ANNALES DE L'I. H. P., SECTION B

ALEKSANDER WERON

Second order stochastic processes and the dilation theory in Banach spaces

Annales de l'I. H. P., section B, tome 16, nº 1 (1980), p. 29-38
<http://www.numdam.org/item?id=AIHPB_1980__16_1_29_0>

Ann. Inst. Henri Poincaré, Section B :
Vol. XVI, n° 1, 1980, p. 29-38. *Calcul des Probabilités et Statistique.*

Second order stochastic processes and the dilation theory in Banach spaces (*)

by

Aleksander WERON
Institute of Mathematics,
Wroclaw Technical University, Wybrzeze Wyspianskiego 27, 50-370 Wroclaw, Poland

SUMMARY. — The study of dilations of operator functions in non-Hilbert spaces has been inspired by the probability theory in Banach spaces. Our aim here is to show that there exists a close connection between the dilation theory and the theory of second order stochastic processes.

1. INTRODUCTION

Simple dilation theorems assert the embeddability of a Hilbert space H into a large Hilbert space H_1 so that a given L(H)-valued function f on a set T can be retrieved by orthogonal projection from a simpler $L(H_1)$-valued function f_1:

$$f(t) = \mathrm{Proj}_H f_1(t), \qquad t \in T.$$

However following [*2*] we consider R-dilations of B-to-B* operator valued functions, where B is a complex Banach space and B* is its topological dual.

By $C\bar{L}(B, B^*)$—denote the space of all continuous anti-linear operators

(*) Partially written during the author's stay at I. N. S. A. Rennes and Southern Illinois University, Carbondale, Illinois 62901.

from B into B* and by CL(B, H)—the space of continuous linear operators from B into a Hilbert space H.

By Banach space valued second order stochastic process (B-process) we mean a family $(X_t)_{t\in T}$ of elements of CL(B, H), cf. [*1*] or [*19*]. Its correlation function is given by the formula

$$K(t, s) = [X_t, X_s] = X_s^* X_t \in \bar{CL}(B, B^*).$$

It is known [*1*] that the space CL(B, H) is a Loynes space with $\bar{CL}(B, B^*)$-valued inner product $[\cdot, \cdot]$. Thus B-process is a trajectory in the Loynes space L(B, H). In the case B is a Hilbert space such processes were detailed consider by R. Payen [*14*]. When B is the topological dual of a Banach space and $H = L^2(\Omega)$ we get the theory of cylindrical processes. The case of 2-cylindrical martingales (H is the space of real square-integrable martingales) was studied by M. Metivier [*9*] and by M. Metivier and J. Pellaumail [*10*]. The important example is cylindrical Brownian motion which is a « white noise in time and in space ».

Let us recall some useful facts on B-processes.

(1.1) PROPOSITION ([*19*]). — *A function* $K : T \times T \to \bar{CL}(B, B^*)$ *is the correlation function of a* B*-process iff it is positive definite:*

$$\sum_{i,j=1}^{N} (K(t_i, t_j)b_i)(b_j) \geq 0 \tag{1.2}$$

for each N, $t_1 \ldots t_N \in T$ *and* $b_1 \ldots b_N \in B$. *Moreover if* B*-processes* X_t *and* Y_t *have the common correlation function then there exists a unitary operator between the time domains* $U : H_X \to H_Y$ *such that* $Y_t = UX_t$.

The above result may be interpreted as the following Aronszajn-Kolmogorov type theorem. For the Hilbert case see ([*14*], p. 350). Cf. also [*8*], 2.10.

(1.3) PROPOSITION. — *For positive definite kernel* $K : T \times T \to \bar{CL}(B, B^*)$ *there is a Hilbert space* H *and an operator valued function* $X : T \to CL(B, H)$ *such that*

$$K(t, s) = X^*(s)X(t) \quad \textit{for} \quad s, t \in T. \tag{1.4}$$

If H *is minimal i. e.,*

$$H = \bigvee_{t\in T} X_t B \equiv \overline{\mathrm{sp}}\{ X_t B, t \in T \} \tag{1.5}$$

then H *and* $X(\cdot)$ *are unique up to unitary equivalence.*

(1.6) PROPOSITION ([*1*], [*13*]). — a) *If* B-*process defined over a group* T *is stationary*, $K(t, s) = K'(t - s)$, *then there exists a unitary representation* U *of* T *in* H_X *such that* $X_t = U_t X_e$.

b) *If* B-*process defined over a unitized semigroup* T *is multiplicatively correlated*, $K(t, s) = K'(t \circ s)$, *then there exists a self-adjoint representation* S_t *of* T *in* H_X *such that* $X_t = S_t X_e$.

2. DILATION THEOREMS

Let $\{ K_t \}$ and $\{ D_t \}$ be two families of operators such that $K_t \in \bar{CL}(B, B^*)$, $D_t \in CL(H, H)$ and $R \in CL(B, H)$. The family $\{ D_t \}$ is called an R-dilation of $\{ K_t \}$ if

$$K_t = R^* D_t R \qquad \text{for every index } t\,.$$

See [*11*] and [*20*]. The first dilation theorem for a B-to-B* valued function was published by N. N. Vakhania (1968) (*) when he studied covariance operators for probability measures in Banach spaces. Namely, he proved the following « lemma on factorization »; for each positive operator $A \in \bar{CL}(B, B^*)$ there exist a Hilbert space H and an operator $R \in CL(B, H)$ such that $A = R^*R$. If H is minimal then the factorization is unique up to unitary equivalence. This result shows that each positive operator $A \in \bar{CL}(B, B^*)$ has a R-dilation to the identity operator in a Hilbert space H:

$$A = R^* I_H R = R^* R\,.$$

In the case of B-to-B* operators the analogues of results of Bochner and Naimark on positive definite functions and positive measures were obtained by the methods of stationary stochastic processes (detailed references in [*20*]). The typical result is the following analogue of Naimark's dilation theorem.

(2.1) EXAMPLE. — Let F be a positive $\bar{CL}(B, B^*)$-valued measure on the σ-algebra of all borelian subsets of a LCA group G, which is the dual group of LCA group T. Let K be its Fourier transform

$$K(t) = \int_G \langle t, g \rangle F(dg)\,, \qquad t \in T$$

where $\langle t, g \rangle$ is a value of a character g on t. Since $K_0(s, t) = K(s - t)$ is positive definite then by (1.1) and (1.6) it is the correlation function

(*) The theorem occurs, however, in an unpublished 1957 report by G. B. PEDRICK, cf. ([*7*], p. 415-416).

of a stationary B-process X_t. Thus there exists a Hilbert space $H = H_X$ and a unitary representation U_g such that

$$K(t) = [X_t, X_e] = [U_t X_e, X_e] = X_e^* U_t X_e = \int_G \langle t, g \rangle X_e^* E(dg) X_e ,$$

where $E(\cdot)$ is a spectral measure of U_g obtained from the Stone's theorem. By the uniqueness of the Fourier transform we have

$$F(\Delta) = X_e^* E(\Delta) X_e , \qquad \Delta \in \Sigma .$$

Since H_X—the time domain of the process X_t is minimal in the sense of (1.5) thus by Prop. 1.3. $H = H_X$, $R = X_e$ and $E(\cdot)$ are unique up to unitary equivalence. So the triple $(H, R, E(\cdot))$ is a minimal and unique R-dilation of $F(\cdot)$. Thus we have proved an analogue of Naimark's dilation theorem.

The reason for which this result was proved by a method of stochastic stationary processes follows immediately if we interpretate the Prop. 1.1 and Prop. 1.6 in the language of dilation theory. Namely, the correlation function $K(t)$ of a stationary B-valued process X_t has a minimal and unique R-dilation (H_X, X_e, U_t).

(2.2) Remark. — The existence of the above close relationship between the dilation theory and stationary B-valued processes is implicit in papers of S. A. Cobanjan (1970), A. I. Ponomarenko (1972) and the writer (1975). (See detailed references in [*20*].) The systematic studies of these ideas in more general setting of semigroups, the theory of so called propagators (stationary process has the unitary propagator) and its connection with Kernel theorem (Prop. 1.3) has been done by P. Masani, cf. [*8*]. However he has used the definition of dilation in which R is a linear isometry on a Hilbert space B into H and consequently consider dilations only for Hilbert case. It is extended by our approach which may be also applied to locally convex spaces. Since the complete analogue of Prop. 1.3 does not hold in all locally convex spaces, therefore one may deduce dilation theorems only for those spaces which are characterized by Kernel theorems, see [*3*].

In [*2*] we proved a B-to-B* analogue of general B. Sz-Nagy's dilation theorem by using the complicated construction which is like the original one for the Hilbert case. Now we give a short proof of the stronger result which is in a spirit of the above example. For non-unital case see [*12*].

Let S be a *-semigroup i. e., S is a unital semigroup with a function $* : S \to S$ such that for each $s, t \in S$

$$(s, t)^* = t^* s^* , \qquad s^{**} = s \qquad \text{and} \qquad e^* = e .$$

A function $K : S \to \bar{CL}(B, B^*)$ is positive-definite if a function $K_1(s, t) = K(t^*s)$ defined on $S \times S$ satisfies (1.2). An operator valued function $D(\cdot) : S \to CL(H)$ is called a *-representation if

$$D(st) = D(s)D(t), \qquad D(e) = I_H, \qquad \text{and} \qquad D(s^*) = D^*(s).$$

If $K(\cdot)$ is the form

$$K(s) = R^*D(s)R, \qquad s \in S,$$

where $R \in CL(B, H)$ then $D(\cdot)$ is an R-dilation of $K(\cdot)$.

(2.3) THEOREM. — a) *Let a* $\bar{CL}(B, B^*)$ *valued kernel* $K(\cdot)$ *on a unitized *-semigroup* S *be positive definite. Then the following are equivalent:*

(1) *There exists a function* $\alpha : S \to \mathbb{R}^+$ *such that for each* $s, t \in S$ *and* $b \in B$

$$(K(t^*s^*st)b)(b) \leq \alpha(s)(K(t^*t)b).$$

(2) $K(\cdot)$ *has an* R-*dilation* $D(\cdot)$ *which is an involutory representation of* S.

(3) *There exists a function* $\sigma : S \to \mathbb{R}^+$ *and a number* $c > 0$ *that for each* $s, t \in S$

$$\| K(s) \| \leq c\sigma(s),$$

where σ *is submultiplicative;* $\sigma(st) \leq \sigma(s)\sigma(t)$.

b) *The minimality condition* $H = \bigvee_{t \in S} \{ D(t)RB \}$ *determines* H *and* $D(\cdot)$ *up to unitary equivalence.*

c) *The minimal space* H *is separable iff* $\bigvee_{t \in S} K(t)B$ *is a separable subspace of* B^*.

Proof. — Since K is positive definite, therefore by Prop. 1.3, there is a Hilbert space H and an operator-valued function $X : S \to CL(B, H)$ such that

$$K(s^*t) = X^*(s)X(t).$$

Now by [7, th. 4.13] (1) is equivalent to the fact that there is a semigroup $\{ D(s), s \in S \}$ on $CL(H, H)$ such that for each $s, t \in S$

$$X(st) = D(s)X(t), \qquad D(t^*) = D^*(t).$$

Hence

$$K(t) = X(e)^*X(t) = X^*(e)D(t)X(e) = R^*D(t)R, \qquad \text{where} \qquad R = X(e).$$

We have thus shown that (1) implies (2). Next, we show that (2) implies (3). Indeed,

$$\| K(s) \| = \| R^*D(s)R \| \leq \| R \|^2 \| D(s) \|.$$

Hence with $c = \|R\|^2$ and $\sigma(s) = \|D(s)\|$ we have (3) since

$$\|D(st)\| \leq \|D(s)\| \, \|D(t)\| .$$

Finally from ([*18*], cf. also [*17*] and [*8*]) we know that (3) is equivalent to (1). This complete the proof of the equivalences.

As for the minimality condition *b*), suppose that for $i = 1, 2$, D_i are involutory representations of S,

$$H_i = \bigvee_{s \in S} D_i(s)R_i(B),$$

where $R_i \in CL(B, H_i)$ and $R_i^* D_i(s) R_i = K(s)$. Let $X_i(s) = D_i(s)R_i$. Then

$$X_1^*(s)X_1(t) = K(s^*t) = X_2^*(s)X_2(t).$$

Hence, cf. (1.1), there is a unitary operator U on H_1 onto H_2 such that $X_2(s) = UX_1(s)$, for each $s \in S$. Thus

$$\begin{aligned} D_2(s)UR_1 &= D_2(s)UX_1(e) = D_2(s)X_2(e) = D_2(s)R \\ &= X_2(s) = UX_1(s) = UD_1(s)R . \end{aligned}$$

This shows that $D_2(s)U = UD_1(s)$ on $R_1(B)$, and hence on its closure, i. e., on H_1. This completes the proof of part *b*).

The part *c*) follows from the fact, that

$$R^*(H) = R^* \bigvee_{t \in S} D(t)R(B) = \bigvee_{t \in S} R^*D(t)R(B) = \bigvee_{t \in S} K(t)B .$$

3. APPLICATIONS TO STOCHASTIC PROCESSES

In this section we want to deduce some theorems for B-processes by using dilation approach. The first result follows immediately from Th. 2.3 *a*).

(3.1) COROLLARY. — *If* B-*process defined over a* *-*semigroup* T *is* *-*stationary;* $K(t, s) = K'(s^*t)$, *then there exists an involuntary representation* D_t *of* T *in* H_X *such that* $X_t = D_t X_e$ *iff the correlation function* $K'(\cdot)$ *satisfies the condition* (1) *or* (3) *from Th. 2.3.*

Next we would like to show that the uniqueness of minimal dilations, Th. 2.3 *b*), suggested a new way for obtaining isomorphism theorems for stationary B-processes. We start from the following example which illustrates the idea in the case $B = \mathbb{C}$.

(3.2) EXAMPLE. — The correlation function $K(\cdot)$ of a continuous stationary complex valued process X_t over an LCA group T is a complex valued positive definite function on T. Since each group is a *-semigroup ($t^* = -t$) and in this case each function trivially satisfies the condition (1) from Th. 2.3, thus $K(\cdot)$ has the minimal unitary dilation (H_X, X_e, U_t).

On the other hand $K(\cdot)$ is continuous and by the Bochner theorem there exists a nonnegative scalar measure F on the Borel σ-algebra of the dual group G such that K is its Fourier-Stieltjes transform. Consider the space $L^2(F)$ of square integrable complex functions on G and the unitary group V_t given by the formula

$$V_t f(g) = \langle t, g \rangle f(g), \qquad t \in T.$$

It is easy to observe that the triplet $(L^2(F), M, V_t)$, where $M : \mathbb{C} \to L^2(F)$ —multiplication by $f(g) \equiv 1$, is another minimal dilation of $K(\cdot)$.

By the uniqueness property in Th. 2.3 there exists a unitary operator $W : L^2(F) \to H_X$ such that $WM = X_e$ and $WV_t = U_tW$. Let's note that W is realized as a stochastic integral

$$\begin{aligned} W \langle t, g \rangle &= WV_tM = U_tWM = U_tX_e \\ &= \int_G \langle t, g \rangle E(dg)X_e = X_t \qquad \text{cf. } (2.1) \end{aligned}$$

Thus we have obtained a new proof of the celebrated isomorphism theorem for stationary processes, see [*15*].

The above construction may not be directly extended to the Banach spaces because a space $L^2(F)$ for $\bar{CL}(B, B^*)$-valued measure F is not defined as far. For a case B is a Hilbert space see [*6*]. But another interesting approach is possible. Let F be $\bar{CL}(B, B^*)$ valued positive measure such that there exists a nonnegative σ-finite measure μ that for each $x, y \in B$ $(F(\cdot)x)(y)$ is absolutely continuous w. r. t. μ. Then by [*5*] there exists a Hilbert space K and an operator $Q \in CL(B, L^2(\mu : K))$ such that

$$\left[\frac{d(Fx)(y)}{d}\right](\cdot) = \langle (Qx)(\cdot), (Qy)(\cdot) \rangle \qquad \text{a. e. } [\mu]$$

Thus $F(\cdot)$ has the following dilation in $L^2(\mu ; K)$:

$$F(\cdot) = Q^*E_0(\cdot)Q,$$

where $E_0(\cdot)$ is the spectral measure in $L^2(\mu : K)$ defined as an operator of multiplication by the indicator 1_Δ. This dilation is minimal by the definition in the subspace

$$L(F) = \overline{\text{sp}} \{ E_0(\Delta)Qb, \Delta \in \Sigma, b \in B \} \subset L^2(\mu : K).$$

On the other hand the measure F defined by the stationary B-process has by (2.1) the minimal spectral dilation $(H_X, X_e, E(.))$. Thus by the uniqueness property there exists a unitary operator $W : H_X \to L$ such that $WX_e = Q$ and $WE(\cdot) = E_0(\cdot)W$. Consequently we obtain

$$WX_t b = \langle t, g \rangle (Qx)(\cdot), \qquad t \in T, \quad b \in B.$$

Thus we have proved the following

(3.3) THEOREM (cf. [*4*]). — *Let* X_t *be a continuous stationary* B-*process over an LCA group such that* F *related to* X_t *as in* (2.1) *has a weak spectral density. Then the time domain* H_X *and the spectral domain* L(F) *of the process are unitary equivalent.*

Finally we shall present some application of dilation approach to 2-integrable martingales which were studied in [*9*] and [*10*].

Let $T = [0, 1]$, $(\Omega, \mathscr{B}, P)$-basic probability space and $\mathscr{M}$ be the space of all square integrable real martingales X_t which are right continuous and with left hand limits and $X_0 = 0$. If martingales X_t and Y_t such that $P[\sup_{t\in T} | X_t - Y_t |] = 0$ are indistinguishable in $\mathscr{M}$, then $\mathscr{M}$ is a Hilbert space with natural L^2 inner product.

Following [*9*, *10*] we say that X is 2-cylindrical martingale on B* if $X \in CL(B^*, \mathscr{M})$. An important role in the theory of such processes plays such called quadratic Doleans' measure M of X which is defined on the σ-algebra $\mathscr{P}$ of predictable subsets of $T \times \Omega$.

Let m be defined by the formula

$$[m(A)](h \otimes g) = \mathbb{E}\{ 1_F [X_t(h) \cdot X_t(g) - X_s(h) \cdot X_s(g)] \},$$

where $A = F \times]s, t]$ is predictable rectangle and $h \otimes g$ belongs to the tensor product $B^* \otimes B^*$. The extension of $[m(A)](h \otimes g)$ defines a mapping M from $\mathscr{P}$ into the dual space $(B^* \otimes_1 B^*)^*$. This mapping M is called quadratic Doleans' measure (see [*10*]).

(3.4) THEOREM. — *Let* M *be the quadratic Doleans' measure of a 2-cylindrical martingale. If* $\bigvee_{A \in \mathscr{P}} M(A)B^*$ *is a separable subspace of* B** *then there exists a nonnegative real valued measure* μ *on* $\mathscr{P}$ *such that for each* $x, y \in B^*$ $[M(\cdot)](x \otimes y)$ *is absolutely continuous w. r. t.* μ.

Proof. — Note that $x \otimes y$ can be identified with a bilinear continuous form on $B^* \times B^*$ or a continuous linear form on $B^* \otimes B^*$ through the formula

$$\langle x \otimes y, x^* \otimes y^* \rangle = \langle x, x^* \rangle \langle y, y^* \rangle .$$

Thus there exists a continuous extension of this linear mapping into an isometry from $(B^* \hat{\otimes}_1 B^*)^*$ onto $CL(B^*, B^{**})$ which to a bounded bilinear form f on $B^* \times B^*$ correspond the bounded lineai operator $F \in CL(B^*, B^{**})$ such that

$$\langle F(x), y \rangle = f(x, y) \qquad \text{(see [10])}.$$

Since by the definition quadratic Doleans' measure is positive valued then we may use the dilation theorem, cf. [20]. Consequently there exists a Hilbert space H, an operator $R \in CL(B^*, H)$ and a spectral measure $E(\cdot)$ on σ-algebra of predictable sets such that

$$M(\cdot) = R^*E(\cdot)R .$$

Moreover by the part *c*) of Th. 2.3, H is separable because $\bigvee_{A \in \mathscr{P}} M(A)B^*$ is separable. Note that $H = \bigvee_{A \in \mathscr{P}} \{ E(A)RB^* \}$.

Let

$$\mu(A) = \sum_{n=1}^{\infty} 2^{-n} \| E(A)e_n \|^2 ,$$

where (e_n) is a CONS in H. Thus μ is nonnegative real valued measurè on $(T \times \Omega, \mathscr{P})$ and for each n $\| E(.)e_n \|^2 \ll \mu$. Hence for each $h, g \in H$ $(E(.)h, g) \ll \mu$ too. Therefore for $h = E(\cdot)Rx$ and $g = E(\cdot)Ry$, where $x, y \in B^*$ we obtain

$$M(\cdot)(x \otimes y) = (R^*E(\cdot)Rx)(y) = \langle E^2(\cdot)Rx, E(\cdot)Ry \rangle \ll \mu ,$$

completing the proof.

ACKNOWLEDGMENT

The author is grateful to the referee for his suggestions.

REFERENCES

[1] S. A. Cobanjan and A. Weron, Banach space valued stationary processes and their linear prediction, *Dissertationes Math.*, t. **125**, 1975, p. 1-45.

[2] J. Górniak and A. Weron, An analogue of Sz.-Nagy's dilation theorem, *Bull. Acad. Polon. Sci.*, t. **24**, 1976, p. 867-872.

[3] J. Górniak and A. Weron, Aronszajn-Kolmogorov type theorems for positive definite kernels in locally convex spaces, *Studia Math.*, t. **69** *(to appear)*.

[4] A. Makagon, An isomorphic theorem for Banach space valued stationary stochastic sequences, *Bull. Acad. Polon. Sci.*, t. **26**, 1978, p. 169-173.

[5] A. Makagon and F. Schmidt, A decomposition of the density of operator valued measure in Banach spaces, *Bull. Acad. Polon. Sci. (to appear)*.

[6] V. Mandrekar and H. Salehi, Subordination of infinite-dimensional stationary stochastic processes, *Ann. Inst. Henri Poincaré*, t. **6**, 1970, p. 115-130.

[7] P. Masani, Dilations as propagators of Hilbertian varieties, *S. I. A. M. J. Anal.*, t. **9**, 1978, p. 414-456.

[8] P. Masani, Propagators and dilations, *Lecture Notes in Math.*, p. 656, 1978, p. 95-117, Springer Verlag.

[9] M. Metivier, *Integrations with respect to processes of linear functionals*, Séminaire de Rennes, 1975.

[10] M. Metivier and J. Pellaumail, *Cylindrical stochastic integral*, Séminaire de Rennes, 1976.

[11] W. Mlak, Dilations of Hilbert space operators (general theory), *Dissertationes Math.*, t. **153**, 1978, p. 1-61.

[12] W. Mlak and A. Weron, Dilations of Banach space operator valued functions, *Ann. Polon. Math. (to appear)*.

[13] Nguyen Van Thu and A. Weron, Examples of non-stationary Banach space valued processes of second order, *Lecture Notes in Math.*, t. **656**, 1978, p. 171-181, Springer Verlag.

[14] R. Payen, Functions aleatoires du second ordre a valeurs dans un espace de Hilbert, *Ann. Inst. Henri Poincaré*, t. 3, 1967, p. 323-396.

[15] M. Rosenberg, Mutual subordination of multivariate stationary processes over any locally compact Abelian group, *Z. Wahrschernlichkeits theorie verw. Geb.*, t. 12, 1969, p. 333-343.

[16] F. H. Szafraniec, On the boundedness condition involved in dilation theory, *Bull. Acad. Polon*, t. **24**, 1976, p. 877-881.

[17] F. H. Szafraniec, Dilations on involution semigroups, *Proc. Amer. Math. Soc.*

[18] F. H. Szafraniec, Apropos Professor Masani's talk, *Lecture Notes in Math.*, t. **656**, 1978, p. 245-249.

[19] A. Weron, Prediction theory in Banach spaces, *Lecture Notes in Math.*, t. **472**, 1975, p. 207-228, Springer Verlag.

[20] A. Weron, Remarks on positive-definite operator valued functions in Banach spaces, *Bull. Acad. Polon.* t. **24**, 1976, p. 873-876.

(Manuscrit reçu le 19 octobre 1979)
The first version was send in July 1978