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# Recurrence in single server queues with impatient customers

by

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ABSTRACT. — Conditions of recurrence and positive recurrence are obtained for a general single server queue with impatient customers. A renewal phenomenon is brought out.

RÉSUMÉ. — Des conditions de récurrence et de récurrence positive sont obtenues pour une file d'attente générale à un seul serveur dans laquelle des clients impatients quittent la file. Un phénomène de renouvellement est dégagé.

## 0. INTRODUCTION

We consider a general single server queue with impatient customers. That is, some customers leave the system if their waiting time exceeds a specified time interval (« rejection interval »).

Relatively few papers have been devoted to this subject. The most important result is that of Daley [*4*] who studied the waiting time distribution in a queue of impatient customers and the probability of rejection of such customers. His assumption is that the rejection interval is the same for all impatient customer.

We assume that these time intervals form a sequence of independent, identically distribued random variables. The purpose of this note is to give conditions of recurrence and positive recurrence (ergodicity) for such a system.

In practice, the most important condition is positive recurrence which physically means the stability of the queue.

A renewal phenomenon, which is very useful for limits theorems, is brought out.

## 1. MODEL AND NOTATIONS

Let:

— $\mathbb{R}_+$ be the set of non-negative real numbers.
— $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{ +\infty \}$.
— $\mathscr{B}_{\mathbb{R}_+}$ (resp. $\mathscr{B}_{\bar{\mathbb{R}}_+}$) be the $\sigma$-field of Borel subsets of $\mathbb{R}_+$ (resp. $\bar{\mathbb{R}}_+$).
— $\alpha$, $\beta$ be two probability measures (p. m.) on $(\mathbb{R}_+, \mathscr{B}_{\mathbb{R}_+})$.
— $\gamma$ be a p. m. on $(\bar{\mathbb{R}}_+, \mathscr{B}_{\bar{\mathbb{R}}_+})$.
— $\mathbb{N}$ be the set of integers: $\mathbb{N} = \{ 0, 1, 2, \dots \}$.
— $\mathbb{N}^*$ the set of positive integers: $\mathbb{N}^* = \{ 1, 2, 3, \dots \}$.

$(\Omega, \mathscr{A}, \mathbb{P})$ is the fundamental probability space of our problem:

— $\Omega = (\mathbb{R}_+ \times \mathbb{R}_+ \times \bar{\mathbb{R}}_+)^{\mathbb{N}^*}$
— $\mathscr{A} = (\mathscr{B}_{\mathbb{R}_+} \otimes \mathscr{B}_{\mathbb{R}_+} \otimes \mathscr{B}_{\bar{\mathbb{R}}_+})^{\otimes \mathbb{N}^*}$
— $\mathbb{P} = (\alpha \times \beta \times \gamma)^{\times \mathbb{N}^*}$

$(A_n, B_n, C_n)$ is the $n$-th coordinate function of $\Omega$. $\Theta$ from $\Omega$ to $\Omega$ is the shift: $\forall n \geqslant 1$, $A_n \circ \Theta = A_{n+1}$, $B_n \circ \Theta = B_{n+1}$, $C_n \circ \Theta = C_{n+1}$. We define $\Theta_0 = \mathrm{Id}_\Omega$, $\Theta_1 = \Theta$, $\Theta_n = \Theta_{n-1} \circ \Theta$.

$\mathscr{A}_0 = \{ \phi, \Omega \}$; $\forall n \geqslant 1$, $\mathscr{A}_n$ is the $\sigma$-field generated by $(A_p, B_p, C_p)$ for $1 \leqslant p \leqslant n$ : $\mathscr{A}_n = \sigma \{ A_p, B_p, C_p ; 1 \leqslant p \leqslant n \}$. If T is an integer valued random variable, $\Theta_T$ is defined by $(\Theta_T)(\omega) = \Theta_{T(\omega)}(\omega)$.

$A_n$ is the inter-arrival time between the $(n-1)$-th and the $n$-th customer.

$B_n$ is the service time of the $(n-1)$-th customer.

$C_n$ is the « impatience » time of the $(n-1)$-th customer.

The assumption is that these three sequences of random variables (r. v.) are independent, and each is a sequence of independent identically distributed random variables.

The customers are indexed by $\mathbb{N} = \{ 0, 1, 2, \dots \}$. The 0-th customer arrives at time 0. The $n$-th customer, for $n \geqslant 1$, arrives at time $\sum_{i=1}^{i=n} A_i$.

Let $W_n$ be the time that the $n$-th customers *has to wait* for beginning his service. $\min(W_n, C_{n+1})$ is then his real waiting time and he leaves the service at time $\min(W_n + B_{n+1}, C_{n+1})$. We have then:

$$W_{n+1} = \begin{cases} (W_n + B_{n+1} - A_{n+1})_+ & \text{if } W_n + B_{n+1} \leqslant C_{n+1} \\ (C_{n+1} - A_{n+1})_+ & \text{if } W_n \leqslant C_{n+1} < W_n + B_{n+1} \\ (W_n - A_{n+1})_+ & \text{if } C_{n+1} < W_n \end{cases}$$

For $x \in \mathbb{R}_+$, we define a stochastic sequence $W^x$ on $(\Omega, \mathscr{A}, \mathbb{P})$ by:

$$W^x(\omega) = \left(W_n^x(\omega)\,;\, n \in \mathbb{N}\right) = (W_n(x, \omega)\,;\, n \in \mathbb{N})$$

$$(1)\quad \begin{cases} W_0(x, \omega) = x \\ W_{n+1}(x, \omega) = [(W_n(x, \omega) + B_{n+1}(\omega)) \wedge (W_n(x, \omega) \vee C_{n+1}(\omega)) - A_{n+1}(\omega)]_+ \\ \qquad = (W_n(x, \omega) + B_{n+1}(\omega) - A_{n+1}(\omega))_+ \\ \qquad\qquad \wedge [(W_n(x, \omega) - A_{n+1}(\omega))_+ \vee (C_{n+1}(\omega) - A_{n+1}(\omega))_+] \end{cases}$$

where $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$, $(a)_+ = a \vee 0$.

LEMMA 1.1. — Define $f : \mathbb{R}_+ \times \mathbb{R}_+ \times \bar{\mathbb{R}}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f(a, b, c, x) = ((x+b) \wedge (x \vee c) - a)_+ = (x+b-a)_+ \wedge [(x-a)_+ \vee (c-a)_+]$$

*i)* $f$ is increasing in $b$, $c$, $x$ and decreasing in $a$.
*ii)* $\forall(a, b, c, x)$, $\forall(a', b', c', x')$

$$|f(a, b, c, x) - f(a', b', c', x')| \leqslant \max(|x-x'|, |a-a'|, |b-b'|, |c-c'|)$$

*Proof.* — *i)* Is trivial and
*ii)* By the mean value theorem [*1*].

PROPOSITION 1.2. — $W^x$ is a homogeneous Markov chain, with respect to $(\mathscr{A}_n, n \geqslant 0)$, on $(\Omega, \mathscr{A}, \mathbb{P})$, starting from $x$, with state space $(\mathbb{R}_+, \mathscr{B}_{\mathbb{R}_+})$. The transition function is given by: for all bounded measurable function $g$ from $\mathbb{R}_+$ to $\mathbb{R}$

$$(Pg)(x) = \int_{\mathbb{R}_+} g(y) P(x, dy) = \int_{\mathbb{R}_+ \times \mathbb{R}_+ \times \bar{\mathbb{R}}_+} \alpha(da)\beta(db)\gamma(dc) g(f(a, b, c, x))$$

For results and notations on Markov chain see [*7*].

The canonical space of the chain is $\left(\mathbb{R}_+^{\mathbb{N}}, \mathscr{B}_{\mathbb{R}_+}^{\otimes \mathbb{N}}, \mathbb{P}_x\right)_{x \in \mathbb{R}_+}$ where $\mathbb{P}_x$ is the distribution of $W^x$. We write again $(W_n, n \geqslant 0)$ the coordinate of this space and, if $\mathscr{F}_n = \sigma\{W_0, W_1, W_2, \ldots, W_n\}$, $W = (W_n, n \geqslant 0)$ is a homogeneous Markov chain with respect to $(\mathscr{F}_n, n \geqslant 0)$ such that for all $x \in \mathbb{R}_+$, $\mathbb{P}_x(W_0 = x) = 1$, the transition function being P. We do not distinguish $W_n$ from $W_n^x$.

## 2. INEQUALITIES, MINIMAL POINT

From (1) we have, for $n \geqslant 0$:

$$W_{n+1} \leqslant (W_n + B_{n+1} - A_{n+1})_+ 1_{\{C_{n+1} = +\infty\}} + (W_n \vee C_{n+1} - A_{n+1})_+ 1_{\{C_{n+1} < +\infty\}}$$
$$W_{n+1} \geqslant (W_n + B_{n+1} 1_{\{C_{n+1} = +\infty\}} - A_{n+1})_+$$

Let:

$$\begin{cases} \bar{W}_0 = W_0 \\ \bar{W}_{n+1} = (\bar{W}_n + B_{n+1} - A_{n+1})_+ 1_{\{C_{n+1} = +\infty\}} + (\bar{W}_n \vee C_{n+1} - A_{n+1})_+ 1_{\{C_{n+1} < +\infty\}} \end{cases}$$

$$\begin{cases} \underline{W}_0 = W_0 \\ \underline{W}_{n+1} = (\underline{W}_n + B_{n+1} 1_{\{C_{n+1} = +\infty\}} - A_{n+1})_+ \end{cases}$$

$(\bar{W}_n ; n \geqslant 0)$ and $(\underline{W}_n ; n \geqslant 0)$ are homogeneous Markov chains with respect to $(\mathscr{A}_n, n \geqslant 0)$, starting from the same point as W. We have for all $n \in \mathbb{N}$, $\underline{W}_n \leqslant W_n \leqslant \bar{W}_n$.

Let:

$$C'_n = C_n 1_{\{C_n < +\infty\}} \qquad (0 \times +\infty = 0 \text{ by convention})$$
$$B'_n = B_n 1_{\{C_n = +\infty\}}$$

Then:

$$\bar{W}_{n+1} = (\bar{W}_n \vee C'_{n+1} + B'_{n+1} - A_{n+1})_+$$

and it is easy to see that:

$$\bar{W}_n = \left( \sum_{i=1}^{i=n} (B'_i - A_i) + \sum_{i=1}^{i=n-1} (C'_{i+1} - \bar{W}_i)_+ + x \vee C'_1 \right)_+ \qquad \mathbb{P}_x \text{ a. e.}$$

We assume that $\gamma(\{+\infty\}) < 1$, $\alpha(\{0\}) < 1$ and that A and B are not $\mathbb{P}$ a. e. the same constant.

Let:

$$a = \text{ess.sup}\, A_1 = \inf \{ t \geqslant 0, \alpha([0, t]) = 1 \}$$
$$b = \text{ess.inf}\, B_1 = \sup \{ t \geqslant 0, \beta([t, +\infty[) = 1 \}$$
$$c = \text{ess.inf}\, C_1$$

LEMMA 2.1. — If $\min(b - a ; c - a) \geqslant 0$, the set $[c - a, +\infty[$ is an absorbing set (i. e. $\forall x \geqslant c - a$, $P(x, [c - a, +\infty[) = 1$) and there is a stopping time $\mathscr{T}$ such as:

i) $\forall k \geqslant 1, \quad \mathbb{E}(\mathscr{T}^k) < +\infty$

*ii)* $\forall x \in \mathbb{R}_+, \quad \forall n \in \mathbb{N}, \quad W_{\mathscr{T}+n} \geqslant c - a \quad \mathbb{P}_x$ a. e.

*Proof.* — Suppose that $c - a > 0$, otherwise there is nothing to prove.
Let $x \geqslant c - a$.
For all $\omega \in \Omega$

$W_1(x, \omega) \geqslant W_1(c - a, \omega) \geqslant (c - a + b) \wedge ((c - a) \vee (c - a) = c - a.$

Let

$$\mathcal{T} = \inf\left(n : \sum_{i=1}^{i=n} (B_i - A_i) \geqslant c - a\right).$$

As $B_i - A_i \geqslant 0$, $\mathcal{T}$ satisfies $\mathbb{E}(\mathcal{T}^k) < +\infty$, $\forall k \geqslant 1$ and it is clear that:

$$W_1(x, \omega) \geqslant W_1(0, \omega) \geqslant B_1 - A_1$$

$$W_{\mathcal{T}-1}(x, \omega) \geqslant W_{\mathcal{T}-1}(0, \omega) \geqslant \sum_{i=1}^{\mathcal{T}-1} (B_i - A_i)$$

$$W_{\mathcal{T}}(x, \omega) \geqslant \left(\sum_{1}^{\mathcal{T}-1} (B_i - A_i) + B_{\mathcal{T}} - A_{\mathcal{T}}\right) \wedge \left(\left(\sum_{1}^{\mathcal{T}-1} (B_i - A_i) - A_{\mathcal{T}}\right) \vee (C_{\mathcal{T}} - A_{\mathcal{T}}) \geqslant c - a\,.\right.$$

## 3. IRREDUCIBILITY

LEMMA 3.1. — *i*) If $\min(b - a, c - a) < 0$, then:

$$\forall x \in \mathbb{R}_+, \qquad \mathbb{P}_x\left(\bigcup_{n \geqslant 0} \{ W_n = 0 \}\right) > 0$$

*ii*) If $\min(b - a, c - a) \geqslant 0$, then:

$$\forall x \in \mathbb{R}_+, \qquad \forall \varepsilon > 0, \qquad \mathbb{P}_x\left(\bigcup_{n \geqslant 0} \{ c - a \leqslant W_n \leqslant c - a + \varepsilon \}\right) > 0\,.$$

*Proof.* — 1) Let $b - a < 0$.
It is clear that $W_{n+1} \leqslant (W_n + B_{n+1} - A_{n+1})_+$.
If we define $(Z_n, n \geqslant 0)$ by:

$$\begin{cases} Z_0 = W_0 \\ Z_{n+1} = (Z_n + B_{n+1} - A_{n+1})_+ \end{cases}$$

then for all $n \in \mathbb{N}$, $Z_n \geqslant W_n$ and the condition, as for the G1/G/1 queue, implies that $\mathbb{P}_x\left(\bigcup_{n \geqslant 0} \{ Z_n = 0 \}\right) > 0$.

2) Let $c - a < 0$. For all $n$ we have:

$$W_{n+1} \leqslant (W_n - A_{n+1})_+ \vee (C_{n+1} - A_{n+1})_+$$

If we define $(Z_n, n \geqslant 0)$ by:

$$\begin{cases} Z_0 = W_0 \\ Z_{n+1} = (Z_n - A_{n+1})_+ \vee (C_{n+1} - A_{n+1})_+ \end{cases}$$

then $\forall n \in \mathbb{N}$, $Z_n \geqslant W_n$.

If $\alpha > 0$, so that $\mathbb{P}(A_1 \geqslant \alpha) > 0$, we define $n = \left[\frac{x}{\alpha}\right] + 1$ for all $x \in \mathbb{R}_+$ and we have:

$$\mathbb{P}_x\left(\bigcup_{n=1}^{x} \{ W_n = 0 \}\right) \geqslant \mathbb{P}_x\left(\bigcup_{n=1}^{x} \{ Z_n = 0 \}\right) \geqslant \mathbb{P}\left(\bigcap_{i=1}^{i=n} \{ C_i - A_i \leqslant 0 ; A_i \geqslant \alpha \}\right)$$

$$\mathbb{P}_x\left(\bigcup_{n=1}^{x} \{ W_n = 0 \}\right) \geqslant [\mathbb{P}(C_1 - A_1 \leqslant 0 ; A_1 \geqslant \alpha)]^n > 0 .$$

3) Let $b - a \geqslant 0$, $c - a \geqslant 0$ and $\varepsilon > 0$ so that $2\varepsilon \leqslant a$. If $x \geqslant c - a$ and $n = \left[\frac{(x - c - \varepsilon)_+}{a - \varepsilon}\right] + 1$ then

$$F = \bigcap_{i=1}^{n+2} \{ C_i - A_i \leqslant c - a + \varepsilon \} \subset \{ \bar{W}_{n+2} \leqslant c - a + \varepsilon \} \subset \{ W_{n+2} \leqslant c - a + \varepsilon \}$$

and

$$\mathbb{P}_x(\{ W_{n+2} \leqslant c - a + \varepsilon \}) \geqslant \mathbb{P}(F) = (\mathbb{P}(C_1 - A_1 \leqslant c - a + \varepsilon))^{n+2} > 0 .$$

LEMMA 3.2. — Assume that $\min(b - a, c - a) \geqslant 0$.
Let $\varepsilon > 0$ be such that $2\varepsilon \leqslant a$, and

$$D_n = \{ B_n \geqslant a + \varepsilon ; C_n - A_n \leqslant c - a + \varepsilon \} .$$

Define S by:

$$\{ S = 0 \} = D_1^c , \qquad \{ S = n \} = D_1 \ldots D_n D_{n+1}^c .$$

Then for every $x$, verifying $c - a \leqslant x \leqslant c - a + \varepsilon$, we have:

*i)* $\{ S \geqslant n \} \subset \{ W_1^x \leqslant c - a + \varepsilon, \ldots, W_n^x \leqslant c - a + \varepsilon \}$
$= \{ W_1^x = C_1 - A_1 \leqslant c - a + \varepsilon, \ldots, W_n^x = C_n - A_n \leqslant c - a + \varepsilon \}$

for all $n \geqslant 1$.

*ii)* the map $v$ from $\mathscr{B}_{\mathbb{R}_+}$ to $\mathbb{R}_+$, defined by

$$v(E) = \mathbb{E}\left(\sum_{p=1}^{p=S} 1_E(W_p^x)\right)$$

is a measure independent of $x$, with $0 < v(\mathbb{R}_+) < +\infty$.

*Proof.* — If

$$\omega \in \{ B_1 \geqslant a + \varepsilon ; C_1 - A_1 \leqslant c - a + \varepsilon \}$$

and if

$$c - a \leqslant x \leqslant c - a + \varepsilon$$

then

$$x + B_1(\omega) - A_1(\omega) \geqslant c - a + \varepsilon$$
$$x - A_1(\omega) \leqslant c - a - (a - 2\varepsilon) \leqslant c - a$$

Therefore

$$W_1(x, \omega) = C_1(\omega) - A_1(\omega) \leqslant c - a + \varepsilon$$

and so the inclusion in the statement of the lemma is proved. S is an integer valued random variable geometrically distribued, so $\mathbb{E}(S) < +\infty$ and the remainder of the proof is easy.

PROPOSITION 3.3. — The chain $W = (W_n, n \geqslant 0)$ is irreducible and aperiodic.

*Proof.* — (For the definition of irreducibility, see [7], p. 71.)

1) If $\min(b - a, c - a) < 0$, the proposition is trivial from lemma 3.1.

2) If $\min(b - a, c - a) \geqslant 0$, let $\nu$ the measure defined in lemma 3.2. It is clear from lemma 3.1, the definition of $\nu$ and the strong Markov property that if $\nu(E) > 0$ for all $x \in \mathbb{R}_+$ there exists $n \in \mathbb{N}$ such that $\mathbb{P}_x(W_n \in E) > 0$.

Indeed the measures $\mu_n$ defined for all $n \in \mathbb{N}^*$ and all $E \in \mathscr{B}_{\mathbb{R}_+}$ by $\mu_n(E) = \mathbb{P}_x(W_n \in E ; S \geqslant n)$ are all equivalent to $\nu$ and $\mu_n(E) = \delta^{n-1}\mu_1(E)$ where $\delta = \mathbb{P}(C_1 - A_1 \leqslant c - a + \varepsilon)$.

For all $x \in [c - a, c - a + \varepsilon]$, $\mathbb{P}_x(W_n \in E) \geqslant \delta^{n-1}\mu_1(E)$ and then, there exists a non empty borel set C, $C \subset [c - a, c - a + \varepsilon]$ so that $\forall n \in \mathbb{N}^*$, $\forall x \in C$, $\forall y \in C$, $\dfrac{dP_n}{\delta\mu_1}(x, y) \geqslant \delta^{n-1}$. The aperiodicity follows from the definition ([7], p. 159).

## 4. TRANSIENCE - RECURRENCE

PROPOSITION 4.1. — 1) If $\mathbb{P}(C_1 = +\infty)\mathbb{E}(B_1) > \mathbb{E}(A_1)$, then $\lim_n W_n = +\infty$ $\mathbb{P}_x$ a. e. for all $x$ and the chain W is transient.

2) If $\mathbb{P}(C_1 = +\infty)\mathbb{E}(B_1) = \mathbb{E}(A_1)$, then

$$\forall t \in \mathbb{R}, \quad \forall x \in \mathbb{R}_+, \qquad \lim_n \mathbb{P}_x(W_n \leqslant t) = 0 .$$

*Proof.* — From $W_n \geqslant \underline{W}_n$ and the classical results on G1/G/1 queues.

LEMMA 4.2. — Let:

$$\begin{cases} V=\{0\} & \text{if} \quad \min(b-a, c-a)<0 \\ V=[c-a, c-a+\varepsilon] \quad \text{with } \varepsilon>0 \text{ such that } \varepsilon<2a & \text{if} \quad \min(b-a, c-a)\geqslant 0 \end{cases}$$

$T=\inf(n\geqslant 1, W_n\in V)$.

If $\forall x\in\mathbb{R}_+$ $\mathbb{P}_x(T<+\infty)=1$, W is recurrent in the Harris sense.

*Proof.* — 1) If $\min(b-a, c-a)<0$, the result is trivial from the definition of recurrence.

2) If $\min(b-a, c-a)\geqslant 0$, note that $T=T_1$ is a $(\mathscr{F}_n; n\geqslant 0)$ (and therefore an $(\mathscr{A}_n; n\geqslant 0)$) stopping time.

Let $S_1$ be defined by:

$$\{S_1=0\}=\{B_{T_1+1}<a+\varepsilon\}\cup\{C_{T_1+1}-A_{T_1+1}>c-a+\varepsilon\}$$

$$\forall n\geqslant 1\ \{S_1=n\}=\bigcap_{i=1}^{i=n}\{B_{T_1+i}\geqslant a+\varepsilon; C_{T_1+i}-A_{T_1+i}\leqslant c-a+\varepsilon\}$$
$$\cap\,\complement\{B_{T_1+n+1}\geqslant a+\varepsilon; C_{T_1+n+1}-A_{T_1+n+1}\leqslant c-a+\varepsilon\}$$

$S_1$ is a geometrically distribued random variable independent of $\mathscr{A}_{T_1}$. Define $(T_2, S_2)\ldots(T_n, S_n)$ as follow:

$$\forall n\geqslant 1 \quad T_{n+1}=T_1\circ\Theta_{T_n+S_n+1}+T_n+S_n+1=\inf(k\geqslant T_n+S_n+1; W_k\leqslant c-a+\varepsilon)$$
$$S_{n+1}=S_1\circ\Theta_{T_n+S_n+1} \qquad \text{i. e.}$$

$$\{S_{n+1}=0\}=\{B_{T_{n+1}+1}\geqslant a+\varepsilon; C_{T_{n+1}+1}-A_{T_{n+1}+1}\leqslant c-a+\varepsilon\}$$

$$\{S_{n+1}=k\}=\bigcap_{i=1}^{i=k}\{B_{T_{n+1}+i}\geqslant a+\varepsilon; C_{T_{n+1}+i}-A_{T_{n+1}+i}\leqslant c-a+\varepsilon\}$$
$$\cap\,\complement\{B_{T_{n+1}+k+1}\geqslant a+\varepsilon; C_{T_{n+1}+k+1}-A_{T_{n+1}+k+1}\leqslant c-a+\varepsilon\}$$

For all $n\in\mathbb{N}^*$, $S_n$ is independent of $\mathscr{A}_{T_n}$ and the sequence $(X_n, n\geqslant 1)$

$$\begin{cases} X_n=0 & \text{if} \quad S_n=0 \\ X_n=(S_n; W_{T_n+1}, \ldots, W_{T_n+S_n}) \end{cases}$$

is, from lemma 3.2, a sequence of i. i. d. r. v.

$(X_n, n\geqslant 1)$ is well defined since for all $x\in\mathbb{R}_+$ and all $n\in\mathbb{N}^*$, $\mathbb{P}_x(T_n<+\infty)=1$, and $S_n$ is geometrically distributed. Then the measure $\nu$ of lemma 3.2 verifies, for all $x\in\mathbb{R}_+$ all $n\in\mathbb{N}^*$ and all $E\in\mathscr{B}_{\mathbb{R}_+}$

$$\nu(E)=\mathbb{E}_x\left(\sum_{p=1}^{p=S_n}1_{\mathbb{E}}(W_{T_n+p})\right)$$

Then if $\nu(E) > 0$

$$\sum_{k=1}^{\infty} 1_E(W_k) \geqslant \sum_{n=1}^{\infty} \left( \sum_{p=1}^{p=S_n} 1_E(W_{T_n+p}) \right) = +\infty$$

since the right term is a sum of non negative i. i. d. r. v. Therefore W is a Harris chain.

For proof of recurrence, we can show that, for all $x \in \mathbb{R}_+$ :

$$\liminf_n W_n^x = 0 \quad \mathbb{P} \text{ a. e.} \quad \text{if} \quad \min(b-a, c-a) < 0$$

$$\liminf_n W_n^x = c - a \quad \mathbb{P} \text{ a. e.} \quad \text{if} \quad \min(b-a, c-a) \geqslant 0$$

Indeed if $\min(b-a, c-a) < 0$, we see easily that the condition $\liminf_n W_n(x, \omega) = 0$ implies that there exists an infinity of $n$ such that $W_n(x, \omega) = 0$. If $\min(b-a, c-a) \geqslant 0$, the condition $\liminf_n W_n(x, \omega) = c - a$ is exactly the recurrence in the set $[c-a, c-a+\varepsilon]$ for all $\varepsilon > 0$.

LEMMA 4.3. — $\liminf_n W_n^x$ is $\mathbb{P}$ a. e. a constant independent of $x$.

*Proof.* — It is easy to see that, if $w(x, \omega) = \liminf_n W_n^x(x, \omega)$, we have:

$$w(x, \omega) = (w(x, \omega) + b - a)_+ \wedge (w(x, \omega) \vee c - a)_+$$

and then, if $w(x, \omega) < +\infty$

$$\min(b-a, c-a) < 0 \quad \text{implies} \quad w(x, \omega) = 0$$
$$\min(b-a, c-a) \geqslant 0 \quad \text{implies} \quad w(x, \omega) = c - a$$

i. e. $w(x, \omega)$ takes only the values $+\infty$ or $(c-a)_+$.

As $x \rightsquigarrow w(x, \omega)$ is continuous (Lemma 1.1) $w(x, \omega)$ is independent of $x$.

On the otherhand: $w(x, \omega) = w(W_n(x, \omega); \Theta_n(\omega))$ and $w(x, \omega)$ is $\mathbb{P}$ a. e. constant from (0, 1) law.

THEOREM 4.4. — If $\mathbb{P}(C_1 = +\infty)\mathbb{E}(B_1) < \mathbb{E}(A_1)$ W is recurrent in the sense of Harris and aperiodic.

If $\mathbb{P}(C_1 = +\infty)\mathbb{E}(B_1) = \mathbb{E}(A_1)$ and if $C_1 1_{\{C_1 < +\infty\}}$ is a bounded random variable, then $W_n$ is recurrent in the sense of Harris and aperiodic.

*Proof.* — From lemma 4.3 it is sufficient to prove

$$\mathbb{P}(\liminf_n W_n^x < +\infty) > 0.$$

On the otherhand we have seen in & 2 that $\mathbb{P}_x$ a. e.

$$\bar{W}_n = \left( \sum_{i=1}^{i=n} (B'_i - A_i) + \sum_{i=1}^{i=n} (C'_{i+1} - \bar{W}_i)_+ + x \vee C'_1 \right)_+$$

If $C'_n$ is a bounded random variable and if $\liminf_n \bar{W}_n = +\infty$ then

$$\sum_{i=1}^{\infty} (C'_{i+1} - \bar{W}_i)_+ < +\infty \qquad \mathbb{P} \text{ a. e.}$$

and therefore $\mathbb{E}(B'_i - A_i) > 0$.

If $C'_n$ is not bounded, we replace $C_i$ by $C_i^k$:

$$\begin{cases} C_i^k = +\infty & \text{if} \quad C_i \geqslant k \\ C_i^k = C_i & \text{if} \quad C_i < k \end{cases}$$

Since $C_i^k \geqslant C_i$ for all $i$, the chain $(W_{k,n}\,;\, n \geqslant 0)$ is greater than the chain $(W_n\,; n \geqslant 0)$ where $(W_{k,n}\,; n \geqslant 0)$ is the chain constructed from $(A_n, B_n, C_n^k)_{n \in \mathbb{N}^*}$ just as the chain $(W_n, n \geqslant 0)$ is constructed from $(A_n, B_n, C_n)_{n \in \mathbb{N}^*}$: $\forall n \in \mathbb{N}$, $W_{k,n} \geqslant W_n$ and so, $\liminf_n W_n^x < +\infty$ $\mathbb{P}$ a. e. if there exists $k$ such that $\mathbb{P}(C_i \geqslant k)\mathbb{E}(B_i) \leqslant \mathbb{E}(A_i)$ and therefore if $\mathbb{P}(C_i = +\infty)\mathbb{E}(B_i) < \mathbb{E}(A_i)$.

## 5. POSITIVE RECURRENCE

LEMMA 5.1. — 1) $F_n(t) = \mathbb{P}(W_n^0 \leqslant t)$ is a decreasing sequence of distribution functions.

2) If W is recurrent then

$$F(t) = \lim_n F_n(t) = \lim_n \mathbb{P}(W_n^x \leqslant t) \qquad \text{for all} \quad x \in \mathbb{R}_+$$

and all continuity point $t$ of F.

*Proof.* — 1) Easy by the Markov property and

$$\mathbb{P}(W_n^x \leqslant t) \leqslant \mathbb{P}(W_n^0 \leqslant t)$$

2) The same as in 2, Theorem 2.8.

THEOREM 5.2. — 1) If $\mathbb{P}(C_1 = +\infty)\mathbb{E}(B_1) < \mathbb{E}(A_1)$, $W = (W_n, n \geqslant 0)$ is positive recurrent,

$$\forall E \in \mathscr{B}_{\mathbb{R}_+} \quad \forall x \in \mathbb{R} \qquad \lim_n \mathbb{P}_x(W_n \in E) = \mu(E),$$

$\mu$ being the unique invariant probability measure of the chain.

2) If $\mathbb{P}(C_1 = +\infty)\mathbb{E}(B_1) = \mathbb{E}(A_1)$ and if the chain is recurrent, the recurrence cannot be positive.

*Proof.* — 1) Follows from Proposition 4.1.

2) We must check that F is a probability distribution function. Let $k$ be such that $\mathbb{P}(C_1 \geqslant k)\mathbb{E}(B_1) < \mathbb{E}(A_1)$.

Then, as in theorem 4.4:

$$\begin{aligned} W_{n+1} &\leqslant (W_n + B_{n+1} - A_{n+1})_+ 1_{\{C_{n+1} \geqslant k\}} + (C_{n+1} W_n - A_{n+1})_+ 1_{\{C_{n+1} < k\}} \\ &\leqslant (W_n + B_{n+1} - A_{n+1})_+ 1_{\{C_{n+1} \geqslant k\}} + (k \vee W_n - A_{n+1})_+ 1_{\{C_{n+1} < k\}} \\ &\leqslant (W_n \vee k + B_{n+1} 1_{\{C_{n+1} \geqslant k\}} - A_{n+1})_+ \end{aligned}$$

Let:

$$\begin{cases} Z_0 = W_0 \vee k \\ Z_{n+1} = (Z_n + B_{n+1} 1_{\{C_{n+1} \geqslant k\}} - A_{n+1}) \vee k \end{cases}$$

As in the G1/G/1 model, it is clear that

$$\lim_n \mathbb{P}(Z_n \leqslant t) = \mathbb{P}\left[\sup_{n \geqslant 0}\left(k + \left(\sum_{i=1}^{i=n} (B_i 1_{\{C_i \geqslant k\}} - A_i)\right)\right) \leqslant t\right]$$

As

$$\sup_{n \geqslant 0}\left(k + \sum_{i=1}^{i=n} (B_i 1_{\{C_i \geqslant k\}} - A_i)\right)$$

is a $\mathbb{P}$ a. e. finite r. v. and as $\mathbb{P}(W_n^0 \leqslant t) \geqslant \mathbb{P}(Z_n \leqslant t)$, F is a probability distribution function.

Let be the p. m. whose the distribution function is F. If $f$ is a bounded continuous function from $\mathbb{R}_+$ to $\mathbb{R}$, $x \rightsquigarrow \mathbb{E}_x(f(W_1))$ is also a bounded continuous function (i. e. P is a Feller kernel).

Then, by the Markov property, the weak convergence of $W_n$ to $\mu$ and the Lebesgue convergence theorem, for all $x \in \mathbb{R}_+$ we have:

$$\begin{aligned} \int_{\mathbb{R}_+} f(x)\mu(dx) &= \lim_n \mathbb{E}_x(f(W_{n+1})) = \lim_n \mathbb{E}_x(\mathbb{E}_{W_n}(f(W_1)) \\ &= \int_{\mathbb{R}_+} \mathbb{E}_x(f(W_1))\mu(dx) = \int_{\mathbb{R}_+} (Pf)(x)\mu(dx) \end{aligned}$$

and therefore $\mu P = \mu$ i. e. $\mu$ is the unique invariant measure.

The end of the theorem is a consequence of Orey's theorem ([7], p. 169).

## 6. RENEWAL PHENOMENON

As in [3] or [5], the chain has a curious renewal phenomenon. This phenomenon allows us to find various kinds of limit theorems as in [6], [3] and [5].

Let $(T_n, S_n ; n \geqslant 1)$ be as in the proof of lemma 4.2, and

$$K_1 = \inf (n \geqslant 1, S_n \geqslant 1) \ldots K_n = \inf (n > K_{n-1}, S_n \geqslant 1)$$

and $L_1 = T_{K_1} + 1 \ldots L_n = T_{K_n} + 1$. It is easy to prove the following [5].

THEOREM 6.1. — The random variables

$$X_1 = (L_1, W_1, \ldots, W_{L_1})$$
$$\vdots$$
$$X_n = (L_n - L_{n-1}, W_{L_{n-1}+1}, \ldots, W_{L_n})$$

is a stationary sequence of two-dependent random variables.

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