ANNALES DE L'I. H. P., SECTION B

WALTER SCHACHERMAYER

# The class of Banach spaces, which do not have c0 as a spreading model, is not L2-hereditary

*Annales de l'I. H. P., section B*, tome 19, nº 1 (1983), p. 1-8
<http://www.numdam.org/item?id=AIHPB_1983__19_1_1_0>

*Ann. Inst. Henri Poincaré,* Section B :
Vol. XIX, n° 1, 1983, p. 1-8. *Calcul des Probabilités et Statistique.*

# The class of Banach spaces, which do not have $c_0$ as a spreading model, is not $L^2$-hereditary

by

**Walter SCHACHERMAYER (LINZ)**
Dr. W. Schachermayer, Universität Linz, Altenberger Str. 69,
A-4045 Linz, Autriche

## 1. INTRODUCTION

In [*4*] the problem was raised wether the fact, that a Banach space E does not have $c_0$ as a spreading model, implies that $L^2([0, 1]; E)$ has the same property. It was conjectured that the answer is no, as the property of not having $c_0$ as a spreading model is somewhat dual to the Banach-Saks property (see [*2*]) and for this latter property J. Bourgain has constructed a counterexample ([*3*]).

The present author has constructed independently of J. Bourgain another space E with the Banach-Saks property and $L^2(E)$ failing it ([*6*]) and it turns out that the dual E′ gives a counterexample to the problem raised in the title.

## 2. THE EXAMPLE

Let $\gamma = \{ n_1, n_2, \ldots, n_k \}$ an increasing finite sequence of natural numbers. Write $n_i = 2^{u_i} + v_i$ where this expression is unique, if we require that $\leqq v_i < 2^{u_i}$. As in [*6*] we associate to every $n_i$ the real number $t(n_i) = v_i/2^{u_i} \in [0, 1[$ and call $\gamma$ *admissible* if

(1) $k \leqq n_1$.

(2) For every $0 \leqq j < 2^{u_1+1}$ there is only one $i$ such that $t(n_i) \in [j/2^{u_1+1}, (j+1)/2^{u_1+1}[$.

For an admissible $\gamma = (n_1, \ldots, n_k)$ and $x \in \mathbb{R}^{(\mathbb{N})}$, the space of finite sequences, we define

$$\| x \|_\gamma = \sum_{i=1}^{k} | x_{n_i} | .$$

For our purposes it will this time be convenient, not to use interpolation but to follow Baernstein's original definition ([*1*]): For $x \in \mathbb{R}^{(\mathbb{N})}$ define

$$\| x \|_{\mathrm{E}} = \sup \left\{ \left( \sum_{l=1}^{\infty} \| x \|_{\gamma_l}^2 \right)^{1/2} \right\}$$

where the sup is taken over all increasing sequences $\{ \gamma_l \}_{l=1}^{\infty}$ of admissible sets (i. e. the last member of $\gamma_l$ is smaller than the first member of $\gamma_{l+1}$).

Let $(\mathrm{E}, \| \ \|_{\mathrm{E}})$ be the completion of $\mathbb{R}^{(\mathbb{N})}$ with respect to this norm. In an analogous way as in [*6*] we shall show that E has the Banach-Saks property, i. e. that it does not have a spreading model isomorphic to $l^1$; we shall also show that E' does not have a spreading model isomorphic to $c_0$. In fact we shall prove a much stronger result.

PROPOSITION 1. — *a*) Every spreading model based on a normalized weak null sequence $(x_n)_{n=1}^{\infty}$ of E is isomorphic to $l^2$.

*b*) Every spreading model based on a normalized weak null sequence $(y_n)_{n=1}^{\infty}$ of E' is isomorphic to $l^2$.

*Proof.* — *a*) Let $(x_n)_{n=1}^{\infty}$ be a normalized weak null sequence in E. As $(x_n)_{n=1}^{\infty}$ converges to zero coordinatewise, we may assume (by a standard perturbation argument) that the $x_n$'s are supported by disjoint blocks, i. e. there is an increasing sequence $(r(n))_{n=1}^{\infty}$ of natural numbers, such that with $r(0) = 0$

$$x_n = \sum_{i=r(n-1)+1}^{r(n)} \lambda_i^{(n)} e_i .$$

Then for every sequence $\alpha_1, \ldots, \alpha_k$ of scalars and $n_1 < \ldots < n_k$, the estimate

$$\left\| \sum_{i=1}^{k} \alpha_i x_{n_i} \right\|_{\mathrm{E}} \geqq \left( \sum_{i=1}^{k} | \alpha_i |^2 \right)^{1/2} \tag{1}$$

holds trivially in view of the definition of $\| \, . \, \|_{\mathrm{E}}$.

For the converse let $1 > \varepsilon > 0$ and choose inductively an increasing sequence $(n_k)_{k=1}^{\infty}$ in $\mathbb{N}$ and infinite subsets $M_k$ of $\mathbb{N}$: Let $M_0 = \mathbb{N}$ and $n_1 = 1$ and suppose $M_{k-1}$ and $n_k$ are defined. Let $p_k$ be such that $2^{p_k - 1} \leqq r(n_k) < 2^{p_k}$ and consider the partition of $[0, 1[$ into the intervals $[j/2^{p_k}, (j+1)/^{2p_k}[$, $j = 0, \ldots, 2^{p_k - 1}$. For $n \geqq n_k$ define

$$\mu_j^{(n)} = \max \{ | \lambda_i^{(n)} | : t(i) \in [j/2^{p_k}, (j + 1)/2^{p_k}[ \}.$$

Note that, for every $n$,

$$\sum_{j=0}^{2^{p_k} - 1} \mu_j^{(n)} \leqq 1$$

in view of the definition of the norm of E and the fact that $\| x_n \|_E = 1$. Find an infinite subset $\overline{M}_k$ of $M_{k-1} \cap [n_k + 1, n_k + 2, \ldots, \infty[$ such that for every $j = 0, \ldots, 2^{p_k} - 1$ the sequence $(\mu_j^{(n)})_{n \in \overline{M}_k}$ converges, to $\mu_j$ say. Clearly

$$\sum_{j=0}^{2^{p_k} - 1} \mu_j \leqq 1 .$$

Finally let $M_k$ be the subset of $\overline{M}_k$ consisting of those $n$ for which for every $j = 0, \ldots, 2^{p_k} - 1$

$$\mu_j^{(n)} \leqq \mu_j + 2^{-p_k} . \varepsilon/3$$

and let $n_{k+1}$ be the first element of $M_k$. This completes the induction.

Note that for an admissible $\gamma = (m_1, \ldots, m_q)$ and $k \in \mathbb{N}$ such that $\inf(\gamma) = m_1 \leqq r(n_k)$ and for every choice of scalars $\alpha_{k+1}, \ldots, \alpha_{k+l}$

$$\left\| \sum_{i=1}^{l} \alpha_{k+i} x_{n_{k+i}} \right\|_{\gamma} \leqq (1 + \varepsilon/3) \sup_{1 \leq i \leq l} \{ | \alpha_{k+i} | \} \leqq \sqrt{1+\varepsilon} \left( \sum_{i=1}^{l} | \alpha_{k+i} |^2 \right)^{1/2} \tag{2}$$

Indeed, as $m_1 \leqq r(n_k) < 2^{p_k}$, we see that $\gamma$ may contain for every $j = 0, \ldots, 2^{p_k - 1}$ at most one index $m_r$ $(1 \leqq r \leqq q)$ with $t(m_r) \in [j/2^{p_k}, (j+1)/2^{p_k}[$; by construction the $m_r$'th entry of each $x_{n_{k+i}}$ $(1 \leqq i \leqq l)$ is bounded in absolut value by $\mu_j + 2^{-p_k} . \varepsilon/3$. As the $x_{n_i}$ are disjointly supported we get

$$\left| \sum_{i=1}^{l} \alpha_{k+1} x_{n_{k+i}}(m_r) \right| \leqq \sup_{1 \leq i \leq l} \{ | \alpha_{k+i} | \} (\mu_j + 2^{-p_k} . \varepsilon/3)$$

Summing over $j$ and recalling the definition of $\| . \|_{\gamma}$ we get the first inequality of (2), while the second is trivial.

We now shall pass to the general case. Fix a sequence $\alpha_1, \ldots, \alpha_k$ of scalars. We shall show

$$\left\| \sum_{i=1}^{k} \alpha_i x_{n_i} \right\|_{\mathrm{E}} \leqq \sqrt{6 + 3\varepsilon}\,.\left( \sum_{i=1}^{k} |\alpha_i|^2 \right)^{1/2} \tag{3}$$

which (in view of (1) and the arbitrarness of $\varepsilon > 0$) will readily prove ($a$). So fix an increasing sequence $\gamma_1 < \gamma_2 < \ldots < \gamma_l$ of admissible sets. For brevity we write

$$x = \sum_{i=1}^{k} \alpha_i x_{n_i}\,.$$

For $i = 1, \ldots, k$ let $\mathrm{J}(i)$ be the set of $j \in \{1, \ldots, l\}$ such that the last element of $\gamma_j$ lies in $]r(n_{i-1}), r(n_i)]$. If $\mathrm{J}(i)$ is not empty denote $j(i)$ the first element of $\mathrm{J}(i)$ and let $s(j(i))$ be the element $s \in \{1, \ldots, k\}$, such that the first element of $\gamma_{j(i)}$ lies $]r(n_{s-1}, r(n_s)]$. Note that for the $j \in \mathrm{J}(i), j > j(i)$ the first and the last element of $\gamma_j$ lie in $]r(n_{i-1}), r(n_i)]$, while for $j(i)$ in general only the last element lies in $]r(n_{i-1}), r(n_i)]$. So we may estimate

$$\left( \sum_{j=1}^{l} \| x \|_{\gamma_j}^2 \right)^{1/2} = \left( \sum_{\substack{i=1 \\ \mathrm{J}(i) \neq \emptyset}}^{k} \sum_{j \in \mathrm{J}(i)} \| x \|_{\gamma_j}^2 \right)^{1/2}$$

$$= \left( \sum_{\substack{i=1 \\ \mathrm{J}(i) \neq \emptyset}}^{k} \left( \| x \|_{\gamma_{j(i)}}^2 + \sum_{\substack{j \in \mathrm{J}(i) \\ j > j(i)}} \| x \|_{\gamma_j}^2 \right) \right)^{1/2}$$

$$= \left( \sum_{\substack{i=1 \\ \mathrm{J}(i) \neq \emptyset}}^{k} \left( \left\| \sum_{s=s(j(i))}^{i} \alpha_s x_{n_s} \right\|_{\gamma_{j(i)}}^2 + \sum_{\substack{j \in \mathrm{J}(i) \\ j > j(i)}} \| \alpha_i x_{n_i} \|_{\gamma_{j(i)}}^2 \right) \right)^{1/2}$$

$$\leqq \left( \sum_{\substack{i=1 \\ \mathrm{J}(i) \neq \emptyset}}^{k} \left( \left( \| \alpha_{s(j(i))} x_{n_{s(j(i))}} \|_{\gamma_{j(i)}} + \left\| \sum_{s=s(j(i))+1}^{i-1} \alpha_s x_{n_s} \right\|_{\gamma_{j(i)}} + \| \alpha_i x_{n_i} \|_{\gamma_{j(i)}} \right)^2 \right.\right.$$
$$\left.\left. + \sum_{\substack{j \in \mathrm{J}(i) \\ j > j(i)}} \| \alpha_i x_{n_i} \|_{\gamma_{j(i)}}^2 \right) \right)^{1/2}.$$

Using (2) and the fact that $(a^{1/2} + b^{1/2} + c^{1/2})^2 \leqslant 3(|a|+|b|+|c|)$ we get

$$\leqq \Bigg( \sum_{\substack{i=1 \\ \mathrm{J}(i) \neq \emptyset}}^{k} \Bigg( \Bigg( \Big( \| \alpha_{s(j(i))} x_{n_{s(j(i))}} \|^2_{\gamma_{j(i)}} \Big)^{\frac{1}{2}} + \sqrt{1+\varepsilon} \Big( \sum_{s=s(j(i))+1}^{i-1} |\alpha_s|^2 \Big)^{\frac{1}{2}}$$

$$+ \big( \| \alpha_i x_{n_i} \|^2_{\gamma_{j(i)}} \big)^{\frac{1}{2}} \Bigg)^2 + \sum_{\substack{j \in \mathrm{J}(i) \\ j > j(i)}} \| \alpha_i x_{n_i} \|^2_{\gamma_{j(i)}} \Bigg) \Bigg)^{\frac{1}{2}}$$

$$\leqq \Bigg( \sum_{\substack{i=1 \\ \mathrm{J}(i) \neq \mathrm{O}}}^{k} 3 \Big( |\alpha_{s(j(i))}|^2 + (1+\varepsilon) \sum_{s=s(j(i))+1}^{i-1} |\alpha_s|^2 + \sum_{j \in \mathrm{J}(i)} \| \alpha_i x_{n_i} \|^2_{\gamma_{j(i)}} \Big) \Bigg)^{\frac{1}{2}}$$

$$\leqq \Big( 3(1 + 1 + \varepsilon) \sum_{i=1}^{k} |\alpha_i|^2 \Big)^{\frac{1}{2}}$$

$$= \sqrt{6 + 3\varepsilon} . \Big( \sum_{i=1}^{k} |\alpha_i|^2 \Big)^{\frac{1}{2}} .$$

Hence we have proved (*3*) and thus part (*a*) of proposition 1.

*Proof of* (*b*). — It is easily seen using (*a*) that the unit vector basis $(e_i)_{i=1}^{\infty}$ of E is shrinking and boundedly complete (see [*1*] or [*6*]), hence the dual unit vectors $(e_i')_{i=1}^{\infty}$ form a basis of E′. So let $(y_n)_{n=1}^{\infty}$ be a normalized sequence tending weakly (and therefore coordinatewise) to zero. Similarly as in (*a*) we may suppose that there is an increasing sequence $(r(n))_{n=1}^{\infty}$ such that

$$y_n = \sum_{i=r(n-1)+1}^{r(n)} \rho_i^{(n)} e_i' .$$

Now choose a sequence $(x_n)_{n=1}^{\infty}$ in E, $\| x_n \| = \langle x_n, y_n \rangle = 1$, which clearly implies that $x_n$ is of the form

$$x_n = \sum_{i=r(n-1)+1}^{r(n)} \lambda_i^{(n)} e_i .$$

As in the prove of (*a*) find a subsequence $(n_k)_{k=1}^{\infty}$ such that $(x_{n_k})_{k=1}^{\infty}$ spans a space $\sqrt{6+\varepsilon}$ isomorphic to $l^2$.

Now fix a sequence $\beta_1, \ldots, \beta_k$ of scalars and find a sequence $\alpha_1, \ldots, \alpha_k$ such that

$$\sum_{i=1}^{k} |\alpha_i|^2 = 1 \quad \text{and} \quad \sum_{i=1}^{k} \alpha_i\beta_i = \left(\sum_{i=1}^{k} |\beta_i|^2\right)^{\frac{1}{2}}.$$

Denote

$$x = \sum_{i=1}^{k} \alpha_i x_{n_i}$$

$$y = \sum_{i=1}^{k} \beta_i y_{n_i}.$$

By (*a*) we know that

$$\| x \|_{\mathrm{E}} \leqq \sqrt{6 + \varepsilon}.$$

Hence

$$\begin{aligned} \| y \|_{\mathrm{E}'} &= \sup \{ | \langle \xi, y \rangle | : \xi \in \mathrm{E}, \| \xi \| \leqq 1 \} \\ &\geqq (6 + \varepsilon)^{-1/2} | \langle x, y \rangle | \\ &= (6 + \varepsilon)^{-1/2} . \sum_{i=1}^{k} \alpha_i\beta_i \langle x_{n_k}, y_{n_k} \rangle \\ &= (6 + \varepsilon)^{-1/2} \left(\sum_{i=1}^{k} |\beta_i|^2\right)^{\frac{1}{2}}. \end{aligned}$$

On the other hand the reverse inequality

$$\| y \|_{\mathrm{E}'} \leqq \left(\sum_{i=1}^{k} |\beta_i|^2\right)^{\frac{1}{2}}$$

is again easily checked directly from the definition of $\| . \|_{\mathrm{E}}$. This proves (*b*) and therefore proposition 1. □

*Remark.* — Consider the sequence of unit-vectors $(e_{2^n-1})_{n=1}^{\infty}$ in E (resp. $(e'_{2^n-1})_{n=1}^{\infty}$ in E′).

It may be checked that every spreading model based on $(e_{2^n-1})_{n=1}^{\infty}$ (resp. $(e'_{2^n-1})_{n=1}^{\infty}$) is isometric to a countable $l^2$-sum of 2-dimensional $l^1$'s, hence in this case the Banach-Mazur distance equals precisely $\sqrt{2}$.

To show that $\mathrm{L}^2(\mathrm{E}')$ does have $c_0$ as spreading model we need a trivial probabilistic lemma, whose proof is left to the reader.

LEMMA. — Let $k \in \mathbb{N}$ and $\varepsilon > 0$; there is $\mathrm{N}(k, \varepsilon)$ such that for $\mathrm{M} > \mathrm{N}(k, \varepsilon)$

and for independent randomvariables $X_1, \ldots, X_k$ taking their values in $\{1, \ldots, M\}$ in a uniformly distributed way, we have

$$P \left\{ \begin{array}{l} \omega : \text{there is} \quad 1 \leqq i < j \leqq k \quad \text{with} \\ \qquad X_i(\omega) = X_j(\omega) \end{array} \right\} < \varepsilon$$

PROPOSITION 2. — $L^2_{(0,1)}(E')$has $c_0$ isometrically as spreading model.

*Proof.* — Similarly as in [6] we let $\{ \vec{f}_u \}_{u=1}^{\infty}$ be an independent sequence in $L^2(E')$ such that $\vec{f}_u$ takes the value $e_{2^u+v}$ with probability $2^{-u}$ (for $v = 0, \ldots, 2^u - 1$). This times the $e_{2^u+v}$ are unit-vectors in $E'$.

Clearly $\| \vec{f}_u \|_{L^2(E')} = 1$ and for every sequence $u_1 < u_2 < \ldots < u_k$ and $\varepsilon_i = \pm 1$

$$\left\| \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i} \right\|_{L^2(E')} \geqq 1$$

Hence the following claim will prove the proposition.

CLAIM. — For every $k \in \mathbb{N}$

$$\lim_{u \to \infty} \sup \left\{ \left\| \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i} \right\| : \begin{array}{l} u \leqq u_1 < \ldots < u_k \\ \varepsilon_i = \pm 1 \end{array} \right\} = 1$$

To prove the claim fix $k$ and $\varepsilon > 0$ and let $u$ be such that $2^u > \max(k, \mathbb{N}(k, \varepsilon))$, where the $N(k, \varepsilon)$ is defined in the preceeding lemma. Now fix $u \leqq u_1 < u_2 < \ldots < u_k$ and a sequence of signs $\varepsilon_1, \ldots, \varepsilon_k$.

To apply the above lemma let $X_1, \ldots, X_k$ be the random variables with values in $\{1, \ldots, 2^{u_1+1}\}$ defined by

$$X_i(\omega) = m \qquad \text{if} \qquad \vec{f}_{u_i}(\omega) = e_{2^{u_i}+v}$$

and

$$t(2^{u_i} + v) = v/2^{u_i} \in [(m-1)/2^{u_1+1} \; ; \qquad m/2^{u_1+1}[$$

It follows form the above lemma and the definition of admissible sets $\gamma$ that there is a subset $A \subseteq [0, 1[$ of measure greater than $1 - \varepsilon$ such that for $\omega \in A$ the set $\gamma_\omega = \{n_1, \ldots, n_k\}$ corresponding to the indices of the unit vectors $\{ \vec{f}_{u_1}(\omega), \ldots, \vec{f}_{u_k}(\omega) \}$ is admissible. Hence for $\omega \in A$ we have

$$\begin{aligned} \left\| \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i}(\omega) \right\|_{E'} &= \sup \left\{ \left\langle \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i}(\omega), x \right\rangle : \| x \|_E \leqq 1 \right\} \\ &\leqq \sup \left\{ \left\langle \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i}(\omega), x \right\rangle : \| x \|_{\gamma_\omega} \leqq 1 \right\} \\ &= 1. \end{aligned}$$

Integrating we obtain

$$\left\| \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i} \right\|_{L^2(E')}^2 \leqq \int_A \left\| \sum_{i=1}^{k} \vec{f}_{u_i}(\omega) \right\|_{E'}^2 d\omega + \int_{[0,1]\setminus A} \left( \sum_{i=1}^{k} \| \vec{f}_{u_i} \|_{E'} \right)^2 d\omega$$
$$\leqq 1 + k^2 \varepsilon .$$

This proves the claim and therefore proposition 2. □

## REFERENCES

[1] A. BAERNSTEIN, On reflexivity and summability; *Studia Math.*, t. **42**, 1972, p. 91-94.

[2] B. BEAUZAMY, Banach Saks properties and spreading models; *Math. Scand.*, t. **44**, 1979, p. 357-384.

[3] S. GUERRE, La propriété de Banach-Saks ne passe pas de E à $L^2(E)$ ; Séminaire sur la géométrie des espaces de Banach, École Polytechnique.

[4] S. GUERRE, J.-Th. LAPRESTÉ, Quelques propriétés des modèles étalés sur les espaces de Banach, *Ann. Inst. H. Poincaré*, t. **XVI**, n° 4, 1980, p. 339-347

[5] J.-Th. LAPRESTÉ, *Sur une propriété des suites asymptotiquement inconditionnelles*, Séminaire sur la géométrie des espaces de Banach, École Polytechnique, 1978/1979, exposé XXX.

[6] W. SCHACHERMAYER, *The Banach-Saks property is not $L^2$-hereditary*, to appear in the Israel Journal.

*(Manuscrit reçu le 22 décembre 1981)*