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Universal distribution for infinitely divisible distributions on Frèchet space

by

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SUMMARY. — In this paper we generalizate to the case of separable Frèchet space the theorem of Doeblin [*3*], which asserts that there exists a distribution belonging to the domain of partial attraction of every one dimensional infinitely divisible distribution. That the distribution is called an *universal distribution in Doeblin sense.* Theremore we show that each distribution concentrated on bounded subset belongs to the domain of attraction of $\delta(0)$ and there exist uncountably many universal distribution, which are not shift equivalent. The main theorem implies also that one distribution is infinitely divisible if *f*, it's domain of partial attraction is nonempty.

INTRODUCTION

Let E be a separable Frèchet space. We denote by $\mathscr{B}$ the σ-algebra generated by the topology in E and by $\mathscr{P}$ the set of all distribution on. We signe by $\Rightarrow$ the weak convergence. It is known that $\mathscr{S}$ with the weak topology is a separable metric space (see [*5*], Th. 6.2). For $q, q_n \in \mathscr{P}$ we say that $\{ q_n \}$ is *shift convergent* to q if there is a sequence $\{ a_n \} \in$ E such that $\delta(a_n) * q_n \Rightarrow q$, and say that q_1 and q_2 are *shift equivalent* if $q_1 = \delta(a) * q_2$ for some $a \in$ E, when $\delta(b)$ is the distribution concentrated at the point $b \in \mathrm{E}$ and $*$ denotes the convolution determined for measures on $\mathscr{B}$ by

$$\mu * \nu(\mathrm{B}) = \int \mu(\mathrm{B} - x)\nu(dx), \qquad \mathrm{B} \in \mathscr{B}.$$

Distribution p is called *infinitely divisible* (*inf. div.*) if for every natural n there exists one distribution p_n such that

$$p = p_n^{*n} = \underbrace{p_n * \ldots * p_n}_{n\text{-times}}$$

For finite measure μ on $\mathscr{B}$ let

$$e(\mu) = e^{-\mu(\mathrm{E})}\left(\delta(0) + \frac{\mu}{1\,!} + \frac{\mu^{*2}}{2\,!} + \ldots\right)$$

This is an *inf. div.* distribution called Poisson distribution with finite canonical measure μ Moreover (see [*2*], Th. 1.9) for each inf. div. distribution p there exist a symetric Gaussian distribution ω, a sequence of finite measures $\{ \mathrm{G}_n \}$ and an $a \in \mathrm{E}$ such that :

$$\delta(a) * \omega * e(\mathrm{G}_n) \Rightarrow p. \tag{1}$$

MAIN RESULTS

For $c > 0$ we define an operator T_c on the set of all measures on $\mathscr{B}$ by

$$\mathrm{T}_c\lambda(\mathrm{B}) = \lambda(c^{-1}\mathrm{B}), \qquad \mathrm{B} \in \mathscr{B}$$

Let $p, q \in \mathscr{P}$. We say that q *belongs to the domain* of *partial attraction* of p if there exist a subsequence $\{ n_k \}$ of natural numbers and a sequence of positive numbers $\{ v_k \}$ such that $\{ \mathrm{T}_{v_k} q^{*v_k} \}$ is shift-convergent to p. When $\{ n_k \}$ coincides with same sequence of all natural numbers we say that q *belongs to the domain* of *attraction* of p.

Distribution q is called *universal* if it belongs to the domain of partial attraction of every inf. div. distribution.

LEMMA 1. — On separable Frèchet space the set of all Poisson distribution is dense in the set of all inf. div. distribution.

Proof. — Let ω be a symetric Gaussian distribution. From theorems 3 [*4*] and 4 [*4*] there exists sequence of Gaussian distribution $\{ p_n \}$ on finite dimensional subspaces weakly convergent to ω. Then for each n there is a sequence of Poisson distribution $\{ p_n^m, m = 1, 2, \ldots \}$ weakly convergent to p_n. Since E is a metric space, we can find for each n one natural $m(n)$ such that sequence $\{ p_n^{m(n)} \}$ weakly converges to ω. Then by virtue of (1) the proof is completed.

LEMMA 2. — Let $A = \{ a_1, a_2, \ldots \}$ be a dense subset of E. Then the set of all Poisson distribution with finite canonical measures concentrated on finite subsets of A is dense in the set of all inf. div. distribution.

Proof. — Let's fix one metric f in E. We can take the sets $V(k, n)$, $k, n = 1, 2, \ldots$ such that for every n.

1) $f(x, a_k) < \frac{1}{n}$ for $x \in V(k, n)$

2) $V(k, n) \cap V(k', n) = \emptyset, \ k \neq k'$

3) $E = \bigcup_{k=n}^{\infty} V(k, n)$.

Let G be an arbitrary finite measure. For every n there is $k(n)$ such that

$$G\Big(\bigcup_{k \geq k(n)} V(k, n)\Big) < 2^{-n} \tag{2}$$

We define the measure μ_n, supported on $\{ a_1, a_2, \ldots, a_{k(n)} \}$ following

$$\mu_n(\{ a_k \}) = G(V(k, n)), \qquad k < k(n)$$

$$\mu_n(\{ a_{k(n)} \}) = G\Big(\bigcup_{k \geq k(n)} V(k, n)\Big)$$

Then $\frac{\mu_n}{G(E)}, \frac{G}{G(E)}$ are probability measures.

Using (2) and Th. II.6.1 [5] we may easily verify that $\frac{\mu_n}{G(E)} \Rightarrow \frac{G}{G(E)}$, which implies

$$\mu_n^{*l} \Rightarrow G^{*l}, \qquad l = 1, 2, \ldots$$

Then, because for $\varepsilon > 0$ arbitrary small there exists $k(\varepsilon)$ such that

$$\sum_{k > k(\varepsilon)} \frac{\mu_n^{*}(E)}{k\,!} = \sum_{k > k(\varepsilon)} \frac{G^{*k}(E)}{k\,!} < \varepsilon, \qquad n = 1, 2, \ldots,$$

we see by some estimations that $e(\mu_n) \Rightarrow e(G)$. Hence by virtue of (1) and lemma 1 this lemma is proved.

Using local convexity of E and the fact that for real α and $a \in E$, $\alpha' a \to 0$ when $\alpha \to 0$ we can easily show the following.

Lemma 3. — Let $A = \{ a_1, a_2, \ldots \}$ be a dense subset of E and let

$$E_k = \text{lin}\{ a_1, \ldots, a_k \}, \qquad k = 1, 2, \ldots$$

$$E_\infty = \bigcup_{k=1}^{\infty} E_k$$

Then there exist a sequence $\{ V_n \}$ of convex neighborhoods of 0 and a base $\{ b_1, b_2, \ldots \}$ of E_∞ such that

a) $V_n = V_n$, $V_{n+1} \subseteq V_n$, $n = 1, 2, \ldots$,

b) $\bigcap_{n=1}^{\infty} V_n = \{ 0 \}$,

c) $\forall(k)\exists(m(k)) : \{ b_1, \ldots, b_{m(k)} \}$ is a base of E_k, and

d) $\forall(n)\forall(j \leqslant n) : 2^{-n+1} b_j \in V_n$.

Theorem 1. — *On separable Frèchet space there exists universal distribution.*

Proof. — We define A, E_k, E_∞, $\{ V_n \}$ and $\{ b_1, b_2, \ldots \}$ as in the lemma 3.

Let $\{ p_1, p_2, \ldots \}$ be a sequence of inf. div. distributions dense in the set of all inf. div. distributions. By virtue of Lemma 2 we may assume that

$$p_n = \delta(d_n) * e(F_n), \qquad n = 1, 2, \ldots$$

with $d_n \in E$, F_n is finite measure supported on $C^n_{2^n}$ and

$$F_n(E) \leqslant 2^n$$

there C^n_r is defined for natural n, real $r > 0$ by

$$C^n_r = \{ \beta_1 b_1 + \ldots + \beta_{m(n)} b_{m(n)} \mid |\beta_1| + \ldots + |\beta_{m(n)}| < r \}.$$

We define

$$G = \sum_{n=1}^{\infty} 2^{-n^2} T_{2^{n^3}} F_n$$

Then

$$G(E) \leqslant \sum_{n=1}^{\infty} 2^{-n^2+n} < \infty$$

Hence distribution $q = e(G)$ is well determined. We shall prove that it is universal distribution.

Let p be any inf. div. distribution. Then there exists subsequence $\{ n_k \}$ of natural numbers such that

$$p_{n_k} \Rightarrow p \tag{3}$$

We show that

$$q_k = \delta(d_{n_k}) * \mathrm{T}_{2^{-n_k^3}}(q^{*2^{n_k^2}}) \Rightarrow p \tag{4}$$

which asserts universalness of p.

For finite measure μ, natural n and real $c > 0$. We have:

$$e(\mu)^{*n} = e(n\mu)$$
$$\mathrm{T}_c(\mu) = e(\mathrm{T}_c\mu)$$

Hence

$$q_k = p_{n_k} * e(\mathrm{N}_k^1) * e(\mathrm{N}_k^2) \tag{5}$$

with

$$\mathrm{N}_k^1 = \sum_{n<n_k} 2^{n_k^2 - n^2} \mathrm{T}_{2^{n^3 - n_k^3}} \mathrm{F}_n$$
$$\mathrm{N}_k^2 = \sum_{n>n_k} 2^{n_k^2 - n^2} \mathrm{T}_{2^{n^3 - n_k^3}} \mathrm{F}_n$$

Because

$$\mathrm{N}_k^2(\mathrm{E}) \leqslant 2^{-n_k} \sum_{n=1}^{\infty} 2^{-n^2+n} \xrightarrow{k} 0$$

then

$$e(\mathrm{N}_k^2) \Rightarrow \delta(0) \tag{6}$$

We prove following that $e(\mathrm{N}_k^1) \Rightarrow \delta(0)$ which together with (3), (5), (6) implies (4).

Let's fix one natural $m_0 > \mathrm{G}(\mathrm{E})$ and let for each natural l the number $\mathrm{H}(l) = 2^{2l+1} m_0^2$. Then

$$\frac{(2^l \mathrm{G}(\mathrm{E}))^{\mathrm{H}(l)}}{\mathrm{H}(l)\,!} \leqslant \frac{(2^l m_0)^{2^{2l+1} m_0^2}}{(2^{2l+1} m_0^2)\,!} = \frac{1}{(2^l m_0^2)\,!} - \frac{2^{2l} m_0^2}{2^{2l} m_0^2 + 1} \cdots \frac{2^{2l} m_0^2}{2.2^{2l} m_0^2} < \frac{1}{2^l}$$

Hence

$$\sum_{n>\mathrm{H}(l)} \frac{(2^l \mathrm{G}(\mathrm{E}))^n}{n\,!} \leqslant \sum_{n=1}^{\infty} \frac{1}{2^l} \frac{1}{2^n} = 2^{-l},$$

and because for each k,

$$N_k^1(E) \leqslant 2^{n_k} T_{2^{-n_k^3}} G(E) = 2^{n_k} G(E)$$

then

$$(7) \qquad \sum_{n > H(n_k)} \frac{(N_k^1)^{*n}(E)}{n\,!} \leqslant \sum_{n > H(n_k)} \frac{(2^{n_k} G(E))^n}{n\,!} < 2^{-n_k}$$

On other hand, if F is any measure supported on C_r^n then for $c > 0$, $T_c F$ is supported on C_{cr}^n, F^{*m} is supported on C_{mr}^n hence for $m \leqslant H(n_k)$; $(N_k^1)^{*m}$ is concentrated on $C(k)$, when

$$C(k) = C^{n_k - 1}_{m_0^2 2^{-3n_k^2 + 6n_k - 1}}$$

Then (7) implies

$$e(.N_k^1)(C(k)) + 2^{-n_k}$$
$$\geqslant e^{-N_k^1(E)} \left(\sum_{0 \leqslant m \leqslant H(n_k)} \frac{(N_k^1)^{*m}(E)}{m\,!} + \sum_{m > H(n_k)} \frac{(N_k^1)^{*m}(E)}{m\,!} \right) = e(N_k^1)(E) = 1$$

Hence

$$(8) \qquad e(N_k^1)(C(k)) \geqslant 1 - 2^{-n_k}$$

Since $m_0^2 2^{-3n_k^2 + 6n_k - 1} \to 0$, there is $k(0)$ such that for $k > k(0)$

$$\beta_k = 2^{-m(n_k - 1) + 1} \geqslant m_0^2 2^{-3n_k^2 + 6n_k - 1}$$

Then from (a)-(d), since every element of $C(k)$ is linear combination of

$$\beta_k b_1, \ldots, \beta_k b_{m(n_k - 1)} \in V_{m(n_k - 1)}$$

with coefficients which absolute values have sum less than 1, we have

$$(9) \qquad C(k) \subseteq V_{m(n_k - 1)} \searrow \{ 0 \}$$

when $k \to \infty$.

For arbitrary $\varepsilon > 0$ there exists $k(\varepsilon)$ such that

$$(10) \qquad 2^{n_k} < \varepsilon\,, \quad k \geqslant k(\varepsilon)$$

We denote $C = \bigcup_{k \geqslant k(\varepsilon)} C(k)$. Let $\{ x_n \} \subseteq C$, then

a) If $\{ x_n \} \quad \bigcup_{n(0) \geqslant k \geqslant k(\varepsilon)} C(k)$ for any natural $n(0)$ then because $C(k)$ are bounded closed subsets in finite-dimensional spaces, they are compact, $\{ x_n \}$ contains any convergent subsequence.

b) If, in inverse, there exist subsequence $\{ k1 \}$ of natural numbers $k' > k(\varepsilon)$ and subsequence $\{ x_{k'} \} \subseteq \{ x_n \}$ such that $x_{k'} \in C(k')$ then (9) implies $x_{k'} \to 0$.

Hence C is relatively compact.

From Theorem II.3.2 [5] there exists a compact set K_1 such that

$$e(N_k^1)(K_1) \geqslant 1 - \varepsilon \qquad \text{for} \qquad k < k(\varepsilon)$$

Let $K = K_1 \cup \bar{C}$. Then from (8) and (10) for each k

$$e(N_k^1)(K) \geqslant 1 - \varepsilon$$

Hence by Prokhorov Theorem (see, for example, Th. II.6.7 [5]), the sequence $\{ e(N_k^1) \}$ is relatively compact.

Simultaneously, if U is any open set containing 0 then from (9) there is natural $k(U)$ such that for $k > k(U)$

$$C_{(k)} \subseteq V_{m(n_k - 1)} \subseteq U$$

Then from (8)

$$e(N_k^1)(U) \geqslant 1 - 2^{-n_k}$$

Hence $e(N_k^1)(U) \to 1$ and if π is any limit-distribution of some subsequence of $\{ e(N_k^1) \}$ then π must be $\delta(0)$. This implies from Th. 2.3 [1] that

$$e(N_k^1) \Rightarrow \delta(0)$$

Using above Theorem we may easily prove following theorem:

THEOREM 2. — *One distribution on separable Frèchet space in inf. div. if it's domain of partial attraction is non empty.*

THEOREM 3. — *If ξ is a distribution concentrated on bounded subset of* E *then ξ belongs to the domain of attraction of* $\delta(0)$ *and hence of* $\delta(a)$ *for every* $a \in E$.

Proof. — We fix any sequence $\{ V_n \}$ of convex neighborhood of 0 satisfying (*a*) and (*b*). Suppose B is a bounded subset on which ξ is supported. Then for each n there is, $k(n)$ such that $k^{-1/2} B\, V_n$ for $k \geqslant k(n)$.

For every k, ξ^{*k} is supported on

$$B_k = \underbrace{B + \ldots + B}_{k\text{-times}}$$

We denote

$$\xi_k = T_{k^{-3/2}} \xi^{*k} \tag{11}$$

Then ξ_k is supported on

$$k^{-3/2}\mathrm{B}_k = \underbrace{\frac{1}{k}(k^{-1/2}\mathrm{B}) + \ldots + \frac{1}{k}(k^{-1/2}\mathrm{B})}_{k\text{-times}} \subseteq \mathrm{V}_k$$

for $k > k(n)$.

Every ξ_k is tight, then for arbitrary $\varepsilon > 0$ there exists a compact set K_k such that

$$\xi_k(\mathrm{K}_k) \geqslant 1 - \varepsilon$$

With next proof part we may suppose that $\mathrm{K}_k \subseteq \mathrm{V}_n$ if $k \geqslant k(n)$.

Similarly as in the proof of Th. 1 we see that $\mathrm{K} = \bigcup_{k=1}^{\infty} \mathrm{K}_k \cup \{ 0 \}$ is compact and for every k

$$\xi_k(\mathrm{K}) \geqslant 1 - \varepsilon$$

Hence from Prokhorop Theorem, sequence $\{ \xi_k \}$ is relatively compact.

As in proof of Th. 1, because

$$\xi_k(\mathrm{V}_n) = 1 \qquad \text{for} \qquad k \geqslant k(m)$$

then

$$\xi_k \Rightarrow \delta(0) \tag{12}$$

From here for every $a \in \mathrm{E}$

$$\delta(a) * \xi_k \Rightarrow \delta(a)$$

This proof is completed.

We see that if q is universal distribution then $\delta(a) * q$ is also universal.

COROLLARY 1. — *Let q be universal distribution defined in proof of theorem 1, and let ξ be a distribution concentrated on bounded subset of* E. *Then $\xi * q$ is also universal. Hence there exist uncountably many universal distributions which are not shift-equivalent.*

Proof. — Let p be any inf. div. distribution. As same as on proof of Th. 1 there are subsequence $\{ n_k \}$ of natural numbers and sequence $\{ d_k \} < \mathrm{E}$ such that

$$q_k = \delta(d_k) * \mathrm{T}_{2^{-n_k^3}}(q^{*2^{n_k^2}}) \Rightarrow p\,.$$

Then from (11) and (12) we have

$$\delta(d_k) * \mathrm{T}_{2^{-n_k^3}}(\xi * q)^{+2^{n_k^2}} = \delta(d_k) * \mathrm{T}_{2^{-n_k^3}}(q^{*2^{n_k^2}}) * \mathrm{T}_{2^{-(n_k^2)\frac{3}{2}}}\xi^{*n_k^2} = q_k * \xi_{n_k^2} \Rightarrow p$$

Hence from Th. 2 we have:

COROLLARY 2. — *Domain of partial attraction of one distribution on* *separable Frèchet space either is empty, if the distribution isn't inf. div., either has uncountably many elements which are not shift-equivalent, if the distribution is inf. div.*

Before here the Theorem 1 has proved for the case of Banach space in my paper « Universal Distribution for inf. div. distributions on Banach space ».

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