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Negative nonsingular transformations

by

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SUMMARY. — By a negative nonsingular transformation T on a finite measure space $(\Omega, \mathscr{A}, \mu)$ we mean a mapping T of Ω into itself, such that T is measurable and $\mu(T^{-1}A) = 0$ if $A \in \mathscr{A}$ and $\mu(A) = 0$. The space Ω is decomposed into several subspaces and the action of T on these subspaces is studied. V. A. Rohlin's tower theorem is established for negative nonsingular transformations. Using Rohlin's theorem it is shown, that for every subset S of natural numbers there exist countable S-generators for aperiodic, negative nonsingular transformations. Furthermore if T is bimeasurable, negative nonsingular and if there exists no nonzero, finite and T-invariant measure absolutely continuous with respect to μ then for every subset S of natural numbers with positive density the sets $B \in \mathscr{A}$, such that the system $\{ T^{-s}B ; s \in S \}$ generates $\mathscr{A}$ mod μ, are dense in $\mathscr{A}$. As a consequence there exist two-set S-generators for T.

PRELIMINARIES

Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and let T denote a measurable transformation from Ω into Ω. T is not assumed to be invertible or measure preserving unless otherwise stated. T is called negative (positive) nonsingular if $\mu(A) = 0$ implies (is implied by) $\mu(T^{-1}A) = 0$ for every measurable subset A of Ω. A transformation T is said to be nonsingular if T is negative and positive nonsingular. A measure ν on $\mathscr{A}$ is called absolutely continuous with respect to μ, written by $\nu \ll \mu$, if $\mu(A) = 0$ implies $\nu(A) = 0$ for every $A \in \mathscr{A}$ or, equivalentely, if for every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that $\mu(A) < \delta$ implies $\nu(A) < \varepsilon$ for every

$A \in \mathscr{A}$. By the equivalence of two measures ν and μ on $\mathscr{A}$ we mean $\nu \ll \mu$ and $\mu \ll \nu$. $T^n(\mu)$ denotes the measure on $\mathscr{A}$ defined by $T^n(\mu)(A) := \mu(T^{-n}A)$ for every $n \geqslant 0$. The transformation T is negative nonsingular iff $T^n(\mu) \ll \mu$ for every $n \geqslant 0$. Set relations are assumed to hold modulo μ. A measurable subset A of Ω is called wandering if $T^{-i}A \cap T^{-j}A = \emptyset$ for $i, j \geqslant 0$, $i \neq j$. A measurable set A is called weakly wandering if there exists a sequence $(n_k)_{k \geqslant 1}$ of natural numbers $n_k \geqslant 0$ such that $T^{-n_k}A \cap T^{-n_l}A = \emptyset$ for $k, l \geqslant 1$, $k \neq l$. We write $\hat{A}$ for $\bigcup_{k \geqslant 0} T^{-k}A$. A measurable set A is called a sweep out set for Ω if $\hat{A} = \Omega$. The set $A \in \mathscr{A}$ is said to be invariant if $T^{-1}A \supset A$. If A is invariant, then by the transformation T_A on $(A, \mathscr{A}_A, \mu_A)$ we understand the restriction of T on the measure space $(A, \mathscr{A} \cap A, \mu|_{\mathscr{A} \cap A})$.

DECOMPOSITION OF Ω

The following decomposition of Ω will be basic for our further considerations.

THEOREM 1. — Let T be a measurable, negative nonsingular transformation on $(\Omega, \mathscr{A}, \mu)$.

Then there exists a unique decomposition of Ω into measurable, invariant and disjoint subsets Ω_1 and Ω_2 (i. e. $\Omega_1, \Omega_2 \in \mathscr{A}$, $\Omega_1 \cap \Omega_2 = \emptyset$, $\Omega_1 + \Omega_2 = \Omega$, $T^{-1}\Omega_i = \Omega_i$ for $i = 1, 2$) with the following properties:

i) For every decreasing sequence $(\varepsilon_i)_{i \geqslant 1} \searrow 0$ of real numbers $\varepsilon_i > 0$ there exists a decreasing sequence $(A_{\varepsilon_i})_{i \geqslant 1} \searrow \emptyset$ of measurable sets A_{ε_i} such that $\mu(A_{\varepsilon_i}) < \varepsilon_i$, A_{ε_i} is invariant and A_{ε_i} is a sweep out set for Ω_1 for all $i \geqslant 1$. Ω_1 is called the purely dissipative part of $(\Omega, \mathscr{A}, \mu)$ and T is called purely dissipative on Ω_1.

ii) There exists an unique measurable subset C of Ω, called the conservative part of Ω, such that C is invariant, C is a sweep out set for Ω_2 and there is no T-wandering subset W of C of positive measure.

Proof. — Exhaust Ω by a sequence $A_1, A_2, \ldots$ of wandering sets such that $\Omega \setminus \bigcup_{i \geqslant 1} A_i$ does not contain any wandering set of positive measure.

For $C := \Omega \setminus \bigcup_{i \geqslant 1} \hat{A}_i$ we have $T^{-1}C \supset C$. Let $\Omega_2 := \hat{C}$ and

$$\Omega_1 := \complement\Omega_2 = \bigcup_{i \geqslant 1} (\hat{A}_i \setminus \Omega_2) = \bigcup_{k \geqslant 1} B_k$$

for a sequence $(B_k)_{k\geq 1}$ of wandering sets. Now

$$W_k := (B_1 \cup B_2 \cup \ldots \cup B_k) \setminus \bigcup_{i\geq 1} T^{-i}(B_1 \cup \ldots \cup B_k)$$

is a wandering set for every $k \geq 1$ and the increasing sequence $\hat{W}_1 \subset \hat{W}_2 \subset \ldots$ converges to Ω_1. Define $A_{\varepsilon_1} := \Omega_1 \setminus \hat{W}_{n_1}$ if $\mu(\Omega_1 \setminus \hat{W}_{n_1}) < \varepsilon_1$ and define $A_{\varepsilon_i} := \Omega_1 \setminus \hat{W}_{n_i}$ if $\mu(\Omega_1 \setminus \hat{W}_{n_i}) < \varepsilon_i$ and if $n_i \geq n_{i-1}$ for $i > 1$. □

Note that the decomposition $\Omega = \Omega_1 + \Omega_2$ depends only on the equivalence class of μ. The restrictions T_{Ω_1} and $T_{A_{\varepsilon_i}}$ ($i \geq 1$) are dissipative or compressible. The restriction T_C is conservative, i. e. there exists no T_C-wandering subset of C of positive measure. Therefore the negative nonsingularity of T_C implies the positive nonsingularity of T_C. For every measurable subset A of C we have $A \subset \bigcup_{i\geq 1} T_C^{-i}A$ because $A \setminus \bigcup_{i\geq 1} T_C^{-i}A$ is a T_C-wandering subset of C. Furthermore $\mu_C(T_C^{-1}A) = 0$ implies $\mu_C(T_C^{-i}A) = 0$ for all $i > 1$ because T_C is negative nonsingular and we obtain $\mu_C(A) = 0$ i. e. T_C is positive nonsingular.

Using T^n instead of T we get an analogous decomposition of Ω with the same conservative part C as in the theorem above because every power of a conservative transformation is itself a conservative transformation. Thus T_C satisfies the strong recurrence theorem. For every $A \in \mathscr{A}_C$ with $\mu(A) > 0$ we have $\bigcap_{k\geq 0} \bigcup_{i\geq k} T_C^{-i}A \supset A$.

We shall need the following special case of a result of J. Neveu [*12*], which may be proved by a exhaustion argument.

THEOREM 2. — Let T be a negative nonsingular transformation on $(\Omega, \mathscr{A}, \mu)$. Then there exists an unique decomposition of the conservative part C into measurable subsets I and $C \setminus I$ such that I is the largest subset of C with the following property

(1) I is invariant and T_I admits a finite invariant measure ν equivalent to μ_I.

From (1) and since T_C is conservative it follows that $\Omega_2 = \hat{C}$ is the disjoint union of the two invariant measurable subsets $\hat{I}$ and $\widehat{C \setminus I}$.

In the following theorem we establish the existence of eventually weakly wandering sequences on $\Omega \setminus \hat{I}$, which were introduced in [*5*] and which turned out to be important for the existence of subset generators of size two for nonsingular invertible transformations, see [*4*]. By means of the following theorem it is shown in Theorem 9 and in Theorem 10 that for bimeasurable, negative nonsingular, non-invertible transformations there exist generators and subset generators of size two on $\Omega \backslash \hat{I}$.

DEFINITION. — A sequence $(s_j)_{j \geqslant 1}$ of natural numbers is called an eventually weakly wandering sequence (e. w. w. s.) for T if for every $\varepsilon > 0$ there exists a natural number $j(\varepsilon)$ and a measurable set E_ε such that $\mu(\complement E_\varepsilon) < \varepsilon$ and E_ε is a weakly wandering set under the sequence $(s_j)_{j \geqslant j(\varepsilon)}$ i. e.

$$T^{-s_i}E_\varepsilon \cap T^{-s_j}E_\varepsilon = \emptyset \qquad \text{for} \qquad j > i \geqslant j(\varepsilon).$$

A set S of natural numbers is said to have positive density if

$$\limsup_n \frac{1}{n} \sum_{k=0}^{n-1} 1_S(k) > 0 .$$

THEOREM 3. — Let T denote a measurable, negative nonsingular transformation on $(\Omega, \mathscr{A}, \mu)$.

The following conditions are equivalent:

i) T admits no nontrivial finite invariant measure ν absolutely continuous with respect to μ.

ii) There is no measurable invariant subset A of Ω such that the transformation T_A admits a nontrivial finite invariant measure ν absolutely continuous with respect to μ_A.

iii) For every $\varepsilon > 0$ there exists a measurable subset B of Ω with $\mu(\complement B) < \varepsilon$ and $\lim_n \left(\sup_{j \geqslant 0} \frac{1}{n} \sum_{k=j}^{j+n-1} \mu(T^{-k}B) \right) = 0.$

iv) Every set S of natural numbers with positive density contains an e. w. w. s. for T.

v) For every $\varepsilon > 0$ there exists a measurable subset B of Ω with $\mu(\complement B) < \varepsilon$ and $\inf_n \mu(T^{-n}B) = 0$.

vi) For every $\varepsilon > 0$ there exists a weakly wandering set W with $\mu(\complement W) < \varepsilon$.

vii) For every $\varepsilon > 0$ there exists a measurable subset B of Ω with $\mu(\complement B) < \varepsilon$ and $B \cap T^{-k}B = \emptyset$ for infinitely many $k \geqslant 1$.

For nonsingular invertible transformations some of these equivalences are shown in [*8*].

Proof. — *i*) $\Rightarrow$ *ii*): Assume there is a measurable subset A of Ω with $T^{-1}A \supset A$ such that T_A admits a nontrivial finite invariant measure ν

absolutely continuous with respect to μ_A. Then the measure ρ on $\mathcal{A}$, defined by $\rho(B) := \nu(B \cap A)$ for $B \in \mathcal{A}$, is a nontrivial finite T-invariant measure absolutely continuous with respect to μ.

ii) $\Rightarrow$ *iii)*: This implication follows from [*12*], Theorem 2.

iii) $\Rightarrow$ *iv)*: As shown in Theorem 1 we decompose Ω into measurable sets $\Omega_1 = \complement\hat{C}$ and $\Omega_2 = \hat{C}$ with the following properties. There is a measurable set C with $T^{-1}C \supset C$ and $\hat{C} = \bigcup_{k \geq 0} T^{-k}C$ such that C contains no T-wandering measurable set of positive measure and for every $\varepsilon > 0$ there exists a measurable set D with $\mu(D) < \varepsilon$, $T^{-1}D \supset D$ and $\complement\hat{C} = \bigcup_{k \geq 0} T^{-k}D$.

Note that $\limsup_n \frac{1}{n} \sum_{k=0}^{n-1} 1_S(k) > 0$ and $\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \mu_C(T_C^{-k}B) = 0$ imply $\inf_{s \in S} \mu_C(T_C^{-s}B) = 0$ for every measurable subset B of C.

Condition *iii)* now guarantees the existence of a sequence $(B_j)_{j>1}$ of measurable subsets B_j of C such that $\mu(B_j) > \mu(C) - \frac{1}{2^j}$ and $\inf_{s \in S} \mu_C(T_C^{-s}B_j) = 0$ for all $j > 1$.

We construct a decreasing sequence $(\varepsilon_j)_{j \geq 1}$ of positive numbers ε_j and an increasing sequence $(s_j)_{j \geq 1}$ of natural numbers $s_j \in S$. Let $\varepsilon_1 = \frac{1}{2}$ and let s_1 be an arbitrary number in S with $s_1 \geq 1$. Since T is negative nonsingular there exists a positive number $\varepsilon_2 < \frac{\varepsilon_1}{2}$ such that $\mu(B) < \varepsilon_2$ implies $\sum_{k=1}^{s_1} \mu(T^{-k}B) < \frac{1}{2^2}$ for every measurable set B. Furthermore since $\inf_{s \in S} \mu_C(T_C^{-s}B_2) = 0$ and since T_C is positive nonsingular there exists a natural number $s_2 \in S$, $s_2 > 2s_1$ with $\mu(T^{-s_2}C) > \mu(\hat{C}) - \frac{1}{2^2}$ and $\sum_{l=1}^{2s_1} \mu_C(T_C^{-s_2+l}B_2) < \varepsilon_2$.

Assume decreasing positive numbers ε_i and increasing natural numbers $s_i \in S$ have been chosen for $1 \leq i \leq j - 1$ such that

1) $$0 < \varepsilon_i < \frac{\varepsilon_{i-1}}{2}$$

2) $$\mu(B) < \varepsilon_i \quad \text{implies} \quad \sum_{k=1}^{s_{i-1}} \mu(T^{-k}B) < \frac{1}{2^i}$$

3) $$s_i > 2s_{i-1}\,, \qquad s_i \in S$$

4) $$\mu(T^{-s_i}C) > \mu(\hat{C}) - \frac{1}{2^i}$$

5) $$\sum_{l=1}^{2s_{i-1}} \mu_C(T_C^{-s_i+l}B_i) < \varepsilon_i \qquad \text{for} \qquad 1 < i \leqslant j-1\,.$$

Since T is negative nonsingular there exists a positive number $\varepsilon_j < \frac{\varepsilon_{j-1}}{2}$ such that $\mu(B) < \varepsilon_j$ implies $\sum_{k=1}^{s_{j-1}} \mu(T^{-k}B) < \frac{1}{2^j}$. Since $\inf_{s\in S} \mu_C(T_C^{-s}B_j) = 0$ and since T_C is positive nonsingular there exists a natural number $s_j \in S$, $s_j > 2s_{j-1}$ such that $\mu(T^{-s_j}C) > \mu(\hat{C}) - \frac{1}{2^j}$ and $\sum_{l=1}^{2s_{j-1}} \mu_C(T_C^{-s_j+l}B_j) < \varepsilon_j$. Therefore the statements 1)-5) hold for $i = j$ and by induction for all $i > 1$.

For $A_j = B_j \setminus \bigcup_{1\leqslant i<j} T_C^{-s_j+s_i}B_j$ $(j > 1)$ we conclude $\mu(A_j) > \mu(C) - 2\frac{1}{2^j}$ and $A_j \cap T^{-s_j+s_i}A_j = \emptyset$ $(1 \leqslant i < j)$. Assume $\varepsilon > 0$. We choose a measurable subset D of $\complement\hat{C}$ with $\mu(D) < \frac{\varepsilon}{6}$, $T^{-1}D \supset D$ and $\complement\hat{C} = \bigcup_{k\geqslant 0} T^{-k}D$. Let $j(\varepsilon)$ denote a natural number such that $\frac{1}{2^{j(\varepsilon)}} < \frac{\varepsilon}{6}$ and

$$\mu(T^{-s_{j(\varepsilon)}}D) > \mu(\complement\hat{C}) - \frac{\varepsilon}{6}\,.$$

We define

$$F_\varepsilon = \bigcap_{j>j(\varepsilon)} A_j$$

$$G_\varepsilon = T^{-s_{j(\varepsilon)}}C \setminus \left(\bigcup_{j>j(\varepsilon)} \left(\bigcup_{k=1}^{s_{j-1}} T^{-k} \left(\bigcup_{l=1}^{2s_{j-1}} T_C^{-s_j+l}B_j \right)\right)\right) \setminus C$$

$$H_\varepsilon = T^{-s_{j(\varepsilon)}}D \setminus D\,.$$

Then we conclude

$$\mu(F_\varepsilon) > \mu(C) - 2\frac{1}{2^{j(\varepsilon)}} > \mu(C) - \frac{\varepsilon}{3}$$

$$\mu(G_\varepsilon) > \mu(\hat{C}\setminus C) - 2\frac{1}{2^{j(\varepsilon)}} > \mu(\hat{C}\setminus C) - \frac{\varepsilon}{3}$$

$$\mu(H_\varepsilon) > \mu(\complement\hat{C}) - \frac{\varepsilon}{3}\,.$$

Therefore the measurable set $E_\varepsilon = F_\varepsilon + G_\varepsilon + H_\varepsilon$ satisfies $\mu(\complement E_\varepsilon) < \varepsilon$.

Finally we will show that the sets $T^{-s_j}E_\varepsilon$ ($j \geqslant j(\varepsilon)$) are pairwise disjoint. Assume $j > i \geqslant j(\varepsilon)$. The condition 3) implies $T^{-s_i}H_\varepsilon \cap T^{-s_j}H_\varepsilon = \emptyset$ and $T^{-s_i}G_\varepsilon \cap T^{-s_j}G_\varepsilon = \emptyset$. From $T^{-s_i}A_j \cap T^{-s_j}A_j = \emptyset$ it follows that $T^{-s_i}F_\varepsilon \cap T^{-s_j}F_\varepsilon = \emptyset$. To show that $G_\varepsilon \cap T^{-s_j+s_i}F_\varepsilon = \emptyset$ we assume $x \in (T^{-s_{j(\varepsilon)}}C \backslash C) \cap T^{-s_j+s_i}F_\varepsilon$. Then there exists a natural number n with $1 \leqslant n \leqslant s_{j(\varepsilon)} \leqslant s_{j-1}$ such that $T^{n-1}x \notin C$, $T^n x \in C$, $T^{n+(s_j-s_i-n)}x \in F_\varepsilon \subset B_j$ and $T^n x \in T_C^{-s_j+s_i+n}B_j$. It follows that $x \in T^{-n}(T_C^{-s_j+s_i+n}B_j)$ and $x \notin G_\varepsilon$. Therefore $T^{-s_i}G_\varepsilon \cap T^{-s_j}F_\varepsilon = \emptyset$ and E_ε is a weakly wandering set under the sequence $(s_j)_{j \geqslant j(\varepsilon)}$.

It is easy to see that on $\complement\hat{C}$ every infinite set of natural numbers contains an e. w. w. s. for T.

iv) $\Rightarrow$ *v*) is obvious.

v) $\Rightarrow$ *vi*): See e. g. [6].

vi) $\Rightarrow$ *vii*) is obvious.

vii) $\Rightarrow$ *v*): Assume $\varepsilon > 0$. Let $(B_i)_{i \geqslant 1}$ be a sequence of measurable subsets of Ω and let $(k_i)_{i \geqslant 1}$ denote an increasing sequence of natural numbers such that $\mu(\complement B_i) < \dfrac{\varepsilon}{2^i}$ and $B_i \cap T^{-k_i}B_i = \emptyset$ for all $i \geqslant 1$. For the intersection $B = \bigcap_{i \geqslant 1} B_i$ we get $\mu(\complement B) \leqslant \sum_{i \geqslant 1} \mu(\complement B_i) < \varepsilon$. Since $T^{-k_i}B_i \subset \complement B_i$ it follows that $\mu(T^{-k_i}B) < \dfrac{\varepsilon}{2^i}$ for $i \geqslant 1$ and therefore $\inf_n \mu(T^{-n}B) = 0$.

v) $\Rightarrow$ *i*) is obvious. □

From Theorem 2 and Theorem 3 we obtain the following

COROLLARY. — For each of the following properties there exists a sequence $(A_i)_{i \geqslant 1}$ of measurable sets such that $\Omega \backslash \hat{I} = \bigcup_{i \geqslant 1} A_i$ and the sets A_i possess one of the following properties:

i) $\liminf_n \mu(T^{-n}A_i) = 0$ for all $i \geqslant 1$

ii) $\liminf_n \dfrac{1}{n} \sum_{j=0}^{n-1} \mu(T^{-j}A_i) = 0$ for all $i \geqslant 1$

iii) $\limsup_n \dfrac{1}{n} \sum_{j=0}^{n-1} \mu(T^{-j}A_i) = 0$ for all $i \geqslant 1$

iv) $$\lim_n \left(\sup_{j \geqslant 1} \frac{1}{n} \sum_{k=j}^{j+n-1} \mu(T^{-k}A_i) \right) = 0 \quad \text{for all} \quad i \geqslant 1$$

v) A_i is weakly wandering for all $i \geqslant 1$

DEFINITION. — Let T denote a negative nonsingular transformation on $(\Omega, \mathscr{A}, \mu)$. T is called aperiodic iff

(2) for every $n \geqslant 1$ and every $A \in \mathscr{A}$ with $\mu(A) > 0$ there exists a measurable subset B of A such that $\mu(B \setminus T^{-n}B) > 0$.

This definition of aperiodicity is used in [7] under the additional assumption of positive nonsingularity. If T is invertible and negative nonsingular then condition (2) is equivalent to

(3) for every $n \geqslant 1$ and every $A \in \mathscr{A}$ with $\mu(A) > 0$ there exists a measurable subset B of A such that $\mu(B \triangle T^{-n}B) > 0$.

To obtain (2) from (3) we fix a number $n \geqslant 1$ and a set $A \in \mathscr{A}$ with $\mu(A) > 0$ and we chose a measurable subset B of A such that $\mu(B \triangle T^{-n}B) > 0$. If $\mu(T^{-n}B \setminus B) > 0$, then since T is negative nonsingular we get $\mu(B \setminus T^n B) > 0$, $B \setminus T^n B \subset A$ and $\mu((B \setminus T^n B) \setminus T^{-n}(B \setminus T^n B)) = \mu(B \setminus T^n B) > 0$.

Furthermore if T is an invertible nonsingular transformation and if $\mathscr{A}$ is countably generated and contains all points of Ω then (2) is equivalent to each of the following two conditions:

(4) $$\mu(\{ \omega \in \Omega \mid T^n \omega = \omega \quad \text{for some} \quad n \geqslant 1 \}) = 0$$

(5) $$\mu(\{ T^m \neq \mathrm{id}_\Omega \}) = 1 \quad \text{for all} \quad m \geqslant 1$$

The difference of (2) and (3) in the case of noninvertible transformations is explained by the following

EXAMPLE. — $\Omega = \mathbb{N}$, $\mathscr{A} = 2^{\mathbb{N}}$, $\mu(\{ m \}) = 2^{-m}$ $(m \geqslant 1)$ and

$$T : \mathbb{N} \to \mathbb{N} : m \mapsto \begin{cases} m - 1 & m \geqslant 2 \\ 1 & m = 1 \end{cases}$$

After a finite number of applications of T every $m \geqslant 1$ arrives at 1 and remains there. Thus T is in a certain sense periodic, T does not satisfy condition (2) but T satisfies condition (3).

DEFINITION. — A negative nonsingular transformation T on $(\Omega, \mathscr{A}, \mu)$ is called periodic on a set $A \in \mathscr{A}$ iff

(6) there is some $n \geqslant 1$ such that $\mu(B \setminus T^{-n}B) = 0$ holds for every measurable subset B of A.

The smallest n satisfying condition (6) is called the period of T on A. T is said to have strict period n on A, if T is periodic with period n on every measurable subset B of A.

THEOREM 4. — Let T denote a negative nonsingular transformation on $(\Omega, \mathscr{A}, \mu)$. Then there is an unique measurable decomposition of I into pairwise disjoint subsets I_i $(i \geqslant 0)$ such that I_i is invariant $(i \geqslant 0)$ and T_{I_i} has strict period i on I_i $(i \geqslant 1)$. For every $i \geqslant 1$ there exists a measurable subset B_i of I_i such that $T^{-k}B_i \cap T^{-l}B_i = \varnothing$ for $0 \leqslant k < l \leqslant i - 1$ and $\bigcup_{k=0}^{i-1} T_{I_i}^{-k}B_i = I_i$. T_{Ω_0} is aperiodic where $\Omega_0 = \Omega_1 + \widehat{C \setminus I} + \hat{I}_0$. Thus also T_{I_0}, $T_{C\setminus I}$ and $T_{A_{\varepsilon_i}}$ are aperiodic.

As a consequence we obtain a measurable partition of Ω into invariant and pairwise disjoint sets: $\Omega = \Omega_1 + \widehat{C \setminus I} + \hat{I}_0 + \hat{I}_1 + \dots$.

Proof. — The theorem follows by an exhaustion procedure on C. The proofs of the lemmas 1.1, 1.2 and 1.3 of [*7*] apply almost without changes. Note that T_C is positive nonsingular on C. $\square$

APERIODICITY, ROHLIN SETS AND SWEEP OUT SETS

THEOREM 5. — Let T be a negative nonsingular transformation on $(\Omega, \mathscr{A}, \mu)$. The following conditions are equivalent:

i) T is aperiodic.

ii) For every $m > 1$ and for every $A \in \mathscr{A}$ with $\mu(A) > 0$ there exists a measurable subset B of A such that $\mu(B) > 0$ and B, $T^{-1}B, \dots, T^{-m+1}B$ are pairwise disjoint.

iii) For every $n > 1$ and every $\varepsilon > 0$ there exists a (n, ε)-Rohlin set D i. e. there exists a measurable subset D of Ω such that D, $T^{-1}D, \dots, T^{-n+1}D$ are pairwise disjoint and $\mu\left(\Omega \setminus \bigcup_{k=0}^{n-1} T^{-k}D\right) < \varepsilon$.

Proof. — In the case of invertible nonsingular transformations the implication *i)* $\Rightarrow$ *iii)* first appeared in [*1*], see also [*8*], Theorem 1.11 and [*9*].

i) $\Rightarrow$ *ii)*: Let $m > 1$ and $A \in \mathscr{A}$ with $\mu(A) > 0$ be fixed. According to the definition of aperiodicity there exists a subset B_1 of A such that $B_1 \setminus T^{-1}B_1 =: A_1 \subset A$ has a positive measure and $A_1 \cap T^{-1}A_1 = \emptyset$. Repeat this argument with A_1 and T^2 instead of A and T and so on, after $m - 1$ steps we get a measurable subset B of A with the desired properties.

ii) $\Rightarrow$ *iii)*: Let $n > 1$ and $\varepsilon > 0$ be fixed. On Ω_1 the assertion follows at once. We choose $A_\varepsilon \in \mathscr{A}$ with $T^{-1}A_\varepsilon \supset A_\varepsilon$, $\mu(A_\varepsilon) < \varepsilon$ and $\widehat{A_\varepsilon} = \Omega_1$. Then $D_1 := \bigcup_{j \geq 0} (T^{-jn-1}A_\varepsilon \setminus T^{-jn}A_\varepsilon)$ is a (n, ε)-Rohlin set on Ω_1. Note that T is always aperiodic on Ω_1.

Now we construct a (n, ε)-Rohlin set on $\Omega_2 = \hat{C}$. We may assume $\mu(C) > 0$. According to condition *ii)* for a fixed number $k > \frac{1}{\varepsilon}$ we find a measurable subset B_1 of C of positive measure such that B_1, $T^{-1}B_1, \ldots, T^{-kn+1}B_1$ are pairwise disjoint. We have $\mu\left(B_1 \setminus \bigcup_{i \geq 1} T^{-i}B_1\right) = 0$ since C contains no wandering subset of positive measure. Now an exhaustion procedure on C yields a measurable subset B of C such that B, $T^{-1}B, \ldots, T^{-kn+1}B$ are pairwise disjoint and B is a sweep out set for Ω_2. For some number i, $0 \leq i < k$ we have $\mu\left(\bigcup_{j=0}^{n-1} T^{-in-j}B\right) < \varepsilon$. $\overline{B} := T^{-in}B$ is again a sweep out set for Ω_2 and $D_2 := \bigcup_{j \geq 1}\left(T^{-jn}\overline{B} \setminus \bigcup_{k=0}^{jn-1} T^{-k}\overline{B}\right)$ is a (n, ε)-Rohlin set on Ω_2.

iii) $\Rightarrow$ *i)*: If T is not aperiodic on Ω then there exists a number $n \geq 1$ and a measurable set A of positive measure such that $\mu(B \setminus T^{-n}B) = 0$ for all measurable subsets B of A. But for $0 < \varepsilon < \mu(A)$ and for a $(n + 1, \varepsilon)$-Rohlin set D on Ω we obtain $\mu(T^{-i}D \cap A) > 0$ for some number i, $0 \leq i \leq n$. Hence $\mu((T^{-i}D \cap A) \setminus T^{-n}(T^{-i}D \cap A)) = 0$ and consequently

$$(T^{-i}D \cap A) \cap T^{-n}(T^{-i}D \cap A) \neq \emptyset$$

which contradicts $D \cap T^{-n}D = \emptyset$. □

The implication *i)* $\Rightarrow$ *iii)* of Theorem 5 can be strengthened in the following way:

COROLLARY. — Let T be an aperiodic, negative nonsingular transformation on $(\Omega, \mathscr{A}, \mu)$.

Then for every $n > 1$ and for every $\varepsilon > 0$ there exists a measurable subset D of Ω such that $T^{-i}D$ is a (n, ε)-Rohlin set and a sweep out set for every i, $0 \leqslant i \leqslant n - 1$. Especially for every $n > 1$ and for every $\varepsilon > 0$ there exists a sweep out set E with $\mu(E) \leqslant \frac{1}{n}$ which is also a (n, ε)-Rohlin set.

Proof. — Since T is negative nonsingular for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mu(A) < \delta$ implies $\mu(T^{-i}A) < \varepsilon$ for $0 \leqslant i \leqslant n - 1$. Therefore $T^{-i}D$ is a (n, ε)-Rohlin set for $0 \leqslant i \leqslant n - 1$ if D is a (n, δ)-Rohlin set. It remains to show that for every $n > 1$ and for every $\varepsilon > 0$ there exists a (n, ε)-Rohlin set which is also a sweep out set for Ω. On Ω_2 this assertion follows by an exhaustion argument. On Ω_1 this assertion is trivial if T is invertible. From Hopf's decomposition we obtain that for every $n > 1$ there exists a set $E \in \mathscr{A}$ with $\mu(E) \leqslant \frac{1}{n}$ such that $E, T^{-1}E, \ldots, T^{-n+1}E$ are pairwise disjoint and $\bigcup_{i=0}^{n-1} T^{-i}E = \Omega_1$. But if T is not invertible some additional considerations are necessary.

For every $\eta > 0$, for a wandering set $W \subset \Omega_1$ and for a set $A \subset \Omega_1$ it is easy to show that $\mu(\widehat{W \setminus A}) > \mu(\hat{W}) - \eta$ if $\mu(A)$ is sufficiently small. First we define inductively a decreasing sequence $(\varepsilon_i)_{i \geqslant 1} \searrow 0$ of real numbers $\varepsilon_i > 0$ and we denote by $(A_{\varepsilon_i})_{i \geqslant 1}$ a decreasing sequence of measurable sets A_{ε_i} corresponding to ε_i according Theorem 1. Let $\varepsilon > 0$ and $n > 1$ be fixed. Assume $\varepsilon_1 := \frac{\varepsilon}{2}$, $A_{\varepsilon_1}, \varepsilon_2, A_{\varepsilon_2}, \ldots, \varepsilon_i, A_{\varepsilon_i}$ be chosen. Let $\delta_i > 0$ such that $\mu(A) < \delta_i$ implies $\mu(\widehat{(T^{-1}A_{\varepsilon_i} \setminus A_{\varepsilon_i}) \setminus A}) > \mu(\widehat{(T^{-1}A_{\varepsilon_i} \setminus A_{\varepsilon_i})}) - \varepsilon_i$. Define $\varepsilon_{i+1} := \min\left\{ \frac{\varepsilon}{2^{i+1}}, \vartheta_{i+1} \right\}$, where $\vartheta_{i+1} > 0$ is such that $\mu(B) < \vartheta_{i+1}$ implies $\mu(T^{-n}B) < \delta_i$. Now

$$D := \bigcup_{i \geqslant 1} (T^{-1}A_{\varepsilon_i} \setminus (A_{\varepsilon_i} \cup T^{-n}A_{\varepsilon_{i+1}})) \cup \bigcup_{j \geqslant 1} (T^{-jn-1}A_{\varepsilon_1} \setminus T^{-jn}A_{\varepsilon_1})$$

is a (n, ε)-Rohlin set and a sweep out set for Ω_1. □

The invertible case of the following theorem was obtained in [*3*].

THEOREM 6. — Let T denote an aperiodic, negative nonsingular transformation on $(\Omega, \mathscr{A}, \mu)$.

Then for every finite set $\{ n_1, \ldots, n_r \}$ of integers with $0 \leqslant n_1 < \ldots < n_r$ there exists a measurable subset A of Ω such that $\bigcup_{i=1}^{r} T^{-n_i}A = \Omega$ and $\mu(A) < \frac{1}{r} \sum_{k=1}^{r} \frac{1}{k}$.

Proof. — Let $R := \{ n_1, \ldots, n_r \}$, $0 \leqslant n_1 < \ldots < n_r \leqslant s$, where s is an integer such that $a := \frac{s}{r} \sum_{k=1}^{r} \frac{1}{k}$ is not an integer, $S := \{ 0, 1, \ldots, s - 1 \}$ and $\varepsilon = a - [a] > 0$. Now choose a measurable subset D of Ω such that $T^{-i}D$ is a $\left(s, \frac{\varepsilon}{s^2} \right)$-Rohlin set for all i, $0 \leqslant i \leqslant s - 1$. By [*3*], Lemma 2.3 there exists a subset E of S such that $(E + R) \bmod s = S$ and $| E | \leqslant [a]$. Then there exists an integer $\bar{j} \in S$ such that for $\bar{E} := (\{ \bar{j} \} + E) \bmod s$ $\mu\left(\bigcup_{i \in \bar{E}} T^{-i}D \right) \leqslant \frac{[a]}{s}$. Define $Y := \Omega \setminus \bigcup_{j=0}^{s-1} T^{-j}D$ and obtain

$$\mu\left(\bigcup_{j=0}^{s-1} T^{-j}Y \right) < \frac{\varepsilon}{s}.$$

Then for $A := \bigcup_{i \in \bar{E}} T^{-i}D \cup \bigcup_{j=0}^{s-1} T^{-j}Y$ we get $\mu(A) < \frac{[a]}{s} + \frac{\varepsilon}{s} = \frac{1}{r} \sum_{k=1}^{r} \frac{1}{k}$ and A sweeps out on R.

First we observe that $\bigcup_{i=1}^{r} T^{-n_i}A = \bigcup_{i \in \bar{E}+R} T^{-i}D \cup \bigcup_{j=n_1}^{n_r+s-1} T^{-j}Y$. Now assume $\omega \in \Omega$, $\omega \notin \bigcup_{i \in \bar{E}+R} T^{-i}D$ and $\omega \notin T^{-s}Y = \Omega \setminus \bigcup_{j=s}^{2s-1} T^{-j}D$. Then there exists an integer j_0, $s \leqslant j_0 \leqslant 2s - 1$ such that $\omega \in T^{-j_0}D$. Since $(\bar{E}+R) \bmod s = S$ and $j_0 \notin \bar{E}+R$ we conclude $j_0 - s \in \bar{E}+R$, $\omega \notin T^{-j_0+s}D$, $n_1 \leqslant j_0 - s$ and therefore

$$\omega \in \Omega \setminus \bigcup_{j=j_0-s}^{j_0-1} T^{-j}D = T^{-j_0+s}Y \subset \bigcup_{i=1}^{r} T^{-n_i}A. \qquad \square$$

Condition *ii*) of the following theorem is investigated in [*2*] for non-singular invertible transformations in a more general set-up and several equivalent formulations are given. We now show that the negative non-singular transformations satisfying condition *ii*) of Theorem 7 are exactly the aperiodic transformations and therefore are identical with the transformations which satisfy the condition *iii*) of Theorem 5.

THEOREM 7. — Let T denote a negative nonsingular transformation on the measure space $(\Omega, \mathscr{A}, \mu)$.

The following conditions are equivalent:

i) T is aperiodic.

ii) For every infinite sequence $0 \leqslant n_1 < n_2 < \ldots$ of natural numbers and for every real number $\varepsilon > 0$ there exists a measurable subset A of Ω with $\mu(A) < \varepsilon$ and a natural number $r \geqslant 2$ such that A sweeps out under the finite sequence $n_1, \ldots, n_r$, i. e. $\Omega = \bigcup_{i=1}^{r} T^{-n_i}A$.

Proof. — The implication *i*) $\Rightarrow$ *ii*) is a consequence of Theorem 6.

ii) $\Rightarrow$ *i*): It follows from Theorem 4 that for every finite measure space $(\Omega, \mathscr{A}, \mu)$ and for every negative nonsingular transformation T there exists a measurable partition $(\Omega_n)_{n \geqslant 0}$ of Ω with $T^{-1}\Omega_n \supset \Omega_n$ for every $n \geqslant 0$ such that T_{Ω_0} is aperiodic on Ω_0, $\Omega_n = \hat{I}_n$ for a measurable invariant subset I_n of Ω and T_{I_n} has strict period n on I_n for $n \geqslant 1$. Furthermore T_{I_n} admits a finite invariant measure equivalent to μ_{I_n} on I_n for $n \geqslant 1$. Therefore, if T is not aperiodic, without loss of generality we assume $\Omega = I_n$ for a $n \geqslant 1$. Then for every measurable subset A of Ω we have $T^{-n}A \supset A$ and $\mu(T^{-n}A \backslash A) = 0$ since Ω contains no weakly wandering sets of positive measure.

Let $0 < \varepsilon < \mu(\Omega)$ and choose $\delta > 0$ such that $\mu(A) < \delta$ implies $\mu\left(\bigcup_{i=0}^{n-1} T^{-i}A\right) < \varepsilon$. This is possible since T^i is negative nonsingular for $0 \leqslant i \leqslant n-1$. Now for every measurable subset A of Ω with $\mu(A) < \delta$ and for every $r \geqslant 1$ we conclude $\bigcup_{i=0}^{r} T^{-i}A \subset \bigcup_{j=0}^{n-1} T^{-j}A \neq \Omega$, which is a contradiction to condition *ii*). □

COUNTABLE GENERATORS AND TWO-SET GENERATORS

A finite or countable set $\xi = \{ A_i ; i \in I \}$ of measurable, pairwise disjoint subsets $A_i \in \mathscr{A}$ with union Ω is called a partition of Ω. Let S denote an infinite subset of $\mathbb{N} = \{ 0, 1, 2, \ldots \}$. A partition $\xi = \{ A_i ; i \in I \}$ is called S-generator for T if $\mathscr{A}$ is mod μ the smallest σ-algebra containing $\{ T^{-s}A_i ; s \in S, i \in I \}$. A S-generator $\xi = \{ A_i ; i \in I \}$ is called countable or finite or a two-set generator if I is countable or finite or a two-set.

If T is a measurable, negative nonsingular transformation on $(\Omega, \mathscr{A}, \mu)$

admitting a countable $\mathbb{N}$-generator then T is isomorphic to the left shift on the sequence space $\Omega' = \{(n_0, n_1, n_2, \ldots); n_i \geqslant 1\}$, i. e. there exists a measure algebra isomorphism between $\mathscr{A}$ and the product σ-algebra on Ω' via the mapping $\varphi : \Omega \to \Omega'$, where $(\varphi(\omega))_k := i$ if $\omega \in T^{-k}A_i$ for $k \geqslant 0$ and $i \geqslant 1$. Furthermore, if T admits a countable S-generator, then the coordinate process $(X_k)_{k \geqslant 0}$ on Ω' is determined by the process $(X_s)_{s\in S}$, i. e. the coordinate mappings X_k are functions of $\{X_s ; s \in S\}$ for every $k \geqslant 0$.

For an introduction and for a review of results on $\mathbb{N}$-generators see [*11*]. V. A. Rohlin [*13*] showed that there exist countable $\mathbb{N}$-generators for measure preserving noninvertible aperiodic transformations if and only if $\mathscr{A} = \mathscr{C} \vee T^{-1}\mathscr{A}$ for a countable partition $\mathscr{C}$. G. Helmberg and F. H. Simons [7] generalised this result for nonsingular noninvertible transformations. Recently M. H. Ellis and N. A. Friedman [*4*] established the existence of countable subset generators for nonsingular invertible aperiodic transformations.

We now show the existence of countable subset generators for arbitrary negative nonsingular aperiodic transformations.

THEOREM 8. — Let T be a negative nonsingular aperiodic transformation on $(\Omega, \mathscr{A}, \mu)$. Let $\mathscr{C}$ denote a countable measurable partition of Ω satisfying $\mathscr{A} = \mathscr{C} \vee T^{-1}\mathscr{A}$ and let $\mathscr{A}$ be generated mod μ by the sets A_i $(i \geqslant 1)$.

Then for every infinite subset $S = \{n_k \mid k \geqslant 0\}$ of natural numbers $0 = n_0 < n_1 < \ldots$ there exists a countable measurable partition $\mathscr{K}$ of Ω such that $\bigvee_{k \geqslant 0} T^{-n_k}\mathscr{K} = \mathscr{A}$ mod μ, i. e. T admits a countable S-generator for $\mathscr{A}$.

In Lebesgue-spaces the condition $\mathscr{A} = \mathscr{C} \vee T^{-1}\mathscr{A}$ means that T is countable to one. Note that the two conditions $\mathscr{A} = \mathscr{C} \vee T^{-1}\mathscr{A}$ and $\mathscr{A}$ countable generated are necessary for the conclusion of the theorem. Furthermore if T is not aperiodic and the measure space $(\Omega, \mathscr{A}, \mu)$ is nonatomic then the conclusion of the theorem is not true.

As in [7], § 3 we need the following lemma:

LEMMA. — Let the assumptions of Theorem 8 be satisfied. Let B denote a measurable set in $\mathscr{A}$ and let $\mathscr{A}_n$ $(n \geqslant 0)$ and $\mathscr{B}$ denote sub-σ-algebras of $\mathscr{A}$. We define

$$U(B, m) = \left\{ \mathscr{D} \subset \mathscr{A} \mid \mathscr{D} \text{ sub-}\sigma\text{-algebra of } \mathscr{A}, \inf_{D \in \mathscr{D}} \mu((A_i \,\triangle\, D) \cap B) < \frac{1}{m} \text{ for } 1 \leqslant i \leqslant m \right\} \quad (1 \leqslant m)$$

$^{\mathrm{B}}\lim_n \mathscr{A}_n = \{ \mathrm{A} \in \mathscr{A} \mid$ there is a sequence $(\mathrm{E}_n)_{n \geq 0}$, $\mathrm{E}_n \in \mathscr{A}_n$ such that $\lim_n \mu((\mathrm{A} \triangle \mathrm{E}_n) \cap \mathrm{B}) = 0 \}$

$$\mathscr{B}^0 = \mathscr{B}, \qquad \mathscr{B}^{n_0, n_1, \ldots, n_l} = \bigvee_{i=0}^{l} \mathrm{T}^{-n_i}\mathscr{B} \vee \bigvee_{j=0}^{n_l - 1} \mathrm{T}^{-j}\mathscr{C} \qquad (l \geq 1)$$

(α)

The following conditions are equivalent:

i) for every $m \geq 1$ there exists a $n(m) \geq 1$ such that $\mathscr{A}_n \in \mathrm{U}(\mathrm{B}, m)$ for all $n \geq n(m)$,

ii) $^{\mathrm{B}}\lim_n \mathscr{A}_n = \mathscr{A}$,

iii) $^{\left(\bigcup_{i=0}^{l} \mathrm{T}^{-n_i}\mathrm{B}\right)}\lim_n \mathscr{A}_n^{n_0, n_1, \ldots, n_l} = \mathscr{A}$ for all $l \geq 0$.

(β)

Let $l \geq 0$ and $\mathrm{B} \in \mathscr{A}$ be fixed. Then for every $m \geq 1$ there exists a $k \geq 1$ such that $\mathscr{D} \in \mathrm{U}(\mathrm{B}, k)$ implies

$$\mathscr{D}^{n_0, n_1, \ldots, n_l} \in \mathrm{U}\left(\bigcup_{i=0}^{l} \mathrm{T}^{-n_i}\mathrm{B}, m\right).$$

The proof of the lemma is similar as in [*7*], § 3 and is omitted.

Proof of Theorem 8. — At first, by analogous arguments as in [*4*], § 2, we construct a sequence $(k_l)_{l \geq 0}$ of natural numbers and a sequence $(\mathrm{B}_l)_{l \geq 0}$ of measurable sets in $\mathscr{A}$ by repeated application of Theorem 6 in the following way.

Let $k_0 = 1$ and $\mathrm{B}_0 = \Omega$. Since T is negative nonsingular, there exist a $\delta_1 > 0$ such that $\mu(\mathrm{B}) < \delta_1$ implies $\mu\left(\bigcup_{i=0}^{k_0} \mathrm{T}^{-n_i}\mathrm{B}\right) < \frac{1}{2}$. Theorem 6 yields a natural number $k_1 > k_0$ and a measurable set B_1 such that $\mu(\mathrm{B}_1) < \frac{1}{k_1}\sum_{k=1}^{k_1}\frac{1}{k} < \delta_1$ and $\bigcup_{i=0}^{k_1} \mathrm{T}^{-n_i}\mathrm{B}_1 = \Omega$. Since T is negative nonsingular there exists a $\delta_2 > 0$ such that $\mu(\mathrm{B}) < \delta_2$ implies $\mu\left(\bigcup_{i=0}^{k_1} \mathrm{T}^{-n_i}\mathrm{B}\right) < \frac{1}{2^2}$. Theorem 6 yields a natural number $k_2 > k_1$ and a measurable set B_2 such that $\mu(\mathrm{B}_2) < \frac{1}{k_2}\sum_{k=1}^{k_2}\frac{1}{k} < \delta_2$ and $\bigcup_{i=0}^{k_2} \mathrm{T}^{-n_i}\mathrm{B}_2 = \Omega$ and so on. We obtain natural numbers k_l and measurable sets B_l ($l \geqslant 0$) such that $k_{l-1} < k_l$,

$$\mu\left(\bigcup_{i=0}^{k_{l-1}} T^{-n_i}B_l\right) < \frac{1}{2^l} \quad \text{and} \quad \bigcup_{i=0}^{k_l} T^{-n_i}B_l = \Omega \quad \text{for} \quad l \geqslant 1 .$$

We now define a sequence $(C_l)_{l \geqslant 0}$ of pairwise disjoint neasurable sets C_l by $C_l = B_l \Big\backslash \bigcup_{j>l} B_j$ and obtain $\sum_{l \geqslant 0} C_l = \Omega$ and

$$\mu\left(\bigcup_{i=0}^{k_l} T^{-n_i}C_l\right) = \mu\left(\bigcup_{i=0}^{k_l} T^{-n_i}B_l \Big\backslash \bigcup_{j>l}\bigcup_{i=0}^{k_l} T^{-n_i}B_j\right) \geqslant 1 - \sum_{j>l}\frac{1}{2^j} = 1 - \frac{1}{2^l} .$$

Let $\mathscr{F}$ denote the countable partition of Ω generated by the sets C_l and $B_j \cap C_l$ ($0 < j < l$) and let $\mathscr{G}$ denote the countable partition of Ω defined on C_l by $\left(\bigvee_{i=0}^{n_{k_{l+1}}} T^{-i}\mathscr{C}\right) \cap C_l$ for $l \geqslant 0$. Note that $\bigvee_{i=0}^{n_{k_1}} T^{-i}\mathscr{C} \subset \mathscr{G}$ and $\left(\bigvee_{i=0}^{n_{k_{l+1}}} T^{-i}\mathscr{C}\right) \cap B_l \subset (\mathscr{F} \vee \mathscr{G})$ for $l > 0$.

Furthermore $\bigvee_{i=0}^{n_{k_l}} T^{-i}\mathscr{C} \subset \bigvee_{j=0}^{k_l} T^{-n_j}(\mathscr{F} \vee \mathscr{G})$ for $l \geqslant 0$ because for $C \in \mathscr{C}$ and for $n_{k_r} \leqslant i \leqslant n_{k_{r+1}}$ ($1 \leqslant r < l$) we obtain

$$\begin{aligned} T^{-i}C &= T^{-i}C \cap \bigcup_{j=0}^{k_r} T^{-n_j}B_r = T^{-i}C \cap \bigcup_{j=0}^{k_r} T^{-n_j}\left(C_r + \sum_{s>r}(B_r \cap C_s)\right) \\ &= \bigcup_{j=0}^{k_r} T^{-n_j}\left((T^{-(i-n_j)}C) \cap C_r + \sum_{s>r}(T^{-(i-n_j)}C) \cap B_r \cap C_s\right) \\ &\in \bigcup_{j=0}^{k_r} T^{-n_j}(\mathscr{F} \vee \mathscr{G}) . \end{aligned}$$

We now construct a sequence $(m_l)_{l \geqslant 0}$ of natural numbers by repeated application of the Lemma. Let $m_0 > 1$ and $m_1 > 2m_0$ such that $\mathscr{D} \in U(B_1, m_1)$ implies $\mathscr{D}^{n_0, n_1, \ldots, n_{k_1}} \in U\left(\bigcup_{i=0}^{k_1} T^{-n_i}B_1, m_0\right)$. Define $m_2 > 2m_1$ such that

$\mathscr{D} \in \mathrm{U}(\mathrm{B}_2, m_2)$ implies $\mathscr{D}^{n_0,n_1,\ldots,n_{k_2}} \in \mathrm{U}\left(\bigcup_{i=0}^{k_2} \mathrm{T}^{-n_i}\mathrm{B}_2, m_1\right)$ and so on. We obtain a sequence $(m_l)_{l\geq 0}$ of natural numbers such that $2m_{l-1} < m_l$ and

$$\mathscr{D} \in \mathrm{U}(\mathrm{B}_l, m_l) \quad \text{implies} \quad \mathscr{D}^{n_0,n_1,\ldots,n_{k_l}} \in \mathrm{U}\left(\bigcup_{i=0}^{k_l} \mathrm{T}^{-n_i}\mathrm{B}_l, m_{l-1}\right) \quad \text{for} \quad l \geq 1\,.$$

Define a countable measurable partition $\mathscr{H}$ in Ω such that $\mathscr{H} \in \mathrm{U}(\mathrm{C}_l, m_{l+1})$ $(l \geqslant 0)$. Then $\mathscr{F} \vee \mathscr{H} \in \mathrm{U}(\mathrm{B}_l, m_l)$ $(l \geqslant 0)$ and from

$$\bigvee_{j=0}^{k_l} \mathrm{T}^{-n_j}(\mathscr{F} \vee \mathscr{G} \vee \mathscr{H}) \supset (\mathscr{F} \vee \mathscr{G} \vee \mathscr{H})^{n_0,n_1,\ldots,n_{k_l}}$$

it follows that

$$\bigvee_{j\geqslant 0}^{k_l} \mathrm{T}^{-n_j}(\mathscr{F} \vee \mathscr{G} \vee \mathscr{H}) \in \mathrm{U}(\Omega, m_{l-1}) \qquad \text{for} \qquad l \geqslant 1$$

and $\bigvee_{j\geqslant 0} \mathrm{T}^{-n_j}(\mathscr{F} \vee \mathscr{G} \vee \mathscr{H}) = \mathscr{A} \bmod \mu.$ $\square$

The existence of two-set ℕ-generators and of two-set S-generators for invertible transformations has been settled by U. Krengel [*10*], L. K. Jones and U. Krengel [*8*] and by M. H. Ellis and N. A. Freidman [*4*].

Nothing is known about the existence of finite ℕ-generators for non-invertible transformations, see [*11*], p. 473. For bimeasurable, negative nonsingular transformations without nontrivial finite invariant measure absolutely continuous with respect to μ we now show the existence of two-set ℕ-generators and we give sufficient conditions for the existence of two-set S-generators.

A measurable subset A of Ω is called a set with a S-dense orbit or S-dense set for T if $\{\mathrm{T}^{-s}\mathrm{A}\,;\, s\in \mathrm{S}\}$ is dense in $\mathscr{A}$. If A is S-dense for T then $\xi = \{\mathrm{A}, \complement\mathrm{A}\}$ is a two-set S-generator for T.

THEOREM 9. — Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and assume $\mathscr{A}$ to be countably generated. Let T denote a measurable, negative non-singular transformation on Ω such that $\mathrm{T}\mathscr{A} \subset \mathscr{A}$ i. e. T maps every measurable set onto a measurable set.

Then the following conditions are equivalent:

i) The ℕ-dense sets for T are dense in $\mathscr{A}$.

ii) T does not admit a nontrivial finite invariant measure absolutely continuous with respect to μ.

Proof. — *i*) $\Rightarrow$ *ii*): If the $\mathbb{N}$-dense sets for T are dense in $\mathcal{A}$ then for every $\varepsilon > 0$ there exists a measurable subset A of Ω with $\mu(\complement A) < \varepsilon$ such that A is a $\mathbb{N}$-dense set and therefore $\inf_n \mu(T^{-n}A) = 0$. The assertion now follows by Theorem 3.

ii) $\Rightarrow$ *i*): We choose a decreasing sequence $(\varepsilon_i)_{i \geq 1}$ of real numbers $\varepsilon_i > 0$, a sequence $(B_i)_{i \geq 1}$ of measurable subsets B_i of Ω and an increasing sequence $(n_i)_{i \geq 1}$ of natural numbers n_i in the following way.

We set $\varepsilon_1 = \frac{1}{2}$. An application of Theorem 3 yields a set $B_1 \in \mathcal{A}$ with $\mu(\complement B_1) < \frac{\varepsilon_1}{2}$ and $B_1 \cap T^{-n_1}B_1 = \emptyset$ for a natural number $n_1 \geq 1$. Assume ε_k, B_k and n_k have been defined for $1 \leq k < i$ such that $0 < \varepsilon_k < \frac{\varepsilon_{k-1}}{2}$ and $\mu(A) < \varepsilon_k$ implies $\mu(T^{-n_{k-1}}A) < \frac{\varepsilon_{k-1}}{2}$ $(A \in \mathcal{A})$, $\mu(\complement B_k) < \frac{\varepsilon_k}{2}$, $n_k > n_{k-1}$ and $B_k \cap T^{-n_k}B_k = \emptyset$ for $1 < k < i$.

Since T is negative nonsingular there exists a real number $\varepsilon_i > 0$ with

1) $$0 < \varepsilon_i < \frac{\varepsilon_{i-1}}{2}$$

such that

2) $$\mu(A) < \varepsilon_i \quad \text{implies} \quad \mu(T^{-n_{i-1}}A) < \frac{\varepsilon_{i-1}}{2} \qquad (A \in \mathcal{A}).$$

Theorem 3 guarantees the existence of a measurable subset B_i of Ω and of a natural number $n_i > n_{i-1}$ such that

3) $$\mu(\complement B_i) < \frac{\varepsilon_i}{2} \quad \text{and} \quad B_i \cap T^{-n_i}B_i = \emptyset .$$

By induction we obtain sequences $(\varepsilon_i)_{i \geq 1}$, $(B_i)_{i \geq 1}$ and $(n_i)_{i \geq 1}$ which satisfy 1), 2) and 3) for every $i > 1$.

From 3) we obtain $B_i \cap T^{n_i}B_i = \emptyset$ and $\mu(T^{n_i}B_i) < \frac{\varepsilon_i}{2}$ for $i \geq 1$. Furthermore from $T^{-n_i}(T^{n_i}B_i) \supset B_i$ we conclude $\mu(\complement T^{-n_i}(T^{n_i}B_i)) < \frac{\varepsilon_i}{2}$ for $i \geq 1$.

Since

$$\mu\Big(\bigcup_{l>i} T^{n_l}B_l\Big) < \sum_{l>i} \frac{\varepsilon_l}{2} < \varepsilon_{i+1} \qquad (i \geq 1)$$

it follows from 2) that

$$4) \qquad \mu\left(T^{-n_i}\left(\bigcup_{l>i} T^{n_l}B_l\right)\right) < \frac{\varepsilon_i}{2} \qquad \text{for} \qquad i \geqslant 1 .$$

Now let $(A_i)_{i\geqslant 1}$ denote a sequence of measurable subsets A_i of Ω such that $\{A_i ; i \geqslant 1\}$ is dense in $\mathscr{A}$ and such that each A_i occurs infinitely often in the sequence $(A_i)_{i\geqslant 1}$.

We define measurable subsets D_j of Ω by

$$D_j := \sum_{i\geqslant j}\left(T^{n_i}(A_i \cap B_i)\Big\backslash \bigcup_{l>i} T^{n_l}B_l\right) \qquad \text{for} \qquad j \geqslant 1 .$$

From 3) and 4) we obtain for every $j \geqslant 1$ and every $k \geqslant j$

$$\mu(T^{-n_k}D_j \triangle A_k)$$
$$\leqslant \mu\left((T^{-n_k}D_j \triangle A_k) \cap \left(B_k\Big\backslash T^{-n_k}\bigcup_{l>k} T^{n_l}B_l\right)\right) + \mu\left(\complement\left(B_k\Big\backslash T^{-n_k}\bigcup_{l>k} T^{n_l}B_l\right)\right)$$
$$< \mu\left(\left[\sum_{i\geqslant j}\left\{T^{-n_k}\left(T^{n_i}(A_i \cap B_i)\Big\backslash \bigcup_{l>i} T^{n_l}B_l\right)\right\} \cap \left(B_k\Big\backslash T^{-n_k}\bigcup_{l>k} T^{n_l}B_l\right)\right]\right.$$
$$\left.\triangle\left(A_k \cap B_k\Big\backslash T^{-n_k}\bigcup_{l>k} T^{-n_l}B_l\right)\right) + \frac{\varepsilon_k}{2} + \frac{\varepsilon_k}{2} .$$

Since

$$\left\{T^{-n_k}\left(T^{n_i}(A_i \cap B_i)\Big\backslash \bigcup_{l>i} T^{n_l}B_l\right)\right\} \cap B_k \begin{cases} = \varnothing & \text{for} \quad i<k \\ \subset T^{-n_k}\left(\bigcup_{l>k} T^{n_l}B_l\right) & \text{for} \quad i>k \end{cases}$$

we conclude

$$\mu(T^{-n_k}D_j \triangle A_k)$$
$$\leqslant \mu\left(\left[\left\{T^{-n_k}\left(T^{n_k}(A_k \cap B_k)\Big\backslash \bigcup_{l>k} T^{n_l}B_l\right)\right\} \cap \left(B_k\Big\backslash T^{-n_k}\bigcup_{l>k} T^{n_l}B_l\right)\right]\right.$$
$$\left.\triangle\left[A_k \cap B_k\Big\backslash T^{-n_k}\bigcup_{l>k} T^{n_l}B_l\right]\right) + \varepsilon_k$$
$$= \mu\left(\left[\{T^{-n_k}(T^{n_k}(A_k \cap B_k))\} \cap B_k\Big\backslash T^{-n_k}\bigcup_{l>k} T^{n_l}B_l\right]\right.$$
$$\left.\triangle\left[A_k \cap B_k\Big\backslash T^{-n_k}\bigcup_{l>k} T^{n_l}B_l\right]\right) + \varepsilon_k$$
$$\leqslant \mu([\{T^{-n_k}(T^{n_k}(A_k \cap B_k))\} \cap B_k] \triangle [A_k \cap B_k]) + \varepsilon_k = \varepsilon_k$$

because $\{T^{-n_k}(T^{n_k}(A_k \cap B_k))\} \cap B_k = A_k \cap B_k$.

The sets D_j therefore are $\mathbb{N}$-dense for T with $\mu(D_j) < \varepsilon_j$ for every $j \geqslant 1$. Now choose an arbitrary measurable subset A of Ω and a real number $\varepsilon > 0$. We obtain a $\mathbb{N}$-dense set $A' \in \mathscr{A}$ with $\mu(A \triangle A') < \varepsilon$ either by

$A' := T^{-k_j}D_j$ for a suitable chosen $k_j > 1$ for every $j \geqslant 1$ or by

$A' := \left(A \setminus \bigcup_{l \geqslant j} T^{n_l}B_l\right) \cup D_j$ for a suitable chosen $j \geqslant 1$. □

COROLLARY 1. — Let T denote a negative nonsingular, conservative and countable to one transformation on a Lebesgue space $(\Omega, \mathscr{A}, \mu)$. Then the assertions of Theorem 9 are valid.

Proof. — For a negative nonsingular, conservative transformation T we have $\mu\left(A \setminus \bigcup_{k \geqslant 1} T^{-k}A\right) = 0$ $(A \in \mathscr{A})$ and therefore $\mu(T^{-1}A) = 0$ implies $\mu(A) = 0$ $(A \in \mathscr{A})$. i. e. T is positive nonsingular.

If T is a negative nonsingular, conservative and countable to one transformation on a Lebesgue space then it is shown in [*14*] that $T\mathscr{A} \subset \mathscr{A}$, i. e. T is bimeasurable, and T satisfies $\mu(TA) = 0$ if $A \in \mathscr{A}$ and $\mu(A) = 0$. □

COROLLARY 2. — Let the hypotheses of Theorem 9 be fulfilled and suppose there is no nonzero finite invariant measure absolutely continuous with respect to μ. Then the sets $A \in \mathscr{A}$, for which the partition $\xi = \{ A, \complement A \}$ is a two-set $\mathbb{N}$-generator, are dense in $\mathscr{A}$.

THEOREM 10. — Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and suppose $\mathscr{A}$ is countably generated. Let T denote a bimeasurable, negative nonsingular transformation on Ω and let S be an infinite subset of $\mathbb{N}$. Then the following conditions are equivalent:

i) The S-dense sets for T are dense in $\mathscr{A}$.
ii) S contains an e. w. w. s. for T.

Proof. — *i*) $\Rightarrow$ *ii*): From condition *i*) it follows that for every $\varepsilon > 0$ there exists a measurable subset B of Ω with $\mu(\complement B) < \varepsilon$ such that $\inf_{s \in S} \mu(T^{-s}B) = 0$. In a similar manner as in the proof of Theorem 3 one can construct an increasing sequence $(s_j)_{j \geqslant 1}$ of natural numbers $s_j \in S$ which is an e. w. w. s. for T.

ii) $\Rightarrow$ *i*): Let $(s_j)_{j \geqslant 1}$ denote an e. w. w. s. for T with $s_j \in S$ $(j \geqslant 1)$ and assume a real number $\varepsilon > 0$. Then there exists a measurable subset E of Ω and a natural number $j(\varepsilon)$ such that $\mu(\complement E) < \frac{\varepsilon}{2}$, E is weakly wandering on $(s_j)_{j \geq j(\varepsilon)}$ and $\mu\left(\bigcup_{j \geq j(\varepsilon)} T^{-s_j}E\right) < \frac{\varepsilon}{2}$.

Define $B := E \setminus \bigcup_{j \geq j(\varepsilon)} T^{-s_j}E$. Then B is a measurable subset of Ω with $\mu(\complement B) < \varepsilon$ and $B \cap T^{-s}B = \emptyset$ for infinitely many $s \in S$. Now after replacing $\mathbb{N}$ by S the second part of the proof of Theorem 9 applies. □

COROLLARY 1. — Let T be a negative nonsingular transformation on Ω. If T is purely dissipative, i. e. Ω is a countable union of wandering sets, then every infinite set S of natural numbers contains an e. w. w. s. for T.

COROLLARY 2. — Let T be a negative nonsingular transformation on Ω and let S be an infinite set of natural numbers with positive density. If T does not admit a nonzero finite invariant measure absolutely continuous with respect to μ then S contains an e. w. w. s. for T.

Proof. — The assertion follows from Theorem 3. □

COROLLARY 3. — Suppose the hypotheses of Theorem 10 are valid and let S contain an e. w. w. s. for T.

Then the sets $A \in \mathscr{A}$, for which the partition $\xi = \{ A, \complement A \}$ is a two-set S-generator for T, are dense in $\mathscr{A}$.

Example. — Set $\Omega = (0, 1]$, let $\mathscr{A}$ be the Lebesgue σ-algebra in $(0, 1]$ and let μ be the Lebesgue measure on $\mathscr{A}$. Let T denote the dyadic translation on Ω, defined by $T(\omega) := \omega - \frac{1}{2^i}$ if $\omega \in \left(\frac{1}{2^i}, \frac{1}{2^{i-1}}\right]$ $(i \geq 1)$. Then T is bimeasurable, negative nonsingular and purely dissipative. Therefore for an arbitrary infinite set S of natural numbers the sets $A \in \mathscr{A}$, such that $\{ A, \complement A \}$ is a S-generator, are dense in $\mathscr{A}$.

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