

ANNALES DE L'I. H. P., SECTION B

A. BELLOW

L. EGGHE

Generalized fatou inequalities

Annales de l'I. H. P., section B, tome 18, n° 4 (1982), p. 335-365

http://www.numdam.org/item?id=AIHPB_1982__18_4_335_0

© Gauthier-Villars, 1982, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section B » (<http://www.elsevier.com/locate/anihpb>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

Generalized fatou inequalities

by

A. BELLOW (*)

Northwestern University, Evanston,
Illinois 60201 U. S. A.

L. EGGHE ()**

L. U. C., Universitaire Campus,
B-3610 Diepenbeek, Belgium.

SUMMARY. — The purpose of this paper is to give a « local version » of Edgar's inequalities [10], holding in any Banach space with the Radon-Nikodym Property (RNP). The inequalities are simultaneously extended in several directions; in fact, we obtain our inequalities in pointwise form. The extension is such that it permits to derive or generalize most of the convergence results for stochastic processes in amart theory (amart, weak sequential amart, uniform amart, mil).

RÉSUMÉ. — Le but de cet article est de donner une « version locale » des inégalités de Edgar [10], valable dans tout espace de Banach ayant la propriété de Radon-Nikodym. Les inégalités sont généralisées simultanément dans plusieurs directions; en fait nos inégalités sont présentées sous forme ponctuelle. L'extension que nous obtenons est telle qu'elle permet de déduire ou de généraliser la plupart des résultats de convergence pour les processus stochastiques dans la théorie des amarts (amart, amart séquentiel faible, amart uniforme, mil).

(*) The research of this author was supported by the National Science Foundation, U. S. A.

(**) The second named author expresses his thanks to the Mathematics Department of Amherst College, Amherst, Massachusetts, for their kind hospitality during the research on this paper, and to the Belgium National Science Foundation (NFWO) for their material support.

The paper is divided as follows:

1. Introduction	336
2. Pointwise inequalities and corollaries	338
3. Applications	351
4. Extension to general directed index sets	357
Appendix	363
References	364

1. INTRODUCTION

Throughout this paper (Ω, \mathcal{F}, P) will be a fixed probability space and E a (real) separable Banach space. We denote by $L^1_E = L^1_E(\Omega, \mathcal{F}, P)$ the space of all Bochner integrable functions $X : \Omega \rightarrow E$. $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ will always denote an adapted sequence (stochastic process), that is, $(\mathcal{F}_n)_{n \in \mathbb{N}}$ is an increasing sequence of sub- σ -algebras of \mathcal{F} (called stochastic basis), and each $X_n : \Omega \rightarrow E$ is \mathcal{F}_n -measurable and Bochner integrable.

The set of all simple stopping times (with respect to the stochastic basis $(\mathcal{F}_n)_{n \in \mathbb{N}}$), that is, the set of all mappings $\tau : \Omega \rightarrow \mathbb{N}$ such that $\{\tau = n\} \in \mathcal{F}_n$ for each $n \in \mathbb{N}$ and τ assumes only finitely many values, is denoted by T . If $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is an adapted sequence and $\tau \in T$, we recall that the random variable X_τ is defined by $(X_\tau)(\omega) = X_{\tau(\omega)}(\omega)$ for $\omega \in \Omega$, and the sub- σ -algebra \mathcal{F}_τ is defined by

$$\mathcal{F}_\tau = \{A \in \mathcal{F} \mid A \cap \{\tau = n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}.$$

If \mathcal{G} is a sub- σ -algebra of \mathcal{F} and $X : \Omega \rightarrow E$ is Bochner integrable, we denote by $E^\mathcal{G}X$ the conditional expectation of X with respect to \mathcal{G} .

If $\sigma \in T$ and $J \subset T$, we write $J(\sigma) = \{\tau \in J \mid \tau \geq \sigma\}$.

The starting point of this paper is the following inequality of Edgar :

THEOREM 1.1 [10]. — *Let E be a separable dual Banach space. Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an adapted sequence of E -valued integrable r. v.'s. Assume that $(X_n)_{n \in \mathbb{N}}$ is L^1_E -bounded. Then :*

$$\int_{\Omega} \limsup_{m, n \in \mathbb{N}} \|X_m - X_n\| dP \leq 2 \limsup_{\sigma \in T} \left(\sup_{\tau \in T(\sigma)} \int_{\Omega} \|E^{\mathcal{F}_\sigma} X_\tau - X_\sigma\| dP \right).$$

It is also remarked in [10] that the above result is in general false in Banach spaces with the Radon-Nikodym Property (RNP). The counter example is based on the work of [18] and [19] (the latter paper shows the

existence of a Banach space having (RNP) which is not isomorphic with a separable dual space).

However, as noted in [14], Theorem 1.1 remains true in Banach spaces with (RNP) if we require $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ to be an amart. In this setting, Edgar's inequality was extended in [14] to the case of a general directed index set.

By imposing certain natural conditions on the adapted sequence $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$, we are able to obtain a « local version » of Edgar's inequalities, valid in any Banach space with (RNP). The inequalities are simultaneously extended in several directions; in fact, we obtain our inequalities in pointwise form. The main results of the paper are contained in Section 2: Theorems 2.1, 2.2 and 2.3 (see also Corollary 2.4 for the « Distributional form » and « Integral form » of our inequalities).

Section 3 is devoted to applications of the main inequalities obtained in Section 2. We show that our inequalities are sharp enough to yield at once most of the important strong and weak a. s. convergence theorems for stochastic processes in amart theory. We thus derive or generalize:

a) The « mil » convergence theorem of Peligrad [23] and of Bellow-Dvoretzky [5].

b) The weak sequential amart (WS-amart) convergence theorem of Brunel-Sucheston [6].

c) The uniform amart convergence of Bellow [3]. (This was already remarked in [10] and [14] to follow immediately from the integral form of the inequalities).

d) The Riesz decomposition theorem for amarts of Edgar-Sucheston [12] can also be obtained, via inequalities. This in turn easily implies the Pettis convergence theorem of Uhl [24].

e) Also the following result can be proved: Let E be a subspace of a separable dual and let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an L_E^1 -bounded mil, consisting of step functions, with $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ for each $n \in \mathbb{N}$. Then $(X_n)_{n \in \mathbb{N}}$ converges strongly a. s.

In Section 4 we consider the general directed index set case. We recall the basic notation and definitions.

Let I be a directed index set and correspondingly let $(\mathcal{F}_i)_{i \in I}$ be a stochastic basis, that is, $(\mathcal{F}_i)_{i \in I}$ is an increasing net of sub- σ -algebras of \mathcal{F} indexed by I ($i \leq j$ implies $\mathcal{F}_i \subset \mathcal{F}_j$ for all $i, j \in I$). $(X_i, \mathcal{F}_i)_{i \in I}$ will always denote an adapted net (stochastic process), that is, each $X_i : \Omega \rightarrow E$ is \mathcal{F}_i -measurable and Bochner integrable.

Again we denote by T the set of all simple stopping times (with respect to the stochastic basis $(\mathcal{F}_i)_{i \in I}$), that is, the set of all stopping times τ for which $\tau(\Omega)$ is finite.

In the discrete case $I = \mathbb{N}$, our inequalities imply convergence of stochastic processes. On the other hand, in the general directed index set case, it is well known that without an additional condition on the stochastic basis $(\mathcal{F}_i)_{i \in I}$ we cannot expect to obtain essential convergence theorems for $(X_i, \mathcal{F}_i)_{i \in I}$. Thus if we want to extend our inequalities to the general directed index set case, we are led to require the Vitali Condition V, the natural (and necessary) condition in this context (see [17] [20] [21]):

Vitali Condition V. — We say that $(\mathcal{F}_i)_{i \in I}$ satisfies the Vitali condition V, if for every adapted family of sets $(A_i)_{i \in I}$ (i. e., $A_i \in \mathcal{F}_i$, for every $i \in I$), for every $A \in \mathcal{F}_\infty = \sigma\left(\bigcup_{i \in I} \mathcal{F}_i\right)$ (the σ -algebra generated by $\bigcup_{i \in I} \mathcal{F}_i$), such that $A \subset e \limsup_{i \in I} A_i$ and for every $\varepsilon > 0$, there exist finitely many indices i_1, \dots, i_n in I and pairwise disjoint sets $B_j \subset A_{i_j}$, $B_j \in \mathcal{F}_{i_j}$ ($j = 1, \dots, n$) such that

$$P\left(A \setminus \bigcup_{j=1}^n B_j\right) \leq \varepsilon.$$

The extension of our inequalities to the general directed index set case is carried out in Section 4.

The paper ends with an Appendix in which we give a simple and transparent proof — via inequalities — of a theorem of Millet-Sucheston dealing with stochastic and essential convergence.

2. POINTWISE INEQUALITIES AND COROLLARIES

In what follows we assume that

- I) E is a separable Banach space, with norm $\| \cdot \|$.
- II) \mathcal{C} is a locally convex Hausdorff topology on E , weaker than the norm topology on E and such that the unit ball

$$E_1 = \{ x \in E \mid \|x\| \leq 1 \}$$

is \mathcal{C} -closed. Let us denote $F = (E, \mathcal{C})$. Then $F' \subset E'$ and to say that E_1 is \mathcal{C} -closed is equivalent to saying that for all $x \in E$,

$$\|x\| = \sup \{ |\langle x', x \rangle| \mid x' \in F', \|x'\| \leq 1 \}$$

(use the separation theorem for closed convex sets).

Examples. — 1) $\mathcal{C} =$ the norm topology on E ; 2) $\mathcal{C} = \sigma(E, E')$, the weak

topology of E ; 3) If E is a dual Banach space, $E = G'$, we may take $\tilde{\mathcal{C}} = \sigma(G', G)$, i. e., the weak*-topology.

We recall several elementary facts about $\tilde{\mathcal{C}}$ (see [15] and [16]):

LEMMA 2.1. — 1) *There is a countable set*

$$D = \{x'_1, x'_2, \dots, x'_j, \dots\} \subset \{x' \in F' \mid \|x'\| \leq 1\}$$

with the property:

$$\|x\| = \sup \{|\langle x', x \rangle| \mid x' \in D\},$$

for all $x \in E$.

2) *The formula*

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\langle x'_j, x - y \rangle|}{1 + |\langle x'_j, x - y \rangle|}, \quad x, y \in E$$

defines a metric on E . The topology $\tilde{\mathcal{C}}_0$ on E , induced by d is weaker than the topology $\tilde{\mathcal{C}}$.

3) $\tilde{\mathcal{C}}_0$ generates the same Borel sets as $(E, \|\cdot\|)$; hence $\tilde{\mathcal{C}}$ does also.

4) If $K \subset E$ is $\tilde{\mathcal{C}}$ -compact, then K is metrizable and hence $\tilde{\mathcal{C}}$ -sequentially compact (i. e., every sequence in K contains a subsequence which is $\tilde{\mathcal{C}}$ -convergent to an element of K).

DEFINITION 2.1. — We denote by $Q_{\tilde{\mathcal{C}}}$ the set of all mappings $q : E \rightarrow \mathbb{R}_+$ satisfying the following two properties:

- i) q is a continuous seminorm on $(E, \|\cdot\|)$;
- ii) The set $\{x \in E \mid q(x) \leq 1\}$ is $\tilde{\mathcal{C}}$ -closed.

Equivalently we may define $Q_{\tilde{\mathcal{C}}}$ as the set of all mappings $q : E \rightarrow \mathbb{R}_+$ satisfying condition i) above and ii') there is a countable set $D_q \subset F' = (E, \tilde{\mathcal{C}})'$ such that

$$q(x) = \sup_{x' \in D_q} |\langle x', x \rangle| \quad \text{for all } x \in E.$$

Examples. — The most important examples of seminorms q belonging to $Q_{\tilde{\mathcal{C}}}$ are: 1) $q(x) = \|x\|$, and 2) $q(x) = |\langle x', x \rangle|$ with $x' \in F' = (E, \tilde{\mathcal{C}})'$.

Before stating and proving our main inequalities, we collect some preliminary results.

LEMMA 2.2. — Let $(Z_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an adapted real valued sequence. There is then a sequence of stopping times $(\sigma_n)_{n \in \mathbb{N}}$ with $\sigma_n \in T(n)$ and $\sigma_n \leq \sigma_{n+1}$ for each $n \in \mathbb{N}$, such that

$$\limsup_{n \in \mathbb{N}} Z_n(\omega) = \lim_{n \in \mathbb{N}} Z_{\sigma_n}(\omega), \quad \text{a. s.}$$

For a proof, see for instance [2] or [11].

LEMMA 2.3. — Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an adapted sequence of E -valued integrable r. v. 's. Let $\sigma \in T$ and $(\gamma_j)_{j \in \mathbb{N}}$ a sequence of stopping times such that $\gamma_j \in T(\sigma)$ for each $j \in \mathbb{N}$. Then, if q denotes an arbitrary continuous seminorm on $(E, \| \cdot \|)$:

(1) There is a sequence $(\beta_j)_{j \in \mathbb{N}}$ of stopping times with $\beta_j \in T(\sigma)$ for each $j \in \mathbb{N}$ and such that

$$q(E^{\mathcal{F}} \sigma X_{\beta_j}(\omega)) \uparrow \sup_{j \in \mathbb{N}} q(E^{\mathcal{F}} \sigma X_{\gamma_j}(\omega)).$$

(2) There is a sequence $(\delta_j)_{j \in \mathbb{N}}$ of stopping times with $\delta_j \in T(\sigma)$ for each $j \in \mathbb{N}$ and such that

$$q(E^{\mathcal{F}} \sigma X_{\delta_j}(\omega) - X_{\sigma}(\omega)) \uparrow \sup_{j \in \mathbb{N}} q(E^{\mathcal{F}} \sigma X_{\gamma_j}(\omega) - X_{\sigma}(\omega)).$$

Proof. — We prove only (2) since the proof of (1) is entirely similar.

For each $n \in \mathbb{N}$, we can find a partition $\{A_1, \dots, A_n\}$ of Ω with $A_i \in \mathcal{F}_{\sigma}$, for $1 \leq i \leq n$ and such that on A_i

$$q(E^{\mathcal{F}} \sigma X_{\gamma_i}(\omega) - X_{\sigma}(\omega)) = \sup_{1 \leq j \leq n} q(E^{\mathcal{F}} \sigma X_{\gamma_j}(\omega) - X_{\sigma}(\omega)).$$

Using the « localization property » in T , we define δ_n by setting $\delta_n = \gamma_i$ on A_i . Clearly, $\delta_n \in T(\sigma)$ and

$$q(E^{\mathcal{F}} \sigma X_{\delta_n}(\omega) - X_{\sigma}(\omega)) = \sup_{1 \leq j \leq n} q(E^{\mathcal{F}} \sigma X_{\gamma_j}(\omega) - X_{\sigma}(\omega))$$

finishing the proof. \square

COROLLARY 2.1. — Under the assumptions of Lemma 2.3, suppose that Z is a finite positive r. v. such that

$$Z(\omega) \leq \sup_{p \in \mathbb{N}} q(E^{\mathcal{F}} \sigma X_{\gamma_p}(\omega) - X_{\sigma}(\omega)) \quad \text{a. s.}$$

Then, for each $n \in \mathbb{N}$, there is a stopping time $\tau_n \in T(\sigma)$ such that

$$P\left(\left\{\omega \in \Omega \mid q(E^{\mathcal{F}} \sigma X_{\tau_n}(\omega) - X_{\sigma}(\omega)) \geq Z(\omega) - \frac{1}{2^n}\right\}\right) \geq 1 - \frac{1}{2^n}.$$

Proof. — By (2) of Lemma 2.3, there is a sequence $(\delta_j)_{j \in \mathbb{N}}$ of stopping times with $\delta_j \in T(\sigma)$ for each $j \in \mathbb{N}$ and

$$V_j(\omega) = q(E^{\mathcal{F}} \sigma X_{\delta_j}(\omega) - X_{\sigma}(\omega)) \uparrow \sup_{p \in \mathbb{N}} q(E^{\mathcal{F}} \sigma X_{\gamma_p}(\omega) - X_{\sigma}(\omega)).$$

Hence

$$\inf(V_j(\omega), Z(\omega)) \uparrow Z(\omega), \quad \text{a. s.}$$

So, given $n \in \mathbb{N}$ there is a j_n large enough such that

$$P\left(\left\{\omega \in \Omega \mid |Z(\omega) - \inf(V_{j_n}(\omega), Z(\omega))| > \frac{1}{2^n}\right\}\right) \leq \frac{1}{2^n}.$$

The proof is completed by putting $\tau_n = \delta_{j_n}$. \square

The next two lemmas and the proposition following them are not needed in the proof of the main theorem, Theorem 2.1 below; they are only used in the proofs of Theorems 2.2 and 2.3 below.

LEMMA 2.4. — *Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an adapted sequence of E -valued integrable r. v.'s and assume that it is of class (B). Let $M = \sup_{\tau \in T} \|X_\tau\|_1$. Let q be any continuous seminorm on $(E, \|\cdot\|)$ and let $c \in \mathbb{R}_+$ such that $q(x) \leq c\|x\|$ for all $x \in E$. Fix $\sigma \in T$. Then we have*

$$(1) \quad \int_{\Omega} e \sup_{\tau \in T(\sigma)} q(E^{\mathcal{F}_\sigma} X_\tau) dP \leq \sup_{\tau \in T(\sigma)} \int_{\Omega} q(E^{\mathcal{F}_\sigma} X_\tau) dP \leq Mc < \infty.$$

In particular, taking $q(\cdot) = \|\cdot\|$, it follows that the set

$$\{E^{\mathcal{F}_\sigma} X_\tau \mid \tau \in T(\sigma)\} \subset L_E^1$$

is uniformly integrable.

$$(2) \quad \int_{\Omega} e \sup_{\tau \in T(\sigma)} q(E^{\mathcal{F}_\sigma} X_\tau - X_\sigma) dP \leq \sup_{\tau \in T(\sigma)} \int_{\Omega} q(E^{\mathcal{F}_\sigma} X_\tau - X_\sigma) dP < \infty.$$

Proof. — We sketch the proof of (1) (the proof of inequality (2) is entirely similar). By [22], Proposition VI-1-1, p. 121 there is a sequence $(\gamma)_{j \in \mathbb{N}}$ of stopping times such that $\gamma_j \in T(\sigma)$ for each $j \in \mathbb{N}$ and such that

$$e \sup_{\tau \in T(\sigma)} q(E^{\mathcal{F}_\sigma} X_\tau(\omega)) = \sup_{j \in \mathbb{N}} q(E^{\mathcal{F}_\sigma} X_{\gamma_j}(\omega)), \quad \text{a. s.}$$

We now only have to apply Lemma 2.3 and the monotone convergence theorem. \square

Remark 2.1. — Lemma 2.4 above extends the analogous Lemma 2 in [10].

LEMMA 2.5. — *Let (Ω, \mathcal{F}, P) be a probability measure space. Suppose $\mathcal{H} \subset L_E^1(\Omega, \mathcal{F}, P)$ is uniformly integrable. Then the set*

$$\{E^{\mathcal{G}}(X) \mid X \in \mathcal{H}, \mathcal{G} \subset \mathcal{F} \text{ is a sub-}\sigma\text{-algebra of } \mathcal{F}\}$$

is also uniformly integrable.

Proof. — For each $a > 0$, $X \in \mathcal{H}$ and \mathcal{G} sub- σ -algebra of \mathcal{F} , we have

$$(1) \quad \int_{\{||E^{\mathcal{G}}X|| > a\}} ||E^{\mathcal{G}}X|| dP \leq \int_{\{||E^{\mathcal{G}}X|| > a\}} ||X|| dP.$$

Let $M = \sup_{X \in \mathcal{H}} ||X||_1$; then $M < \infty$. It follows that

$$aP(\{||E^{\mathcal{G}}X|| > a\}) \leq \int_{\{||E^{\mathcal{G}}X|| > a\}} ||E^{\mathcal{G}}X|| dP \leq \int_{\{||E^{\mathcal{G}}X|| > a\}} ||X|| dP \leq M,$$

whence

$$P(\{||E^{\mathcal{G}}X|| > a\}) \leq \frac{M}{a}$$

and thus $\lim_{a \rightarrow \infty} P(\{||E^{\mathcal{G}}X|| > a\}) = 0$, uniformly for $X \in \mathcal{H}$ and \mathcal{G} a sub- σ -algebra of \mathcal{F} . Since \mathcal{H} is uniformly integrable, we have

$$\lim_{\delta \rightarrow 0} \left(\sup_{P(A) \leq \delta} \sup_{X \in \mathcal{H}} \int_A ||X|| dP \right) = 0.$$

The result follows from this and (1). □

The following proposition (as noted earlier) is not needed in the proof of Theorem 2.1 below, but indicates an important case in which Theorem 2.1 is valid; see Theorems 2.2 and 2.3.

PROPOSITION 2.1. — *Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of integrable E -valued r. v. on the probability space (Ω, \mathcal{G}, P) . For each $h \in L_{\mathbb{R}}^{\infty}(\Omega, \mathcal{G}, P)$, define*

$$o(h) = \left\{ \int_{\Omega} h U_n dP \mid n \in \mathbb{N} \right\}.$$

We assume that

- a) *For each $h \in L_{\mathbb{R}}^{\infty}(\Omega, \mathcal{G}, P)$, the set $o(h)$ is \mathcal{C} -relatively compact.*
- b) *The sequence $(U_n)_{n \in \mathbb{N}}$ is uniformly integrable.*

Then there is a subsequence $(U_{n_k})_{k \in \mathbb{N}}$ such that the \mathcal{C} -limit of $\left(\int_{\Omega} h U_{n_k} dP \right)_{k \in \mathbb{N}}$ exists for all $h \in L_{\mathbb{R}}^{\infty}(\Omega, \mathcal{G}, P)$.

Proof. — Let $\mathcal{G}_0 = \sigma(U_1, U_2, \dots, U_n, \dots)$, the smallest σ -algebra that makes every U_n measurable. Using (a), (4) of Lemma 2.1 and a diagonal argument, we can find a sub-sequence $(U_{n_k})_{k \in \mathbb{N}}$ such that

$$\left(\int_{A_i} U_{n_k} dP \right)_{k \in \mathbb{N}}$$

\mathcal{C} converges for each A_i , where $\{A_1, A_2, \dots, A_n, \dots\}$ is a countable set, dense in \mathcal{G}_0 (\mathcal{G}_0 is indeed separable).

Let now A in \mathcal{G}_0 be arbitrary. Let $\varepsilon > 0$. Using the density of $(A_i)_{i \in \mathbb{N}}$ and (b), there exists A_i , with

$$\int_{A_i \Delta A} \|U_n\| dP < \varepsilon$$

for each $n \in \mathbb{N}$, whence

$$(1) \quad \left\| \int_A U_{n_k} dP - \int_{A_i} U_{n_k} dP \right\| < \varepsilon$$

for all $k \in \mathbb{N}$. By (a) again, the sequence $\left(\int_A U_{n_k} dP\right)_{k \in \mathbb{N}}$ has a \mathcal{C} -convergent subsequence. Hence to show that the sequence $\left(\int_A U_{n_k} dP\right)_{k \in \mathbb{N}}$ is \mathcal{C} -convergent it is enough to show that it is \mathcal{C} -Cauchy. This is proved in a straightforward way using inequality (1). Thus $\left(\int_{\Omega} g U_{n_k} dP\right)_{k \in \mathbb{N}}$ is \mathcal{C} -convergent for each $g \in L_{\mathbb{R}}^{\infty}(\Omega, \mathcal{G}_0, P)$, g a step function. Using (a) again we deduce that $\left(\int_{\Omega} h U_{n_k} dP\right)_{k \in \mathbb{N}}$ is \mathcal{C} -convergent for every $h \in L_{\mathbb{R}}^{\infty}(\Omega, \mathcal{G}_0, P)$, (since such an h is approximable uniformly by \mathcal{G}_0 -simple functions). Let now $h \in L_{\mathbb{R}}^{\infty}(\Omega, \mathcal{G}, P)$. Then $E^{\mathcal{G}_0} h = h_0 \in L_{\mathbb{R}}^{\infty}(\Omega, \mathcal{G}_0, P)$ and thus

$$\int_{\Omega} h U_{n_k} dP = \int_{\Omega} h_0 U_{n_k} dP.$$

So also $\left(\int_{\Omega} h U_{n_k} dP\right)_{k \in \mathbb{N}}$ is \mathcal{C} -convergent. \square

COROLLARY 2.2 [10]. — *Let E be a separable dual Banach space and let $(U_n)_{n \in \mathbb{N}}$ be a sequence of E -valued r. v.'s on the probability space (Ω, \mathcal{G}, P) which is uniformly integrable. Then there is a subsequence $(U_{n_k})_{k \in \mathbb{N}}$ such that the weak*-limit of*

$$\left(\int_{\Omega} h U_{n_k} dP\right)_{k \in \mathbb{N}}$$

exists, for each $h \in L_{\mathbb{R}}^{\infty}(\Omega, \mathcal{G}, P)$.

We recall that a set $\Gamma \subset E'$ is called *norm determining* if

$$\|x\| = \sup_{x' \in \Gamma} |\langle x', x \rangle| \quad \text{for each } x \in E.$$

Note that if Γ is norm determining then $\|x'\| \leq 1$ for all $x' \in \Gamma$.

LEMMA 2.6. — *Let (Ω, \mathcal{G}) be a measurable space. Let $\Gamma \subset E'$ be a norm*

determining set. Let $v : \mathcal{G} \rightarrow E$ be a finitely additive set function and assume that v is Γ -countably additive, i. e. if $(A_j)_{j \in \mathbb{N}}$ is a disjoint sequence of sets in \mathcal{G} , then

$$\left\langle x', v\left(\bigcup_{j \in \mathbb{N}} A_j\right) \right\rangle = \lim_n \left\langle x', \sum_{j=1}^n v(A_j) \right\rangle,$$

for each $x' \in \Gamma$. Then v is strongly countably additive.

Proof. — We reason by contradiction following the argument of Pettis (see [9], p. 318-319):

If v is not strongly countably additive, then there is a decreasing sequence $(A_n)_{n \in \mathbb{N}}$ in \mathcal{G} with $\bigcap_{n \in \mathbb{N}} A_n = \phi$ and there is $\varepsilon > 0$ such that

$$\|v(A_n)\| > \varepsilon \quad \text{for } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, there is $x'_n \in \Gamma$ such that

$$(1) \quad |\langle x'_n, v(A_n) \rangle| > \varepsilon.$$

Let $H = \{x_1, x_2, \dots, x_k, \dots\}$ be a countable dense set in E . By the Cantor diagonal procedure, there is a subsequence $(y'_j)_{j \in \mathbb{N}} = (x'_{n_j})_{j \in \mathbb{N}}$ of $(x'_n)_{n \in \mathbb{N}}$ such that $\lim_{j \in \mathbb{N}} \langle y'_j, x_k \rangle$ exists for all $k \geq 1$. Since $\|y'_j\| \leq 1$ for all $j \in \mathbb{N}$, it follows that

$$\lim_{j \in \mathbb{N}} \langle y'_j, x \rangle \quad \text{exists for all } x \in E$$

and hence that

$$\lim_{j \in \mathbb{N}} \langle y'_j, v(B) \rangle \quad \text{exists for each } B \in \mathcal{G}.$$

By Nikodym's theorem (see [9], Corollary III.7.4, p. 160), the set of scalar-valued measures $\{\mu_j(\cdot) = \langle y'_j, v(\cdot) \rangle \mid j \in \mathbb{N}\}$ is uniformly countably additive; this contradicts (1), since

$$|\mu_j(A_{n_j})| = |\langle x'_{n_j}, v(A_{n_j}) \rangle| > \varepsilon \quad \text{for } j = 1, 2, \dots$$

and hence the lemma is proved. \square

LEMMA 2.7. — Let $(U_k)_{k \in \mathbb{N}}$ be a sequence of E -valued integrable r. v.'s which is L^1_E -bounded. Let $\mathcal{G} \subset \mathcal{F}$ be a sub- σ -algebra and assume that

$$v(A) = \tilde{c} - \lim_{k \in \mathbb{N}} \int_A U_k dP$$

exists in E for each $A \in \mathcal{G}$. Then we have

$$(1) \quad |v|(A) \leq \sup_{k \in \mathbb{N}} \int_A \|U_k\| dP \quad \text{for each } A \in \mathcal{G}.$$

In particular, $|v|(\Omega) \leq \sup_{k \in \mathbb{N}} \|U_k\|_1$. Thus v is of bounded total variation.

Also v is strongly countably additive and absolutely continuous w. r. t. $P|\mathcal{G}$.

Proof. — We first prove (1). Fix A in \mathcal{G} . Let $A_i \in \mathcal{G}$ be disjoint sets, with $A_i \subset A$, for each $i = 1, \dots, p$. Let $\varepsilon > 0$. Choose $x'_i \in D$ with

$$\|v(A_i)\| - \frac{\varepsilon}{p} \leq |\langle x'_i, v(A_i) \rangle|, \quad i = 1, \dots, p.$$

Then if k is large enough we have for each $i = 1, \dots, p$,

$$\left| \langle x'_i, v(A_i) \rangle - \left\langle x'_i, \int_{A_i} U_k dP \right\rangle \right| \leq \frac{\varepsilon}{p}$$

and hence

$$\begin{aligned} \sum_{i=1}^p \|v(A_i)\| &\leq \sum_{i=1}^p |\langle x'_i, v(A_i) \rangle| + \varepsilon \\ &\leq \sum_{i=1}^p \left| \left\langle x'_i, \int_{A_i} U_k dP \right\rangle \right| + 2\varepsilon \leq \int_A \|U_k\| dP + 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$|v|(A) \leq \sup_{k \in \mathbb{N}} \int_A \|U_k\| dP \leq M < \infty,$$

where we denoted by M the L^1_E -bound of $(U_k)_{k \in \mathbb{N}}$. In particular, $|v|(\Omega) < \infty$.

Now the set D of Lemma 2.1 is a norm determining set. For each $x' \in F'$, $\langle x', v(\cdot) \rangle$ is countably additive on \mathcal{G} , by the real Vitali-Hahn-Saks theorem. Using Lemma 2.6, we see that v is strongly countably additive: v is also obviously absolutely continuous w. r. t. $P|\mathcal{G}$ (by inequality (1)). \square

From Lemma 2.7 we easily obtain:

COROLLARY 2.3. — Assume that E has (RNP). Let $(U_k)_{k \in \mathbb{N}}$ be a sequence of E -valued integrable r. v.'s which is L^1_E -bounded. Let $(\mathcal{G}_n)_{n \in \mathbb{N}}$ be an increasing sequence of sub- σ -algebras of \mathcal{F} and assume that for each $m \in \mathbb{N}$,

$$v_m(A) = \mathcal{C} - \lim_{k \in \mathbb{N}} \int_A U_k dP$$

exists in E , for all $A \in \mathcal{G}_m$. Let

$$Y_m = \frac{dv_m}{dP} \quad \text{for } m \in \mathbb{N}.$$

Then $(Y_m, \mathcal{G}_m)_{m \in \mathbb{N}}$ is an L^1_E -bounded martingale and hence there is $Y \in L^1_E$ such that $Y_m \rightarrow Y$ a. s. Furthermore, if $(U_k)_{k \in \mathbb{N}}$ is uniformly integrable, then $(Y_m)_{m \in \mathbb{N}}$ is also, and hence we have $Y_m \rightarrow Y$ in L^1_E also.

Proof. — By Lemma 2.7, v_m is of bounded total variation, strongly countably additive and absolutely continuous w. r. t. $P|_{\mathcal{G}_m}$; thus Y_m is well-defined. The L^1_E -boundedness of $(Y_m)_{m \in \mathbb{N}}$ (resp. the uniform integrability of $(Y_m)_{m \in \mathbb{N}}$) follow immediately from the inequalities of Lemma 2.7. \square

We are now in a position to state and prove our main result:

THEOREM 2.1. — Assume that the Banach space E has (RNP) and that $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is an adapted sequence of E -valued integrable r. v.'s.

(I) Suppose that

(A $_{\mathcal{G}}$) There is a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ such that $(X_{n_k})_{k \in \mathbb{N}}$ is L^1_E -bounded and such that for each $A \in \bigcup_{m \in \mathbb{N}} \mathcal{F}_m$,

$$\mathcal{G} - \lim_{k \in \mathbb{N}} \int_A X_{n_k} dP$$

exists in E .

Then we have almost surely

$$(1) \quad \limsup_{m, n \in \mathbb{N}} \|X_n(\omega) - X_m(\omega)\| \leq 2 \limsup_{m \in \mathbb{N}} \left(\sup_{\substack{n \in \mathbb{N} \\ n \geq m}} \|E^{\mathcal{F}_m} X_n(\omega) - X_m(\omega)\| \right).$$

(II) Suppose that

(B $_{\mathcal{G}}$) There is a sequence of stopping times $(\gamma_n)_{n \in \mathbb{N}}$ such that $n \leq \gamma_n \leq \gamma_{n+1}$, $\gamma_n \in T$ for each $n \in \mathbb{N}$, $(X_{\gamma_n})_{n \in \mathbb{N}}$ is L^1_E -bounded and such that for each $A \in \bigcup_{m \in \mathbb{N}} \mathcal{F}_m$,

$$\mathcal{G} - \lim_{n \in \mathbb{N}} \int_A X_{\gamma_n} dP$$

exists in E .

Then there exists a martingale $(Y_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ which is L^1_E -bounded, with almost sure limit Y , such that: For every seminorm q belonging to $Q_{\mathcal{G}}$, there are increasing sequences of stopping times $(\sigma_n)_{n \in \mathbb{N}}$ and $(\tau_n)_{n \in \mathbb{N}}$ (depending on q), with $n \leq \sigma_n \leq \tau_n$, $\sigma_n, \tau_n \in T$ for each $n \in \mathbb{N}$, and such that almost surely

$$(2) \quad \limsup_{m \in \mathbb{N}} q(Y_m(\omega) - X_m(\omega)) \leq \liminf_{n \in \mathbb{N}} q(E^{\mathcal{F}_{\sigma_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)).$$

Hence also:

$$(3) \quad \limsup_{m \in \mathbb{N}} q(Y(\omega) - X_m(\omega)) \leq \liminf_{n \in \mathbb{N}} q(E^{\mathcal{F}_{\sigma_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)),$$

$$(4) \quad \limsup_{m, n \in \mathbb{N}} q(X_n(\omega) - X_m(\omega)) \leq 2 \liminf_{n \in \mathbb{N}} q(E^{\mathcal{F}_{\sigma_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega))$$

almost surely.

Proof. — (I) Let $D \subset F'$ be the set of Lemma 2.1. Define

$$\mu_m(A) = \bar{c} - \lim_{k \in \mathbb{N}} \int_A X_{n_k} dP,$$

for each $A \in \mathcal{F}_m$ and each $m \in \mathbb{N}$. From Lemma 2.7 it follows that μ_m is of bounded total variation on \mathcal{F}_m , is strongly countably additive and absolutely continuous w. r. t. $P|_{\mathcal{F}_m}$. Define

$$Y_m = \frac{d\mu_m}{dP}$$

or more correctly $Y_m = \frac{d\mu_m}{d(P|_{\mathcal{F}_m})}$ for each $m \in \mathbb{N}$. By Corollary 2.3 $(Y_m, \mathcal{F}_m)_{m \in \mathbb{N}}$ is an L^1_E -bounded martingale converging a. s. to an integrable function.

Fix $m \in \mathbb{N}$ and $A \in \mathcal{F}_m$. We have for each $x' \in D$:

$$\begin{aligned} (5) \quad \left| \int_A \langle x', Y_m - X_m \rangle dP \right| &= \left| \left\langle x', \int_A (Y_m - X_m) dP \right\rangle \right| \\ &= \lim_{k \in \mathbb{N}} \left| \left\langle x', \int_A (X_{n_k} - X_m) dP \right\rangle \right| \\ &= \lim_{k \in \mathbb{N}} \left| \left\langle x', \int_A (E^{\mathcal{F}_{\sigma_{n_k}}} X_{n_k} - X_m) dP \right\rangle \right| \\ &\leq \sup_{k \geq m} \int_A \| E^{\mathcal{F}_{\sigma_{n_k}}} X_{n_k} - X_m \| dP \\ &\leq \int_A \left(\sup_{\substack{n \geq m \\ n \in \mathbb{N}}} \| E^{\mathcal{F}_{\sigma_n}} X_n - X_m \| \right) dP. \end{aligned}$$

Consider the function $G_m = \sup_{\substack{n \geq m \\ n \in \mathbb{N}}} \| E^{\mathcal{F}_{\sigma_n}} X_n - X_m \|$ and let

$$C_m = \{ \omega \in \Omega \mid G_m(\omega) = +\infty \}.$$

Then $C_m \in \mathcal{F}_m$. We shall show that a. s. on Ω , we have:

$$(6) \quad \| Y_m(\omega) - X_m(\omega) \| \leq \sup_{n \in \mathbb{N}(m)} \| E^{\mathcal{F}_m} X_n(\omega) - X_m(\omega) \| = G_m(\omega).$$

Clearly (6) is satisfied on C_m . For $\lambda > 0$, let

$$\Omega_\lambda = \{ \omega \in \Omega \mid G_m(\omega) \leq \lambda \}.$$

Since inequality (5) holds for each $A \in \mathcal{F}_m$, $A \subset \Omega_\lambda$ and $\Omega_\lambda \in \mathcal{F}_m$, we have

$$| \langle x', Y_m(\omega) - X_m(\omega) \rangle | \leq G_m(\omega)$$

a. s. on Ω_λ , for each $x' \in D$. So

$$\| Y_m(\omega) - X_m(\omega) \| \leq G_m(\omega)$$

a. s. on Ω_λ . Now $\bigcup_{j \in \mathbb{N}} \Omega_j = \{ \omega \in \Omega \mid G_m(\omega) < +\infty \} = C_m^c$; so we have proved (6). From (6), the desired conclusion follows, if we remark that

$$\begin{aligned} \limsup_{m, n \in \mathbb{N}} \| X_n(\omega) - X_m(\omega) \| & \leq \limsup_{n \in \mathbb{N}} \| X_n(\omega) - Y_n(\omega) \| + \limsup_{m, n \in \mathbb{N}} \| Y_n(\omega) - Y_m(\omega) \| \\ & \quad + \limsup \| Y_m(\omega) - X_m(\omega) \| \\ & \leq 2 \limsup_{n \in \mathbb{N}} \| X_n(\omega) - Y_n(\omega) \|, \quad \text{a. s. ,} \end{aligned}$$

since $(Y_n)_{n \in \mathbb{N}}$ converges strongly a. s. This finishes the proof of part (I). (II) Define

$$\mu_m(A) = \mathcal{C} - \lim_{n \in \mathbb{N}} \int_A X_{\gamma_n} dP, \quad \text{for } A \in \mathcal{F}_m.$$

As in part (I), using Lemma 2.7 and Corollary 2.3 we may define

$$Y_m = \frac{d\mu_m}{dP}$$

for each $n \in \mathbb{N}$; this yields and $L^1_{\mathbb{F}}$ -bounded martingale.

Let now q be a seminorm belonging to $Q_{\mathcal{C}}$ and let D_q be the corresponding countable set in statement *ii'*) of Definition 2.1. Let $\sigma \in T$. For each $A \in \mathcal{F}_\sigma$ and $x' \in D_q$ we have, by an argument similar to that used in the proof of (5),

$$(7) \quad \left| \left\langle x', \int_A (Y_\sigma - X_\sigma) dP \right\rangle \right| \leq \int_A^* \left(\sup_{n \in \mathbb{N}(\sigma)} q(E^{\mathcal{F}_\sigma} X_{\gamma_n} - X_\sigma) \right) dP.$$

As in part (I) we see here also that

$$(8) \quad q(Y_\sigma(\omega) - X_\sigma(\omega)) \leq \sup_{p \in \mathbb{N}(\sigma)} q(E^{\mathcal{F}_p} X_{\gamma_p}(\omega) - X_\sigma(\omega)) \quad \text{a. s.}$$

Now apply Lemma 2.2 to $Z_n = q(Y_n - X_n)$. Hence there is a sequence of stopping times $(\sigma_n)_{n \in \mathbb{N}}$, with $n \leq \sigma_n \leq \sigma_{n+1}$, $\sigma_n \in \mathbb{T}$, for each $n \in \mathbb{N}$, such that

$$(9) \quad \limsup_{n \in \mathbb{N}} q(Y_n(\omega) - X_n(\omega)) = \lim_{n \in \mathbb{N}} q(Y_{\sigma_n}(\omega) - X_{\sigma_n}(\omega)), \quad \text{a. s.}$$

By (8) and Corollary 2.1 of Lemma 2.3, we find, for each $n \in \mathbb{N}$ a stopping time $\tau_n \in \mathbb{T}(\sigma_n)$, such that

$$P\left(\left\{\omega \in \Omega \mid q(E^{\mathcal{F}_{\tau_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)) \geq q(Y_{\sigma_n}(\omega) - X_{\sigma_n}(\omega)) - \frac{1}{2^n}\right\}\right) \geq 1 - \frac{1}{2^n}.$$

Let

$$B_n = \left\{\omega \in \Omega \mid q(E^{\mathcal{F}_{\tau_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)) < q(Y_{\sigma_n}(\omega) - X_{\sigma_n}(\omega)) - \frac{1}{2^n}\right\}$$

and

$$B_\infty = \limsup_{n \in \mathbb{N}} B_n.$$

Then $P(B_\infty) = 0$. If $\omega \notin B_\infty$, then, for all n large enough (depending on ω) we have

$$(10) \quad q(E^{\mathcal{F}_{\tau_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)) \geq q(Y_{\sigma_n}(\omega) - X_{\sigma_n}(\omega)) - \frac{1}{2^n}$$

and by eventually going to a subsequence, we may insure that both $(\sigma_n)_{n \in \mathbb{N}}$ and $(\tau_n)_{n \in \mathbb{N}}$ are increasing. This, together with (9) yields

$$\limsup_{n \in \mathbb{N}} q(Y_n(\omega) - X_n(\omega)) \leq \liminf_{n \in \mathbb{N}} q(E^{\mathcal{F}_{\tau_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)),$$

almost surely. Hence (2) is proved. Since E has (RNP), Y_n converges strongly a. s. to an integrable r. v. Y . We deduce that $q(Y_n - Y)$ converges to 0 a. s. (q is continuous on $(E, \|\cdot\|)$), and thus the inequalities (3) and (4) follow. This finishes the proof of part (II). \square

We now state some variants of Theorem 2.1, giving situations in which condition $(A_{\mathcal{E}})$ or $(B_{\mathcal{E}})$ is satisfied.

THEOREM 2.2. — *Assume that the Banach space E has (RNP). Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an adapted sequence of E -valued integrable r. v.'s. Let $(X_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(X_n)_{n \in \mathbb{N}}$. We suppose that:*

a) For each $m \in \mathbb{N}$ and $h \in L_{\mathbb{R}}^\infty(\Omega, \mathcal{F}_m, P)$, the set

$$o(h) = \left\{\int_{\Omega} h X_{n_k} dP \mid k \geq m\right\}$$

is \mathcal{C} -relatively compact. We also suppose that either b), b') or b'') below is satisfied:

- b) The sequence $(X_{n_k})_{k \in \mathbb{N}}$ is uniformly integrable.
- b') $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is of class (B).
- b'') For each $m \in \mathbb{N}$, $(E^{\mathcal{F}_m} X_{n_k})_{k \in \mathbb{N}(m)}$ is uniformly integrable.

Then condition $(A_{\mathcal{C}})$ of Theorem 2.1 is satisfied and hence we have almost surely:

$$\limsup_{m, n \in \mathbb{N}} \|X_n(\omega) - X_m(\omega)\| \leq 2 \limsup_{m \in \mathbb{N}} \left(\sup_{\substack{n \geq m \\ n \in \mathbb{N}}} \|E^{\mathcal{F}_m} X_n(\omega) - X_m(\omega)\| \right).$$

Proof. — Note first that b) \Rightarrow b'') by Lemma 2.5, and b') \Rightarrow b'') by Lemma 2.4. Hence it suffices to consider the case when b'') is satisfied.

Fix $m \in \mathbb{N}$ and let $U_p = E^{\mathcal{F}_m} X_{n_p}$, for $p \geq m$. Applying Proposition 2.1 (on the probability space $(\Omega, \mathcal{F}_m, P)$) for each $m \in \mathbb{N}$, yields after a diagonalization procedure the final subsequence appearing in condition $(A_{\mathcal{C}})$. \square

THEOREM 2.3. — Assume that E has (RNP). Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an adapted sequence of E-valued integrable r. v.'s. Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of stopping times such that $n \leq \alpha_n \leq \alpha_{n+1}$, $\alpha_n \in T$, for each $n \in \mathbb{N}$. We suppose that:

- a) For each $m \in \mathbb{N}$ and $h \in L_{\mathbb{R}}^{\infty}(\Omega, \mathcal{F}_m, P)$, the set

$$o(h) = \left\{ \int_{\Omega} h X_{\alpha_n} dP \mid n \geq m \right\}$$

is \mathcal{C} -relatively compact.

We also suppose that either b), b') or b'') below is satisfied:

- b) The sequence $(X_{\alpha_n})_{n \in \mathbb{N}}$ is uniformly integrable.
- b') $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is of class (B).
- b'') For each $m \in \mathbb{N}$, $(E^{\mathcal{F}_m} X_{\alpha_p})_{p \in \mathbb{N}(m)}$ is uniformly integrable.

Then condition $(B_{\mathcal{C}})$ of Theorem 2.1 is satisfied. Hence there is an r. v. $Y \in L_E^1$ such that: For every seminorm q belonging to $Q_{\mathcal{C}}$, there are increasing sequences of stopping times $(\sigma_n)_{n \in \mathbb{N}}$ and $(\tau_n)_{n \in \mathbb{N}}$ (depending on q), with $n \leq \sigma_n \leq \tau_n$, $\sigma_n, \tau_n \in T$ for each $n \in \mathbb{N}$, and such that almost surely

$$\limsup_{m \in \mathbb{N}} q(Y(\omega) - X_m(\omega)) \leq \liminf_{n \in \mathbb{N}} q(E^{\mathcal{F}_{\sigma_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)).$$

Hence also:

$$\limsup_{m, n \in \mathbb{N}} q(X_m(\omega) - X_n(\omega)) \leq 2 \liminf_{n \in \mathbb{N}} q(E^{\mathcal{F}_{\sigma_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)).$$

Proof. — Again it suffices to consider the case when b'') is satisfied (as before b) \Rightarrow b'') by Lemma 2.5, and b') \Rightarrow b'') by Lemma 2.4). Fix

$m \in \mathbb{N}$ and let $U_p = E^{\mathcal{F}_m} X_{\alpha_p}$, for $p \geq m$. An application of Proposition 2.1 and the diagonal procedure conclude the argument. \square

Remark 2.2. — In both Theorems 2.2 and 2.3, conditions $b)$, $b')$ imply condition b''); nevertheless conditions $b)$ and $b')$ seem more natural to us.

We close this section with a corollary to Theorems 2.1 and 2.3

COROLLARY 2.4. — *Suppose the assumptions in Theorem 2.1, part (II) are satisfied, so that the inequalities (3) and (4) are valid. Then we also have:*

3a) For each $\lambda > 0$:

$$P(\limsup_{n \in \mathbb{N}} q(Y(\omega) - X_n(\omega)) > \lambda) \leq \liminf_{n \in \mathbb{N}} P(q(E^{\mathcal{F}_{\sigma_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)) > \lambda).$$

4a) For each $\lambda > 0$:

$$P(\limsup_{m, n \in \mathbb{N}} q(X_n(\omega) - X_m(\omega)) > \lambda) \leq \liminf_{n \in \mathbb{N}} P(2q(E^{\mathcal{F}_{\sigma_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)) > \lambda)$$

(« Distributional form of (3) and (4) »).

$$3b) \quad \int_{\Omega} \limsup_{n \in \mathbb{N}} \|Y - X_n\| \leq \liminf_{n \in \mathbb{N}} \int_{\Omega} \|E^{\mathcal{F}_{\sigma_n}} X_{\tau_n} - X_{\sigma_n}\|$$

$$4b) \quad \int_{\Omega} \limsup_{m, n \in \mathbb{N}} \|X_n - X_m\| \leq 2 \liminf_{n \in \mathbb{N}} \int_{\Omega} \|E^{\mathcal{F}_{\sigma_n}} X_{\tau_n} - X_{\sigma_n}\|$$

(« Integral form of (3) and (4) » — see also [10]).

Proof. — a) We only prove 3a); the proof of 4a) is entirely similar.

With the notation of Theorem 2.1, part (II), inequality (3) yields for each $\lambda > 0$:

$$\begin{aligned} P(\limsup_{n \in \mathbb{N}} q(Y(\omega) - X_n(\omega)) > \lambda) &\leq P(\liminf_{n \in \mathbb{N}} q(E^{\mathcal{F}_{\sigma_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)) > \lambda) \\ &\leq \liminf_{n \in \mathbb{N}} P(q(E^{\mathcal{F}_{\sigma_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)) > \lambda). \end{aligned}$$

b) 3b) and 4b) follow readily from inequalities (3) and (4), by Fatou's lemma. \square

3. APPLICATIONS

This section is devoted to applications of Theorems 2.1 and 2.2. We begin with an easy lemma:

LEMMA 3.1. — *Consider the following conditions on an adapted sequence $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ of E-valued integrable r. v.'s:*

(A₀) There is a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ such that $(X_{n_k})_{k \in \mathbb{N}}$ is L^1_E -bounded, and such that for each $A \in \bigcup_{m \in \mathbb{N}} \mathcal{F}_m$, the sequence

$$\left(\int_A X_{n_k} dP \right)_{k \in \mathbb{N}}$$

is strongly convergent in E .

(A₁) There is a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ such that $(X_{n_k})_{k \in \mathbb{N}}$ is L^1_E -bounded and such that, for every fixed $k \in \mathbb{N}$, the sequence

$$(E^{\mathcal{F}_{n_l}} X_{n_l})_{l \geq k}$$

is Cauchy in L^1_E .

(A₂) There is a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ such that $(X_{n_k})_{k \in \mathbb{N}}$ is L^1_E -bounded, and such that

$$\limsup_{k \in \mathbb{N}} \left(\sup_{\substack{l \geq k \\ l \in \mathbb{N}}} \| E^{\mathcal{F}_{n_l}} X_{n_l} - X_{n_k} \|_1 \right) = 0.$$

Then $(A_2) \Rightarrow (A_1) \Rightarrow (A_0) \Rightarrow (A_{\mathcal{E}})$.

Proof. — This is proved in a straightforward way. From Theorem 2.1, part (I), and Lemma 3.1 we immediately derive:

COROLLARY 3.1 [23]. — Assume that the Banach space E has (RNP). Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an E -valued L^1_E -bounded mil satisfying

$$\limsup_{m \in \mathbb{N}} \left(\sup_{\substack{n \geq m \\ n \in \mathbb{N}}} \| E^{\mathcal{F}_n} X_n - X_m \|_1 \right) = 0.$$

Then $(X_n)_{n \in \mathbb{N}}$ converges strongly a. s.

THEOREM 3.1. — Assume that the Banach space E has (RNP). Suppose $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is an adapted sequence of E -valued integrable r. v.'s and that there is a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ with the following properties:

- i) $(X_{n_k})_{k \in \mathbb{N}}$ is uniformly integrable.
- ii) $\limsup_{k \in \mathbb{N}} \left(\sup_{\substack{l > k \\ l \in \mathbb{N}}} \| E^{\mathcal{F}_{n_l}} X_{n_l}(\omega) - X_{n_k}(\omega) \| \right) = 0$, a. s.

Then,

$$\limsup_{m, n \in \mathbb{N}} \| X_n(\omega) - X_m(\omega) \| \leq 2 \limsup_{\substack{m \in \mathbb{N} \\ n \geq m}} \left(\sup_{m \in \mathbb{N}} \| E^{\mathcal{F}_m} X_n(\omega) - X_m(\omega) \| \right).$$

Proof. — By i) and Lemma 2.5,

$$(E^{\mathcal{F}_{n_l}} X_{n_l} - X_{n_k})_{\substack{(k, l) \in \mathbb{N} \times \mathbb{N} \\ k \leq l}}$$

is uniformly integrable.

From *ii*) it then follows that (A_2) is satisfied. By Lemma 3.1, $(A_{\mathcal{C}})$ is satisfied, and hence the posted inequality. \square

COROLLARY 3.2 [5]. — *Assume that the Banach space E has (RNP). Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an E -valued mil with a uniformly integrable subsequence. Then $(X_n)_{n \in \mathbb{N}}$ converges strongly a. s.*

COROLLARY 3.3 [10]. — *Every mil $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ of class (B) taking values in a subspace of a separable dual converges strongly a. s.*

Proof. — Taking for \mathcal{C} the weak*-topology of the separable dual Banach space, we see that $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ satisfies hypotheses *a*) and *b'*) of Theorem 2.2. The mil assumption now yields strong a. s. convergence. \square

We next derive the weak sequential amart ((WS) amart) convergence theorem of Brunel-Sucheston [6]. In fact a slight generalization is proved.

THEOREM 3.2. — *Let E be a Banach space with (RNP). Suppose $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is a (WS) amart, such that there is a sequence $(\gamma_n)_{n \in \mathbb{N}}$ in T , $n \leq \gamma_n \leq \gamma_{n+1}$ for each $n \in \mathbb{N}$, for which $(X_{\gamma_n})_{n \in \mathbb{N}}$ is L^1_E -bounded. Then $(X_n)_{n \in \mathbb{N}}$ converges scalarly a. s. to a strongly integrable r. v.*

Proof. — Take $\mathcal{C} = \sigma(E, E')$ in Theorem 2.1, part (II). Fix $A \in \mathcal{F}_m$ (m arbitrary in \mathbb{N}). Define

$$\delta_n = \begin{cases} \gamma_n & \text{on } A \\ m & \text{on } \Omega \setminus A \end{cases}$$

for $n \geq m$. Then $(\delta_n)_{n \in \mathbb{N}(m)}$ is a sequence in T and increasing. By the (WS) amart assumption:

$$\left(\int_{\Omega} X_{\delta_n} \right)_{n \in \mathbb{N}(m)}$$

converges weakly. This means

$$\left(\int_A X_{\gamma_n} + \int_{\Omega \setminus A} X_m \right)_{n \in \mathbb{N}(m)}$$

converges weakly; hence $\left(\int_A X_{\gamma_n} \right)_{n \in \mathbb{N}(m)}$ converges weakly. From Theorem 2.1, part (II), we now have: There exists an integrable r. v. Y such that for every $x' \in E'$, there are increasing sequences $(\sigma_n^{x'})_{n \in \mathbb{N}}$ and $(\tau_n^{x'})_{n \in \mathbb{N}}$ in T , $n \leq \sigma_n^{x'} \leq \tau_n^{x'}$, for each $n \in \mathbb{N}$, satisfying a. s.:

$$\limsup_{n \in \mathbb{N}} |x'(Y(\omega) - X_n(\omega))| \leq \liminf_{n \in \mathbb{N}} |x'(E^{\mathcal{F}_{\sigma_n^{x'}}} X_{\tau_n^{x'}}(\omega) - X_{\sigma_n^{x'}}(\omega))|.$$

Hence (Fatou):

$$\int_{\Omega} \limsup_{n \in \mathbb{N}} |x'(Y - X_n)| dP \leq \liminf_{n \in \mathbb{N}} \int_{\Omega} |x'(E^{\mathcal{F}_{\tau_n^{x'}}} X_{\tau_n^{x'}} - X_{\sigma_n^{x'}})| dP.$$

Since $(x'(X_n), \mathcal{F}_n)_{n \in \mathbb{N}}$ is obviously a scalar amart, hence a real « uniform amart » [3], for every $x' \in E'$, the integral in the right hand side of the last inequality tends to 0 as $n \rightarrow \infty$ (both $(\sigma_n^{x'})_{n \in \mathbb{N}}$ and $(\tau_n^{x'})_{n \in \mathbb{N}}$ are cofinal in T , for each $x' \in E'$). This proves the theorem. \square

COROLLARY 3.4 [6]. — *Let E be a Banach space with (RNP) and suppose that E' is separable. Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be a (WS) amart of class (B). Then $(X_n)_{n \in \mathbb{N}}$ converges to an integrable r. v. Y, weakly a. s.*

Proof. — We have already proved in Theorem 3.2 that there exists an integrable r. v. Y such that for each $x' \in E'$:

$$(1) \quad \lim_{n \in \mathbb{N}} \langle x', X_n(\omega) \rangle = \langle x', Y(\omega) \rangle, \quad \text{a. s.}$$

By the maximal lemma for processes of class (B) ([7], [11]), we have for each $\lambda > 0$:

$$P(\sup_{n \in \mathbb{N}} \|X_n\| > \lambda) \leq \frac{1}{\lambda} \sup_{\tau \in T} \int_{\Omega} \|X_{\tau}\|.$$

Hence $\sup_n \|X_n(\omega)\| < \infty$ a. s. This, together with (1), proves the theorem (since in this case E' contains a countable dense set). \square

We recall that the Pettis-norm $\|\cdot\|_P : L^1_E \rightarrow \mathbb{R}_+$ is defined by

$$\|X\|_P = \sup_{\substack{x' \in E' \\ \|x'\| \leq 1}} \int_{\Omega} |\langle x', X \rangle| dP.$$

Our next application is the Riesz-decomposition theorem of Edgar-Sucheston for amarts; in fact we prove a generalization:

THEOREM 3.3. — *Assume that E has (RNP) and let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an adapted sequence of E-valued integrable r. v.'s. Then, with the notation of Theorem 2.1 and $\mathcal{C} = \sigma(E, E')$, we have:*

If condition $(A_{\mathcal{C}})$ is satisfied, then for each $\sigma \in T$:

$$(1) \quad \|Y_{\sigma} - X_{\sigma}\|_P \leq 2 \liminf_{k \in \mathbb{N}(\sigma)} \|E^{\mathcal{F}_{\tau_k}} X_{\tau_k} - X_{\sigma}\|_P.$$

If condition $(B_{\mathcal{C}})$ is satisfied, then for each $\sigma \in T$:

$$(2) \quad \|Y_{\sigma} - X_{\sigma}\|_P \leq 2 \liminf_{n \in \mathbb{N}(\sigma)} \|E^{\mathcal{F}_{\tau_n}} X_{\tau_n} - X_{\sigma}\|_P.$$

Proof of (1). — Following the notation and proof of Theorem 2.1, we have, for each $x' \in E'$ and $A \in \mathcal{F}_\sigma$:

$$\begin{aligned} \left\langle x', \int_A (Y_\sigma - X_\sigma) \right\rangle &= \lim_{k \in \mathbb{N}(\sigma)} \left\langle x', \int_A (X_{n_k} - X_\sigma) \right\rangle \\ &= \lim_{k \in \mathbb{N}(\sigma)} \int_A \langle x', E^{\mathcal{F}_\sigma} X_{n_k} - X_\sigma \rangle \\ &\leq \liminf_{k \in \mathbb{N}(\sigma)} \int_A |\langle x', E^{\mathcal{F}_\sigma} X_{n_k} - X_\sigma \rangle| \\ &\leq \liminf_{k \in \mathbb{N}(\sigma)} \left(\sup_{\substack{x' \in E' \\ \|x'\| \leq 1}} \int_\Omega |\langle x', E^{\mathcal{F}_\sigma} X_{n_k} - X_\sigma \rangle| \right) \\ &= \liminf_{k \in \mathbb{N}(\sigma)} \| E^{\mathcal{F}_\sigma} X_{n_k} - X_\sigma \|_P. \end{aligned}$$

Since this inequality holds for each $x' \in D$, we have for all $A \in \mathcal{F}_\sigma$:

$$\left\| \int_A (Y_\sigma - X_\sigma) \right\| \leq \liminf_{k \in \mathbb{N}(\sigma)} \| E^{\mathcal{F}_\sigma} X_{n_k} - X_\sigma \|_P.$$

Recall now that for an E -valued r. v. Z on the probability space (Ω, \mathcal{F}, P) which is \mathcal{G} -measurable (\mathcal{G} a sub- σ -algebra of \mathcal{F}):

$$\| Z \|_P \leq 2 \sup_{A \in \mathcal{G}} \left\| \int_A Z dP \right\|$$

(see [12], p. 86 or [9], p. 97, Lemma 5 for the scalar-valued case, from which the vector-valued case follows easily); then inequality (1) follows.

The proof of inequality (2) is entirely similar. \square

COROLLARY 3.5 [12]. — *Let E be a Banach space with (RNP). Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an E -valued amart such that*

$$\liminf_{n \in \mathbb{N}} \int_\Omega \| X_n \| < \infty.$$

Then $(X_n)_{n \in \mathbb{N}}$ can be uniquely written as the sum of an L^1_E -bounded martingale $(Y_n)_{n \in \mathbb{N}}$ and a potential $(Z_n)_{n \in \mathbb{N}}$ (that is, $\lim_T \| Z_\tau \|_P = 0$). Furthermore, if $(X_n)_{n \in \mathbb{N}}$ is uniformly integrable, then $(Y_n)_{n \in \mathbb{N}}$ is also.

Proof. — The conclusion follows readily from Theorem 3.3 if we use the known fact (see [1]) that $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is an E -valued amart if and only if

$$\limsup_{\sigma \in T} \left(\sup_{\tau \in T(\sigma)} \| E^{\mathcal{F}_\sigma} X_\tau - X_\sigma \|_P \right) = 0;$$

for the uniform integrability statement make use of Corollary 2.3. \square

Uhl's Pettis-convergence theorem for amarts is an easy consequence of Corollary 3.5. We include it here for the sake of completeness:

COROLLARY 3.6 [24]. — *Let E be a Banach space with (RNP). Then every uniformly integrable amart $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ converges in the Pettis-norm to a Bochner integrable function Y .*

Proof. — By Corollary 3.5, $X_n = Y_n + Z_n$, where $(Y_n)_{n \in \mathbb{N}}$ is uniformly integrable. Thus there is $Y \in L^1_E$ such that $Y_n \rightarrow Y$ in L^1_E and the conclusion follows from the inequality

$$\|X_n - Y\|_P \leq \|X_n - Y_n\|_P + \|Y_n - Y\|_P \leq \|Z_n\|_P + \|Y_n - Y\|_1. \quad \square$$

That also the uniform amart convergence theorem of Bellow follows from the inequality of Edgar was already mentioned in [10] and [14]. We include it here too for the sake of completeness:

THEOREM 3.4 [3]. — *Let E be a Banach space with (RNP). Suppose that $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is an L^1_E -bounded uniform amart, that is,*

$$\limsup_{\sigma \in T} \left(\sup_{\tau \in T(\sigma)} \|E^{\mathcal{F}_\sigma} X_\tau - X_\sigma\|_1 \right) = 0.$$

Then $(X_n)_{n \in \mathbb{N}}$ converges strongly a. s.

Proof. — This follows readily from Corollary 2.4, inequality (3b). \square

Our last application does not occur in the literature, and extends Proposition 3.3 in [13]: In what follows \mathcal{C} is the topology in Theorem 2.2.

THEOREM 3.5. — *Assume that each σ -algebra \mathcal{F}_n of the stochastic basis is finite. Let E be a Banach space with (RNP). Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an adapted sequence of E -valued r. v.'s and suppose there exists a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ which is L^1_E -bounded and such that, for each $m \in \mathbb{N}$ and $h \in L^\infty_{\mathbb{R}}(\Omega, \mathcal{F}_m, P)$, the set*

$$o(h) = \left\{ \int_{\Omega} h X_{n_k} dP \mid k \geq m \right\}$$

is \mathcal{C} -relatively compact. Then we have almost surely:

$$\limsup_{m, n \in \mathbb{N}} \|X_m(\omega) - X_n(\omega)\| \leq 2 \limsup_{m \in \mathbb{N}} \left(\sup_{n \in \mathbb{N}(m)} \|E^{\mathcal{F}_m} X_n(\omega) - X_m(\omega)\| \right).$$

Proof. — From the L^1_E -boundedness of $(X_{n_k})_{k \in \mathbb{N}}$ and the finiteness of \mathcal{F}_m , it follows that the sequence $(E^{\mathcal{F}_m} X_{n_k})_{k \in \mathbb{N}(m)}$ is uniformly bounded, hence uniformly integrable (for each $m \in \mathbb{N}$); thus condition b'') of Theo-

rem 2.2 is satisfied. Since condition a) is clearly satisfied, Theorem 2.2 applies yielding the desired inequality.

COROLLARY 3.7. — Assume that each σ -algebra \mathcal{F}_n of the stochastic basis is finite. Let E be a Banach space with (RNP). Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an E -valued mil and suppose there exists a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ which is L_E^1 -bounded and such that, for each $m \in \mathbb{N}$ and $h \in L_{\mathbb{R}}^\infty(\Omega, \mathcal{F}_m, P)$, the set

$$o(h) = \left\{ \int_{\Omega} h X_{n_k} dP \mid k \geq m \right\}$$

is \mathcal{C} -relatively compact. Then $(X_n)_{n \in \mathbb{N}}$ converges strongly a. s. In particular, this is the case for every E -valued mil with an L_E^1 -bounded subsequence, if E is a subspace of a separable dual Banach space.

4. EXTENSION TO GENERAL DIRECTED INDEX SETS

In this section we will try to generalize Theorem 2.1 to the case of adapted nets $(X_i, \mathcal{F}_i)_{i \in I}$, i. e., I is a directed set, every $X_i \in L_E^1$ and X_i is \mathcal{F}_i -measurable.

As is obvious from the preceding results, we cannot extend Theorem 2.1 without supposing the Vitali condition V (see Section 1 for the definition). We will therefore consider the inequalities appearing in Theorem 2.1 in the case that $(\mathcal{F}_i)_{i \in I}$ satisfies the Vitali condition V.

We begin by examining the lemmas preliminary to Theorem 2.1. The extension of Lemma 2.2 to the general index setting is proved in [20] (Theorem 5.1).

LEMMA 4.1 [20]. — Let $(Z_i, \mathcal{F}_i)_{i \in I}$ be an adapted net of real-valued r. v.'s. Suppose the stochastic basis $(\mathcal{F}_i)_{i \in I}$ satisfies the Vitali condition V. Then given an arbitrary sequence $(s_n)_{n \in \mathbb{N}}$ of elements of I , there exists an increasing sequence $(\sigma_n)_{n \in \mathbb{N}}$ of stopping times in T with $\sigma_n \in T(s_n)$ for each $n \in \mathbb{N}$, and such that

$$e \limsup_{i \in I} Z_i(\omega) = \lim_{n \in \mathbb{N}} Z_{\sigma_n}(\omega) \quad \text{a. s.}$$

From Lemma 4.1 we derive the following corollary which generalizes Lemma 2.1 in [14] and which will be used in the appendix of this paper:

COROLLARY 4.1. — Under the assumptions of Lemma 4.1 we have for each $\lambda \in \mathbb{R}$:

$$P(e \limsup_{\sigma \in T} Z_\sigma > \lambda) \leq \limsup_{\sigma \in T} P(Z_\sigma > \lambda).$$

Proof. — We note first that

$$(1) \quad e \limsup_{\sigma \in T} Z_\sigma = e \limsup_{i \in I} Z_i$$

(this follows easily from the observation that $e \sup_{i \in I(i_0)} Z_i = e \sup_{\sigma \in T(i_0)} Z_\sigma$, for each $i_0 \in I$).

Fix now $\sigma_0 \in T$. By Lemma 4.1 there exists an increasing sequence of stopping times $(\sigma_n)_{n \in \mathbb{N}}$ with $\sigma_n \in T(\sigma_0)$ for each $n \in \mathbb{N}$, such that

$$(2) \quad e \limsup_{i \in I} Z_i(\omega) = \lim_{n \in \mathbb{N}} Z_{\sigma_n}(\omega) \quad \text{a. s.}$$

From (1) and (2) we deduce for each $\lambda \in \mathbb{R}$:

$$\{ e \limsup_{\sigma \in T} Z_\sigma > \lambda \} \subset \liminf_{n \in \mathbb{N}} \{ Z_{\sigma_n} > \lambda \}$$

a.s.

and by Fatou's lemma,

$$\begin{aligned} P(e \limsup_{\sigma \in T} Z_\sigma > \lambda) &\leq P(\liminf_{n \in \mathbb{N}} \{ Z_{\sigma_n} > \lambda \}) \\ &\leq \liminf_{n \in \mathbb{N}} P(Z_{\sigma_n} > \lambda) \leq \sup_{\sigma \in T(\sigma_0)} P(Z_\sigma > \lambda). \end{aligned}$$

Since $\sigma_0 \in T$ was arbitrary, the corollary is proved. □

The following lemma and its corollary are the generalized versions of Lemma 2.3 and its Corollary 2.1:

LEMMA 4.2. — *Let $(X_i, \mathcal{F}_i)_{i \in I}$ be an adapted net of E -valued integrable r. v.'s. Let $\sigma \in T$. Let $\alpha \in T, \alpha \geq \sigma$ and let $(\gamma_h)_{h \in H}$ be a family of stopping times such that $\gamma_h \in T(\alpha)$, for each $h \in H$. Then if q denotes an arbitrary continuous seminorm on $(E, \| \cdot \|)$:*

1) *There is a sequence $(\beta_j)_{j \in \mathbb{N}}$ of stopping times with $\beta_j \in T(\alpha)$ for each $j \in \mathbb{N}$ such that almost surely*

$$q(E^{\mathcal{F}_\sigma} X_{\beta_j})(\omega) \uparrow e \sup_{h \in H} q(E^{\mathcal{F}_\sigma} X_{\gamma_h})(\omega).$$

2) *There is a sequence $(\delta_j)_{j \in \mathbb{N}}$ of stopping times with $\delta_j \in T(\alpha)$ for each $j \in \mathbb{N}$ such that almost surely*

$$q(E^{\mathcal{F}_\sigma} X_{\delta_j}(\omega) - X_\sigma(\omega)) \uparrow e \sup_{h \in H} q(E^{\mathcal{F}_\sigma} X_{\gamma_h}(\omega) - X_\sigma(\omega)).$$

COROLLARY 4.2. — *Under the assumptions of Lemma 4.2, suppose that Z is a positive r. v. such that*

$$Z(\omega) \leq e \sup_{h \in H} q(E^{\mathcal{F}_\sigma} X_{\gamma_h}(\omega) - X_\sigma(\omega))$$

almost surely. Then, for each $n \in \mathbb{N}$, there is a stopping time $\tau_n \in T(\alpha)$ such that

$$P\left(\left\{\omega \in \Omega \mid q(E^{\mathcal{F}} \circ X_{\tau_n}(\omega) - X_{\sigma}(\omega)) \geq Z(\omega) - \frac{1}{2^n}\right\}\right) \geq 1 - \frac{1}{2^n}.$$

We finally restate Lemma 2.4 in this general setting:

LEMMA 4.3. — *Let $(X_i, \mathcal{F}_i)_{i \in I}$ be an adapted net of E -valued integrable r. v.'s, and assume that it is of class (B). Let $M = \sup_{\sigma \in T} \|X_{\sigma}\|_1$. Let q be any continuous seminorm on $(E, \|\cdot\|)$ and let $c \in \mathbb{R}_+$ such that $q(x) \leq c \|x\|$ for all $x \in E$. Fix $\sigma \in T$. Then we have*

$$(1) \quad \int_{\Omega} e \sup_{\tau \in T(\sigma)} q(E^{\mathcal{F}} \circ X_{\tau}) dP \leq \sup_{\tau \in T(\sigma)} \int_{\Omega} q(E^{\mathcal{F}} \circ X_{\tau}) dP \leq Mc < \infty.$$

In particular, taking $q(\cdot) = \|\cdot\|$, it follows that the set

$$\{E^{\mathcal{F}} \circ X_{\tau} \mid \tau \in T(\sigma)\} \subset L_E^1$$

is uniformly integrable.

$$(2) \quad \int_{\Omega} e \sup_{\tau \in T(\sigma)} q(E^{\mathcal{F}} \circ X_{\tau} - X_{\sigma}) dP \leq \sup_{\tau \in T(\sigma)} \int_{\Omega} q(E^{\mathcal{F}} \circ X_{\tau} - X_{\sigma}) dP < \infty.$$

We are now in a position to state and prove the generalized versions of Theorem 2.1 (see also Theorems 2.2 and 2.3):

THEOREM 4.1. — *Suppose the stochastic basis $(\mathcal{F}_i)_{i \in I}$ satisfies the Vitali condition V. Let E be a Banach space with (RNP) and let $(X_i, \mathcal{F}_i)_{i \in I}$ be an adapted net of E -valued integrable r. v.'s. Suppose that:*

(A_σ) *There is $J \subset I$ cofinal in I such that the subnet $(X_j)_{j \in J}$ is L_E^1 -bounded and such that for each $A \in \bigcup_{i \in I} \mathcal{F}_i$,*

$$\mathcal{C} - \lim_{j \in J} \int_A X_j dP$$

exists in E .

We also suppose that either a), a') or a'') below is satisfied:

a) $(X_j)_{j \in J}$ *is uniformly integrable.*

a') $(X_i, \mathcal{F}_i)_{i \in I}$ *is of class (B).*

a'') *For each $i \in I$, $(E^{\mathcal{F}_i} X_j)_{j \in J(i)}$ is uniformly integrable.*

Then we have almost surely

$$(1) e \limsup_{i, j \in I} \|X_i(\omega) - X_j(\omega)\| \leq 2 e \limsup_{i \in I} (e \sup_{j \in J(i)} \|E^{\mathcal{F}_i} X_j(\omega) - X_i(\omega)\|).$$

Proof. — Note first that $a) \Rightarrow a''$) by Lemma 2.5, and $a') \Rightarrow a''$) by Lemma 4.3. Hence it suffices to consider the case when a'') is satisfied.

For $i \in I$ fixed and each $A \in \mathcal{F}_i$ define

$$\mu_i(A) = \bar{c} - \lim_{j \in J} \int_A X_j dP.$$

It is obvious that $\mu_i : \mathcal{F}_i \rightarrow E$ is finitely additive. Since $(X_j)_{j \in J(i)}$ is L^1_E -bounded, an argument similar to that used in (the first part of) the proof of Lemma 2.7 shows that μ_i is of bounded total variation.

Now for each $x' \in D$ we have

$$\langle x', \mu_i(A) \rangle = \lim_{j \in J} \int_A \langle x', X_j \rangle dP = \lim_{j \in J} \int_A \langle x', E^{\mathcal{F}_i} X_j \rangle dP.$$

Since by a''), $(\langle x', E^{\mathcal{F}_i} X_j \rangle)_{j \in J(i)}$ is uniformly integrable, the scalar-valued set function $\langle x', \mu_i(\cdot) \rangle$ is countably additive. It then follows from Lemma 2.6 that μ_i is strongly countably additive. Now

$$Y_i = \frac{d\mu_i}{dP}$$

is defined. This yields the martingale $(Y_i, \mathcal{F}_i)_{i \in I}$. The rest of the proof is the same as the proof of the corresponding part of Theorem 2.1. \square

THEOREM 4.2. — *Suppose the stochastic basis $(\mathcal{F}_i)_{i \in I}$ satisfies the Vitali condition V. Let E be a Banach space with (RNP) and let $(X_i, \mathcal{F}_i)_{i \in I}$ be an adapted net of E -valued integrable r. v.'s. Suppose that:*

$(B_{\bar{c}})$ *There is an increasing net $(\gamma_k)_{k \in K}$ in T (i. e., $k \leq k'$ implies $\gamma_k \leq \gamma_{k'}$), which is cofinal in T , such that the net $(X_{\gamma_k})_{k \in T}$ is L^1_E -bounded and such that for each $A \in \bigcup_{i \in I} \mathcal{F}_i$,*

$$\bar{c} - \lim_{k \in K} \int_A X_{\gamma_k} dP$$

exists in E .

We also suppose that either b), b') or b'') below is satisfied:

b) $(X_{\gamma_k})_{k \in K}$ is uniformly integrable.

b') $(X_i, \mathcal{F}_i)_{i \in I}$ is of class (B).

b'') For each $i \in I$, there is $k^(i) \in K$ with $\gamma_{k^*(i)} \geq i$ such that*

$$\{ E^{\mathcal{F}_i} X_{\gamma_k} \mid k \geq k^*(i) \}$$

is uniformly integrable.

Then there exists a martingale $(Y_i, \mathcal{F}_i)_{i \in I}$ which is L^1_E -bounded, with almost sure limit Y , such that: For every seminorm q belonging to $Q_{\bar{\sigma}}$ and for any given sequence $(s_n)_{n \in \mathbb{N}}$ of elements of I , there exist increasing sequences $(\sigma_n)_{n \in \mathbb{N}}$ and $(\tau_n)_{n \in \mathbb{N}}$ of stopping times in T (depending on q and (s_n)) with $s_n \leq \sigma_n \leq \tau_n$ for each $n \in \mathbb{N}$ and such that almost surely

$$(2) \quad e \limsup_{i \in I} q(Y_i(\omega) - X_i(\omega)) \leq \liminf_{n \in \mathbb{N}} q(E^{\mathcal{F}_{\sigma_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)).$$

Hence also

$$(3) \quad e \limsup_{i \in I} q(Y(\omega) - X_i(\omega)) \leq \liminf_{n \in \mathbb{N}} q(E^{\mathcal{F}_{\sigma_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega))$$

$$(4) \quad e \limsup_{i, j \in I} q(X_i(\omega) - X_j(\omega)) \leq 2 \liminf_{n \in \mathbb{N}} q(E^{\mathcal{F}_{\sigma_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)),$$

almost surely.

Proof. — The existence of the martingale $(Y_i, \mathcal{F}_i)_{i \in I}$ is obtained by an argument similar to that given in the proof of Theorem 4.1.

As in the proof of Theorem 2.1, part (II), we have for each $\sigma \in T$, $A \in \mathcal{F}_{\sigma}$ and $x' \in D_q$,

$$(5) \quad \left| \left\langle x', \int_A (Y_{\sigma} - X_{\sigma}) dP \right\rangle \right| \leq \int_A^* e \sup_{k \in K(k^*)} q(E^{\mathcal{F}_{\sigma}} X_{\gamma_k} - X_{\sigma}) dP$$

where k^* is any index in K such that $\gamma_{k^*} \geq \sigma$. We again have almost surely

$$(6) \quad q(Y_{\sigma}(\omega) - X_{\sigma}(\omega)) \leq e \sup_{k \in K(k^*)} q(E^{\mathcal{F}_{\sigma}} X_{\gamma_k}(\omega) - X_{\sigma}(\omega)).$$

By Lemma 4.1 applied to $Z_i = q(Y_i - X_i)$ for $i \in I$, there is an increasing sequence $(\sigma_n)_{n \in \mathbb{N}}$ of stopping times in T with $\sigma_n \in T(s_n)$ for each $n \in \mathbb{N}$, such that almost surely

$$(7) \quad e \limsup_{i \in I} q(Y_i(\omega) - X_i(\omega)) = \lim_{n \in \mathbb{N}} q(Y_{\sigma_n}(\omega) - X_{\sigma_n}(\omega)).$$

Now define the increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times in T , by induction, as follows: by (6) above and Corollary 4.2 of Lemma 4.2, there is $\tau_1 \in T(\sigma_1)$ such that

$$P\left(\left\{ \omega \in \Omega \mid q(E^{\mathcal{F}_{\tau_1}} X_{\tau_1}(\omega) - X_{\sigma_1}(\omega)) \geq q(Y_{\sigma_1}(\omega) - X_{\sigma_1}(\omega)) - \frac{1}{2} \right\}\right) \geq 1 - \frac{1}{2}.$$

Assume $\tau_1 \leq \tau_2 \leq \dots \leq \tau_{n-1}$ constructed with $\sigma_j \leq \tau_j$ ($1 \leq j \leq n-1$). Let k^* be an index in K such that $\gamma_{k^*} \geq \sigma_n$ and $\gamma_{k^*} \geq \tau_{n-1}$. By (6) above and Corollary 4.2, there is $\tau_n \in T(\gamma_{k^*})$ such that

$$P\left(\left\{ \omega \in \Omega \mid q(E^{\mathcal{F}_{\tau_n}} X_{\tau_n}(\omega) - X_{\sigma_n}(\omega)) \geq q(Y_{\sigma_n}(\omega) - X_{\sigma_n}(\omega)) - \frac{1}{2^n} \right\}\right) \geq 1 - \frac{1}{2^n};$$

clearly $\tau_n \geq \sigma_n$ and $\tau_n \geq \tau_{n-1}$. The proof is now concluded as in Theorem 2.1, part (II).

Remark 4.1. — The Vitali condition V was used in applying Lemma 4.1 and for the essential convergence of the martingale $(Y_i, \mathcal{F}_i)_{i \in I}$ (see [20], Theorem 12.8).

Remark 4.2. — 1) From the proof of Theorem 4.1 it is clear that instead of assuming $(A_{\mathcal{E}})$ and a, a' or a'' , it suffices to assume $(A_{\mathcal{E}})$ and the following weaker — scalar version — of a'' :

a''_s) For each $x' \in D$ and $i \in I$, $(\langle x', E^{\mathcal{F}_i} X_j \rangle)_{j \in I(i)}$ is uniformly integrable.

2) In the case $I = \mathbb{N}$, condition a''_s follows from $(A_{\mathcal{E}})$ (by the scalar version of the Vitali-Hahn-Saks theorem; see [9], p. 158). Hence in this formulation, assuming $(A_{\mathcal{E}})$ and a''_s , Theorem 4.1 is an extension of Theorem 2.1, part (I).

3) Analogous remarks hold for Theorem 4.2.

APPENDIX

In this appendix we give a simple and direct proof of a theorem of Millet and Sucheston (Theorem 4.1 in [20]) dealing with stochastic and essential convergence. Our proof, via inequalities, seems to us transparent and very natural. The theorem in question is the following (it appears that a « localization property » in the second variable, such as condition (1) below, is also needed in this theorem):

THEOREM A1. — *Let $(\mathcal{F}_i)_{i \in I}$ be a stochastic basis satisfying the Vitali condition V. Let E be an arbitrary Banach space. For each $\sigma, \tau \in T$ with $\sigma \leq \tau$, let $f(\sigma, \tau)$ be an E-valued, \mathcal{F}_σ -measurable r. v. Assume that the family of r. v.'s $(f(\sigma, \tau))_{\substack{\sigma \leq \tau \\ \sigma, \tau \in T}}$ satisfies the following two conditions:*

- 1) *If $A \in \mathcal{F}_\sigma$ and $\tau', \tau'' \in T(\sigma)$ are such that $\tau'(\omega) = \tau''(\omega)$ on A, then*

$$1_A f(\sigma, \tau') = 1_A f(\sigma, \tau'')$$

(i. e., localization property in the second variable).

- 2) *For every $\sigma, \tau \in T$ with $\tau \geq \sigma$ and $\tau \geq i$,*

$$1_{\{\sigma=i\}} f(i, \tau) = 1_{\{\sigma=i\}} f(\sigma, \tau)$$

(i. e., localization property in the first variable).

If $(f(\sigma, \tau))_{\substack{\sigma \leq \tau \\ \sigma, \tau \in T}}$ converges stochastically to f_∞ , then $(f(\sigma, \tau))_{\substack{\sigma \leq \tau \\ \sigma, \tau \in T}}$ converges essentially to f_∞ .

We first need a proposition in which condition 1) is indispensably used.

PROPOSITION A1. — *Let E be an arbitrary Banach space. Let $(\mathcal{F}_i)_{i \in I}$ be a stochastic basis. Let $\sigma \in T$ and let $(g(\tau))_{\tau \in T(\sigma)}$ be a family of E-valued, \mathcal{F}_σ -measurable r. v.'s. Assume that:*

- (1) *If $A \in \mathcal{F}_\sigma$, $\tau', \tau'' \in T(\sigma)$ and $\tau'(\omega) = \tau''(\omega)$ for $\omega \in A$, then*

$$1_A g(\tau') = 1_A g(\tau'').$$

Then for each $\lambda > 0$ we have

$$P(e \sup_{\tau \in T(\sigma)} \|g(\tau)\| > \lambda) \leq \sup_{\tau \in T(\sigma)} P(\|g(\tau)\| > \lambda).$$

Proof. — Applying [22], Proposition VI-1-1, p. 121, there is a sequence $(\sigma_n)_{n \in \mathbb{N}}$ in $T(\sigma)$ such that

$$e \sup_{\tau \in T(\sigma)} \|g(\tau)\| = \sup_{n \in \mathbb{N}} \|g(\sigma_n)\|.$$

The same localization argument as in Lemma 2.3, based now on condition (1), yields: There exists a sequence $(\beta_j)_{j \in \mathbb{N}}$ in $T(\sigma)$ such that $(\|g(\beta_j)\|)_{j \in \mathbb{N}}$ increases to $\sup_{n \in \mathbb{N}} \|g(\sigma_n)\|$, a. s. We now have, for each $\lambda > 0$:

$$\begin{aligned} P(e \sup_{\tau \in T(\sigma)} \|g(\tau)\| > \lambda) &= P(\lim_{j \in \mathbb{N}} \|g(\beta_j)\| > \lambda) \\ &= \lim_{j \in \mathbb{N}} P(\|g(\beta_j)\| > \lambda) \\ &\leq \sup_{\tau \in T(\sigma)} P(\|g(\tau)\| > \lambda). \end{aligned} \quad \square$$

Proof of Theorem A1. — For each $\varepsilon > 0$ there exists $i_0 \in I$ and an \mathcal{F}_{i_0} -measurable function f_{i_0} such that

$$(3) \quad P\left(\|f_{i_0} - f_\infty\| > \frac{\varepsilon}{4}\right) \leq \frac{\varepsilon}{4}.$$

Also, there is a $\sigma_0 \in T$, and we may assume $\sigma_0 \geq i_0$, such that for every $\sigma \in T(\sigma_0)$ and $\tau \in T(\sigma)$:

$$P\left(\|f(\sigma, \tau) - f_\infty\| > \frac{\varepsilon}{4}\right) \leq \frac{\varepsilon}{4}$$

whence

$$(4) \quad P\left(\|f(\sigma, \tau) - f_{i_0}\| > \frac{\varepsilon}{2}\right) \leq \frac{\varepsilon}{2}.$$

Now for $i \in I(i_0)$ define

$$(5) \quad Z_i = e \sup_{\tau \in T(i)} \|f(i, \tau) - f_{i_0}\|.$$

Then Z_i is \mathcal{F}_i -measurable and it is not hard to check, using conditions (1) and (2), that for $\sigma \in T(\sigma_0)$ we have

$$(6) \quad Z_\sigma = e \sup_{\tau \in T(\sigma)} \|f(\sigma, \tau) - f_{i_0}\|.$$

By Corollary 4.1, using (5) and (6) above, we have

$$(7) \quad P\left(e \limsup_{\sigma \in T} (e \sup_{\tau \in T(\sigma)} \|f(\sigma, \tau) - f_{i_0}\|) > \frac{\varepsilon}{2}\right) \leq \limsup_{\sigma \in T} P\left(e \sup_{\tau \in T(\sigma)} \|f(\sigma, \tau) - f_{i_0}\| > \frac{\varepsilon}{2}\right).$$

For $\sigma \in T(\sigma_0)$, let $(g(\tau))_{\tau \in T(\sigma)}$ be defined by $g(\tau) = f(\sigma, \tau) - f_{i_0}$. Because of assumption (1), Proposition A1 applies and by (4) above we have for $\sigma \in T(\sigma_0)$:

$$(8) \quad P\left(e \sup_{\tau \in T(\sigma)} \|f(\sigma, \tau) - f_{i_0}\| > \frac{\varepsilon}{2}\right) \leq \sup_{\tau \in T(\sigma)} P\left(\|f(\sigma, \tau) - f_{i_0}\| > \frac{\varepsilon}{2}\right) \leq \frac{\varepsilon}{2}.$$

From (7) and (8) we deduce

$$P\left(e \limsup_{\sigma \in T} (e \sup_{\tau \in T(\sigma)} \|f(\sigma, \tau) - f_{i_0}\|) > \frac{\varepsilon}{2}\right) \leq \frac{\varepsilon}{2}$$

and finally by the triangle inequality and (3) above

$$P(e \limsup_{\sigma \in T} (e \sup_{\tau \in T(\sigma)} \|f(\sigma, \tau) - f_\infty\|) > \varepsilon) \leq \varepsilon$$

proving that $(f(\sigma, \tau))_{\substack{\sigma \leq \tau \\ \sigma, \tau \in T}}$ converges essentially to f_∞ . \square

Remark A1. — The assertion that $(f(\sigma, \tau))_{\substack{\sigma \leq \tau \\ \sigma, \tau \in T}}$ converges essentially to f_∞ implies of course that $(f(i, j))_{\substack{i \leq j \\ i, j \in I}}$ converges essentially to f_∞ .

Remark A2. — We recall the most important applications of the above theorem. Below $(X_i, \mathcal{F}_i)_{i \in I}$ is an adapted net of E-valued r. v.'s (in case ii) and iii) also assumed integrable). For $\sigma \in T$ and $\tau \in T(\sigma)$ set:

- i) $f(\sigma, \tau) = X_\sigma$;
- ii) $f(\sigma, \tau) = E^{\mathcal{F}_\sigma} X_\tau$;
- iii) $f(\sigma, \tau) = E^{\mathcal{F}_\sigma} X_\tau - X_\sigma$.

REFERENCES

[1] K. ASTBURY, Amarts indexed by directed sets, *Ann. Prob.*, t. **6**, 2, 1978, p. 267-278.
 [2] D. G. AUSTIN, G. A. EDGAR, A. IONESCU TULCEA, Pointwise convergence in terms of expectations, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, t. **30**, 1974, p. 17-26.

- [3] A. BELLOW, Uniform amarts: a class of asymptotic martingales for which strong almost sure convergence obtains, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, t. **41**, 1978, p. 177-191.
- [4] A. BELLOW, A. DVORETZKY, A characterization of almost sure convergence, *Probability in Banach spaces II*, Oberwolfach 1978, ed. A. Beck, *Lecture Notes in Math.*, Springer-Verlag, t. **709**, 1979, p. 45-65.
- [5] A. BELLOW, A. DVORETZKY, *A note on the strong convergence of mils in Banach spaces*, Preprint, 1979.
- [6] A. BRUNEL, L. SUCHESTON, Sur les amarts faibles à valeurs vectorielles, *C. R. Acad. Sci. Paris*, t. **282**, série A, 1976, p. 1011-1014.
- [7] R. V. CHACON, L. SUCHESTON, On convergence of vector-valued asymptotic martingales, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, t. **33**, 1975, p. 55-59.
- [8] J. DIESTEL, J. J. UHL, Jr., *Vector Measures*, *A. M. S.-Mathematical Surveys*, t. **15**, 1977.
- [9] N. DUNFORD, J. SCHWARTZ, *Linear Operators I*, Interscience Publ., 1957.
- [10] G. A. EDGAR, Uniform semiamarts, *Ann. Inst. H. Poincaré*, sect. B, 15, t. **3**, 1979, p. 197-203.
- [11] G. A. EDGAR, L. SUCHESTON, Amarts, a class of asymptotic martingales, A. Discrete parameter, *J. Multivariate Anal.*, t. **6**, 1976, p. 193-221.
- [12] G. A. EDGAR, L. SUCHESTON, The Riesz decomposition theorem for vector-valued amarts, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, t. **36**, 1976, p. 85-92.
- [13] L. EGGHE, Strong convergences of pramarts in Banach spaces, to appear in *Can. J. Math.*
- [14] L. EGGHE, Some new Chacon-Edgar type inequalities for stochastic processes and characterizations of Vitali conditions, to appear in *Ann. Inst. H. Poincaré*, sect. B, t. **16**, n° 4, 1980, p. 327-338.
- [15] J. HOFFMANN-JØRGENSEN, Sums of independent Banach space valued random variables, *Studia Mathematica*, t. LII, 1974, p. 159-186.
- [16] A. KORZENIOWSKI, Martingales in Banach spaces for which the convergence with probability one, in probability and in law coincide, *Coll. Math.*, t. **39**, n° 1, 1978, p. 153-159.
- [17] K. KRICKEBERG, *Convergence of martingales with a directed index set*, *Trans. Amer. Math. Soc.*, t. 83, 1956, p. 313-337.
- [18] P. MCCARTNEY, Neighborly bushes and the Radon-Nikodym Property for Banach spaces, *Pacific J. Math.*, t. **87**, n° 1, 1980, p. 157-168.
- [19] P. MCCARTNEY, R. O'BRIEN, A separable Banach space with the Radon-Nikodym Property, which is not isomorphic to a subspace of a separable dual, *Proc. Amer. Math. Soc.*, t. **78**, n° 1, 1980, p. 40-42.
- [20] A. MILLET, L. SUCHESTON, Convergence of classes of amarts indexed by directed sets, *Can. J. Math.*, t. **32**, n° 1, 1980, p. 86-125.
- [21] A. MILLET, L. SUCHESTON, Characterizations of Vitali conditions with overlap in terms of convergence of classes of amarts, *Can. J. Math.*, t. **31**, n° 5, 1979, p. 1033-1046.
- [22] J. NEVEU, *Discrete Parameter Martingales*. North-Holland, 1975.
- [23] M. PELIGRAD, A limit theorem for martingale-like sequences, *Rev. Roum. Math. Pures et Appl.*, t. **21**, n° 6, 1976, p. 733-736.
- [24] J. J. UHL, Jr., Pettis mean convergence of vector-valued asymptotic martingales, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, t. **37**, 1977, p. 291-295.

(Manuscrit reçu le 20 novembre 1981)