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Section B :
Calcul des Probabilités et Statistique.

Embeddings in Brownian motion

by

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SUMMARY. — We prove the following Skorohod's type embedding theorem. If (B_t) is a Brownian motion and T is any stopping time, then there exists a standard stopping time S before T such that B_T and B_S have the same distribution. We also give an example of a descending martingale which does not embed in Brownian motion by means of standard non randomized stopping times.

RÉSUMÉ. — On montre le résultat suivant de « type Skorohod ». Si (B_t) est un mouvement brownien linéaire et T un temps d'arrêt quelconque, alors il existe un temps d'arrêt quelconque, alors il existe un temps d'arrêt standard S avant T tel que B_T et B_S aient la même loi. On donne aussi un exemple de martingale descendante qui ne peut être réalisée par un mouvement brownien arrêté en une suite décroissante de temps d'arrêt standards et non flous.

I. INTRODUCTION

Let $\{ B_t ; t \geq 0 \}$ be standard Brownian motion starting at the origin. We recall that the potential of a measure μ on the Borel field of $\mathbb{R}$ is defined by $P_\mu(x) = - \int | x - y | \mu(dy)$. If T is any stopping time we denote by P_{B_T}

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the potential of the distribution of B_T. A stopping time is said to be *standard* (see [*3*]) provided there exists a sequence T_n of bounded stopping times such that $\lim T_n = T$ with probability one and $\lim P_{B_{T_n}}(x) = P_{B_T}(x)$ for every x. It is implicit here that $P_{B_T}(x)$ is finite. We recall that the Skorohod embedding theorem as extended in [*3*] asserts that if μ is a probability measure of mean zero, there exists a standard stopping time S such that B_S has distribution μ. In theorem 1, we give a refinement of Skorohod's result, that is if μ is the distribution of Brownian motion stopped at T where T is any stopping time, then there exists a standard stopping time S such that $S \leq T$ and B_S has distribution μ.

We also recall that a stopping time T is said to be *minimal* (see [*5*]) if for every stopping time S such that $S \leq T$ and B_S and B_T have the same distribution then $S = T$ a. e. In [*5*], Monroe uses the theory of barriers and fairly difficult arguments to establish the regularity properties of minimal stopping times. These regularity properties are easy to obtain for standard stopping times (see [*3*]). An immediate corollary of theorem 1, is that the two notions essentially coincide, thus yielding the properties of minimal stopping times from those of the standard ones.

In Section III, we given an example of a descending martingale which does not embed in standard Brownian motion by means of standard stopping times, and on the other hand, we remark that a result of Monroe [*5*] combined with the results of Section II implies the embedding of such martingales by means of standard stopping times in a distributional enlargement of Brownian motion, a motion introduced and studied in [*1*]. Finally, we conjecture that the embedding is not possible using only a product enlargement. (Every product enlargement is a distributional enlargement, but the converse is not true.)

II. A REFINEMENT OF SKOROHOD'S EMBEDDING THEOREM

We first recall a few facts about potential theory on the line and we refer for more details to [*3*].

1) If μ_1 is a unit measure with Center of mass at x_0 and if μ_0 is the unit mass at x_0 then $P_{\mu_1}(x) \leq P_{\mu_0}(x)$.

2) If ν, μ, μ_n are unit measures such that $\mu_n \to \mu$ weakly, $P_{\mu_n}(x) \geq P_\nu(x)$ and $\lim P_{\mu_n}(x)$ exists for one x then $\lim P_{\mu_n}(x) = P_\mu(x)$, $\forall x$.

3) If μ_1 and μ_2 have finite and equal potentials then $\mu_1 = \mu_2$.

We also recall a fact about the balayage of potentials with respect to a finite open interval I.

4) If T is a standard stopping time, then the stopping time S given by $S(w) = \inf \{ t \geq T(w) ; B_t \in I^c \}$ is a standard stopping time as well and

i) $P_{B_T}(x) = P_{B_S}(x)$ on I^c.

ii) $P_{B_S}(x)$ is linear on $\bar{I}$.

We shall also need the following domination principle.

5) Let μ_1 and μ_2 be finite measures on $\mathbb{R}$ with the same total mass and having finite potentials. If

a) $P_{\mu_1}(x) \geq P_{\mu_2}(x)$ $\forall x \in \mathbb{R}$ and $P_{\mu_1}(x) = P_{\mu_2}(x)$ on the support of μ_1 then $P_{\mu_1}(x) = P_{\mu_2}(x)$ $\forall x \in \mathbb{R}$.

We are now ready to prove the following

THEOREM 1. — For any stopping time T such that $E[B_T] = 0$ there exists a standard stopping time $S \leq T$ such that $P_{B_S} = P_{B_T}$.

Proof. — Let C be the class of all standard stopping times S less than T such that $P_{B_S} \geq P_{B_T}$. The class is not empty since by (1), zero belongs to C. Any chain in C has an upper bound by (2). Hence C is a Zorn class and it admits a maximal element S. Now, we show that $P_{B_S} = P_{B_T}$.

Suppose not. Consider the open set $\mathcal{O} = \{ x ; P_{B_S}(x) > P_{B_T}(x) \}$. We claim that $P(\{ S < T \} \cap \{ B_S \in \mathcal{O} \}) > 0$. In fact, if $S = T$ a. e. on $\{ B_S \in \mathcal{O} \}$, let μ_1 (resp ν_1) be the restriction of the distribution of B_S on $\mathcal{O}$ (resp $\mathcal{O}^c$). Similarly, let μ_2 (resp ν_2) be the restriction of the distribution of B_T on $\mathcal{O}$ (resp $\mathcal{O}^c$). Clearly

$$P_{B_S} = P_{\mu_1} + P_{\nu_1}, \qquad P_{B_T} = P_{\mu_2} + P_{\nu_2}$$

and

$$P_{\mu_1} = P_{\mu_2}, \qquad \text{hence} \qquad P_{\nu_1} \geq P_{\nu_2}.$$

On the other hand, $P_{B_T} = P_{B_S}$ on $\mathcal{O}^c$, thus $P_{\nu_1} = P_{\nu_2}$ on $\mathcal{O}^c$ which carries ν_1. Also, ν_1 and ν_2 have the same total mass. Thus, the domination principle applies to give that $P_{\nu_1} = P_{\nu_2}$ everywhere, therefore $P_{B_T} = P_{B_S}$ everywhere which is a contradiction.

Since $\mathcal{O}$ is an open set, there exists an open interval $I \subseteq 0$ and a constant $c > 0$ such that

i) $$P(\{ S < T \} \cap \{ B_S \in I \}) > 0$$

and

ii) $$P_{B_S}(x) > c > P_{B_T}(x) \quad \forall x \in 1.$$

Let now $S'(w) = \inf \{ t \geq S(w) : B_t(w) \notin I \}$. By (4) S′ is a standard stopping time and $P_{B_S} \geq P_{B_{S'}} > c > P_{B_T}$ on I. The property *i*) translates to $P(\{ S < T \} \cap \{ S < S' \}) > 0$. That is $S \neq S' \wedge T$. Finally, we get that $S' \wedge T$ is a standard stopping time greater than S and less than T such that $P_{B_{S' \wedge T}} \geq P_{B_S} \geq P_{B_T}$ which contradicts the maximality of S.

We note that a similar and simpler reasoning as above gives a proof of the Skorohod embedding theorem, that if μ is a probability measure of mean Zero, there exists a standard stopping time S such that B_S has distribution μ. For that, it is enought to consider the class of standard stopping times T such that $P_{B_T} \geq P_\mu$ and pick a maximal element S. If there exists a constant $c > 0$ and an interval I such that $P_{B_S} > c > P_\mu$ on I, then the stopping time $S' = \inf \{ t \geq S ; B_t \notin I \}$ contradicts the maximality of S.

COROLLARY — For a stopping time T, the following properties are equivalent:

a) T is standard,

b) T is a minimal and $E[B_T] = 0$.

Proof. — *b*) $\Rightarrow$ *a*) follows immediately from theorem 1 and (3). For the reverse implication we notice that if T is standard and $S \leq T$ then for any non-negative Borel function f on $\mathbb{R}$ we have:

$$E\left(\int_S^T f(B_t)dt\right) = c\int f(x)(P_{B_T}(x) - P_{B_S}(x))dx$$

where c is some constant different from zero. Taking $f > 0$ on $\mathbb{R}$, we see that if $P_{B_T} = P_{B_S}$, then $S = T$ a. e.

III. EMBEDDING A DESCENDING MARTINGALE IN BROWNIAN MOTION

It is well known ([*3*], [*4*], [*5*]) that if $(Y_n)_n$ is an ascending martingale and if (B_t, F_t) is standard Brownian motion with initial distribution equal to the distribution of Y_0, then there exists standard stopping times (with respect to F_t) $0 = \sigma_0 \leq \sigma_1 \leq \ldots$ such that $(Y_n)_n$ and $(B_{\sigma_n})_n$ have the same joint distribution. We then say that $(Y_n)_n$ embeds in Brownian motion by means of standard stopping times. In the following, we show that, unless the fields $(F_t)_t$ are enlarged, one cannot hope for a similar result in the case of descending martingales.

THEOREM II. — There exists a descending martingale which does not embed in standard Brownian motion (B_t, F_t) by means of standard stopping times with respect to F_t.

Proof. — Let (Ω, F, P, B_t, F_t) be standard Brownian motion starting at the origin. For each n, define the stopping times

$$\sigma_n = \inf\left\{ t\,;\, B_t \notin \left[-\frac{1}{n}, \frac{1}{n} \right] \right\} \quad \text{and} \quad \sigma'_n = \inf\left\{ t\,;\, B_t \notin \left[-\frac{2}{n}, \frac{2}{n} \right] \right\}.$$

Let $(\bar{\Omega}, \bar{F}, \bar{P})$ and $(\bar{\bar{\Omega}}, \bar{\bar{F}}, \bar{\bar{P}})$ be two copies of (Ω, F, P). Denote by $\tilde{\Omega}$ the disjoint union of $\bar{\Omega}$ and $\bar{\bar{\Omega}}$, by $\tilde{F}$ the natural σ-field on $\tilde{\Omega}$ and by $\tilde{P}$ the probability measure on $\tilde{F}$ equal to $\frac{1}{2}(\bar{P} + \bar{\bar{P}})$.

We now define for each $n \in \mathbb{N}$, the random variables

$$Y_{-n} : (\tilde{\Omega}, \tilde{F}, \tilde{P}) \to \mathbb{R}$$

by

$$Y_{-n}(\tilde{w}) = \begin{cases} B_{\sigma_n}(w) & \text{if } \tilde{w} \in \bar{\Omega} \\ B_{\sigma'_n}(w) & \text{if } \tilde{w} \in \bar{\bar{\Omega}} \end{cases}$$

Clearly, $(Y_{-n})_n$ is a descending martingale with respect to the σ-fields $(\tilde{F}_{-n})$ generated by $\{ Y_{-m}\,;\, m \geq n \}$. We also note that the distribution μ_n of Y_{-n} converges when n goes to infinity to the distribution of B_0.

Suppose now, there exists a decreasing sequence of standard stopping times (τ_n) such that (B_{τ_n}) and $(Y_{-n})_n$ have the same joint distributions. The potentials $P_{B_{\tau_n}}$ converges to P_{B_0} and since the τ_n's are standard, lemma 7.2 of [3] gives $\int \tau_n dP = \int (P_{B_0}(x) - P_{B_{\tau_n}}(x))dx$. Therefore $\int \tau_n dP$ converges to zero and so the sequence (τ_n) decreases to zero, so by the zero-one law, one obtains that the germ σ-field $\bigcap_n \sigma(B_{\tau_k}\,;\, k \geq n)$ is trivial. On the other hand, the σ-field $\bigcap_n \tilde{F}_{-n}$ contains at least the sets $\bar{\Omega}$ and $\bar{\bar{\Omega}}$ and this yields a contradiction.

In [1], the notion of distributional stopping times is introduced and studied. The idea is to sufficiently enlarge the σ-fields of Brownian motion to get a richer class of stopping times but without letting the enlargement alter the properties of the process in question. By combining the results of Section II and a theorem of Monroe [5], the following result may be proved.

THEOREM III. — Every descending martingale may be embedded in a distributional enlargement of Brownian motion by means of standard stopping times.

One may ask if the embedding is possible using only a product enlargement (and standard stopping times, of course). We don't have an answer to this question, but conjecture that a product enlargement is not enough for the embedding. This question is closely related to the problem of representing a decreasing sequence of stopping times in a distributional enlargement by a decreasing sequence of stopping times in a product enlargement. (For the case of an increasing sequence see [*1*].) This question is also related to the problem of whether randomization is enough to assure the compactness of time changes in the topology of Baxter and Chacon [*1*].

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