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Local nondeterminism and local times of general stochastic processes

by

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ABSTRACT. — Let $X(t)$, $0 \leqslant t \leqslant 1$, be a real stochastic process, and let $g(t)$, $0 \leqslant t \leqslant 1$, be a nonnegative integrable function. $X(t)$ is said to be locally g-nondeterministic if for every $k \geqslant 2$, there exists $c_k > 0$ such that the joint density of the $k-1$ increments, $X(t_{j+1})-X(t_j)$, $j=1, \ldots, k-1$, $t_1 < \ldots < t_k$, evaluated at the origin, is bounded above by

$$c_k g(t_2 - t_1) \ldots g(t_k - t_{k-1}) .$$

This condition implies the validity of key estimates in the analysis of the local time of the process. The latter imply specific irregularity properties of the sample functions. Such properties have been studied for several years in the context of Gaussian processes. The contribution of this work is the demonstration that local nondeterminism can be usefully defined even for processes that are not necessarily Gaussian, and that the comprehensive theory of sample function irregularity for the latter processes can be extended to more general processes. Applications to Markov processes are exhibited.

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1. INTRODUCTION

The purpose of this work is to show how the concepts and calculations used in the analysis of local times of Gaussian processes can be fruitfully extended to a much larger class of processes. One of the central ideas in the Gaussian context is « local nondeterminism », introduced by the author in [5]. There it signifies that the value of the process at a given time point is relatively unpredictable on the basis of a finite set of observations from the immediate past. This means that there is a permanent element of uncertainty in the local evolution of the sample function. In the Gaussian case, the process is called locally nondeterministic if the conditional variance of an increment, given the values of a finite set of observations from the immediate past, is bounded below by a constant positive multiple of the unconditional variance. In the general case, the incremental variance, as a measure of local unpredictability, is replaced by a measure of local predictability, namely, the value of the incremental density function at the origin. More precisely, local nondeterminism involves a suitable bound on the joint density function of increments over nonoverlapping intervals, where the density is evaluated at the origin.

The formal definition of this concept in the general case leads to an extension of the methods and results in the Gaussian case. The theme of our research in this area has been that the smoothness of the local time, as a function of of the time parameter and the space variable, implies specific irregularity properties of the sample functions. Local nondeterminism provides the basis of a set of computations required to establish the smoothness of the local time. There is a major difference between the general method of this paper, and the previous work for the specific Gaussian process: the irregularity of the sample functions is based on the smoothness of the local time only as a function of the time parameter. Smoothness in the space variable is replaced by the assumption of the higher order integrability of the local time in the space variable. Previous results requiring smoothness in the space variable were so restricted that they had to be limited to the Gaussian case, or to cases that were very close to the Gaussian.

We prove two results about the behavior of the sample functions. The first is: the approximate lim sup (for $s \to t$) of the ratio, $|X(t)-X(s)|/\delta(|s-t|)$ is infinite for all t, almost surely, for a class of functions determined by the hypothesis of local nondeterminism. The first result of this type was proved by the author [3]: if a function has a jointly continuous local time, then, the statement above holds with $\delta(s)=s$. The ramifications of this result, and its relations to Jarnik functions, were considered by Geman and Horowitz; an extensive discussion is contained in their survey [*11*]. The novelty of our current result is that the local property holds at every point almost surely without the requirement that the local time is continuous or Holder continuous in the space variable.

Our second result on the behavior of the sample functions is about the magnitude of the level sets. Let $w(t)$ be a Hausdorff measure function, and consider the w-measure of the set $\{ s : X(s) = X(t) \}$ for each t. Our theorem states that if $w(t)$ is at least as large as a function specified by the condition of local nondeterminism, then the w-measure of the indicated set is almost surely infinite, for almost all t. This represents a generalization of earlier theorems in the Gaussian case on the Hausdorff dimension of the level sets, where $w(t)$ is of the special form $| t |^{\alpha}$. There the lower bound on the dimension of the level set was obtained by local time methods. Marcus [*14*] showed that the capacity methods used in the Gaussian case could also be used in a more general situation. Our present method, which is not restricted to the measure functions $| t |^{\alpha}$, uses only the smoothness and integrability properties of the local time mentioned above.

The concept of local time was first formulated in the case of the Brownian motion process by Levy [*13*]; his work was continued by Trotter [*20*]. Since that time, the subject developed in two distinct directions, namely, for processes with independent increments, and for Gaussian processes. Geman and Horowitz made a survey of the two fields, and indicated some of the common features. The primary purpose of our work, as noted above, is to show how local nondeterminism, a central concept of Gaussian local times, can be formulated in a manner so general that it can be applied to other processes of interest. Our first result above, on the approximate lim sup of the difference quotient, appears to be new even in the case of processes with independent increments. Our second result on the lower bound for the measure of the level set, although not new for the latter processes, is new for the general class of Markov processes satisfying the simple conditions described in example 7.2.

We close with a brief survey of local nondeterminism. After our introduction of the concept in [*5*], Pitt [*17*] extended it to random fields in the Gaussian case. Cuzick [*9*] introduced the modification of local « ϕ-nondeterminism », which motivated our definition of local g-nondeterminism. (We use a different symbol because $g = 1/\phi$.) These results are all described in the survey of Geman and Horowitz [*11*]. A recent addition in book form is that of Adler [*1*]. Since the publication of these results, the author has also introduced the concept of local nondeterminism for a class of processes which are not necessarily Gaussian, but are strongly related to them [1].

2. LOCAL TIMES FOR REAL VALUED MEASURABLE FUNCTIONS

For the convenience of the reader, we restate some basic definitions. Let $x(t)$, $0 \leqslant t \leqslant 1$, be a real valued measurable function. For every pair of linear Borel sets $A \subset (-\infty, \infty)$, and $I \subset [0, 1]$, define

$$\nu(A, I) = \text{Lebesgue measure}\{ s : x(s) \in A, s \in I \}.$$

If, for fixed I, $\nu(., I)$ is absolutely continuous as a measure of sets A, then its Radon-Nikodym derivative, which we denote as $\alpha_I(x)$, is called the local time of $x(t)$ relative to I. It satisfies

$$\nu(A, I) = \text{Lebesgue measure}\{ s : x(s) \in A, s \in I \}. \tag{2.1}$$

We say that the local time is square integrable if $\alpha_I^2(x)$ is integrable over all x.

LEMMA 2.1. — Let $x(t)$ have the local time α_I. Then, for every A and I, and every $p \geqslant 1$ and $q \geqslant 1$ satisfying $p^{-1} + q^{-1} = 1$,

$$\nu(A, I) \leqslant (\text{mes } A)^{1/q}\left(\int_{-\infty}^{\infty} \alpha_I^p(x)dx\right)^{1/p}, \tag{2.2}$$

where the integral may be finite or infinite.

Proof. — Let $\chi_A(x)$ be the indicator of the set A; then write ν in (2.1) as the integral of the product $\chi_A \alpha_I$, and apply Holder's inequality.

For $p \geqslant 1$, define

$$\xi(I) = \left(\int_{-\infty}^{\infty} \alpha_I^p(x)dx\right)^{1/p}. \tag{2.3}$$

Then $\xi(.)$ is a nonnegative, nondecreasing, subadditive set function. Indeed, if I and I′ are disjoint, then, by definition,

$$\alpha_{I \cup I'}(x) = \alpha_I(x) + \alpha_{I'}(x)$$

for almost all x. If the sets are not disjoint, it follows that the equality sign in the equation above is replaced by the inequality sign $\leqslant$. The subadditivity of ξ now follows by integration of $(\alpha_I + \alpha_{I'})^p$ and the application of Minkowski's inequality.

LEMMA 2.2. — Let $\delta(s)$, $s > 0$,be a positive function such that

$$\liminf_{s\to 0} \frac{(\delta(s))^{1/q}}{s} \sup_{0\leqslant a<b\leqslant 1, b-a\leqslant 2s} \xi([a, b]) = 0; \tag{2.4}$$

then

$$\text{approx.} \limsup_{s\to t} \frac{|x(s) - x(t)|}{\delta(|s - t|)} = \infty, \tag{2.6}$$

for every $0 \leqslant t \leqslant 1$.

(*Remark*: We recall the following definition; see, for example [*11*]. The approximate lim sup of the function $f(s)$ for $s \to t$ is at least y if t is not a point of dispersion for the set $\{ s : f(s) \geqslant y \}$. The approximate lim sup is $+\infty$ if the latter is true for every $y > 0$.)

Proof. — For arbitrary $M > 0$, $0 \leqslant t \leqslant 1$, and $s > 0$

$$(2s)^{-1} \text{ mes } \{ t' : |t - t'| \leqslant s, |x(t') - x(t)| < M\delta(|t' - t|) \}$$

is, by Lemma 2.1, at most equal to

$$\frac{(2M\delta(s))^{1/q}}{2s} \sup_{0\leqslant a<b\leqslant 1, b-a\leqslant 2s} \xi([a, b]) .$$

By (2.4), the latter has the lim inf 0. Therefore, t is not a density point for the set,

$$\left\{ s: \frac{|x(s) - x(t)|}{\delta(|s - t|)} < M \right\}.$$

Therefore, it is not a dispersion point for the complementary set,

$$\left\{ s: \frac{|x(s) - x(t)|}{\delta(|s - t|)} \geq M \right\}.$$

Therefore, the approximate lim sup of the ratio is at least equal to M. Since M is arbitrary, the conclusion follows.

The next result is about the relation between the smoothness of α_I as a function of I, and the magnitude of the level sets $\{ s: x(s) = x(t) \}$ for $0 \leqslant t \leqslant 1$. The magnitude will be expressed in terms of a Hausdorff measure. We recall the definition of such a measure. Let $w(t)$, $0 \leqslant t \leqslant 1$, be an increasing, right continuous function such that $w(0) = 0$; such a function is called a measure function. For an arbitrary subinterval B of $[0, 1]$, put $|B| =$ length of B. For arbitrary $\delta > 0$, and a subset $J \subset [0, 1]$, define

$$\mu_0(J) = \inf \sum_n w(|B_n|),$$

where the infimum is taken over all sequences of intervals B_n whose length is at most δ, and whose union contains J. Then we define the w-measure of J as $\lim_{\delta \to 0} \mu_\delta(J)$.

Suppose that w satisfies the additional condition:

$$(2.7) \qquad w(t)/t \text{ is decreasing, and } w(t)/t \to \infty \qquad \text{for } t \downarrow 0\,.$$

If, in the definition of the w-measure, we restrict the class of intervals B to the dyadic rational intervals,

$$(2.8) \qquad I_{n,k} = [k2^{-n}, (k+1)^{2^{-n}}] \qquad 1 \leqslant k \leqslant 2^n, \quad n \geqslant 0\,,$$

then we obtain a possibly larger measure for a given set through the same covering process. Let $\mu(J)$ and $\mu^*(J)$ be the measures calculated from the class of all intervals and all dyadic intervals, respectively: then,

$$(2.9) \qquad \mu(J) \leqslant \mu^*(J) \leqslant 2\mu(J)\,.$$

This was originally proved by Besicovitch [*8*] for the special functions $w(t) = t^\alpha$, but the proof is valid for all measure functions w satisfying (2.7); see Hawkes [*12*].

LEMMA 2.3..—.Let w be a measure function satisfying (2.7), and let $\alpha_{n,k}$ be the local time of the function $x(t)$, $0 \leqslant t \leqslant 1$, relative to the interval $I_{n,k}$ in (2.8). If for some $p \geqslant 1$,

$$(2.10) \qquad \sum_{n=0}^{\infty} w^{-p}(2^{-n}) \sum_{k=1}^{2^n} \int_{-\infty}^{\infty} \alpha_{n,k}^p(x)dx < \infty\,,$$

then there exists a set N of Lebesgue measure 0 such that

$$(2.11) \qquad \alpha_J(x) = 0, \qquad \text{for} \quad x \in {}^c N\,,$$

for every Lebesgue measurable set J of finite w-measure.

Proof. — The assumption (2.10) and Fubini's theorem imply the existence of a set N of measure 0 such that

$$\sum_{n=0}^{\infty} \sum_{k=1}^{2^n} w^{-p}(2^{-n})\alpha_{n,k}^p(x) < \infty\,,$$

for all $x \in {}^c N$. Therefore,

$$(2.12) \qquad \max_{1 \leqslant k \leqslant 2^n} \alpha_{n,k}(x) = \text{l. c.}\,(w(2^{-n})), \qquad \text{for} \quad n \to \infty\,,$$

for all $x \in {}^c\mathrm{N}$. If J is a measurable set of finite w-measure, then, for $x \in {}^c\mathrm{N}$, $\alpha_{\mathrm{J}}(x)$ is dominated by a sum $\Sigma\alpha_{n,k}(x)$ over an arbitrary covering subfamily of $\{ \mathrm{I}_{n,k} \}$. (We have implicitly used the fact that there is a version of the local time which is a measure in sets I [*3*].) By (2.12) there is such a sum which is arbitrarily small, and this proves (2.11).

THEOREM 2.1. — Let w be a measure function satisfying (2.7) and (2.10). Then, for almost all t, $0 \leqslant t \leqslant 1$, the set $\{ s: x(s) = x(t) \}$ is of infinite w-measure.

Proof. — Put $\mathrm{M} = \{ y: \{ s: x(s) = y \}$ is of finite w-measure $\}$. Let N be as in Lemma 2.3. If $x \in \mathrm{M} \cap {}^c\mathrm{N}$, then

$$\alpha_{\{s:x(s)=x\}}(x) = 0 .$$

By the result of Geman and Horowitz [*11*], Theorem (6.4), it follows that if another fixed set of measure 0 is ignored, then the equation above implies $\alpha_{[0,1]}(x) = 0$. Thus, with the exception of a null set, we have $\mathrm{M} \subset \{ x: \alpha_{[0,1]}(x) = 0 \}$. The latter inclusion also holds for the corresponding preimages because the inverse image of a null set for a function with a local time is also a null set.

This means that

$$\{ t: \{ s: x(s)=x(t) \} \quad \text{is of finite} \quad w\text{-measure} \} \subset \{ t: \alpha_{[0,1]}(x(t))=0 \},$$

except for a null t-set. But the second set displayed above has t-measure 0; see [*11*], formula (6.7). The proof is complete.

3. SUFFICIENT CONDITIONS FOR THE EXISTENCE AND SQUARE INTEGRABILITY OF THE LOCAL TIME OF THE SAMPLE FUNCTION OF A STOCHASTIC PROCESS

Let $\mathrm{X}(t)$, $0 \leqslant t \leqslant 1$, be a real valued, separable and measurable stochastic process. Suppose that for each s and t, $s \neq t$, the random variables $\mathrm{X}(s)$ and $\mathrm{X}(t)$ have a joint density function $p(x, y; s, t)$.

THEOREM 3.1. — If the function

$$q(x, y) = \int_0^1 \int_0^1 p(x, y; s, t)dsdt \tag{3.1}$$

is continuous in (x, y), and

$$\int_{-\infty}^{\infty} q(x, x)dx < \infty, \tag{3.2}$$

then the local time $\alpha_I(x)$ exists almost surely, and

$$\mathrm{E}\int_{-\infty}^{\infty} \alpha_I^2(x)dx \leqslant \int_I\int_I\int_{-\infty}^{\infty} p(x, x; s, t)dxdsdt, \tag{3.3}$$

for every measurable $I \subset [0, 1]$.

Proof. — First we show that $q(x, y)$ is a positive definite kernel on R^2. Indeed, for any bounded measurable function $g(x)$, we have, by (3.1) and Fubini's theorem,

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(x, y)g(x)g(y)dxdy &= \int_0^1\int_0^1 \mathrm{E}[g(X(s))g(X(t))]dsdt \\ &= \mathrm{E}\left\{\int_0^1 g(X(t))dt\right\}^2 \geqslant 0.\end{aligned}$$

Since, by assumption, q is continuous, the Cauchy-Schwarz inequality implies

$$q(x, y) \leqslant [q(x, x)q(y, y)]^{\frac{1}{2}} \tag{3.4}$$

for all x and y.

Let $\chi_A(x)$ be the indicator function of the set A, and define, for $\varepsilon > 0$,

$$\delta_\varepsilon(x) = (2\varepsilon)^{-1}\chi_{[-1,1]}(x/\varepsilon). \tag{3.5}$$

According to [11], Theorem (21.15), the existence almost surely of a square integrable local time is implied by

$$\liminf_{\varepsilon\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_\varepsilon(x-y)q(x, y)dxdy < \infty. \tag{3.6}$$

By (3.4), the double integral in (3.6) is at most equal to

$$\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} \delta_\varepsilon(x-y)q^{\frac{1}{2}}(x, x)dx\right\} q^{\frac{1}{2}}(y, y)dy.$$

Since δ_ε is a density function, the second moment inequality implies that the inner integral above is at most equal to

$$\left\{\int_{-\infty}^{\infty} \delta_\varepsilon(x-y)q(x, x)dx\right\}^{\frac{1}{2}},$$

so that the preceding double integral is at most equal to

$$\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\delta_\varepsilon(x-y)q(x,x)dx\right\}^{\frac{1}{2}}q^{\frac{1}{2}}(y,y)dy\,.$$

By the Cauchy-Schwarz inequality, the latter is at most equal to the square root of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta_\varepsilon(x-y)q(x,x)dxdy\,.\int_{-\infty}^{\infty}q(y,y)dy=\left\{\int_{-\infty}^{\infty}q(y,y)dy\right\}^2.$$

This completes the proof of (3.6), and so a square integrable local time exists. The inequality (3.3) now follows by using the calculations above in the representation of $\int\alpha_{\mathrm{I}}^2(x)dx$ as (see [*11*], Theorem (7.2))

$$\int_{\mathrm{I}}\lim_{\varepsilon\to 0}\int_{\mathrm{I}}\delta_\varepsilon(\mathrm{X}(s)-\mathrm{X}(t))dsdt\,,$$

and then employing Fatou's lemma. Here the set I simply replaces [0, 1] as the domain of integration.

Remark. — Theorem 3.1 is related to that of Pitt [*17*], but the hypothesis and conclusion are different. He notes that his hypothesis A_k (here for $k = 2$) can actually be weakened to the assumption of the boundedness of our function $q(x, y)$; see [*17*], page 314. However, his conclusion is that the local time is square integrable only over compact sets.

We also note that Theorem 3.1 is also more general than our corresponding result in [*6*], Theorem 3.1, because there we supposed that the densities are positive definite.

4. AN INEQUALITY FOR THE HIGHER MOMENTS OF THE LOCAL TIME

Assume that, for every $k \geqslant 1$, the k-dimensional distributions of the process have a density with respect to Lebesgue measure; and let $p(x_1, \ldots, x_k; t_1, \ldots, t_k)$ be the joint density function of random variables $\mathrm{X}(t_1), \ldots, \mathrm{X}(t_k)$ at the point $(x_1, \ldots, x_k)$. Define

$$(4.1)\quad q_{\mathrm{I}}(x_1,\ldots,x_k)=\int_{\mathrm{I}}\cdots\int_{\mathrm{I}}p(x_1,\ldots,x_k;t_1,\ldots,t_k)dt_1\ldots dt_k\,.$$

We now obtain an inequality for the latter function for a certain subsequence of integer values k.

LEMMA 4.1. — If k is a positive integral power of 2, and q_{I}, defined by (4.1), is continuous on R^k, then

$$q_{\mathrm{I}}^k(x_1, \ldots, x_k) \leqslant \prod_{i=1}^{k} q_{\mathrm{I}}(x_i, \ldots, x_i). \tag{4.2}$$

Proof. — For typographical convenience, we suppress the index set I, and put $q = q_{\mathrm{I}}$. We give the proof first for $k = 4$ where the computation is relatively simple, but sufficiently illustrative of the general case. We note that (3.4) covers the case $k = 2$.

We observe that $q(x_1, x_2, x_3, x_4)$ is a positive definite kernel in the two (2-dimensional) variables (x_1, x_2) and (x_3, x_4). Indeed, if $g(x, y)$ is an arbitrary bounded measurable function on R^2, then, as in the proof of Theorem 3.1,

$$\int_{\mathrm{R}^4} q(x_1, x_2, x_3, x_4) g(x_1, x_2) g(x_3, x_4) dx_1 dx_2 dx_3 dx_4$$
$$= \mathrm{E}\left[\int_{\mathrm{I}}\int_{\mathrm{I}} g(\mathrm{X}(s), \mathrm{X}(t)) ds dt\right]^2 \geqslant 0.$$

Then it follows from the assumed continuity of q and the Cauchy-Schwarz inequality that

$$q^2(x_1, x_2, x_3, x_4) \leqslant q(x_1, x_2, x_1, x_2) q(x_3, x_4, x_3, x_4). \tag{4.3}$$

It is clear from (4.1) that q is a symmetric function of its variables; hence, by another application of (4.3),

$$q(x_1, x_2, x_1, x_2) = q(x_1, x_1, x_2, x_2) \leqslant q^{\frac{1}{2}}(x_1, x_1, x_1, x_1) q^{\frac{1}{2}}(x_2, x_2, x_2, x_2),$$

and a similar bound holds for $q(x_3, x_4, x_3, x_4)$. This completes the proof of (4.2) for $k = 4$.

The proof of the general case $k = 2^m$ for some integer $m \geqslant 1$ is conceptually similar but notationally more complex. The positive definiteness and continuity of q imply

$$q^2(x_1, \ldots, x_{k/2}, x_{k/2+1}, \ldots, x_k)$$
$$\leqslant q(x_1, \ldots, x_{k/2}, x_1, \ldots, x_{k/2}) q(x_{k/2+1}, \ldots, x_k, x_{k/2+1}, \ldots, x_k)$$

By symmetry, the latter is equal to

$$q(x_1, x_1, \ldots, x_{k/2}, x_{k/2}) q(x_{k/2+1}, x_{k/2+1}, \ldots, x_k, x_k).$$

Another application of positive definiteness yields, through the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&q^2(x_1, x_1, \ldots, x_{k/2}, x_{k/2}) \\
&\qquad \leqslant q(x_1, x_1, \ldots, x_{k/4}, x_{k/4}, x_1, x_1, \ldots, x_{k/4}, x_{k/4}) \\
&\qquad \times q(x_{k/4+1}, x_{k/4+1}, \ldots, x_{k/2}, x_{k/2}, x_{k/4+1}, x_{k/4+1}, \ldots, x_{k/2}, x_{k/2}),
\end{aligned}
$$

which, by symmetry, is equal to

$$
\begin{aligned}
&q(x_1, x_1, x_1, x_1, \ldots, x_{k/4}, x_{k/4}, x_{k/4}, x_{k/4}) \\
&\qquad\qquad \times q(x_{k/4+1}, x_{k/4+1}, x_{k/4+1}, x_{k/4+1}, \ldots, x_{k/2}, x_{k/2}, x_{k/2}, x_{k/2}).
\end{aligned}
$$

This procedure is repeated until k identical variables are obtained in the right hand bound for the q-functions.

THEOREM 4.1. — Under the conditions of Theorem 3.1 and Lemma 4.1, we have

$$
\mathrm{E}\int_{-\infty}^{\infty} \alpha_{\mathrm{I}}^{k}(x)dx \leqslant \int_{-\infty}^{\infty} q_{\mathrm{I}}(x, \ldots, x)dx, \tag{4.4}
$$

for every I, and where the right hand member may be finite or not.

Proof. — By the results of [*11*], Section 7, we may express the left hand member of (4.4) as

$$
\mathrm{E}\int_{\mathrm{I}} \alpha_{\mathrm{I}}^{k-1}(\mathrm{X}(t))dt, \qquad \text{or} \qquad \mathrm{E}\int_{\mathrm{I}} \lim_{\varepsilon\to 0}\left\{\int_{\mathrm{I}} \delta_\varepsilon(\mathrm{X}(s) - \mathrm{X}(t))ds\right\}^{k-1} dt,
$$

which, by Fatou's lemma, is at most equal to the lim inf, for $\varepsilon \to 0$, of

$$
\int_{\mathrm{I}} \ldots \int_{\mathrm{I}} \mathrm{E}\prod_{i=1}^{k-1} \delta_\varepsilon(\mathrm{X}(t_i) - \mathrm{X}(t_k))dt_1 \ldots dt_k .
$$

By (4.5), the latter is equal to

$$
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \delta_\varepsilon(x_i - x_k)q_{\mathrm{I}}(x_1, \ldots, x_k)dx_1 \ldots dx_k .
$$

By Lemma 4.1, the latter is at most equal to

$$
\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} \delta_\varepsilon(x - y)q_{\mathrm{I}}^{1/k}(x, \ldots, x)dx\right\}^{k-1} q_{\mathrm{I}}^{1/k}(y, \ldots, y)dy .
$$

Since δ_ε is a probability density, the kth moment inequality implies that the integral above is at most equal to

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \delta_\varepsilon(x - y) q_{\mathrm{I}}(x, \ldots, x) dx \right\}^{(k-1)/k} q_{\mathrm{I}}^{1/k}(y, \ldots, y) dy .$$

Now apply the Holder inequality with $p = k$; then the expression above is at most equal to

$$\left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_\varepsilon(x - y) q_{\mathrm{I}}(x, \ldots, x) dx dy \right\}^{(k-1)/k}$$
$$\times \left\{ \int_{-\infty}^{\infty} q_{\mathrm{I}}(y, \ldots, y) dy \right\}^{1/k} = \int_{-\infty}^{\infty} q_{\mathrm{I}}(x, \ldots, x) dx .$$

This completes the proof.

Remark. — The comment following the proof of Theorem 3.1 regarding Pitt's result is also appropriate here. We also note that we do not require his conditions of the type B_k, concerning Hölder conditions in the space variables of the joint density.

5. LOCAL NONDETERMINISM

Let $X(t)$, $0 \leqslant t \leqslant 1$, be a real valued stochastic process whose finite-dimensional distributions have densities with respect to Lebesgue measure. For $0 \leqslant t_1 < \ldots < t_k \leqslant 1$, let $\rho_k(t_1, \ldots, t_k)$ be the joint density function of the $k - 1$ increments $X(t_{j+1}) - X(t_j)$, $j = 1, \ldots, k - 1$, at the origin in R^{k-1}. By an elementary transformation of variables, ρ_k is obtained from the joint density function of $X(t_i)$, $i = 1, \ldots, k$, by means of the formula,

$$\rho_k(t_1, \ldots, t_k) = \int_{-\infty}^{\infty} p(x, \ldots, x; t_1, \ldots, t_k) dx . \tag{5.1}$$

DÉFINITION 5.1. — Let $g(t)$ be a nonnegative measurable function such that

$$\int_0^1 g(t) dt < \infty .$$

$X(t)$ is said to be locally g-nondeterministic if there is a sequence of positive numbers c_k such that

$$\rho_k(t_1, \ldots, t_k) \leqslant c_k \prod_{j=1}^{k-1} g(| t_{j+1} - t_j |), \tag{5.2}$$

for all $0 \leqslant t_1 < \ldots < t_k \leqslant 1$, and all $k \geqslant 2$.

When $k = 2$, $\rho_2(t_1, t_2)$ is the density of $X(t_2) - X(t_1)$ at 0, and (5.2) asserts that ρ_2 is bounded by $g(|t_2 - t_1|)$. If X has stationary increments, then ρ_2 is a function of $t_2 - t_1$, and we write $\bar{\rho}_2(t_2 - t_1) = \rho_2(t_1, t_2)$. In such cases, $\bar{\rho}_2$ itself often serves in the role of the function g; here (5.2) becomes

$$(5.3) \qquad \rho_k(t_1, \ldots, t_k) \leqslant c_k \prod_{j=1}^{k-1} \rho_2(t_{j+1}, t_j).$$

By writing the joint density of the increments as the product of the successive conditional densities, we see that the relation (5.3), for every $k \geqslant 2$, is equivalent to

Conditional density of $X(t_{j+1}) - X(t_j)$ at 0, given $X(t_{i+1}) - X(t_i) = 0$ for $i = 1, \ldots, j-1$, $\leqslant$ Unconditional density of $X(t_{j+1}) - X(t_j)$ at 0, *times* c_j, for all $j \geqslant 2$.

This signifies that the « likelihood » that the increment is equal to 0, given that successive increments in the recent past have been equal to 0, is of the same order of magnitude as when there is no information about the values of past increments. This is an extension of the concept of local nondeterminism in the Gaussian case, where the density at the origin is the square root of the reciprocal of the determinant of the covariance matrix. The idea of using a function g other than the density ρ_2 was introduced in the Gaussian case as local ϕ-nondeterminism by Cuzick [9].

The key estimate of this work is in the following lemma:

LEMMA 5.1. — Under the conditions of Theorem 4.1, if $X(t)$ is locally g-nondeterministic, then the right hand member of (4.4) is at most equal to

$$(5.4) \qquad k!\, c_k \left\{ \int_0^{|I|} g(s)ds \right\}^{k-1},$$

for any interval I, and any k which is a positive integral power of 2.

Proof. — It is clear that $p(x, \ldots, x; t_1, \ldots, t_k)$ is a symmetric function of the t's; hence the integral over I^k is equal to $k!$ times the integral over $t_1 < \ldots < t_k$, $t_i \in I$, $i = 1, \ldots, k$; thus,

$$q_I(x, \ldots, x) = k! \int \ldots \int_{t_1 < \ldots < t_k} p(x, \ldots, x; t_1, \ldots, t_k)dt_1 \ldots dt_k.$$

Integrate over x, and then apply Fubini's theorem; then the bound (5.4) follows from (5.1) and (5.2).

Local nondeterminism can be extended to the case of a random field $X(t)$, where $t \in R^N$ and $X(t) \in R^d$, for $d \geqslant 1$, $N \geqslant 1$. The condition on the density of the increments is independent of the dimensionality of the state space, and so it is the same for all d. In the place of the restrictions $t_j < t_{j+1}$ on the parameter values, we may use the same conditions introduced by Pitt [17] in the Gaussian case, namely, $\| t_{j+1} - t_j \| \leqslant \| t_{j+1} - t_i \|$, for $i \leqslant j$.

6. LOCAL OSCILLATION OF THE SAMPLE FUNCTIONS

In this section we apply our results in Section 3 to the sample functions of the stochastic process.

LEMMA 6.1. — Let $\xi(\mathrm{I})$, where I is an arbitrary subinterval of $[0, 1]$, be a nonnegative, subadditive and nondecreasing random interval function. Then, for every $z > 0$, $n \geqslant 1$, and $p \geqslant 1$,

$$(6.1)\quad \mathrm{P}\Big(\sup_{0 \leqslant a < b \leqslant 1, b-a \leqslant n^{-1}} \xi([a, b]) > z\Big) \leqslant 2(2/z)^p \sum_{j=1}^{n} \mathrm{E}\xi^p\left(\left[\frac{j-1}{n}, \frac{j}{n}\right]\right).$$

Proof. — Since ξ is subadditive, if $\xi([a, b]) > z$, for $0 < b-a < n^{-1}$, then there is an integer j, $1 \leqslant j \leqslant n-1$, such that $(j-1)/n \leqslant a < b \leqslant (j+1)/n$, and

$$\text{Either}\quad \xi\left(\left[\frac{j-1}{n}, \frac{j}{n}\right]\right) > \frac{1}{2} z, \quad\text{or}\quad \xi\left(\left[\frac{j}{n}, \frac{j+1}{n}\right]\right) > \frac{1}{2} z.$$

Thus the left hand member of (6.1) is at most equal to

$$2 \sum_{j=1}^{n} \mathrm{P}\left(\xi\left(\left[\frac{j-1}{n}, \frac{j+1}{n}\right]\right) > \frac{1}{2} z\right),$$

which, by the Markov inequality, is at most equal to the right hand member of (6.1)

THEOREM 6.1. — Let $\mathrm{X}(t)$ be locally g-nondeterministic, and satisfy the conditions of Theorem 4.1. Let $\delta(s)$, $s > 0$, be a positive function such that for some $\eta > 0$,

$$(6.2)\qquad \delta(s) = 0\left(\frac{s^{1+\eta}}{\int_0^{2s} g(r)dr}\right), \quad \text{for} \quad s \to 0.$$

Then, with probability 1,

$$\text{approx.} \limsup_{s \to t} \frac{|X(s) - X(t)}{\delta(|s-t|)} = \infty , \tag{6.3}$$

for all $0 \leqslant t \leqslant 1$.

Proof. — Let ξ be defined in terms of the local time as in (2.3). As indicated following that formula, ξ satisfies the conditions of our lemma. According to Lemma 2.2, it suffices to show that (2.4) holds almost surely for some p and q such that $p^{-1} + q^{-1} = 1$. The latter holds if the random variable

$$(\delta(s))^{1/q} s^{-1} \sup_{0<b-a<2s} \xi([a, b]) \tag{6.4}$$

converges to 0 in probability for $s \to 0$. (For the purpose of applying Lemma 6.1 we may even restrict the variable s to the sequence for which $2s$ is the reciprocal of an integer.) For arbitrary $\varepsilon > 0$, put $z = \varepsilon s/\delta^{1/q}(s)$, and apply (6.1); then the probability that (6.4) exceeds ε is at most equal to

$$2^{p+1}\delta^{p/q}(s)\varepsilon^{-p}s^{-p}(2s)^{-1} \sup_{0<b-a<2s} E\xi^{p}([a, b]) .$$

By Theorem 5.1, if p is of the form 2^m for some integer $m \geqslant 1$, then the expression displayed above is at most

$$2^{p} c_p p\,!\, \varepsilon^{-p}\delta^{p/q}(s)s^{-p-1}\left(\int_0^{2s} g(r)dr\right)^{p-1} . \tag{6.5}$$

Since $p/q = p - 1$, it follows from (6.2) that the expression (6.5) is of the order $s^{\eta(p-1)-2}$, which, for $p > 1 + 2/\eta$, converges to 0 for $s \to 0$. Therefore, (6.4) converges in probability to 0.

Remark. — It follows immediately that if g has the bound $g(t) \leqslant Ct^{-\beta}$ for small t, then $X(t)$ nowhere satisfies a Holder condition of order $> \beta$.

THEOREM 6.2. — Let $X(t)$ be locally g-nondeterministic and satisfy the conditions of Theorem 4.1. Let $w(t)$ be a measure function satisfying (2.7) and which, for sufficiently small t, is at least equal to some constant positive multiple of the function

$$\left(\frac{|\log t|^{1+\varepsilon}}{t}\right)^{1/p}\left(\int_0^t g(s)ds\right)^{(p-1)/p} \tag{6.6}$$

for some $\varepsilon > 0$ and some p of the form 2^m for $m \geqslant 1$. Then, with probability 1, the set $\{ s: X(s) = X(t) \}$ is of infinite w-measure for almost all t, $0 \leqslant t \leqslant 1$.

Proof. — Consider the series appearing in (2.10), where the function $x(t)$ is now the process $X(t)$. Take the expected value of the series, and then change the order of summation and expectation. By Lemma 5.1, the expected value is at most equal to a constant times

$$\sum_{n=0}^{\infty} 2^n \left(\int_0^{2^{-n}} g(s)ds \right)^{p-1} w^{-p}(2^{-n}).$$

The series converges because w has the lower asymptotic bound (6.6). It follows that the series (2.10) for $X(t)$ converges with probability 1. The assertion of this theorem now follows from Theorem 2.1.

COROLLARY. — If there is a number β, $0 < \beta < 1$, such that $g(t) \leqslant Ct^{-\beta}$ for t near 0, then, with probability 1, the Hausdorff dimension of the set $\{ s: X(s) = X(t) \}$ is at least equal to $1 - \beta$, for almost all t, $0 \leqslant t \leqslant 1$.

Proof. — For arbitrary b, $0 \leqslant b < 1 - \beta$, consider the measure function $w(t) = t^b$. If p is sufficiently large, then $b < -1/p+(1-\beta)(p-1)/p$. It follows that the expression (6.6) is at most equal to a constant times t^b for small t. Theorem 6.2 implies that the w-measure is infinite, which implies that the Hausdorff dimension of the set is greater than b.

We remark that local time methods give only a lower bound on the magnitude of the level sets: indeed, they furnish a measure of the irregularity of the sample function. The upper bound on the magnitude of the level sets is based on the regularity of the sample function, and has to be obtained by the corresponding methods.

7. APPLICATIONS

In applying the idea of local g-nondeterminism, it is useful to know the form of the likely candidate for the role of the function g. As we noted in Section 5, in the case of stationary increments we often consider the density function of the increment at 0. Even if the increments are not stationary, we might still be able to find a function g such that the density of the random variable $(X(s)-X(t)) \,.\, g(|s-t|)$ at the origin is bounded for $|s - t| \to 0$. If X is stochastically continuous, then we necessarily have $g(t) \to \infty$ for $t \to 0$.

EXAMPLE 7.1. — As noted earlier, local nondeterminism was introduced by the author first in the context of Gaussian processes. In the case of stationary increments, the function g took the form of the reciprocal of the incremental standard deviation. Early cases of Theorems 6.1 and 6.2 are those in [*3*], [*4*] and [*15*]. Again we refer to [*11*] for a complete survey.

EXAMPLE 7.2. — Let $X(t)$ be a time homogeneous Markov process with a transition density function $p(t; x, y)$ representing the conditional density of $X(t)$ at y, given $X(0) = x$. Suppose that $X(0) = x_0$. Then the joint density of $X(s)$ and $X(t)$ is

$$\begin{aligned} p(x, y; s, t) &= p(s; x_0, x)p(t - s; x, y), \qquad s < t \\ &= p(t; x_0, y)p(s - t; y, x), \qquad s > t. \end{aligned}$$

It follows that the function q in (3.1) is

$$\begin{aligned} q(x, y) = &\int_0^1 \int_0^t p(s; x_0, x)p(t - s; x, y)dsdt \\ &+ \int_0^1 \int_0^s p(t; x_0, y)p(s - t; y, x)dtds. \end{aligned}$$

The condition (3.2) becomes

$$\int_0^1 \int_0^t \int_{-\infty}^{\infty} p(s; x_0, x)p(t - s; x, x)dxdsdt < \infty. \tag{7.1}$$

For arbitrary $t_1 < \ldots < t_k$, the joint density of $X(t_1), \ldots, X(t_k)$ at $(x_1, \ldots, x_k)$ is

$$\prod_{j=1}^{k} p(t_j - t_{j-1}; x_{j-1}, x_j), \qquad \text{with} \qquad t_0 = 0.$$

Therefore the joint density of $X(t_{j+1}) - X(t_j), j = 1, \ldots, k - 1$, at $(0, \ldots, 0)$ is

$$\int_{-\infty}^{\infty} p(t_1; x_0, x)p(t_2 - t_1; x, x) \ldots p(t_k - t_{k-1}; x, x)dx. \tag{7.2}$$

Put

$$g(t) = \sup_x p(t; x, x), \tag{7.3}$$

and suppose that $\int_0^1 g(t)dt < \infty$. Then (7.2) is dominated by

$$g(t_2 - t_1) \ldots g(t_k - t_{k-1}),$$

and so the process is locally g-nondeterministic. The implications of Theorems 6.1 and 6.2 are new for the general processes considered here.

EXAMPLE 7.3. — Let us specialize the previous example to the case where the process has stationary, independent increments. Let

$$\mathrm{E}e^{iu(\mathrm{X}(t)-\mathrm{X}(s))} = e^{-(t-s)\Psi(u)}, \qquad s < t,$$

be the Levy representation of the characteristic function. According to [2] a sufficient condition for the existence and square integrability of the local time for almost all sample functions is

$$\frac{1-e^{-\Psi u)}}{\Psi(u)} du < \infty . \tag{7.4}$$

If

$$\int_{-\infty}^{\infty} | e^{-t\Psi(u)} | \, du < \infty, \qquad \text{for} \qquad 0 < t \leqslant 1,$$

then the density of $\mathrm{X}(t) - \mathrm{X}(s)$ exists and is continuous, and is equal to

$$\int_{-\infty}^{\infty} e^{-(t-s)\Psi(u)} du \tag{7.5}$$

at the origin. Since the increments over nonoverlapping intervals are mutually independent, it follows that the joint density of the increments is equal to the product of the functions (7.5) for

$$t - s = t_{j+1} - t_j, \qquad j = 1, \ldots, k-1 .$$

The transition density of the process is

$$p(t\,; x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iu(y-x)-t\Psi(u)} du \,;$$

hence, the function $g(t)$ defined by (7.3) takes the form (7.5). The implications of Theorem 6.1, even in this special case, appear to be new.

The results of Theorem 6.2 are consistent with earlier results on the exact dimension of the level sets for processes with stationary independent increments, but not as refined. If $\Psi(u) = \mathrm{C} \,|\, u \,|^{\alpha}$, $1 < \alpha \leqslant 2$, that is, the process is stable of index α, then it has been shown by Taylor and Wendel [*19*] that $w(t) = t^{1-1/\alpha}(\log\log t)^{1/\alpha}$ is the exact measure function for a fixed level set in the range.

Extensions of this result to the general class of processes with stationary independent increments are described in the surveys of Taylor [*18*] and Fristedt [*10*]. More recently, Perkins [*16*] proved a very exact result in the case of the Brownian motion process: If $w(t) = (2t \log\log t)^{\frac{1}{2}}$, then the w-measure of $(s\colon \mathrm{X}(s) = x,\ 0 \leqslant s \leqslant t)$ is equal to the local time at x, and this holds for *all* x, almost surely.

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