \lim_{n \to \infty} \sum_{1_U \leq j_U \leq r(n)_U} (\Delta_{J_U^{j_U,n}} I_{0,(.,1_{\bar U})}^{(\mathscr{T},\phi)}(Y_0))^2 = \int_{\mathbb{I}^{\mathscr{T}}} Y_0^2(., s)ds.$$

Proof. — Taking $\bar U = \emptyset$, we can avoid some unessential technicalities. Further, omitting n as an index will cause no confusion, as it is kept fix during the following arguments. Note first that by linearity

$$\Delta_{J^j} I_{0,.}^{(\mathscr{T},\phi)}(Y_0) = I_0^{(\mathscr{T},\phi)}(Y_0 [1_{\Omega \times J^j}]^{\mathscr{T}}) = \sum_{1 \leq k^T \leq q} \alpha_{\boldsymbol{k}} \prod_{T \in \mathscr{T}} \Delta_{K^{k^T} \cap (J^j)^T} W^{\phi(T)}.$$

Therefore, putting

$$Z_j^{\mathscr{S}} := \sum_{1 \leq k^T \leq q, T \in \bar{\mathscr{S}}} \left(\sum_{1 \leq k^T \leq q, T \in \mathscr{S}} \alpha_{\boldsymbol{k}} \prod_{T \in \mathscr{S}} \Delta_{K^{k^T} \cap (J^j)^T} W^{\phi(T)} \right)^2 \prod_{T \in \bar{\mathscr{S}}} 1_{K^{k^T} \cap (J^j)^T},$$

$1 \leq j \leq r$, and $Z^{\mathscr{S}} := \sum_{1 \leq j \leq r} Z_j^{\mathscr{S}}$, $\mathscr{S} \subset \mathscr{T}$,

the triangle inequality implies that it is enough to show

(2.2)

$$\left\| \int_{\mathbb{I}^{\bar{\mathscr{S}}}} Z^{\mathscr{S}}(., s)ds - \int_{\mathbb{I}^{\bar{\mathscr{S}} \cup \{S\}}} Z^{\mathscr{S} \setminus \{S\}}(., u)du \right\|_2 \to 0 \ (n \to \infty) \text{ for } S \in \mathscr{S} \subset \mathscr{T}.$$

Let $S \in \mathscr{S} \subset \mathscr{T}$. Since W has independent, centered increments, we have

$$(2.3) \quad E\left(\prod_{k=i,j} \left[\int_{\mathbb{I}^{\bar{\mathscr{S}}}} Z_k^{\mathscr{S}}(., s)ds - \int_{\mathbb{I}^{\bar{\mathscr{S}} \cup \{S\}}} Z_k^{\mathscr{S} \setminus \{S\}}(., u)du \right] \right) = 0, \quad \text{if } i_S \neq j_S.$$

As in addition for $J \in \mathscr{I}$, $1 \leqq i \leqq d$, $(\Delta_J W^i)^4$ has variance $c\lambda^N(J)^2$ with a suitable constant c independent of J, we get

$$(2.4)\quad E\left(\left[\int_{\mathbb{I}^{\mathscr{S}}} Z_j^{\mathscr{S}}(.,\,s)ds - \int_{\mathbb{I}^{\mathscr{S}\cup\{S\}}} Z_j^{\mathscr{S}\setminus\{S\}}(.,\,u)du\right]^2\right)$$

$$\leqq \prod_{T\in\overline{\mathscr{S}}} \lambda^{|T|}(J_T^j) \int_{\mathbb{I}^{\mathscr{S}}} E\left(\left[Z_j^{\mathscr{S}}(.,\,s) - \int_{\mathbb{I}} Z_j^{\mathscr{S}\setminus\{S\}}(.,\,s,\,u^S)du^S\right]^2\right)ds$$

(Jensen's inequality)

$$\leqq c \prod_{T\in\overline{\mathscr{S}}\cup\{S\}} \lambda^{|T|}(J_T^j) \int_{\mathbb{I}^{\mathscr{S}\cup\{S\}}} E([Z_j^{\mathscr{S}\setminus\{S\}}(.,\,s)]^4)ds$$

$$\leqq c \prod_{T\in\mathscr{C}} \lambda^{|T|}(J_T^j) \prod_{1\leqq i\leqq N} q_i^{4|\mathscr{S}\setminus\{S\}|} \int_{\Omega\times\mathbb{I}} [1_{\Omega\times J^j}]^{\mathscr{C}} \, |Y_0|^4 d(P\times\lambda^N).$$

Combining (2.3) and (2.4) yields

$$\left\|\int_{\mathbb{I}^{\mathscr{S}}} Z^{\mathscr{S}}(.,\,s)ds - \int_{\mathbb{I}^{\mathscr{S}\cup\{S\}}} Z^{\mathscr{S}\setminus\{S\}}(.,\,u)du\right\|_2^2$$

$$= \sum_{1_S\leqq j_S\leqq r_S} E\left(\left[\sum_{1_{\overline{S}}\leqq j_{\overline{S}}\leqq r_{\overline{S}}} \int_{\mathbb{I}^{\mathscr{S}}} Z_j^{\mathscr{S}}(.,\,s)ds - \int_{\mathbb{I}^{\mathscr{S}\cup\{S\}}} Z_j^{\mathscr{S}\setminus\{S\}}(.,\,u)du\right]^2\right) \quad ((2.3))$$

$$\leqq \left(\prod_{i\in\overline{S}} r_i\right) c \left(\frac{1}{n}\right)^N \prod_{1\leqq i\leqq N} q_i^{4|\mathscr{S}\setminus\{S\}|} \int_{\Omega\times\mathbb{I}} |Y_0|^4 d(P\times\lambda^N).\ \text{(Cauchy-Schwartz, (2.4))}$$

By choice of $(J^j : 1 \leqq j \leqq r)$, $r_i \leqq n + 1 + q_i$ for $1 \leqq i \leqq N$. This implies (2.2). □

PROPOSITION 1. — For $1 < p < \infty$ there exist real constants $A_p, B_p > 0$, such that for $(\mathscr{C}, \phi) \in \Psi$, $Y \in L_{(\mathscr{C},\phi)}$

$$A_p E\left(\left[\int_{\mathbb{I}^{\mathscr{C}^1}} \left(\int_{\mathbb{I}^{\mathscr{C}^0}} Y(.,\,s)ds_{\mathscr{C}^0}\right)^2 ds_{\mathscr{C}^1}\right]^{p/2}\right) \leqq E(|I^{(\mathscr{C},\phi)}(Y)|^p)$$

$$\leqq B_p E\left(\left[\int_{\mathbb{I}^{\mathscr{C}^1}} \left(\int_{\mathbb{I}^{\mathscr{C}^0}} Y(.,\,s)ds_{\mathscr{C}^0}\right)^2 ds_{\mathscr{C}^1}\right]^{p/2}\right).$$

Proof.— Due to the density of $\mathscr{E}_{\mathscr{C}}$ in $L_{(\mathscr{C},\phi)}$, the asserted inequalities need to be established only for $Y_0 \in \mathscr{E}_{\mathscr{C}}$. Evidently, we can assume $\mathscr{C}^1 = \mathscr{C}$. Using the notations of lemma 3, put

$$V_n(Y_0) := \sum_{1_{\mathscr{C}}\leqq j_{\mathscr{C}}\leqq r(n)_{\mathscr{C}}} (\Delta_{J_{\mathscr{C}}^{j_{\mathscr{C}},n}} I^{(\mathscr{C},\phi)}_{(0,(.,1_{\overline{\mathscr{C}}})}(Y_0))^2, \quad n \in \mathbb{N}.$$

(2.1) implies

$$(2.5)\quad A_p E([V_n(Y_0)]^{p/2}) \leqq E(|I^{(\mathscr{T},\phi)}(Y_0)|^p) \leqq B_p E([V_n(Y_0)]^{p/2}), \quad n \in \mathbb{N}.$$

Moreover, by (2.5), the sequence $(V_n(Y_0))^{p/2}$, $n \in \mathbb{N}$, is uniformly integrable for $p > 1$, and, by lemma 3, it converges at least in probability. Therefore, Vitali's theorem completes the proof. □

Remarks. — 1. Doob's maximal inequalities can be used to sharpen the right inequality of proposition 1.

2. For $1 < p < \infty$, $(\mathscr{T}, \phi) \in \Psi_U$, $Y \in \mathscr{M}(\mathscr{P}^U, \mathscr{B}(\mathbb{R}))$, proposition 1 yields the weaker inequality $E(|I^{(\mathscr{T},\phi)}(Y^{\mathscr{T}})|^p) \leqq B_p E\left(\left[\int_{\mathbb{I}} |Y|^2(., s)ds\right]^{p/2}\right)$.

3. CONTINUITY OF THE LOCAL TIME OF W IN (t, x), DIFFERENTIABILITY IN x

We now come back to the study of smoothness properties of local time and its distributional derivatives $K^{(k)}$. As will be seen, this amounts essentially to the study of the finiteness of the moments of local time. The following « moment lemma » plays a central role.

LEMMA 4. — Let $-dv - 2N < l \in \mathbb{R}, p \in \mathbb{N}, 0 < u^0 \in \mathbb{I}$ be given. Then there exists $c_1 \in \mathbb{R}$ such that for all $J =]s, t] \in \mathscr{I}$, $s \geqq u^0$, $x \in \mathbb{R}^d$

$$E\left(\int_{J^p} \prod_{1 \leqq i \leqq p} |W_{u^i} - x|^l \prod_{1 \leqq i \leqq p} du^i\right) \leqq \begin{cases} c_1 \lambda^N(J)^p(1 + |x|^{lp}), & \text{if } l \geqq 0, \\ c_1 \lambda^N(J)^{p(1 + l/2N)}, & \text{if } l < 0. \end{cases}$$

Proof. — Since $l > -d$, $\beta_l := E(|W_{\underline{1}}|^l)$ is finite. In case $l \geqq 0$, note that Hölder's inequality implies for $x \in \mathbb{R}^d$, $u^i \in \mathbb{I}$, $1 \leqq i \leqq p$

$$E\left(\prod_{1 \leqq i \leqq p} |W_{u^i} - x|^l\right) \leqq \prod_{1 \leqq i \leqq p} [E(|W_{u^i} - x|^{lp})]^{1/p}$$

$$\leqq \prod_{1 \leqq i \leqq p} [2^{lp-1}(E(|W_{u^i}|^{lp} + |x|^{lp}))]^{1/p}.$$

The desired conclusion follows easily. Let $l < 0$. First observe that it is enough to show

(3.1) there exists $c_2 \in \mathbb{R}$ such that for $u^0 \leqq u^i \in \mathbb{I}$, $1 \leqq i \leqq p$, with pairwise different coordinates u_j^i, $1 \leqq j \leqq \mathrm{N}$, $1 \leqq i \leqq p$, and $x \in \mathbb{R}^d$

$$\mathrm{E}\Big(\prod_{1 \leqq i \leqq p} |\mathrm{W}_{u^i} - x|^l\Big) \leqq c_2 \prod_{1 \leqq i \leqq p} \prod_{1 \leqq j \leqq \mathrm{N}} (u_j^i - r_j^i)^{l/2\mathrm{N}},$$

where $r_j^i := \max\{u_j^q : 1 \leqq q \leqq p,\ u_j^q < u_j^i\} \vee u_j^0$, $1 \leqq j \leqq \mathrm{N}$, $1 \leqq i \leqq p$. Indeed, integrating (3.1) over J^p gives the desired conclusion: introduce new variables $v_j^i := u_j^i - r_j^i$, $1 \leqq j \leqq \mathrm{N}$, $1 \leqq i \leqq p$, observe $l > -2\mathrm{N}$ and keep in mind that the set of all $(u^1, \ldots, u^p)$, not all of whose coordinates are pairwise different, is a zero-set w. r. t. $\lambda^{\mathrm{N}p}$.

To prove (3.1), we proceed by induction on p. For $u > u^0$ we first decompose W_u in the following way. Consider the σ-fields

$$\mathscr{G} := \sigma\Big(\Delta_{\mathrm{K}}\mathrm{W} : \mathscr{I} \ni \mathrm{K} \subset \bigcup_{\mathrm{T} \in \Pi_{\mathrm{N}}, |\mathrm{T}| \neq 1}]u^0, \underline{1}]^{\mathrm{T}}\Big),$$

$$\mathscr{G}^j := \sigma(\Delta_{\mathrm{K}}\mathrm{W} : \mathscr{I} \ni \mathrm{K} \subset]u^0, \underline{1}]^{\{j\}}), \quad 1 \leqq j \leqq \mathrm{N}$$

and write

$$\mathrm{W}_u = \mathrm{V}^0(u) + \sum_{1 \leqq j \leqq \mathrm{N}} \mathrm{V}^j(u), \quad \text{putting} \quad \mathrm{V}^j(u) := \Delta_{]u^0, u]^{\{j\}}}\mathrm{W}, \quad 1 \leqq j \leqq \mathrm{N}.$$

Then

(3.2) $\mathrm{V}^0(u) \in \mathscr{M}(\mathscr{G}, \mathscr{B}(\mathbb{R}^d))$, $\mathrm{V}^j(u) \in \mathscr{M}(\mathscr{G}^j, \mathscr{B}(\mathbb{R}^d))$, $(\mathscr{G}, \mathscr{G}^1, \ldots, \mathscr{G}^{\mathrm{N}})$ is independent.

Now let $p = 1$. For $1 \leqq j \leqq \mathrm{N}$ set $a_j := \Big[\prod_{1 \leqq q \leqq \mathrm{N}, q \neq j} u_q^0 (u_j^1 - u_j^0)\Big]^{1/2}$. Infer from (3.2) that $\sum_{1 \leqq j \leqq \mathrm{N}} \mathrm{V}^j(u^1)$ is centered Gaussian with variance $\sum_{1 \leqq j \leqq \mathrm{N}} a_j^2$.

Consequently,

$$\mathrm{E}(|\mathrm{W}_{u^1} - x|^l) \leqq \mathrm{E}\Big(\Big|\sum_{1 \leqq j \leqq \mathrm{N}} \mathrm{V}^j(u^1)\Big|^l\Big) \quad ((3.2),\ \mathrm{E}(|\xi - y|^l) \leqq \mathrm{E}(|\xi|^l),\ y \in \mathbb{R}^d, \text{ for a Gaussian unit vector } \xi)$$

$$= \beta_l \Big[\sum_{1 \leqq j \leqq \mathrm{N}} a_j^2\Big]^{l/2} \leqq \beta_l \prod_{1 \leqq j \leqq \mathrm{N}} a_j^{l/\mathrm{N}} \mathrm{N}^{l/2} \quad (\text{« arithm. mean »} \leqq \text{« geom. mean »}).$$

This is (3.1) for $p = 1$.

Now assume (3.1) is valid for p. Set $T:=\{ j: 1 \leqq j \leqq N, u_j^{p+1} = \max_{1 \leqq i \leqq p+1} u_j^i \}$ and let q_j, r_j be chosen such that $u_j^{q_j} = \max \{ u_j^q : 1 \leqq q \leqq p, u_j^q < u_j^{p+1} \} \vee u_j^0$, $u_j^{r_j} = \min \{ u_j^q : 1 \leqq q \leqq p, u_j^q > u_j^{p+1} \}$, if $j \notin T$. For $1 \leqq j \leqq N$, $V^j(u^{p+1})$ can be derived from $V^j(u^{q_j})$ and $V^j(u^{r_j})$ by « interpolation » resp. « extrapolation » with some Gaussian unit vector ξ_j such that (cf. (3.2))

$$(3.3) \qquad (\mathcal{G}, \mathcal{G}^1, \ldots, \mathcal{G}^N, \xi_1, \ldots, \xi_N) \quad \text{is independent},$$

and, putting $b_j := \left[\prod_{1 \leqq q \leqq N, q \neq j} u_q^0 (u_j^{r_j} - u_j^{p+1})(u_j^{p+1} - u_j^{q_j})(u_j^{r_j} - u_j^{q_j})^{-1} \right]^{1/2}$ for $j \notin T$,

resp. $b_j := \left[\prod_{1 \leqq q \leqq N, q \neq j} u_q^0 (u_j^{p+1} - u_j^{q_j}) \right]^{1/2}$ for $j \in T$,

$$(3.4) \qquad (V^j(u^i): 1 \leqq i \leqq p+1, 1 \leqq j \leqq N) \text{ is equal in law to}$$
$$(V^j(u^i), b_j \xi_j + d_j^1 V^j(u^{r_j}) + d_j^2 V^j(u^{q_j}), \quad 1 \leqq i \leqq p, \quad 1 \leqq j \leqq N)$$

with suitable $d_j^k \in \mathbb{R}$.
Now we are ready for the induction step. We proceed in a similar way as for $p = 1$, the role of $\sum_{1 \leqq j \leqq N} V^j(u^1)$ being taken by $\sum_{1 \leqq j \leqq N} b_j \xi_j$:

$$\begin{aligned}
&E\Big(\prod_{1 \leqq i \leqq p+1} | W_{u^i} - x |^l \Big) \\
&= E\Big(\prod_{1 \leqq i \leqq p} | W_{u^i} - x |^l \Big| \sum_{1 \leqq j \leqq N} b_j \xi_j + d_j^1 V^j(u^{r_j}) + d_j^2 V^j(u^{q_j}) + V^0(u^{p+1}) - x \Big|^l \Big) && ((3.4)) \\
&\leqq E\Big(\prod_{1 \leqq i \leqq p} | W_{u^i} - x |^l \Big) E\Big(\Big| \sum_{1 \leqq j \leqq N} b_j \xi_j \Big|^l \Big) && ((3.3), \text{cf. « } p=1 \text{ »}) \\
&\leqq E\Big(\prod_{1 \leqq i \leqq p} | W_{u^i} - x |^l \Big) \beta_l \prod_{1 \leqq j \leqq N} b_j^{l/N} N^{l/2} && (\text{cf. « } p = 1 \text{ »})
\end{aligned}$$

To complete the proof, it remains to apply the induction hypothesis and to look at the definition of b_j, $1 \leqq j \leqq N$. □

Remark. — Essential use is made of the hypothesis « $u^0 > 0$ » in the proof of lemma 4. This is the reason why our smoothness results (theorems 2 and 3) contain no statement for intervals which « touch » the boundary $\partial\mathbb{R}_+^N \cap \mathbb{I}$.

As a direct consequence of lemma 4 we can prove now (by a rather crude estimation) that the moments of $K^{(k)}$ are bounded.

PROPOSITION 2. — Let $d \in \mathbb{N}$, $k \in \mathbb{N}_0^d$ be such that $2|k| + d < 2N$. Further, let $p \in \mathbb{N}$, $0 < u^0 \in \mathrm{II}$, and a product ρ on $\mathscr{B}(\mathbb{I})$ of finite measures ρ_i, $1 \leqq i \leqq N$, be given. Then there exist $c_i \in \mathbb{R}$, $i = 1, 2$, $c_2 > 0$, such that for $\rho^2 \times \lambda^d$ – a. e. $(s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$, $s \geqq u^0$,

$$E(|K^{(k)}(., s, t, x)|^p) \leqq c_1 \exp(-c_2 |x|^2),$$

where $K^{(k)}$ is given by theorem 1.

Proof. — We proceed in two steps. First we use Tanaka's formula and Burkholder's inequalities in order to establish

(3.5) there exists $c_3 \in \mathbb{R}$ such that for $(s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$, $s \geqq u^0$,

$$E(|M^{(k)}(., s, t, x)|^{2p}) \leqq c_3(1 + |x|^{2p(2N-d-|k|+1)}).$$

For $(\mathscr{C}, \phi) \in \Lambda$ with order m, $J =]s, t] \in \mathscr{I}$, $s \geqq u^0$, $u \geqq u^0$, remark 2 after proposition 1 yields

$$E(|I^{(\mathscr{C},\phi,u_{\mathscr{C}})}([1_{\Omega\times J_{\mathscr{C}}} D^{(k)}D^{(\mathscr{C},\phi)}F^{N,d}(x, W_{(.,u_{\mathscr{C}})})]^{\mathscr{C}})|^{2p})$$
$$\leqq B_{2p}E\left(\left[\int_{\mathbb{I}_{\mathscr{C}}} 1_{J_{\mathscr{C}}}(D^{(k)}D^{(\mathscr{C},\phi)}F^{N,d}(x, W_{(.,u_{\mathscr{C}})}))^2 d\lambda^{|\mathscr{C}|}\right]^p\right).$$

Therefore, by (1.4) and Tanaka's formula (3.5) follows once we have shown that

(3.6) for $0 < \delta < 1/2$, $(\mathscr{C}, \phi) \in \Lambda$ with order m, $l := 2(2N - d - |k| - m \pm \delta)$

there exists $c_4 \in \mathbb{R}$ such that for $J =]s, t] \in \mathscr{I}$, $s \geqq u^0$, $u \geqq u^0$

$$E\left(\left[\int_{\mathbb{I}_{\mathscr{C}}} 1_{J_{\mathscr{C}}} |W_{(.,u_{\mathscr{C}})} - x|^l d\lambda^{|\mathscr{C}|}\right]^p\right) \leqq \begin{cases} c_4(1 + |x|^{pl})\lambda^{|\mathscr{C}|}(J_{\mathscr{C}})^p, & \text{if } l \geqq 0, \\ c_4\lambda^{|\mathscr{C}|}(J_{\mathscr{C}})^{p(1 + l/2|\mathscr{C}|)}, & \text{if } l < 0. \end{cases}$$

But $l > -d \vee -2|\mathscr{C}|$. Consequently, (3.6) follows from scaling ($u \geqq u^0$) and lemma 4. Now remember that $K^{(0)}(., s, t, .)$ is a local time of W over $]s, t]$ for ρ^2 – a. e. $(s, t) \in \hat{\mathbb{I}}^2$. Using Fubini's theorem, we infer from this

$$K^{(0)}(., s, t, x) = K^{(0)}(., s, t, x)1_{\{\sup_{s<u\leqq t} |W_u| \geqq 1/2|x|\}}$$

for $\rho^2 \times \lambda^d$ – a. e. $(s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$.

Apply theorem 1, *i*) and *iii*) and the inequality of Cauchy-Schwartz. Thus

$$E(|K^{(k)}(.,s,t,x)|^p) \leqq [E(|M^{(k)}(.,s,t,x)|^{2p})]^{1/2}[P(\sup_{s<u\leqq t}|W_u| \geqq 1/2|x|)]^{1/2}$$

for $\rho^2 \times \lambda^d$ – a. e. $(s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$

But from Paranjape, Park [15] we have

$$P(\sup_{0\leqq u\leqq 1}|W_u| \geqq 1/2|x|) \leqq c_5 \exp\left(-1/2d\left|\frac{x}{2}\right|^2\right). \tag{3.7}$$

Combine (3.5) with (3.7) to complete the proof. □

To be able to verify Kolmogorov's criterion for $M^{(k)}$ we need to investigate the Hölder continuity of

$$x \to D^{(k)}D^{(\mathscr{C},\phi)}F^{N,d}(x,y), \qquad y \in \mathbb{R}^d, \qquad (\mathscr{C},\phi)\in\Lambda, \qquad k \in \mathbb{N}_0^d.$$

LEMMA 5. — Let $q \in \mathbb{N}_0^d$, $0 < \delta, \eta < 1$. For $y, z \in \mathbb{R}^d$ put $A_{y,z} := \{x \in \mathbb{R}^d : |x - z| \geqq 2|y - z|\}$, for $\gamma > 0$ put $g_\gamma : \mathbb{R}_+ \to \mathbb{R}$, $r \to r^{2N-d-\gamma+\delta} + r^{2N-d-\gamma-\delta}$. Then there exists $c_1 \in \mathbb{R}$ such that for $y, z \in \mathbb{R}^d$

$$|D^{(q)}F^{N,d}(.,y) - D^{(q)}F^{N,d}(.,z)| \leqq c_1[(g_{|q|}(|\,.-z|) + g_{|q|}(|\,.-y|))1_{\overline{A_{y,z}}} + |y-z|^\eta g_{|q|+\eta}(|\,.-z|)1_{A_{y,z}}].$$

Proof. — Fix $y, z \in \mathbb{R}^d$. On $\overline{A_{y,z}}$, use (1.3). Let $x \in A_{y,z}$. Then for each w on the line segment connecting $y-x$ and $z-x$ we have

$$1/2|x-z| \leqq |w| \leqq 3/2|x-z|. \tag{3.8}$$

Therefore,

$$\begin{aligned}
&|D^{(q)}(F^{N,d}(x,y) - F^{N,d}(x,z))| \\
&\leqq c_2[g_{|q|}(|x-y|) + g_{|q|}(|x-z|)]^{1-\eta} \\
&\qquad \left|\sum_{1\leqq j\leqq d}(y_j - z_j)\int_0^1 D^{(q+e_j)}F^{N,d}(x, y+s(z-y))ds\right|^\eta && ((1.3)) \\
&\leqq c_3|y-z|^\eta g_{|q|}(|x-z|)^{1-\eta}\left[\int_0^1 g_{|q|+1}(|x-(y+s(z-y))|)ds\right]^\eta && ((1.3),\ (3.8)) \\
&\leqq c_4|y-z|^\eta g_{|q|}(|x-z|)^{1-\eta} g_{|q|+1}(|x-z|)^\eta && ((3.8)),
\end{aligned}$$

with $c_2, .., c_4$ independent of $x, y, z \in \mathbb{R}^d$.

This gives the desired inequality on $A_{y,z}$. □

We are now prepared to verify Kolmogorov's criterion for $M^{(k)}$. This will be done separately for the time (proposition 3) and space (proposition 4) variables.

PROPOSITION 3. — Let $d \in \mathbb{N}$, $k \in \mathbb{N}_0^d$ be such that $2|k| + d < 2N$. Further, let $p \in \mathbb{N}$, $0 < u^0 \in \mathbb{I}$, $0 < \eta < 1/2$ be given. Then there exists $c_1 \in \mathbb{R}$ such that for all $x \in \mathbb{R}^d$, $(s, t), (s', t') \in \hat{\mathbb{I}}^2$, $s, s' \geqq u^0$

$$E(|M^{(k)}(., s, t, x) - M^{(k)}(., s', t', x)|^{2p}) \leqq c_1 |(s, t) - (s', t')|^{p\eta/N}.$$

Proof. — Fix $0 < \delta$ such that $\delta + \eta < 1/2$ and put

$$g^{(\mathscr{C},\phi)}(s, t, x) := \Delta_{J_{\underline{\mathscr{C}}}} I^{(\mathscr{C},\phi,.)}([1_{\Omega \times J_{\underline{\mathscr{C}}}} D^{(k)} D^{(\mathscr{C},\phi)} F^{N,d}(x, W_{(.,.)})]^{\mathscr{C}}), \quad J =]s, t] \in \mathscr{I},$$
$$x \in \mathbb{R}^d, \quad (\mathscr{C}, \phi) \in \Lambda.$$

We will show for each $(\mathscr{C}, \phi) \in \Lambda$ with order m

(3.9) there exists $c_2 \in \mathbb{R}$ such that for $x \in \mathbb{R}^d$, (s, t), $(s', t') \in \hat{\mathbb{I}}^2$, $s, s' \geqq u^0$

$$E(|g^{(\mathscr{C},\phi)}(s, t, x) - g^{(\mathscr{C},\phi)}(s', t', x)|^{2p}) \leqq c_2 |(s, t) - (s', t')|^{p\eta/|\underline{\mathscr{C}}|}$$

Once this is done, the assertion follows from Tanaka's formula. Compare $g^{(\mathscr{C},\phi)}(s, t, x)$ and $g^{(\mathscr{C},\phi)}(s', t', x)$ coordinatewise in s, s', t, t' to conclude that it is enough to find a constant c_3 such that $E(|g^{(\mathscr{C},\phi)}(s, t, x)|^{2p})$ can be estimated by $c_3 |s_i - t_i|^{p\eta/|\underline{\mathscr{C}}|}$ for $x \in \mathbb{R}^d$, $(s, t) \in \hat{\mathbb{I}}^2$, and all $1 \leqq i \leqq N$. Hereby it is essential to distinguish between $i \in \underline{\mathscr{C}}$ and $i \notin \underline{\mathscr{C}}$. Therefore, like in the proof of proposition 2, an application of Burkholder's inequalities reduces (3.9) to

(3.10) there exists $c_4 \in \mathbb{R}$ such that for $x \in \mathbb{R}^d$, $(s, t) \in \hat{\mathbb{I}}^2$, $s \geqq u^0$, $u \geqq u^0$

$$E\left(\left[\int_{\mathbb{I}_{\underline{\mathscr{C}}}} 1_{J_{\underline{\mathscr{C}}}} (D^{(k)} D^{(\mathscr{C},\phi)} F^{N,d}(x, W_{(.,u_{\overline{\mathscr{C}}})}))^2 d\lambda^{|\underline{\mathscr{C}}|}\right]^p\right) \leqq c_4 |s_i - t_i|^{p\eta/|\underline{\mathscr{C}}|}, \quad \text{if } i \in \underline{\mathscr{C}},$$

$$E\Big(\Big[\int_{\mathbb{I}_{\underline{\mathscr{C}}}} 1_{J_{\underline{\mathscr{C}}}} [D^{(k)} D^{(\mathscr{C},\phi)}(F^{N,d}(x, W_{(.,u_{\overline{\mathscr{C}}\setminus\{i\}},t_i)})$$
$$- F^{N,d}(x, W_{(.,u_{\overline{\mathscr{C}}\setminus\{i\}},s_i)}))]^2 d\lambda^{|\underline{\mathscr{C}}|}\Big]^p\Big) \leqq c_4 |s_i - t_i|^{p\eta/|\underline{\mathscr{C}}|},$$

if $i \notin \underline{\mathscr{C}}$, where $J =]s, t]$.

First consider the case $i \in \underline{\mathscr{C}}$. Estimate the integrand with the help of (1.4) and conclude by (3.6), observing that $l = 2(2N - d - |k| - m \pm \delta) > 2\eta - 2|\underline{\mathscr{C}}|$. The case $i \notin \underline{\mathscr{C}}$ is more difficult, since lemma 5 has to be used for estimating the integrand. As $u \geqq u^0 > 0$, by scaling we may suppose $\overline{\mathscr{C}} = \{i\}$. Then, according to lemma 5, (3.10) is a consequence of

(3.11) there exists $c_5 \in \mathbb{R}$ such that for $x \in \mathbb{R}^d$, $(s, t) \in \hat{\mathbb{I}}^2$, $s \geqq u^0$

i)

$$E\left(\left[\int_{\mathbb{I}_{\underline{\mathscr{C}}}} 1_{J_{\underline{\mathscr{C}}}} |W_{(.,s_i)} - W_{(.,t_i)}|^{2\eta} |W_{(.,s_i)} - x|^{l-2\eta} d\lambda^{|\underline{\mathscr{C}}|}\right]^p\right) \leqq c_5 |s_i - t_i|^{p\eta/|\underline{\mathscr{C}}|},$$

ii)

$$E\left(\left[\int_{\mathbb{I}_{\overline{\mathscr{C}}}} 1_{J_{\overline{\mathscr{C}}}} \mid W_{(.,s_i)}-x\mid^l 1_{\{|W_{(.,s_i)}-x|<2|W_{(.,t_i)}-W_{(.,s_i)}|\}} d\lambda^{|\overline{\mathscr{C}}|}\right]^p\right) \leqq c_5 \mid s_i - t_i \mid^{p\eta/|\overline{\mathscr{C}}|},$$

iii)

$$E\left(\left[\int_{\mathbb{I}_{\overline{\mathscr{C}}}} 1_{J_{\overline{\mathscr{C}}}} \mid W_{(.,t_i)}-x\mid^l 1_{\{|W_{(.,s_i)}-x|<2|W_{(.,t_i)}-W_{(.,s_i)}|\}} d\lambda^{|\overline{\mathscr{C}}|}\right]^p\right) \leqq c_5 \mid s_i - t_i \mid^{p\eta/|\overline{\mathscr{C}}|},$$

where $J =]s, t]$.

To argue (3.11), *i*), use independence of increments to single out a factor $| t_i - s_i |^{2p\eta}$ and observe that the remainder can be treated by lemma 4, since by choice of δ, $l - 2\eta > -d \vee -2|\overline{\mathscr{C}}|$. To argue (3.11), *ii*) and *iii*), we make use of the boundedness of the moments of local time (proposition 2). To infer *ii*), we will show

(3.12) there exists $c_6 \in \mathbb{R}$ such that for $x \in \mathbb{R}^d$, $(s, t) \in \hat{\mathbb{I}}^2$, $q \in \mathbb{R}_+$

$$E\left(\left[\int_{\mathbb{I}_{\overline{\mathscr{C}}}} 1_{J_{\overline{\mathscr{C}}}} \mid W_{(.,s_i)}-x\mid^l 1_{\{|W_{(.,s_i)}-x|<q\}} d\lambda^{|\overline{\mathscr{C}}|}\right]^p\right) \leqq c_6 q^{p(l+d)}, \text{ with } J =]s, t].$$

Note first that (3.12) implies (3.11), *ii*). Indeed, $W_{(.,t_i)} - W_{(.,s_i)}$ is independent of $W_{(.,s_i)}$. Consequently, by (3.12), the left side of (3.11), *ii*) is less or equal to

$$E\Big(\sup_{u_{\overline{\mathscr{C}}} \in \mathbb{I}_{\overline{\mathscr{C}}}} \mid W_{(u_{\overline{\mathscr{C}}},t_i)} - W_{(u_{\overline{\mathscr{C}}},s_i)} \mid^{p(l+d)}\Big),$$

which, by Doob's inequality, is $c_7 \mid t_i - s_i \mid^{p(l+d)/2}$, with a suitable $c_7 \in \mathbb{R}$. But

$$(3.13) \qquad l + d \geqq 2N - d - 2|k| \pm 2\delta \geqq 1 \pm 2\delta > 2\eta$$

evidently implies (3.11), *ii*). To prove (3.12), for familiar reasons, we may and do assume $\overline{\mathscr{C}} = \emptyset$. Let $L(., s, t, .)$ be a local time of W over $J =]s, t]$, $(s, t) \in \hat{\mathbb{I}}^2$, $s \geqq u^0$. For $x \in \mathbb{R}^d$, (0.1) gives

$$\int_{\mathbb{I}} 1_J(u) \mid W_u - x \mid^l 1_{\{|W_u - x| < q\}} du \leqq \left(\prod_{1 \leqq i \leqq N} u_i^0\right)^{1-N} \int_{K_q(0)} |z|^l L(., s, t, x+z) dz.$$

Therefore, proposition 2 $\left(\text{with } \rho = \bigtimes_{1 \leqq i \leqq N} (\varepsilon_{\{s_i\}} + \varepsilon_{\{t_i\}}),\ \varepsilon_v \text{ being the point mass in } v \in \mathbb{I}\right)$ yields a constant $c_8 \in \mathbb{R}$, such that for $x \in \mathbb{R}^d$, $(s, t) \in \hat{\mathbb{I}}^2$, $q \in \mathbb{R}_+$

$$E\left(\left[\int_{\mathbb{I}} 1_J \mid W_u - x \mid^l 1_{\{|W_u - x| < q\}} du\right]^p\right) \leqq c_8 \left(\int_{K_q(0)} \mid z \mid^l dz\right)^p .$$

As $l > -d$, the integral on the right side exists and (3.12) follows.
Finally, for (3.11), *iii*) independence of increments can be used in nearly the same way as it has just been done. Consider the process

$$X(\omega, u) := u_i W(\omega, 1/u_i, u_{\{\bar{i}\}}), \quad \omega \in \Omega, \quad u \in \mathbb{R}_+^N ,$$

which is again an (N, d)-Wiener process.
Observe that

$$\{ \mid W_{(.,s_i)} - x \mid < 2 \mid W_{(.,t_i)} - W_{(.,s_i)} \mid \}$$
$$\subset \{ \mid W_{(.,t_i)} - x \mid < 3 \mid W_{(.,t_i)} - W_{(.,s_i)} \mid \}, \quad s_i, t_i \in \mathbb{I},$$

use this to write (3.11), *iii*) in terms of X and carry out an analogous calculation to the one which proved *ii*). This gives the desired conclusion. □

PROPOSITION 4. — Let $d \in \mathbb{N}$, $k \in \mathbb{N}_0^d$ be such that $2|k| + d < 2N$. Further, let $p \in \mathbb{N}$, $0 < u^0 \in \mathbb{I}$, $0 < \eta < 1/2$ be given. Then there exists $c_1 \in \mathbb{R}$ such that for $x, y \in \mathbb{R}^d$, $(s, t) \in \hat{\mathbb{I}}^2$, $s \geqq u^0$

$$E(\mid M^{(k)}(., s, t, x) - M^{(k)}(., s, t, y) \mid^{2p}) \leqq c_1 \mid x - y \mid^{2p\eta} .$$

Proof. — Fix $0 < \delta$ such that $\delta + \eta < 1/2$. Following the proof of proposition 3, we can, due to Tanaka's formula, fix $(\mathscr{C}, \phi) \in \Lambda$ with order m and apply Burkholder's inequalities (in the form of remark 2 after proposition 1) to see that it suffices to show

(3.14) there exists $c_2 \in \mathbb{R}$ such that for $x, y \in \mathbb{R}^d$, $(s, t) \in \hat{\mathbb{I}}^2$, $s \geqq u^0$, $u \geqq u^0$

$$E\left(\left[\int_{\mathbb{I}_{\mathscr{C}}} 1_{J_{\mathscr{C}}} [D^{(k)} D^{(\mathscr{C},\phi)} (F^{N,d}(x, W_{(.,u_{\mathscr{C}})}) - F^{N,d}(y, W_{(.,u_{\mathscr{C}})}))]^2 d\lambda^{|\mathscr{C}|}\right]^p\right)$$
$$\leqq c_2 \mid x - y \mid^{2p\eta}, \quad \text{where} \quad J =]s, t].$$

Put again $l := 2(2N - d - |k| - m \pm \delta)$. Use lemma 5 to estimate the integrand, and scaling, in order to trace back (3.14) to

(3.15) there exists $c_3 \in \mathbb{R}$ such that for $x, y \in \mathbb{R}^d$, $(s, t) \in \hat{\mathbb{I}}^2$

$$i)\quad E\left(\left[\int_{\mathbb{I}} 1_J\,|W-x|^{l-2\eta}\,|x-y|^{2\eta} d\lambda^N\right]^p\right) \leqq c_3\,|x-y|^{2p\eta},$$

$$ii)\quad E\left(\left[\int_{\mathbb{I}} 1_J\,|W-x|^{l}\,1_{\{|W-x|<2|x-y|\}} d\lambda^N\right]^p\right) \leqq c_3\,|x-y|^{2p\eta},$$

$$iii)\quad E\left(\left[\int_{\mathbb{I}} 1_J\,|W-y|^{l}\,1_{\{|W-x|<2|x-y|\}} d\lambda^N\right]^p\right) \leqq c_3\,|x-y|^{2p\eta},$$

where $J =]s, t]$.

By choice of δ, $l - 2\eta > -d \vee -2N$. Therefore, *i*) follows from lemma 4, whereas *ii*) and *iii*) are consequences of (3.12) and (3.13) □

We are now ready to state the first smoothness theorem for the local time of W.

THEOREM 2. — Let $d < 2N$. Then for each $k \in \mathbb{N}_0^d$ such that $2|k|+d<2N$ there exists $L^{(k)} \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\hat{\mathbb{I}}^2) \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$ which satisfies

i) $L^{(k)}(., s, t, x) = M^{(k)}(., s, t, x)$ for λ^{2N+d} – a. e. $(s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$,

ii) $(s, t, x) \to L^{(k)}(\omega, s, t, x)$ is continuous on $\hat{\mathbb{I}}_0^2 \times \mathbb{R}^d$ for P – a. e. $\omega \in \Omega$,

iii) $D^{(k)}L^{(0)}(., s, t, x) = L^{(k)}(., s, t, x)$ for all $(s, t, x) \in \hat{\mathbb{I}}_0^2 \times \mathbb{R}^d$.

$L^{(0)}(., s, t, .)$ is a local time of W over $]s, t]$ for all $(s, t) \in \hat{\mathbb{I}}_0^2$.

Proof. — For $k \in \mathbb{N}_0^d$ such that $2|k| + d < 2N$ let $K^{(k)}$ be given according to theorem 1 with $\rho = \lambda^N$. Fix $p \in \mathbb{N}$, $0 < u^0 \in \mathbb{I}$, $0 < \eta < 1/2$. Eventually alter $K^{(k)}$ on a $P \times \lambda^{2N+d}$-zero set to infer from propositions 3 and 4: there exists $c_1 \in \mathbb{R}$ such that for $x, y \in \mathbb{R}^d$, $(s, t), (s', t') \in \hat{\mathbb{I}}^2$, $s, s' \geqq u^0$

$$E(|K^{(k)}(., s, t, x) - K^{(k)}(., s', t', y)|^{2p}) \leqq c_1\,|(s, t, x) - (s', t', y)|^{p\eta/N}.$$

As we can take $p > (2N+d)N/\eta$ in the preceding inequality, we obtain Kolmogorov's continuity criterion for $K^{(k)}$ (see for example Bernard [*4*]). Thus there exists $L_{u^0}^{(k)} \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\hat{\mathbb{I}}^2 \cap]u^0, \underline{1}]^2) \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$ with (in (s, t, x)) continuous trajectories such that

$$(3.16)\quad L_{u^0}^{(k)}(., s, t, x) = K^{(k)}(., s, t, x) \quad \text{for} \quad (s, t, x) \in \hat{\mathbb{I}}^2 \cap]u^0, \underline{1}]^2 \times \mathbb{R}^d.$$

Consequently we can (P – a. s.) uniquely define processes $L^{(k)} \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\hat{\mathbb{I}}^2) \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$ which coincide with $L_{u^0}^{(k)}$ on $\hat{\mathbb{I}}^2 \cap]u^0, \underline{1}] \times \mathbb{R}^d$ and have continuous trajectories in $(s, t, x) \in \hat{\mathbb{I}}_0^2 \times \mathbb{R}^d$. *i*) follows from (3.16) and theorem 1, *i*); *iii*) is a consequence of *ii*) and theorem 1, *iii*). □

Remark. — The method used to prove lemma 4 does not allow to extend the results of theorem 2 to all intervals of $\mathcal{I}$. Finer estimates, however, should be possible (cf. proof of lemma 1, see also Ehm [*8*]).

4. DIFFERENTIABILITY OF THE LOCAL TIME OF W IN (t, x)

Let $\emptyset \neq V \in \Pi_N$, $d \in \mathbb{N}$ be such that $d < 2|V|$. Then for all $t_{\overline{V}} \in \mathbb{I}_{\overline{V}}$, local times for the $(|V|, d)$-processes $W_{(.,t_{\overline{V}})}$ exist. Integrating them over $t_{\overline{V}}$ produces a local time of the (N, d)-Wiener process. We will now use this observation to study differentiability of local time in t. It also makes clear what the t-derivatives look like. We proceed like in 1.–3.: starting with an appropriate version of Ito's formula we derive Tanaka's formula for (x, t)-derivatives and establish Kolmogorov's criterion for continuity.

PROPOSITION 5. — Let $\emptyset \neq V \in \Pi_N$, $f \in C^{2|V|}(\mathbb{R}^d)$ be such that

$$D^{(\mathscr{C},\phi)} f(W) \in L^2(\Omega \times \mathbb{I}, \mathscr{P}, P \times \lambda^N), \quad (\mathscr{C}, \phi) \in \Psi_V .$$

Further, let a product ρ on $\mathscr{B}(\mathbb{I})$ of finite measures ρ_i, $1 \leqq i \leqq N$, satisfy

$$(4.1) \quad \int_{\mathbb{I}_{\overline{\mathscr{C}}}} [\| \; D^{(\mathscr{C},\phi)} f(W_{(.,t_{\overline{\mathscr{C}}})})^{\mathscr{C}} \|_{(\underline{\mathscr{C}},\phi)}^{t_{\mathscr{C}}}]^2 d\rho_{\overline{\mathscr{C}}} t_{\overline{\mathscr{C}}}) < \infty, \ (\mathscr{C}, \ \phi) \in \Psi, \quad \underline{\mathscr{C}} \subset V .$$

Then for each $(\mathscr{C}, \phi) \in \Lambda$, $\underline{\mathscr{C}} \subset V$, there exists
$X^{(\mathscr{C},\phi)} \in \mathscr{M}(\mathscr{F} \times \mathscr{B}(\hat{\mathbb{I}}_V^2) \times \mathscr{B}(\mathbb{I}_{\overline{V}}), \mathscr{B}(\mathbb{R}))$ such that

i) $X^{(\mathscr{C},\phi)}(., s_V, ., .) \in \mathscr{M}(\mathscr{P}, \mathscr{B}(\mathbb{R}))$, $\quad s_V \in \mathbb{I}_V$,

ii) $X^{(\mathscr{C},\phi)}(., s_V, t_V, u_{\overline{V}})$

$$= \Delta_{]s_V,t_V]_{\overline{\mathscr{C}}} \setminus \overline{V} \times]0,u_{\overline{V}}]} I^{(\mathscr{C},\phi,.)}([1_{\Omega \times]s_V,t_V]_{\underline{\mathscr{C}}}} D^{(\mathscr{C},\phi)} f(W_{(...)})]^{\mathscr{C}})$$

for $\rho_V^2 \times \rho_{\overline{V}}$ – a. e. $(s_V, t_V, u_{\overline{V}}) \in \hat{\mathbb{I}}_V^2 \times \mathbb{I}_{\overline{V}}$,

iii)
$$\Delta_{]s_V,t_V]} f(W_{(.,u_{\overline{V}})}) = \sum_{(\mathscr{C},\phi) \in \Lambda, \underline{\mathscr{C}} \subset V} \frac{1}{2^{|\mathscr{C}^0|}} \alpha_{(\mathscr{C},\phi)} X^{(\mathscr{C}.\phi)}(., s_V, t_V, u_{\overline{V}})$$
$$+ \frac{1}{2^{|V|}} \int_{]s_V,t_V]} \mathbb{D}^{|V|} f(W_{(.,u_{\overline{V}})}) \prod_{i \in \overline{V}} u_i^{|V|} \prod_{i \in V} u_i^{|V|-1} du_V$$

for $\rho_V^2 \times \rho_{\overline{V}}$ – a. e. $(s_V, t_V, u_{\overline{V}}) \in \hat{\mathbb{I}}_V^2 \times \mathbb{I}_{\overline{V}}$, with $\alpha_{(\mathscr{C},\phi)}$ according to (1.1).

Proof. — By (4.1), the existence of $X^{(\mathscr{C},\phi)} \in \mathscr{M}(\mathscr{F} \times \mathscr{B}(\hat{\mathbb{I}}_V^2) \times \mathscr{B}(\mathbb{I}_{\overline{V}}), \mathscr{B}(\mathbb{R}))$

satisfying *i*) and *ii*) follows from lemma 5 of [*10*] in the same way as the corollary of it. Fix $(\mathscr{C}, \phi) \in \Lambda$, $\underline{\mathscr{C}} \subset V$, and $u_{\overline{V}} \in \mathbb{I}_{\overline{V}}$ such that

$$\int_{\mathbb{I}_{\underline{\mathscr{C}}\setminus\overline{V}}} [\, \| D^{(\mathscr{C},\phi)} f(W_{(.,t_{\underline{\mathscr{C}}\setminus\overline{V}},u_{\overline{V}})})^{\mathscr{C}} \|_{(\mathscr{C},\phi)}^{t_{\underline{\mathscr{C}}}\setminus\overline{V}}]^2 d\rho_{\underline{\mathscr{C}}\setminus\overline{V}}(t_{\underline{\mathscr{C}}\setminus\overline{V}}) < \infty .$$

which is true for $\rho_{\overline{V}}$=a. e. $u_{\overline{V}} \in \mathbb{I}_{\overline{V}}$. Apply theorem 4 of [*10*] to the $(\,|V|, d)$-Wiener process $\prod_{i\in\overline{V}} u_i^{-1/2} W_{(.,u_{\overline{V}})}$ to obtain *iii*) (cf. (4.6) in the proof of lemma 6 of [10]). □

Now let $\emptyset \neq V \in \Pi_N$, $d \in \mathbb{N}$, $k \in \mathbb{N}_0^d$ such that $2|k|+d<2|V|$, a product ρ on $\mathscr{B}(\mathbb{I})$ of finite measures ρ_i, $1 \leqq i \leqq N$, and $(\mathscr{C}, \phi) \in \Lambda$, $\underline{\mathscr{C}} \subset V$, be given. Then, the proof of lemma 1 gives

$$(4.2) \qquad (t_{\overline{\mathscr{C}}}, x) \to \| [D^{(k)} D^{(\mathscr{C},\phi)} F^{|V|,d}(x, W_{(.,t_{\overline{\mathscr{C}}})})]^{\mathscr{C}} \|_{(\mathscr{C},\phi)}^{t_{\overline{\mathscr{C}}}}$$

is locally bounded on $\mathbb{I}_{\overline{\mathscr{C}}} \times \mathbb{R}^d$.

Proposition 5 in place of (1.1) and (1.2) with $F^{|V|,d}$ instead of $F^{N,d}$ motivate the following definition, which makes sense in consequence of (4.2).

DÉFINITION 2. — Let $\emptyset \neq V \in \Pi_N$, $d \in \mathbb{N}$, $k \in \mathbb{N}_0^d$ be such that $2|k| + d < 2|V|$. For $x \in \mathbb{R}^d$, $(s_V, t_V) \in \hat{\mathbb{I}}_V^2$, $u_{\overline{V}} \in \mathbb{I}_{\overline{V}}$ let, setting $J_V =]s_V, t_V]$

$$M^{(k,\overline{V})}(., u_{\overline{V}}, s_V, t_V, x) := 2^{|V|} [\Delta_{J_V} D^{(k)} F^{|V|,d}(x, W_{(.,u_{\overline{V}})})$$
$$- \sum_{(\mathscr{C},\phi)\in\Lambda, \underline{\mathscr{C}}\subset V} \frac{1}{2^{|\mathscr{C}^0|}} \alpha_{(\mathscr{C},\phi)} \Delta_{J_{\overline{\mathscr{C}}}\setminus\overline{V} \times]0,u_{\overline{V}}]} I^{(\mathscr{C},\phi,.)}([1_{\Omega\times(J_V)_{\underline{\mathscr{C}}}} D^{(k)} D^{(\mathscr{C},\phi)} F^{|V|,d}(x, W_{(.,.)})]^{\mathscr{C}})].$$

Now observe that the proofs of propositions 3 and 4 go through without essential modifications for $M^{(k,\overline{V})}$ instead of $M^{(k)}$. Therefore, we obtain for V, d, k as above, $p \in \mathbb{N}$, $0 < u^0 \in \mathbb{I}$, $0 < \eta < 1/2$

(4.3) there exists $c_1 \in \mathbb{R}$ such that for $x, y \in \mathbb{R}^d$, $u_{\overline{V}}, u'_{\overline{V}} \in \mathbb{I}_{\overline{V}}$,

$$(s_V, t_V), \quad (s'_V, t'_V) \in \mathbb{I}_V^2, \quad u_{\overline{V}}, u'_{\overline{V}} \geqq u_{\overline{V}}^0, \quad s_V, s'_V \geqq u_V^0$$
$$E(\,| M^{(k,\overline{V})}(., u_{\overline{V}}, s_V, t_V, x) - M^{(k,\overline{V})}(., u'_{\overline{V}}, s'_V, t'_V, y)\,|^{2p})$$
$$\leqq c_1 \,|\, (u_{\overline{V}}, s_V, t_V, x) - (u'_{\overline{V}}, s'_V, t'_V, y)\,|^{p\eta/|V|} .$$

With the help of (4.3), the second smoothness theorem for local times can now be proved.

THEOREM 3. — For each $\emptyset \neq V \in \Pi_N$, $\overline{V} \neq \emptyset$, $d \in \mathbb{N}$, $k \in \mathbb{N}_0^d$ such that $2|k| + d < 2|V|$ there exists $L^{(k,\overline{V})} \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\mathbb{I}_{\overline{V}}) \times \mathcal{B}(\hat{\mathbb{I}}_V^2) \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$ which satisfies

i) $L^{(k,\overline{V})}(., u_{\overline{V}}, s_V, t_V, x) = M^{(k,\overline{V})}(., u_{\overline{V}}, s_V, t_V, x)$
for $\lambda^{|\overline{V}|+2N+d}$ – a. e. $(u_{\overline{V}}, s_V, t_V, x) \in \mathbb{I}_{\overline{V}} \times \hat{\mathbb{I}}_V^2 \times \mathbb{R}^d$,

ii) $(u_{\overline{V}}, s_V, t_V, x) \to L^{(k,\overline{V})}(\omega, u_{\overline{V}}, s_V, t_V, x)$
is continuous on $(\mathbb{I}_{\overline{V}})_0 \times (\hat{\mathbb{I}}_V^2)_0 \times \mathbb{R}^d$, for P – a. e. $\omega \in \Omega$,

iii) $L^{(k,\overline{V})}(., u_{\overline{V}}, s_V, t_V, x) = D^{(k)}L^{(0,\overline{V})}(., u_{\overline{V}}, s_V, t_V, x)$
on $(\mathbb{I}_{\overline{V}})_0 \times (\hat{\mathbb{I}}_V^2)_0 \times \mathbb{R}^d$.

Let $L^{(k)}$ be given according to theorem 2. Then

iv)
$$L^{(k)}(., s, t, x) = |\overline{V}|^{|V|} \int_{]s,t]_{\overline{V}}} \int_{\mathbb{I}_V} L^{(k,\overline{V})}(., u_{\overline{V}}, s_V \vee u_V, t_V, x) \prod_{1 \leq i \leq N} u_i^{|\overline{V}|-1} du, \qquad (s, t) \in \hat{\mathbb{I}}_0^2, \quad x \in \mathbb{R}^d.$$

In particular, P – a. s. for all $x \in \mathbb{R}^d$, $0 < s \in \mathbb{I}$
$t \to L^{(k)}(., s, t, x)$ is continuously partially differentiable in $(t_i, i \in \overline{V})$ and

$$\frac{\partial^{|\overline{V}|}}{\partial(t_i, i \in \overline{V})} L^{(k)}(., s, t, x) = |\overline{V}|^{|V|} \int_{\mathbb{I}_V} L^{(k,\overline{V})}(., u_{\overline{V}}, s_V \vee u_V, t_V, x) \prod_{i \in V} u_i^{|\overline{V}|-1} du_V \prod_{i \in \overline{V}} t_i^{|\overline{V}|-1}.$$

Proof. — To argue *i)-iii)*, we proceed like in the proofs of theorems 1 and 2: we make use of an obvious generalization of lemma 2 which rests upon (4.1) instead of lemma 1; (4.3) takes the place of propositions 3 and 4. To prove *iv)*, employing proposition 5 instead of theorem 4 of [*10*], we derive the following analogon of (1.6)

$$(4.4) \quad \int_{\mathbb{R}^d} L^{(0,\overline{V})}(., u_{\overline{V}}, s_V \vee u_V, t_V, x) h(x) dx = \int_{]s_V \vee u_V, t_V]} h(W_{(v_V, u_{\overline{V}})}) \prod_{i \in \overline{V}} u_i^{|V|} \prod_{i \in V} v_i^{|V|-1} dv_V,$$
$$h \in C_c^\infty(\mathbb{R}^d), \qquad 0 < u \in \mathbb{I}, \qquad (s_V, t_V) \in (\hat{\mathbb{I}}_V^2)_0.$$

Now integrate both sides of (4.4) to get

$$|\overline{V}|^{|V|}\int_{\mathbb{R}^d}\int_{]s,t]_{\overline{V}}}\int_{\mathbb{I}_V} L^{(0,\overline{V})}(.,u_{\overline{V}},s_V\vee u_V,t_V,x)\prod_{1\leq i\leq N} u_i^{|\overline{V}|-1}du\,h(x)dx$$

$$=\int_{]s,t]} h(W_u)\prod_{1\leq i\leq N} u_i^{N-1}du,\qquad h\in C_c^{\infty}(\mathbb{R}^d),\quad (s,t)\in\hat{\mathbb{I}}_0^2.$$

Considering (0.1), this implies the validity of *iv*) for $k = \underline{0}$. Apply *iii*) and theorem 2, *iii*) to infer *iv*) for all $k \in \mathbb{N}_0^d$ such that $2|k|+d < 2|V|$. What remains to be done is an easy consequence of *iv*). □

REFERENCES

[1] R. J. ADLER, A Hölder condition for the local time of the Brownian sheet, *Indiana Univ. Math. J.*, t. **29**, 1980, p. 793-798.

[2] J. AZEMA, M. YOR, Temps locaux, *Astérisque*, t. **52-53**, Société Math. de France, Paris, 1978.

[3] S. M. BERMAN, Gaussian processes with stationary increments: Local times and sample function properties, *Ann. Math. Stat.*, t. **41**, 1970, p. 1260-1272.

[4] P. BERNARD, Quelques propriétés des trajectoires des fonctions aléatoires stables sur $\mathbb{R}^k$, *Ann. Inst. Henri Poincaré*, t. **6** (2), 1970, p. 131-151.

[5] K. BICHTELER, Stochastic integration and L^p-theory of semimartingales. *Ann. Prob.*, t. **9**, 1981, p. 49-89.

[6] R. CAIROLI, J. WALSH, Stochastic integrals in the plane, Acta *Mathematica*, t. **134**, 1975, p. 111-183.

[7] R. CAIROLI, J. WALSH, Régions d'arrêt, localisations et prolongements de martingales, *Z. W.*, t. **44**, 1978, p. 279-306.

[8] W. EHM, Sample function properties of multi-parameter stable processes, *Z. W.*, t. **56**, 1981, p. 195-228.

[9] D. GEMAN, J. HOROWITZ, Occupation densities, *Ann. Prob.*, t. **8**, 1980, p. 1-67.

[10] P. IMKELLER, *Ito's formula for continuous* (N, *d*)*-processes* (to appear in Z. W.).

[11] P. IMKELLER, *Local times for a class of multi-parameter processes* (to appear in Stochastics).

[12] E. MERZBACH, Processus stochastiques à indices partiellement ordonnés, *Rapp. interne*, t. **55**, École Polytechnique, 1979.

[13] C. METRAUX, Quelques inégalités pour martingales à paramètre bidimensionnel. Sém. de Prob. XII, Univ. de Strasbourg, *Lecture Notes in Mathematics*, t. **649**, p. 170-179, Springer, Berlin, 1978.

[14] P. A. MEYER, Un cours sur les intégrales stochastiques. Sém. de Prob. X, Univ. de Strasbourg, *Lecture Notes in Mathematics*, t. **511**, Springer, Berlin, 1976.

[15] S. R. PARANJAPE, C. PARK, Laws of the iterated logarithm of multiparameter Wiener processes, *J. Multiv. Anal.*, t. **3**, 1973, p. 132-136.

[16] L. SCHWARTZ, *Théorie des distributions*, Hermann, Paris, 1966.

[17] L. T. TRAN, On a problem posed by Orey and Pruitt related to the range of the N-parameter Wiener process in $\mathbb{R}^d$, *Z. W.*, t. **37**, 1976, p. 27-33.

[*18*] S. USTUNEL, Stochastic integration on nuclear spaces and its applications, *Ann. Inst. Henri Poincaré*, t. **18**, 1982, p. 165-200.

[*19*] J. WALSH, The local time of the Brownian sheet, *Astérisque*, t. **52-53**, 1978, p. 47-61.

[*20*] E. WONG, M. ZAKAI, Differentiation formulas for stochastic integrals in the plane, *Stoch. Proc. Appl.*, t. **6**, 1978, p. 339-349.

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