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Domains of partial attraction in several dimensions

by

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SOMMAIRE. — On montre que, comme dans $\mathbb{R}$, une mesure de probabilité sur un espace de Banach séparable quelconque a un domaine d'attraction partial non-vide si et seulement si elle est infiniment divisible, et que tous les Banach (séparables) possèdent des lois universelles au sens de Doeblin. On prouve aussi des résultats sur l'ensemble $N(X)$ des suites $n_i \uparrow \infty$ telles que $\{ L(S_{n_i}/a_{n_i}) \}_{i=1}^{\infty}$ est tendue à un centrage près pour quelque suite $a_{n_i} \uparrow \infty$; ils constituent des généralisations de résultats récents de Jain et Orey à $\mathbb{R}^n$ et à quelques Banach (quelques propriétés sont satisfaites si et seulement si B est de cotype 2, et d'autres si et seulement si dim B $< \infty$).

1. INTRODUCTION

The theory of domains of partial attraction was developed in the late thirties by Khinchin, Lévy and Doeblin, particularly by this last author. Recently, Jain and Orey [*9*], almost fourty years after Doeblin's famous paper [*3*], made also very significant contributions to the subject. In this note we try to extend most of the results of the theory of domains of partial attraction to several dimensions, some to $\mathbb{R}^n$ and some to general Banach spaces. For this we use a combination of classical and modern methods: the proofs of Khinchin's and Doeblin's theorems in Section 2 use these authors ideas (as described by Feller [*5*], p. 589-591) together with more recent devices such as BL* distances, etc.; in Section 3 we work on the results of [*9*] and the methods and results of that article are crucial here.

Let us recall the main definitions and describe some notation. A B-valued (B a Banach space) random variable (r. v.) X is in the *domain of partial attraction* of a probability measure (p. m.) ρ, $X \in \mathrm{DPA}(\rho)$, if there exist a subsequence $\{ n_i \} \in \mathbb{N}$, $n_i \uparrow \infty$ and sequences $a_{n_i} \uparrow \infty$, $\{ b_{n_i} \} \subset \mathrm{B}$ such that

$$L(\mathrm{S}_{n_i}/a_{n_i} - b_{n_i}) \rightarrow_w \rho,$$

where $\mathrm{S}_n = \Sigma_{i=1}^n \mathrm{X}_i$, X_i independent identically distributed (i. i. d.) with $L(\mathrm{X}_i) = L(\mathrm{X})$, and $\rightarrow_w$ denoting weak convergence. $N(\mathrm{X})$ is the family of sequences $\{ n_i \} \subset \mathbb{N}$, $n_i \uparrow \infty$, such that there exists a sequence of positive real numbers $a_{n_i} \uparrow \infty$ making $\{ L(\mathrm{S}_{n_i}/a_{n_i}) \}$ shift tight with only non-degenerate (shift-) limit p. m.'s. Then $\{ a_{n_i} \}$ is said to be *admissible* for $\{ n_i \}$ and X (or for one of them if the other one is obvious from the context). $N_0(\mathrm{X})$ is the subset of $N(\mathrm{X})$ consisting of those sequences $\{ n_i \}$ for which $\{ L(\mathrm{S}_{n_i}/a(n_i)) \}$ is shift tight, with $a(n)$ defined by the equation:

$$n\mathrm{P}\{ \| \mathrm{X} \| > a(n) \} + na(n)^{-2}\mathrm{E} \| \mathrm{X} \|^2 \mathrm{I}_{\{\|\mathrm{X}\| \leq a(n)\}} = 1.$$

The first definition is classical and the other ones were introduced by Jain and Orey [*9*]. They also introduced the following one: let $\mathrm{L}(x) \downarrow 0$ as $x \uparrow \infty$; then a set $\mathrm{A} \subset \mathbb{R}_+$ is of *uniform decrease* for L if it is unbounded and if $\lim_{\lambda \to \infty} \sup_{x \in \mathrm{A}} \mathrm{L}(\lambda x)/\mathrm{L}(x) = 0$.

In Section 2 we will use the $d_{\mathrm{BL}*}$ distance on the set of finite Borel measures on the Banach space B ([*4*]). Let us recall that

$$d_{\mathrm{BL}*}(\mu, \nu) = \sup \left\{ \left| \int f d(\mu - \nu) \right| : \| f \|_{\mathrm{BL}} \leq 1 \right\}$$

where $f : \mathrm{B} \to \mathbb{R}$ and

$$\| f \|_{\mathrm{BL}} = \sup_{x \in \mathrm{B}} | f(x) | + \sup_{x,y \in \mathrm{B}, x \neq y} | f(x) - f(y) | / \| x - y \|.$$

Let us also recall that $d_{\mathrm{BL}*}$ metrizes weak convergence on the set of Borel p. m.'s on B ([*4*]) and that

$$d_{\mathrm{BL}*}(a\mu, a\nu) = a d_{\mathrm{BL}*}(\mu, \nu),\ a \geq 0,$$

$$d_{\mathrm{BL}*}(\Sigma_i \mu_i, \Sigma_i \nu_i) \leq \Sigma_i d_{\mathrm{BL}*}(\mu_i, \nu_i)$$

$$d_{\mathrm{BL}*}(\Pi_{i=1}^n \mu_i, \Pi_{i=1}^n \nu_i) \leq | \mu_1 |^{n-1} \Sigma_i d_{\mathrm{BL}*}(\mu_i, \nu_i) \quad \text{if} \quad | \mu_1 | = \mu_i(\mathrm{B}) = \nu_i(\mathrm{B}),\ i = 1, \ldots, n,$$

where Π denotes convolution product.

We will also use the notation Pois ν for $e^{-|\nu|}\Sigma_{n=0}^{\infty} \nu^n/n\,!$ where $| \nu | = \nu(\mathrm{B})$, powers are convolution powers and ν is a finite Borel measure on B. For the definitions of Lévy measures and generalized Poisson measures see [*1*], [*13*] or [*15*].

A random variable in a Banach space is pregaussian if there exists a Gaussian p. m. with the same covariance. See [*8*] for the definition of cotype 2 spaces and the proof of the following property: a Banach space is of cotype 2 if and only if, for B-valued r. v.'s X, $E \| X \|^2 < \infty$ whenever X is in the domain of normal attraction of a Gaussian law. We will use [*1*] as the main reference for the general central limit theorem in Banach spaces even if they are finite dimensional.

The notation is standard: w-lim or $\to_w$ for « weak limit », L for « law », δ_0 for the unit mass at zero, I_A for the indicator function of the set A, S_n for $\Sigma_{i=1}^n X_i$ with X_i i. i. d. and $L(X_i) = L(X)$ and it is always clear from the text what the random variable X is, B for *separable* Banach space, etc.

2. KHINCHIN'S THEOREM AND DOEBLIN'S UNIVERSAL LAWS

We need two lemmas.

2.1. LEMMA. — If ν is a finite measure on a Banach space B then

$$d_{BL^*}(\text{Pois}\ \nu, \delta_0) \leq \int_B \min(2, \| x \|) d\nu(x) \leq 2\nu(B). \tag{2.1}$$

If for some $r > 0$, $\{ X_i \}_{i=1}^n$, $n \in \mathbb{N}$, are i. i. d. B-valued r. v.'s with $L(X_i) = \text{Pois}(\nu/r)$, then for every $a \in \mathbb{R}$,

$$d_{BL^*}[L(a\Sigma_{i=1}^n X_i), \delta_0] \leq 2nr^{-1}\nu(B). \tag{2.2}$$

Proof. — Using the previously mentioned properties of d_{BL^*} we have

$$\begin{aligned} d_{BL^*}(\text{Pois}\ \nu, \delta_0) &= d_{BL^*}(e^{-|\nu|}\Sigma_{k=0}^\infty \nu^k/k\,!, e^{-|\nu|}\Sigma_{k=0}^\infty | \nu |^k \delta_0/k\,!) \\ &\leq e^{-|\nu|}\Sigma_{k=0}^\infty d_{BL^*}(\nu^k, | \nu |^k \delta_0)/k\,! \\ &\leq e^{-|\nu|}\Sigma_{k=1}^\infty | \nu |^{k-1} k d_{BL^*}(\nu, | \nu | \delta_0)/k\,! \\ &= d_{BL^*}(\nu, | \nu | \delta_0), \end{aligned}$$

and

$$d_{BL^*}(\nu, | \nu | \delta_0) = \sup\left\{\left| \int (f - f(0)) d\nu \right| : \| f \|_{BL} \leq 1 \right\} \leq \int \min(2, \| x \|) d\nu(x).$$

Inequality (2.1) is thus proved. As for (2.2) observe that if $\tau_a\nu(A) = \nu(A/a)$, then

$$d_{BL^*}[L(a\Sigma_{i=1}^n X_i), \delta_0] = d_{BL^*}(\text{Pois}\ (nr^{-1}\tau_a\nu), \delta_0) \leq 2nr^{-1}\nu(B). \quad \square$$

2.2. LEMMA. — Let $\{ x_i \}$ be a countable dense subset of B. Then the set $\{ \text{Pois } \nu : \nu = \Sigma_{\text{finite}} a_i \delta_{x_i},\ a_i \in \mathbb{Q}_+ \cup \{ 0 \} \}$ is weakly sequentially dense in the set of all infinitely divisible laws on B; in particular, the infinitely divisible laws on B are a separable set for the weak topology.

Proof. — As shown in [2], if ρ is infinitely divisible and $\rho^{1/n}$ is its n-th convolution root, then

$$w\text{-}\lim_{n\to\infty} \text{Pois } n\rho^{1/n} = \rho.$$

Now the lemma follows from this and the fact that if $\nu_n \to_w \nu$ then also Pois $\nu_n \to_w$ Pois ν. □

Now we can prove the analog of Kinchin's theorem ([10]) in Banach spaces.

2.3. THEOREM. — A Borel p. m. ρ on B has a non-void domain of partial attraction if and only if it is infinitely divisible.

Proof. — Assume first that ρ is infinitely divisible. By Lemma 2.2 there exists a sequence $\{ \nu_k \}$ of finite measures such that Pois $\nu_k \to_w \rho$. Let $\{ n_k \}$ be such that $n_k > 2^{k+1}\nu_k(\text{B})n_{k-1}$, $n_1 = 1$, and let $\{ \text{X}_k \}$ be independent r. v.'s with $L(\text{X}_k) = \text{Pois } (\nu_k/n_k)$. Then the series $\Sigma_{k=1}^{\infty} a_k \text{X}_k$ converges in distribution for any sequence $\{ a_k \}$ because $\Sigma_k \tau_{a_k}\nu_k(\text{B})/n_k = \Sigma_k \nu_k(\text{B})/n_k < \infty$. For some sequence $\{ a_k \}$ to be specified below, define

$$\text{Y}_i = \Sigma_{k=1}^{\infty} a_k \text{X}_{ki}$$

where X_{ki}, $i = 1, \ldots$, are independent copies of X_k, $k = 1, \ldots$ We will show that

$$d_{\text{BL}^*}[L(\Sigma_{i=1}^{n_k}\text{Y}_i/a_k), \text{Pois } \nu_k] \to 0, \tag{2.3}$$

hence that $L(\Sigma_{i=1}^{n_k}\text{Y}_i/a_k) \to_w \rho$, thus showing that $\text{Y} \in \text{DPA}(\rho)$.

Since $L(\Sigma_{i=1}^{n_r}\text{X}_{ri}) = \text{Pois } \nu_r$, we have

$$\begin{aligned} d_{\text{BL}^*}&[L(\Sigma_{i=1}^{n_r}\text{Y}_i/a_r), \text{Pois } \nu_r] \\ &\leq d_{\text{BL}^*}[L(\Sigma_{i=1}^{n_r}\Sigma_{k=1}^{r-1} a_k \text{X}_{ki}/a_r), \delta_0] + d_{\text{BL}^*}[L(\Sigma_{i=1}^{n_r}\Sigma_{k=r+1}^{\infty} a_k \text{X}_{ki}/a_r), \delta_0]. \end{aligned} \tag{2.4}$$

The second summand in the right hand side term of this inequality is bounded by

$$\Sigma_{k=r+1}^{\infty} d_{\text{BL}^*}[L(\Sigma_{i=1}^{n_r} a_k \text{X}_{ki}/a_r), \delta_0] \leq 2\Sigma_{k=r+1}^{\infty} n_k^{-1} n_r \nu_k(\text{B}) \leq \Sigma_{k=r+1}^{\infty} 2^{-k} \to 0 \quad \text{as} \quad r \to \infty$$

independently of $\{ a_k \}$, by Lemma 2.1. Now we choose $\{ a_k \}$ so that the first summand in the right hand side of (2.4) tends to zero: choose $a_1 = 1$ and, given $a_1, \ldots, a_{r-1}$, choose r so that this summand be bounded by, say, $1/r$ (which is possible because for Z fixed, $d_{\mathrm{BL}^*}(L(Z/s_k), \delta_0) \to 0$ if $s_k \to \infty$).

Conversely, let now

$$\rho = w\text{-}\lim_{n\to\infty} L(\Sigma_{i=1}^{n_k} X_i/a_{n_k} - b_{n_k})$$

for some subsequences $n_k \uparrow \infty$ and $a_k \uparrow \infty$. By [1], Theorem 2.10, we may take $b_{n_k} = n_k \lambda_{n_k}$ and $\lambda_{n_k} = \mathrm{EXI}_{\{\|X\| \le ca_{n_k}\}}/a_{n_k}$. If $0 < r < n_k$ is an integer, set

$$\nu_{n_k}^{(r)} = L(\Sigma_{i=1}^{[n_k/r]} X_i/a_{n_k} - [n_k/r]\lambda_{n_k}),$$

where $[n_k/r]$ is the largest integer not exceeding n_k/r. Then $\nu_{n_k}^{(1)} = (\nu_{n_k}^{(r)})^r * \sigma_{n_k}$, and σ_{n_k} has the law of a sum of at most r independent summands distributed like $X_1/a_{n_k} - \lambda_{n_k}$. Therefore $\sigma_{n_k} \to_w \delta_0$ and

$$\nu_{n_k}^{(1)} \simeq_w (\nu_{n_k}^{(r)})^r \quad \text{as} \quad r \to \infty$$

Since $\{ \nu_{n_k}^{(1)} \}$ is convergent, it follows from [11], Theorem III.2.2 that $\{ \nu_{n_k}^{(r)} \}$ is shift tight, hence tight by [1], Theorem 2.5. Hence, if ρ_r is a subsequential limit of $\{ \nu_{n_k}^{(r)} \}$, we obtain that $\rho = (\rho_r)^r$ that is, ρ is infinitely divisible. □

We see as a corollary that the following result of Doeblin [3] also holds in Banach spaces.

2.4. COROLLARY. — Let X be a B-valued r. v. such that $X \in \mathrm{DPA}(\rho)$. If $\widehat{\rho}(f) = e^{\psi(f)}, f \in B'$ (ψ exists because ρ is infinitely divisible), then for every $t > 0$, $e^{t\psi(f)}$ is the characteristic function of a tight p. m. on B, ρ_t, and $X \in \mathrm{DPA}(\rho_t)$.

Proof. — If $\rho = N(a, \Phi) * c\mathrm{Pois}\ \mu$ (Φ is the covariance of the Gaussian law $N(a, \Phi)$ and a its expectation, and $c\mathrm{Pois}\ \mu$ is a centered Poisson p. m. with Lévy measure μ, see *e. g.* [1]), then $\rho_t = N(ta, t\Phi) * c\mathrm{Pois}\ t\mu$ is obviously a tight p. m. The second part of the previous proof proves the corollary for $t = 1/r$, $r \in \mathbb{N}$. It is obvious that the corollary is true for $t \in \mathbb{N}$. Hence it is true for rational t. It is also obvious that if $X \in \mathrm{DPA}(\sigma_n)$ and if $\sigma_n \to_w \sigma$ then $X \in \mathrm{DPA}(\sigma)$, and the result follows at once. □

Next we prove that there exist « Doeblin's universal laws » in any separable Banach space (Doeblin [3]).

2.5. THEOREM. — Let B be a separable Banach space. Then there are p. m.'s on B which belong to the domain of partial attraction of every infinitely divisible law.

Proof. — By Lemma 2.2, there exists a countable dense set $\{ \mathrm{Pois}\ \mu_k \}$, μ_k finite, in the set of all infinitely divisible laws. Since if $X \in \mathrm{DPA}(\mathrm{Pois}\ \mu_{k_l})$

and Pois $\mu_{k_l} \to \rho$ then $X \in \mathrm{DPA}(\rho)$, it is enough to find X in the DPA of Pois μ_k for all k.

Let now $\{ \nu_r \}$ be a sequence of finite measures such that ν_r equals μ_k for infinitely many r.'s, and this for every k, and define Y for the sequence $\{ \nu_r \}$ just as in the first part of the proof of Theorem 2.3, *i. e.* satisfying (2.3). It is then obvious that $Y \in \mathrm{DPA}$ (Pois μ_k) for all k. □

As mentioned in the introduction, the proofs of Theorems 2.3 and 2.5 essentially follow the patterns of the proofs of the one dimensional results as given in [5] (the proof of the converse part of 2.3 given here is perhaps more direct).

3. ON THE SET N(X)

One of the main results in Jain and Orey [9] states that $N(X) \neq \phi$ if and only if the function $x \to P\{ |X| > x \}$ admits a set of uniform decrease. It is obvious that this can not be true in infinite dimensions: except in particular cases no conditions only on the distribution of $\| X \|$ can imply shift tightness of $\{ L(S_{n_i}/a_{n_i}) \}$. Nevertheless we will give an explicit example valid for every infinite dimensional Banach space. In finite dimensions the situation is as in $\mathbb{R}$. Concretely we have:

3.1. THEOREM. — (1) If X is a $\mathbb{R}^n$-valued r. v., then $N(X) = N(\| X \|)$ and there is also equality among the corresponding sets of admissible norming constants.

(2) For every B-valued r. v. X, B any Banach space, $N(X) \neq \phi$ implies $N(\| X \|) \neq \phi$.

(3) If $N(\| X \|) \neq \phi$ implies $N(X) \neq \phi$ for every B-valued r. v. X, then B is finite dimensional.

Proof. — (1) Let X be $\mathbb{R}^n$-valued. If $E \| X \|^2 < \infty$ the proposition is trivially verified. So we may assume $E \| X \|^2 = \infty$. In this case it is well known that

$$(E \| X \| I_{\{ \|X\| \le t \}})^2 = o(E \| X \|^2 I_{\{ \|X\| \le t \}}) \tag{3.1}$$

as $t \to \infty$ ([6], p. 173). It is easy to deduce from the classical theory that for any infinitesimal array $\{ X_{nj} : j = 1, \ldots, k_n, n \in \mathbb{N} \}$ of $\mathbb{R}^n$-valued row-wise independent r. v.'s, $\{ L(\Sigma_j X_{nj}) \}$ is shift tight if and only if the set of finite measures

$$d\nu_n(x) = \begin{cases} \Sigma_j (x - EX_{nj1})^2 dL(X_{nj1})(x) & \text{for} \quad \| x \| \le 1 \\ \Sigma_j dL(X_{nj})(x) & \text{for} \quad \| x \| > 1 \end{cases} \tag{3.2}$$

is uniformly bounded and tight, where $X_{nj1} = X_{nj}I_{\{\|X_{nj}\| \leq 1\}}$. But (3.1) reduces this criterion in our case to: shift tightness of $\{ L(S_{n_i}/a_{n_i}) \}$ is equivalent to uniform boundedness and tightness of the set of measures

$$d\nu_{n_i}(x) = n_i \min(1, \| x \|^2) dL(X/a_{n_i})(x). \tag{3.3}$$

This condition, on the other hand, is also equivalent to shift tightness of

$$\{ L(\Sigma_{j=1}^{n_i} \| X_j \| / a_{n_i}) \}.$$

(2) Let $\{ n_i \} \in N(X)$ and let $\{ a_{n_i} \}$ be admissible. We may assume that $\{ L(S_{n_i}/a_{n_i}) \}$ is shift convergent to a non-degenerate law. Then, for all $c > 0$ except perhaps for a countable set,

$$n_i P \{ \| X \| > c a_{n_i} \} \to_w \mu \{ \| X \| > c \} \tag{3.4}$$

where μ is the Lévy measure of the limit law. (3.3) and (3.4) immediately give that if $\mu \{ \| x \| > c \} \neq 0$ for some $c > 0$ satisfying (3.4), then $\{ c a_{n_i} \}$ is of uniform decrease for the function $L(x) = P \{ \| X \| > x \}$ (see the argument in the proof of Proposition 1.10 of [9]), hence that $N(\| X \|) \neq \phi$ (by Proposition 1.6 in the same article). Since the limit of shifts of $\{ L(S_{n_i}/a_{n_i}) \}$ is non-degenerate, there exists $f \in B'$, $\| f \| = 1$, such that

$$\inf_i [n_i P \{ \| X \| > a_{n_i} \} + n_i a_{n_i}^{-2} E f^2(X) I_{\{ \|X\| \leq a_{n_i} \}}] > 0$$

(just apply the one dimensional central limit theorem); therefore, if $\mu \{ \| x \| > 1/2 \} = 0$, we have

$$P \{ \| X \| > a_{n_i} \} / a_{n_i}^{-2} E \| X \|^2 I_{\{ \|X\| \geq a_{n_i} \}} \to 0 \qquad \text{as} \qquad i \to \infty,$$

and $N(\| X \|) \neq \phi$ again by Proposition 1.6 in [9].

(3) Assume now that B is infinite dimensional. We will give an example of a B-valued r. v. X such that $\mathbb{N} \in N(\| X \|)$ but $N(X) = \phi$. Let

$$S = \{ x \in B : \| x \| = 1 \} \quad \text{and} \quad \{ x_i \} \subset S$$

an infinite sequence with no cluster points (note that S is not compact). Define a random variable X with distribution concentrated on

$$\bigcup_{n=1}^{\infty} \{ \lambda x_n : n^{1/\alpha} < \lambda \leq (n+1)^{1/\alpha} \}$$

for some $\alpha \in (0, 2)$ and such that

$$P[X \in \{ \lambda x_n : \lambda \in A \subset (n^{1/\alpha}, (n+1)^{1/\alpha}] \}] = \alpha \int_A t^{-\alpha-1} dt.$$

Then, for every $t \geq 1$,

$$t^\alpha P \{ \| X \| > t \} = 1. \tag{3.5}$$

In particular, $\{ L(\Sigma_{j=1}^{n} \| X_j \|/n^{1/\alpha}) \}$ (where the X_i are i. i. d. and $L(X_i) = L(X)$) is shift convergent. Hence $\mathbb{N} \in N(\| X \|)$.

Let us suppose now that $\{ L(S_{n_i}/a_{n_i}) \}$ is shift convergent to a non-degenerate limit. Then (3.5) implies that there exists $c \in (0, \infty)$ such that

$$a_{n_i}/n_i^{1/\alpha} \to c. \tag{3.6}$$

In fact $n_i P \{ \| X \| > a_{n_i}\delta \} = n_i a_{n_i}^{-\alpha}\delta^{-\alpha}$ must converge to a finite limit for all but a countable number of positive numbers $\delta > 0$, but if $n_i a_{n_i}^{-\alpha} \to 0$ then

$$n_i a_{n_i}^{-2} E \| X \|^2 I_{\{ \|X\| \leq a_{n_i}\delta \}} \leq 2(2 - \alpha)^{-1} \delta^{2-\alpha} n_i a_{n_i}^{-\alpha} \to 0$$

and the shift limit of $\{ \mathscr{L}(S_{n_i}/a_{n_i}) \}$ is degenerate. Hence (3.6) holds.

By (3.6) there is then no loss of generality in assuming

$$a_{n_i} = n_i^{1/\alpha}. \tag{3.6$'$}$$

By taking a subsequence if necessary, we may also assume that

$$\Sigma_i n_i/n_{i+1} < \infty.$$

A necessary condition for $\{ \mathscr{L}(S_{n_i}/n_i^{1/\alpha}) \}$ to be shift convergent is that

$$n_i \mathscr{L}(X_i/n_i^{1/\alpha}) \mid \{ \| x \| > \delta \} \to_w \tau \mid \{ \| x \| > \delta \} \tag{3.7}$$

for some Lévy measure τ (finite outside the origin) and every $\delta > 0$ such that $\tau \{ \| x \| = \delta \} = 0$ (note that (3.5) implies that this is satisfied for every $\delta > 0$). We will see that (3.7) is impossible. Let $G_i \subset S$, $i = 1, \ldots,$ be a collection of disjoint sets open in S and such that $x_i \in G_i$ for every $i \in \mathbb{N}$, and define:

$$A_1 = [\lambda x : x \in \cup_{i=1}^{\infty}(G_{n_{2i}} \cup G_{n_{2i}+1} \cup \ldots \cup G_{n_{2i+1}-1}), \lambda > 1]$$
$$A_2 = [\lambda x : x \in \cup_{i=1}^{\infty} \{ x_{n_{2i}}, x_{n_{2i}+1}, \ldots, x_{n_{2i+1}-1} \}, \lambda \geq 1].$$

Then A_1 is an open set, A_2 is a closed set, $\tau(A_1) \geq \tau(A_2)$ (as $\tau(S) = 0$) and

$$n_k P \{ X/n_k^{1/\alpha} \in A_1 \} = n_k P \{ X/n_k^{1/\alpha} \in A_2 \}$$

$$\geq \alpha n_{2i} \int_{n_{2i}^{1/\alpha}}^{n_{2i+1}^{1/\alpha}} t^{-\alpha-1} dt = n_{2i}(n_{2i}^{-1} - n_{2i+1}^{-1}) \to 1 \quad \text{if} \quad k = 2i \to \infty$$

and

$$\leq n_{2i+1} \Sigma_{r=i+1}^{\infty} (n_{2r}^{-1} - n_{2r+1}^{-1}) \leq \Sigma_{r=i+1}^{\infty} n_{2i+1}/n_{2r} \to 0 \quad \text{if} \quad k = 2i + 1 \to \infty.$$

We thus have, by (3.7), that

$$1 \leq \limsup_{k\to\infty} n_k P \{ X/n_k^{1/\alpha} \in A_2 \} \leq \tau(A_2)$$
$$\leq \tau(A_1) \leq \liminf_{k\to\infty} n_k P \{ X/n_k^{1/\alpha} \in A_1 \} = 0,$$

contradiction. Hence (3.7) does not hold and $\{ L(S_{n_i}/n_i^{1/\alpha}) \}$ is not shift convergent. We have proved $N(X) = \phi$. □

We do not know whether the inclusion $N(X) \subset N(\| X \|)$ holds in Banach spaces in general; see Theorem 3.3 below for a partial answer to this question.

Theorem 3.1 together with Theorems 2.1, 2.5 and 2.8 of [9] give:

3.2. THEOREM. — Let B be a Banach space. Then the following are equivalent:

i) B is finite dimensional.

ii) $N(X) \neq \phi$ if and only if there exists a set of uniform decrease for the function $L(x) = P\{ \| X \| > x \}$.

iii) $\mathbb{N} \in N(X)$ if and only if $\lim_{\lambda\to\infty} nP\{ \| X \| > \lambda a(n) \} = 0$ uniformly in n, and then $\mathbb{N} \in N_0(X)$.

iv) $\{ n_i \} \in N_0(X)$ if and only if $\lim_{\lambda\to\infty} n_i P\{ \| X \| > \lambda a(n_i) \} = 0$ uniformly in i.

In some cases we can be more precise about statement (2) in Theorem 3.1:

3.3. THEOREM. — The following are equivalent for a Banach space B:

i) B is of cotype 2.

ii) For every B-valued r. v. X, $\{ n_i \} \in N(X)$ if and only if $\{ kn_i \} \in N_0(X)$ for some positive integer k.

iii) For every B-valued r. v. X, $N(X) \subset N(\| X \|)$ and if $\{ n_i \} \in N(X)$ then $\{ a_{n_i} \}$ is admissible for X (and $\{ n_i \}$) if and only if it is admissible for $\| X \|$ (and $\{ n_i \}$).

Proof. — If B is not of cotype 2 then there exists a B-valued r. v. X such that $E \| X \|^2 = \infty$ and X is in the domain of normal attraction of a non-degenerate Gaussian law. Since $nP\{ \| X \| > n^{1/2} \} \to 0$ ([1], Corollary 2.11), it is easy to see that $\| X \|$ belongs to the domain of (non normal) attraction of N(0, 1) (just apply Corollary 1, XVII.5 from [5]). In particular,

$$nP\{ \| X \| > a(n) \} \to 0 \quad \text{and} \quad na(n)^{-2} E \| X \|^2 \, I_{\{\|X\| \leq a(n)\}} \to 1.$$

This, together with $E \| X \|^2 = \infty$ implies $na(n)^{1/2} \to 0$. Hence $L(S_n/a(n)) \to_w \delta_0$ and $N_0(X) = \phi$. So, *ii*) implies *i*). In this example, $\{ n^{1/2} \}$ is admissible for X but not for $\| X \|$, hence *iii*) implies *i*) too.

Next we see that *iii*) implies *ii*). If $\{ n_i \} \in N(X)$ then $\{ n_i \} \in N(\| X \|)$ by *iii*), and the one dimensional result ([9], Theorem 2.1) implies that for some $k \in \mathbb{N}$, $\{ kn_i \} \in N_0(\| X \|)$. But obviously $\{ kn_i \} \in N(X)$, and therefore *iii*) also implies that the set $\{ a(kn_i) \}$ is admissible for $\{ kn_i \}$

and X, as it is for $\{ kn_i \}$ and $\| X \|$. Hence, $\{ kn_i \} \in N_0(X)$. If $\{ kn_i \} \in N(X)$, then $\{ n_i \} \in N(X)$ by [*11*], Theorem III.2.2.

Finally we show that *i*) implies *iii*). Assume B of cotype 2. Let $\{ n_i \} \in N(X)$ and let $\{ a_{n_i} \}$ be admissible. Then, by Theorem 6.7 in [*1*] (note that conclusion (4) there is also valid in cotype 2, see also Theorem 6.6 [*1*]) we have that $[n_i L(X/a_{n_i}) \mid \{ \| x \| > \delta \}]_{i=1}^{\infty}$ is uniformly bounded and tight for every $\delta > 0$ and that there exists $c \in (0, \infty)$ such that

$$(3.8) \qquad c^{-1} \leq n_i P\{ \| X \| > a_{n_i} \} + n_i a_{n_i}^{-2} E \| X \|^2 I_{\{ \|X\| \leq a_{n_i} \}} \leq c.$$

Therefore, by the one dimensional central limit theorem (as described in the proof of Theorem 3.1 (1)), $\{ n_i \} \in N(\| X \|)$ and $\{ a_{n_i} \}$ is admissible for $\| X \|$. Assume now that $\{ n_i \} \in N(X)$ ($\subset N(\| X \|)$) and that $\{ a_{n_i} \}$ is admissible for $\| X \|$. The proof will be finished is we show that $\{ a_{n_i} \}$ is also admissible for X. It is enough to prove that if $\{ a'_{n_i} \}$ is admissible for X then $\{ a_{n_i} \} \simeq \{ a'_{n_i} \}$ in the sense that $0 < \underline{\lim}\, a_{n_i}/a'_{n_i} \leq \overline{\lim}\, a_{n_i}/a'_{n_i} < \infty$. Set

$$L(x) = P\{ \| X \| > x \} \qquad \text{and} \qquad Q(x) = L(x) + x^{-2} E \| X \|^2 I_{\{ \|X\| \leq x \}}$$

(as in [*9*]), and define $Q_{f(X)}(x)$ in the same way with $\| X \|$ replaced by $| f(X) |$, $f \in B'$. If $\underline{\lim}\, a_{n_i}/a'_{n_i} = 0$, then one of the shift limits of $\Sigma_{j=1}^{n_i} \| X_j \| / a'_{n_i}$ is δ_0 (X_j are independent copies of X) and therefore $n_i Q(a'_{n_i}) \to 0$ through a subsequence (by the one dimensional central limit theorem). Hence, for every $f \in B'$, $\| f \| = 1$, $n_i Q_{f(X)}(a'_{n_i}) \leq n_i Q(a'_{n_i}) \to 0$, and since also $n_i P\{ | f(X) | > a'_{n_i} \} \leq n_i L(a'_{n_i}) \to 0$ (through the same subsequence), we conclude that δ_0 is also a shift limit of $\{ L(S_{n_i}/a'_{n_i}) \}$, a contradiction. So *lim* $a_{n_i}/a'_{n_i} > 0$, and we may in fact assume that $a_{n_i} \geq a'_{n_i}$.

By (3.8) and by Proposition 1.10 in [*9*] and its proof, it easily follows that there is either a subsequence $\{ n_{i_k} \}$ such that $\{ a'_{n_{i_k}} \}$ is a set of uniform decrease for L (hence for Q, see [*9*]) or else there is a subsequence such that $L(a'_{n_{i_k}})/Q(a'_{n_{i_k}}) \to 0$ or both. We will assume without loss of generality that these subsequences are $\{ n_i \}$. Let us assume now that $\overline{\lim}\, a_{n_i}/a'_{n_i} = \infty$. If $\{ a'_{n_i} \}$ is a set of uniform decrease for L, hence for Q, then

$$Q(a_{n_i})/Q(a'_{n_i}) \to 0$$

and (3.8) implies that $n_i Q(a_{n_i}) \to 0$; therefore $L(\Sigma_{j=1}^{n_i} \| X_j \| / a_{n_i} - b_{n_i}) \to_w \delta_0$, contradiction; if $L(a'_{n_i})/Q(a'_{n_i}) \to 0$ and if we let $s_i = a_{n_i}/a'_{n_i} \geq 1$, then the inequality

$$Q(a_{n_i})/Q(a'_{n_i}) \leq s_i^{-2} + (1 - s_i^{-2}) L(a'_{n_i})/Q(a'_{n_i})$$

(see the proof of 1.10 in [*9*]), implies that $\{ s_i \}$ is uniformly bounded because by (3.8), $\{ Q(a_{n_i})/Q(a'_{n_i}) \}$ is bounded above and below. This gives also a contradiction and we can conclude that $\{ a_{n_i} \}$ and $\{a'_{n_i} \}$ are equivalent. □

Finally we consider the relation between domains of attraction and domains of partial attraction, as in Theorem 2.12 of Jain and Orey [9].

3.4. THEOREM. — Let X be a $\mathbb{R}^n$-valued r. v. in the domain of partial attraction of ρ. If it is only in the DPA of laws of the same type of ρ then ρ is stable and X is actually in its domain of attraction.

Proof. — Corollary 2.4 implies that ρ is stable. If ρ is stable of order $\alpha < 2$ then any limit of shifts of $\{ L(\Sigma_{j=1}^{n_i} \| X_j \|/a_{n_i}) \}$, X_j i. i. d. with $L(X_j) = L(X)$, will be stable with a Lévy measure ν of the form

$$d\nu(x) = 0 \quad \text{for} \quad x < 0, \qquad d\nu(x) = c dx/x^{1+\alpha} \quad \text{for} \quad x > 0.$$

In fact if $\{ L(\Sigma_{j=1}^{n_i} \| X_j \|/a_{n_i}) \}$ is shift convergent then by Theorem 3.1 a subsequence of $\{ L(\Sigma_{j=1}^{n_i} X_j/a_{n_i}) \}$ is shift convergent too, hence shift convergent to a stable measure of the type of ρ; therefore by the converse central limit theorem and the form of the Lévy measure of ρ, we have that for such a subsequence $\{ n_{i_k} \}$,

$$n_{i_k} P \{ \| X \| > \lambda a_{n_{i_k}} \} \to c/\lambda^\alpha,$$

and

$$\lim_{\delta \downarrow 0} \limsup_k n_{i_k} a_{n_{i_k}}^{-2} E \| X \|^2 I_{\{ \|X\| \leq \delta a_{n_{i_k}} \}} = 0.$$

which imply that the shift limit of $\{ L(\Sigma_{j=1}^{n_i} \| X_j \|/a_{n_i}) \}$ is as stated. Since Poisson measures with Lévy measures which are scalar multiples of each other belong to the same type, the one dimensional result of [9], Theorem 2.12, applies and shows that $\| X \|$ is in the domain of attraction of a stable measure of order α. Then, Theorem 3.1 *i*) shows that $\mathbb{N} \in N(X)$ and the result follows as in the first part of the proof of 2.12 in [9], which is independent of the dimension (take d_{BL^*} distances instead of Lévy distances there).

If ρ is Gaussian, then all the possible limits of shifts of $\{ L(\Sigma_{j=1}^{n_i} \| X_j \|/a_{n_i}) \}$ are Gaussian because by the converse central limit theorem and Theorem 3.1 *i*), $n_i P \{ \| X \| > a_{n_i} \} \to 0$. Hence, the result in [9] gives that $\| X \|$ is in the domain of attraction of N(0, 1) so that by 3.1 *i*), $\mathbb{N} \in N(X)$ and the result follows as before. □

The following example shows that Theorem 3.4 is not true in l_p, $p > 2$. Then the question remains as to whether it is true in any infinite dimensional Banach space at all (possibly it is not).

3.5. EXAMPLE. — Here is an example of a l_p-valued r. v. X, $p > 2$, such that: *i*) it belongs to the domain of partial attraction of a single type, which is Gaussian, and *ii*) it does not belong to its domain of attraction. This example is just a modification of one in Pisier and Zinn [*12*]. Let N be an integer valued r. v. with distribution

$$\begin{aligned} &P\{N=n\}=c/n^{2/p+1}, && n \in [2, 2^2) \cup [2^{2^2}, 2^{2^3}) \cup \ldots \cup [2^{2^{2k}}, 2^{2^{2k+1}}) \cup \ldots \\ &P\{N=n\},=0, && n \in [0, 2) \cup [2^2, 2^{2^2}) \cup \ldots \cup [2^{2^{2k-1}}, 2^{2^{2k}}) \cup \ldots \end{aligned}$$

for some (any) $p > 2$ and a suitable constant $c > 0$, and set

$$X = \varepsilon \Sigma_{N^2 < k \leq N^2 + N} e_k$$

where ε is a Bernoulli r. v. independent of N and $\{ e_k \}$ is the canonical basis of l_p. Denote by $X_{(k)}$ the k-th coordinate of X. Then,

$$(3.9) \quad \Sigma_k (EX_{(k)}^2)^{p/2} = \Sigma_k P\{ N^2 < k \leq N^2 + N \}^{p/2} = \Sigma_n n P\{ N = n \}^{p/2} \leq \Sigma_n c/n^{p/2} < \infty .$$

Then, by a result of Vakhania [*14*], X is pregaussian. Note that in particular the finite dimensional distributions of $S_n/n^{1/2}$ converge to normal laws (as usual, $S_n = \Sigma_{i=1}^n X_i$, X_i i. i. d., $L(X_i) = L(X)$). This implies that in order that $\{ L(S_{n_i}/a_{n_i}) \}$ be tight with only non-degenerate limits there must exist $c > 0$ such that $c^{-1} n_i^{1/2} \leq a_{n_i} \leq c n_i^{1/2}$ and that if it is tight then $\{ L(S_{n_i}/n_i^{1/2}) \}$ is convergent. So, in order to determine $N(X)$ we need only look at the behavior of $\{ L(S_{n_i}/n_i^{1/2}) \}$ and *the only possible non-degenerate limits of such sequences belong to the type of the Gaussian law with the covariance of* X, *say* G_X.

Consider the subsequence

$$n_k = [2^{2^{2k}(2/p)}] - 1 .$$

Since $\| X \| = N^{1/p}$, we have

$$(3.10) \quad n_k P\{ \| X \| > n_k^{1/2} \} = n_k P\{ N > n_k^{p/2} \} \geq (2^{2^{2k}(2/p)} - 2) \Sigma_{u=2^{2^{2k}}}^{2^{2^{2k+1}}-1} c/n^{2/p+1} \geq c' > 0,$$

with c' independent of k. Therefore, no subsequential limit of $\{ L(S_{n_k}/n_k^{1/2}) \}$ can be Gaussian; hence, by the previous considerations, this sequence is not tight and therefore, X *is in the domain of attraction of no non-degenerate law.*

Consider now the subsequence

$$n_k = [2^{2^{2k+1}(2/p)}] + 1 .$$

Then,

$$(3.11)\quad n_k P\{\|X\| > n_k^{1/2}\} = n_k P\{N > n_k^{p/2}\} \leq (2^{2^{2k+1}(2/p)} + 1)\Sigma_{n=2^{2^{2k+2}}}^{\infty} c/n^{2/p+1} \to 0$$

as $k \to \infty$. Note also that (as in the original example of Pisier and Zinn),

$$(3.12)\qquad \sup_{c>0} c^2 P\{\|X\| > c\} < \infty.$$

But then (3.9) (3.11) (3.12) imply that $\{L(S_{n_k}/n_k^{1/2})\}$ converges to G_X, that is, X *is in the domain of partial attraction of* G_X. This last proposition follows from Theorem 3.2 in [7]: that theorem has as an immediate consequence that $\{L(S_{n_k}/n_k^{1/2})\}$ is tight if and only if *i*) $\{L(f(S_{n_k}/n_k^{1/2}))\}$ is tight for every $f \in l'_p$; *ii*) $n_k P\{\|X\| > n_k^{1/2}\} \to 0$; *iii*) X is pregaussian; *iv*) $\lim_{R\to\infty} \sup_{n_i} \Sigma_{k=R+1}^{\infty} n^{1-p/2} E|X_{(k)}|^p I_{\{\|X\| \leq n_i^{1/2}\}} = 0$. Now, *i*)-*iii*) hold in our case by previous arguments and *iv*) follows from

$$nE\|n^{-1/2} X I_{\{\|X\| \leq \delta n^{1/2}\}}\|^p \leq p(p-2)^{-1}\delta^{p-2} \sup_{c>0} c^2 P\{\|X\| > c\}$$

(true by Fubini's theorem) and from (3.12).

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