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Ergodic theory for inner functions of the upper half plane

by

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ABSTRACT. — The real restriction of an inner function of the upper half plane leaves Lebesgue measure quasi-invariant. It may have a finite or infinite invariant measure. We give conditions for the rational ergodicity and exactness of such restrictions.

ABSTRAIT. — La restriction à la droite réelle d'une fonction intérieure du demi-plan supérieur laisse la mesure de Lebesgue quasi-invariante, et peut avoir une mesure invariante finie ou infinie. Nous donnons les conditions pour l'ergodicité rationnelle et l'exactitude de telles transformations.

§ 0. INTRODUCTION

In this paper, we consider the ergodic properties of the real restrictions of inner functions on the open upper half plane:

$$\mathbb{R}^{2+} = \{ x + iy : x, y \in \mathbb{R}, y > 0 \} .$$

Let $f : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ be an analytic function. We say that f is an *inner function on* $\mathbb{R}^{2+}$ if for λ-a. e. $x \in \mathbb{R}$ the limit $\lim_{y \downarrow 0} f(x + iy)$ exists, and is real. (Here, and throughout the paper, λ denotes Lebesgue measure on $\mathbb{R}$). Consider the limit $\lim_{y \downarrow 0} f(x + iy) = \mathrm{T}x$. This is defined λ-a. e. on $\mathbb{R}$. We call this limit the (real) *restriction of* f, and will sometimes write this as $\mathrm{T} = \mathrm{T}(f)$.

We will denote the class of inner functions on $\mathbb{R}^{2+}$ by $I(\mathbb{R}^{2+}) = I$, and their real restrictions by $M(\mathbb{R})$. We note that $f \in I(\mathbb{R}^{2+})$ iff $\emptyset^{-1} f \emptyset(z)$ is an inner function of the unit disc, according to the definition on p. 370 of [9] $\left(\text{where } \emptyset(z) = i\left(\frac{1+z}{1-z}\right)\right)$.

The following characterisation of $I(\mathbb{R}^{2+})$ appears in [6] and [17].

$f \in I(\mathbb{R}^{2+})$ iff

$$f(\omega) = \alpha\omega + \beta + \int_{-\infty}^{\infty} \frac{1 + t\omega}{t - \omega} d\mu(t) \tag{0.1}$$

where $\alpha \geq 0$, $\beta \in \mathbb{R}$ and μ is a bounded, positive Borel measure, singular w. r. t. λ. Since we shall be refering to (0.1) rather a lot, we shall denote the class of bounded, positive, singular measures on $\mathbb{R}$ by $S(\mathbb{R})$.

G. Letac ([6]) has shown that a measurable transformation T of $\mathbb{R}$ preserves the class of Cauchy distributions iff either $T \in M(\mathbb{R})$ or $-T \in M(\mathbb{R})$. In particular, if $dP_{a+ib}(x) = \frac{b}{\pi} \frac{dx}{(x-a)^2 + b^2}$ for $a + ib \in \mathbb{R}^{2+}$ and $T = T(f) \in M(\mathbb{R})$, then:

$$P_\omega \circ T^{-1} = P_{f(\omega)} \quad \text{for} \quad \omega \in \mathbb{R}^{2+} \tag{0.2}$$

This equation shows that $M(\mathbb{R})$ is a class of non-singular transformations of the measure space $(\mathbb{R}, \mathbb{B}, \lambda)$, and is therefore an object of ergodic theory.

Let $f \in I(\mathbb{R}^{2+})$ have a fixed point $\omega_0 \in \mathbb{R}^{2+}$. By (0.2), $T(f)$ preserves the Cauchy distribution P_{ω_0}. It was shown in [16], that if f is $1-1$, then $T(f)$ is conjugate to a rotation of the circle, and shown in [15] that otherwise, $T(f)$ is mixing. We show in § 1 that if f is not $1-1$ then $T(f)$ is exact.

In § 2 we recall some well known facts about inner functions of $\mathbb{R}^{2+}$. The Denjoy-Wolff theorem (see [13], [14] and [18]) adapted to $\mathbb{R}^{2+}$ shows that when studying the ergodic properties of $T(f)$, for $f \in I(\mathbb{R}^{2+})$ with no fixed points in $\mathbb{R}^{2+}$, we may assume that $\alpha(f) \geq 1$. In case $\alpha(f) > 1$, $T(f)$ is dissipative, and when $\alpha(f) = 1$, $T(f)$ preserves Lebesgue measure.

In § 3, we consider the case $\alpha(f) = 1$. Here, the conservativity of a restriction $T(f)$ is sufficient for its rational ergodicity ([1]) (ergodicity was established in [15]). We also give sufficient conditions for exactness, and discuss the similarity classes ([1]) of restrictions.

The ergodic theory of certain restrictions has been considered in [2], [5], [7], [10], [11], [15] and [16].

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§ 1. MIXING RESTRICTIONS PRESERVING FINITE MEASURES

The purpose of this section is to prove

THEOREM 1.1. — Let $f \in \mathrm{I}(\mathbb{R}^{2+})$ and assume that f is not $1-1$. If f has a fixed point $\omega_0 \in \mathbb{R}^{2+}$, then $(\mathrm{R}, \mathscr{B}, \mathrm{P}_{\omega_0}, \mathrm{T}(f))$ is an exact measure preserving transformation.

$$\text{i.e.} \quad \bigcap_{n \geq 1} \mathrm{T}^{-n}\mathscr{B} = \{ \phi, \mathbb{R} \} \bmod \lambda .$$

Before proving theorem 1.1, we shall need some auxiliary results. The first of these is Lin's criterion for exactness of Markov operators (theorem 4.4 in [8]) as applied to our case. To state this, we shall need some extra notation:

Let $\mathrm{T} \in \mathrm{M}(\mathbb{R})$, then $(\mathbb{R}, \mathbb{B}, \lambda, \mathrm{T})$ is a non-singular transformation, and so $g \in \mathrm{L}^\infty(\mathbb{R}, \mathbb{B}, \lambda)$ iff $g \circ \mathrm{T} \in \mathrm{L}^\infty(\mathbb{R}, \mathbb{B}, \lambda)$. We define the dual operator of T, $\hat{\mathrm{T}} : \mathrm{L}^1(\mathbb{R}, \mathbb{B}, \lambda) \to \mathrm{L}^1(\mathbb{R}, \mathbb{B}, \lambda)$ by

$$\int_{\mathrm{R}} \hat{\mathrm{T}}h . g d\lambda = \int_{\mathrm{R}} h . g \circ \mathrm{T} d\lambda \qquad \text{for} \quad h \in \mathrm{L}^1 \quad \text{and} \quad g \in \mathrm{L}^\infty$$

If we write, for $\omega = a + ib \in \mathbb{R}^{2+}$

$$\frac{d\mathrm{P}_\omega}{d\lambda}(x) = \phi_\omega(x) = \frac{b}{\pi} \cdot \frac{1}{(x-a)^2 + b^2}$$

then equation (0.2) translates to:

$$(1.1) \qquad \hat{\mathrm{T}}\phi_\omega = \phi_{f(\omega)} \qquad \text{for} \quad \mathrm{T} = \mathrm{T}(f) \in \mathrm{M}(\mathbb{R})$$

Clearly, $\hat{\mathrm{T}}$ is a positive linear operator, $\int_{\mathrm{R}} \hat{\mathrm{T}} h d\lambda = \int_{\mathbb{R}} h d\lambda$ for $h \in \mathrm{L}^1$.

Lin's Criterion (for restrictions). — Let $\mathrm{T} = \mathrm{T}(f) \in \mathrm{M}(\mathbb{R})$.

T is exact iff

(1.2) $\| \hat{\mathrm{T}}^n u \|_1 \to 0$ for every $u \in \mathrm{L}^1$, $\int_{\mathbb{R}} u d\lambda = 0$. Here, and throughout, $\| u \|_1 = \int_{\mathbb{R}} | u | d\lambda$.

We shall also need the following (elementary) lemma.

LEMMA 1.2. — If $\omega_n \in \mathbb{R}^{2+}$ and $\omega_n \to \omega \in \mathbb{R}^{2+}$ then:

$$\| \phi_{\omega_n} - \phi_\omega \|_1 \to 0$$

Proof of theorem 1.1. — We first show that $f^n(\omega) \to \omega_0 \ \forall \omega \in \mathbb{R}^{2+}$, where $f^1(\omega) = f(\omega)$ and $f^{n+1}(\omega) = f(f^n(\omega))$.

Let ϕ: $U = [\,|Z| < 1] \to R^{2+}$ be a conformal map. Then $g = \phi^{-1} f \phi$: $U \to U$ is analytic, and $g(\phi(\omega_0)) = \phi(\omega_0)$. By the Schwartz lemma ([9]): $|g'(\phi(\omega_0))| < 1$ as g is not $1-1$. It is now not hard to see that

$$g^n(Z) \to \phi(\omega_0) \qquad \forall z \in U,$$

and hence that $f^n(\omega) \to \omega_0 \ \forall \omega \in \mathbb{R}^{2+}$.
Hence, by lemma 1.2

$$\| \hat{T}^n \phi_\omega - \phi_{\omega_0} \|_1 = \| \phi_{f^n(\omega)} - \phi_{\omega_0} \|_1 \to 0 \qquad \text{for} \quad \omega \in \mathbb{R}^{2+}.$$

We will now establish that

$$\| \hat{T}^n u \|_1 \to 0 \qquad \text{for} \quad u \in L^1$$

with $\int_{\mathbb{R}} u d\lambda = 0$ which, by Lin's criterion, will ensure the exactness of T.

Let $u \in L^1$ with $\int_{\mathbb{R}} u d\lambda = 0$ and let $\varepsilon > 0$. By Wiener's Tauberian theorem (see [12], p. 357), there exist $\alpha_1 \dots \alpha_N$, $a_1 \dots a_N \in \mathbb{Q}$ such that

$$\left\| u - \sum_{j=1}^{N} \alpha_j \phi_{a_j + i} \right\|_1 < \varepsilon/2$$

Clearly, this implies that $\left| \sum_{j=1}^{N} \alpha_j \right| < \varepsilon/2$ and so:

$$\| \hat{T}^n u \|_1 \leq \left\| \hat{T}^n \left(u - \sum_{j=1}^{N} \alpha_j \phi_{a_j+i} \right) \right\|_1$$

$$+ \left\| \hat{T}^n \left(\sum_{j=1}^{N} \alpha_j (\phi_{a_j+i} - \phi_{\omega_0}) \right) \right\|_1 + \left\| \sum_{j=1}^{N} \alpha_j \phi_{\omega_0} \right\|_1 \leq \left\| u - \sum_{j=1}^{N} \alpha_j \phi_{a_j+i} \right\|_1$$

$$+ \sum_{j=1}^{N} |\alpha_j| \, \| \hat{T}^n \phi_{a_j+i} - \phi_{\omega_0} \|_1 + \left| \sum_{j=1}^{N} \alpha_j \right| < \varepsilon + o(1) \quad \text{as } k \to \infty \qquad \square$$

Since $\varepsilon > 0$ was arbitrary: $\| \hat{T}^n u \|_1 \to 0$. $\square$

§ 2. BASIC CLASSIFICATION

PROPOSITION 2.1. [17]. — Let $f \in \mathrm{I}(\mathbb{R}^{2+})$. Then

$$\frac{f(ib)}{ib} \to \begin{cases} \alpha(f) = \alpha \in [0, \infty) \text{ as } b \to \infty \ (\alpha \text{ as in } 0.1) \\ \gamma(f) \in [\alpha, \infty] \text{ as } b \downarrow 0 . \end{cases}$$

Moreover

$$\alpha = \gamma \qquad \text{iff} \quad f(\omega) = \alpha\omega$$

Proof. — From the representation 0.1, we immediately calculate that:

$$(2.1) \quad \frac{f(ib)}{ib} = \alpha + \frac{\beta}{ib} + \frac{1 - b^2}{ib} \int_{-\infty}^{\infty} \frac{t d\mu(t)}{t^2 + b^2} + \int_{-\infty}^{\infty} \frac{1 + t^2}{t^2 + b^2} d\mu(t)$$

It follows from elementary integration theory that

$$\frac{f(ib)}{ib} \to \alpha = \alpha(f) \qquad \text{as} \quad b \to \infty .$$

To check the limit as $b \to 0$, we « flip » f to get:

$$\tilde{f}(\omega) = - 1/f(- 1/\omega)$$

Since $\tilde{f} \in \mathrm{I}(\mathbb{R}^{2+})$, we have that

$$\frac{\tilde{f}(ib)}{ib} \to \alpha(\tilde{f}) \in [0, \infty) \qquad \text{as} \quad b \to \infty$$

but this decodes to:

$$\frac{f(ib)}{ib} \to \gamma(f) = \frac{1}{\alpha(\tilde{f})} \in (0, \infty] \qquad \text{as} \quad b \downarrow 0 .$$

Now, if $\gamma(f) < \infty$ then, by 2.1:

$$\gamma(f) = \alpha + \int_{-\infty}^{\infty} \frac{1 + t^2}{t^2} d\mu(t)$$

Hence $\gamma(f) \geq \alpha(f)$ with equality iff $\mu \equiv 0$. □

PROPOSITION 2.2. — Let $f \in \mathrm{I}(\mathbb{R}^{2+})$ and $\mathrm{T} = \mathrm{T}(f)$.
If $\alpha(f) > 1$ then T is dissipative.

Proof. — Write $f^n(\omega) = u_n(\omega) + iv_n(\omega)$.

From the representation (0.1), we have:

$$v_{n+1}(\omega) = \alpha v_n(\omega) + v_n(\omega) \int_{-\infty}^{\infty} \frac{(1 + t^2)d\mu(t)}{(t - u_n)^2 + v_n^2} \geq \alpha v_n$$

Hence $v_n(i) \geq \alpha^n$ for $n \geq 1$, and

$$\hat{T}^n \phi_i(t) = \frac{v_n(i)}{\pi((t - u_n)^2 + v_n^2)} \leq \frac{1}{\pi \alpha^n}$$

Clearly

$$\sum_{n=1}^{\infty} \hat{T}^n \phi_i(t) \leq \frac{1}{(\alpha - 1)} \qquad \forall t \in \mathbb{R}$$

and so

$$\sum_{n=1}^{\infty} 1_A \circ T^n < \infty \qquad \text{a. e. } \forall A \in \mathbb{B}\,;\ \lambda(A) < \infty \qquad \square$$

PROPOSITION 2.3 (Letac [6]). — Let $f \in I(\mathbb{R}^{2+})$, $T = T(f)$. If $\alpha(f) = 1$ then $\lambda \circ T^{-1} = \lambda$.

Proof. — Let $f(ib) = u(b) + iv(b)$ we have:

$$\frac{u(b)}{b} \to 0 \quad \text{and} \quad \frac{v(b)}{b} \to 1 \quad \text{as} \quad b \to \infty .$$

Hence, for $A \in B$:

$$\pi b P_{ib}(A) \to \lambda(A)$$

and

$$\pi b P_{f(ib)}(A) \to \lambda(A) \qquad \text{as} \quad b \to \infty .$$

Since $P_{ib}(T^{-1}A) = P_{f(ib)}(A)$, we have that

$$\lambda(T^{-1}A) = \lambda(A) \qquad \text{for} \quad A \in \mathbb{B} \qquad \square$$

The next result is the Denjoy-Wolff theorem stated on $\mathbb{R}^{2+}$, which shows that if $f \in I(\mathbb{R}^{2+})$ has no fixed point in $\mathbb{R}^{2+}$, then $\exists \tilde{f} \in I(R^{2+})$ with $\alpha(\tilde{f}) \geqslant 1$, and such that $(\mathbb{R}, \mathcal{B}, \lambda, T(f))$ and $(R, B, \lambda, T(\tilde{f}))$ are conjugate, (and therefore have the same ergodic properties).

THEOREM 2.4. — Let $f \in I(\mathbb{R}^{2+})$ have no fixed points in $\mathbb{R}^{2+}$, and assume that $\alpha(f) < 1$; then

$$\exists\, !\ t \in R \qquad \text{such that} \qquad \alpha(\phi_t f \phi_t^{-1}) \geq 1$$

where

$$\phi_t(\omega) = \frac{1 + t\omega}{t - \omega}.$$

(Note that $\alpha(\phi_0^{-1} f \phi_0) = 1/\gamma(f)$).

Proof. — Let $\phi(z) = i \dfrac{1+Z}{1-Z}$. Then $g = \phi^{-1} f \phi$: $U \to U$ is analytic, and has no fixed points in U. The Denjoy-Wolff theorem on U (see [*13*] or [*14*]) shows that $\exists\,!\ \rho \in T$ such that

$$(*) \qquad \operatorname{Re}\left(\frac{\rho + g(Z)}{\rho - g(Z)}\right) \geq \operatorname{Re}\left(\frac{\rho + Z}{\rho - Z}\right) \qquad \forall Z \in U$$

Now let $t = \phi(\rho)$, $\psi = i \dfrac{\rho + Z}{\rho - Z}$ and $\tilde{f} = \psi g \psi^{-1} \in I(\mathbb{R}^{2+})$. It follows that $\phi\psi^{-1} = \phi_t^{-1}$ and hence that $\tilde{f} = \phi_t f \phi_t^{-1}$. Also, $(*)$ means that $\operatorname{Im} \psi g(Z) \geq \operatorname{Im} \psi(Z)$ for $Z \in U$, and hence $\operatorname{Im} \tilde{f}(\omega) \geq \operatorname{Im} \omega$ for $\omega \in \mathbb{R}^{2+}$, which implies $\alpha(\tilde{f}) \geq 1$. $\square$

If $\alpha(\phi_t f \phi_t^{-1}) > 1$ for some t, then by proposition 2.2, $T(f)$ is dissipative. If $\alpha(\phi_t f \phi_t^{-1}) = 1$, then, by proposition 2.3, $T(\phi_t f \phi_t^{-1}) = \phi_t T(f) \phi_t^{-1}$ preserves Lebesgue measure. Hence $T(f)$ preserves the measure ν_t, where $d\nu_t(x) = dx/(x - t)^2$. The rest of this section is devoted to odd restrictions.

(We say that a restriction T is *odd* if $T(-x) = -T(x)$).

LEMMA 2.5. — Let $f \in I(\mathbb{R}^{2+})$ and let $T = T(f)$. The following are equivalent:

i) T is odd *ii*) $\operatorname{Re} f(ib) = 0$ for $b > 0$

iii) $f(-\bar{\omega}) = -\overline{f(\omega)}$ for $\omega \in \mathbb{R}^{2+}$

iv) $$f(\omega) = \alpha\omega + \int_{-\infty}^{\infty} \frac{1 + t\omega}{t - \omega} d\mu(t)$$

where $\mu \in S(\mathbb{R})$ is symmetric

Proof. — The implications *iv*) $\Rightarrow$ *iii*) $\Rightarrow$ *i*) and *iii*) $\Rightarrow$ *ii*) are elementary. That *ii*) $\Rightarrow$ *iii*) is because of the Schwartz reflection principle (see [*9*]). The fact that for $t \geq 0$:

$$e^{itf(\omega)} = \int_{-\infty}^{\infty} e^{itT(x)} \phi_\omega(x) dx$$

gives the implication *i*) $\Rightarrow$ *iii*).

We show that *iii*) $\Rightarrow$ *iv*). Assume *iii*). It is evident that $\beta = 0$ in the representation 0.1, so we have

$$f(\omega) = \alpha\omega + \int_{-\infty}^{\infty} \frac{1 + t\omega}{t - \omega} d\mu(t) \qquad \text{where} \quad \alpha \geq 0 \quad \text{and} \quad \mu \in \mathrm{S}(\mathbb{R}).$$

We must show that μ is symetric. To see this, we first rewrite the equation $v(-a + ib) = v(a + ib)$ (implied by *iii*)) as:

$$(2.2) \qquad \int_{\infty}^{\infty} \phi_b(t - a)(1 + t^2)d\mu(t) = \int_{\infty}^{\infty} \phi_b(t + a)(1 + t^2)d\mu(t)$$

Next, we take $g(t)$ a continuous function of compact support and let $g_b(t) = \phi_{ib} * g$ for $b > 0$. It follows from (2.2) that

$$\int_{\infty}^{\infty} g_b(-t)(1 + t^2)d\mu(t) = \int_{\infty}^{\infty} g_b(t)(1 + t^2)d\mu(t).$$

The symmetry of μ is established by the (elementary) facts that

$$g_b(t) \to g(t) \qquad \text{as} \quad b \to 0$$

$$\sup_{\substack{t \in \mathbb{R} \\ b > 0}} (1 + t^2)\,|\,g_b(t)\,| < \infty \qquad \square$$

We denote the collection of those inner functions on $\mathbb{R}^{2+}$ satisfying the conditions of the above lemma by $\mathrm{I}_0(\mathbb{R}^{2+})$, and remark that $f \in \mathrm{I}_0(\mathbb{R}^{2+})$ iff $\emptyset^{-1} f \emptyset$ is an essentially real inner function of U. $\left(\text{Here } \emptyset(z) = i\left(\frac{1 + z}{1 - z}\right)\right)$.

Theorem 2.6. — Let $f \in \mathrm{I}_0(\mathbb{R}^{2+})$ and $\mathrm{T} = \mathrm{T}(f)$.
If $\alpha(f) < 1 < \gamma(f)$ then T preserves a Cauchy distribution. Moreover, if $\omega f(\omega)$ is not constant, then T is exact.

Proof. — If $f \in \mathrm{I}_0(\mathbb{R}^{2+})$ then it follows from the lemma

$$\gamma(f) = \alpha(f) + \int_{\infty}^{\infty} \frac{1 + t^2}{t^2} d\mu(t).$$

Now since $\alpha(f) < 1 < \gamma(f)$, we have that

$$\int_{\infty}^{\infty} \frac{1 + t^2}{t^2} d\mu(t) > 1 - \alpha > 0.$$

But $\int_{\infty}^{\infty} \frac{1 + t^2}{t^2 + b^2} d\mu(t) \downarrow 0$ as $b \to \infty$ so there is a $b_0 > 0$ such that $\int_{\infty}^{\infty} \frac{1 + t^2}{t^2 + b_0^2} d\mu(t) = 1 - \alpha$, i.e. $f(ib_0) = ib_0$, hence $P_{ib_0} \circ T^{-1} = P_{ib_0}$.

The result now follows from theorem 1.1. □

To illustrate the results of this section, we consider $Tx = \alpha x + \beta \tan x$ where $\alpha, \beta > 0$. Here, $\alpha(T) = \alpha$, and $\gamma(T) = \alpha + \beta$.

If either $\alpha > 1$, or $\alpha + \beta < 1$, T is dissipative.

If $\alpha < 1 < \alpha + \beta$, then T preserves a Cauchy distribution and is exact. (This was established in [5] for $\alpha = 0$, $\beta > 1$).

The remaining cases ($\alpha = 1$ and $\alpha + \beta = 1$) are contained in the discussion of:

§ 3. RESTRICTIONS PRESERVING INFINITE MEASURES

In this section, we consider those restrictions preserving infinite measures with $\alpha = 1$, or $\alpha(\phi_t f \phi_t^{-1}) = 1$ for some t.

We will see that for these transformations, conservativity is sufficient for ergodicity and rational ergodicity ([1]), a stronger property (example 1.2 in [1]). We then give sufficient conditions for exactness.

Firstly, we recall the definition of rational ergodicity. Let $(X, \mathbb{B}, m, \tau)$ be a conservative, ergodic, measure preserving transformation of a non-atomic, σ-finite measure space. We say that τ is *rationally ergodic* if there is a set A, of positive finite measure and $K < \infty$ such that

$$\text{(B)} \qquad \int_A \left(\sum_{k=0}^{n-1} 1_A \circ \tau^k \right)^2 dm \leq K \left(\sum_{k=0}^{n-1} m(A \cap \tau^{-k} A) \right)^2 \qquad \text{for} \quad n \geq 1$$

For a rationally ergodic transformation τ, we let $B(\tau)$ denote the collection of sets with the property (B). It was shown in [1] that there is a sequence $\{ a_n(\tau) \}$ such that

$$\frac{1}{a_n(\tau)} \sum_{k=0}^{n-1} m(A \cap \tau^{-k} A) \to m(A)^2 \qquad \text{for every} \quad A \in B(\tau)$$

The sequence $\{ a_n(\tau) \}_n$ is known as a *return sequence for* τ and the collection of all sequences asymptotically proportional to $a_n(\tau)$

$$\left(\text{i.e. } \frac{a_n}{a_n(\tau)} \to c \in (0, \infty) \right)$$

is known as the *asymptotic type of* τ and denoted by $\mathscr{A}(\tau)$. It was shown in [*1*] (theorem 2.4) that if τ_1 and τ_2 are rationally ergodic transformations which are both factors of the same measure preserving transformation, then

$$\mathscr{A}(\tau_1) = \mathscr{A}(\tau_2) \qquad \left(\text{i. e. } \exists \lim_{n\to\infty} \frac{a_n(\tau_1)}{a_n(\tau_2)} \in (0, \infty)\right).$$

We commence with the case $\alpha(f) = 1$.

LEMMA 3.1. — Let $f \in \mathrm{I}(\mathbb{R}^{2+})$ be non-linear and let $\mathrm{T} = \mathrm{T}(f)$,

$$f^n(\omega) = u_n(\omega) + iv_n(\omega) \qquad \text{for} \quad n \geq 1 \ \ \omega \in \mathbb{R}^{2+}.$$

If $\alpha = 1$ then T is conservative

$$\text{iff} \quad \sum_{n=1}^{\infty} \frac{v_n(\omega)}{|f^n(\omega)|^2} = \infty \qquad \forall \omega \in \mathbb{R}^{2+}.$$

Proof. — It will be more comfortable to work on the unit disc U. Accordingly, we let $\mathrm{M}(z) = \emptyset^{-1}f\emptyset(z)$ where $\emptyset(z) = i\left(\frac{1+z}{1-z}\right)$. Then M is an inner function on U. Let $\mathrm{M}(\mathrm{r}e^{i\theta}) \to \tau e^{i\theta}$ as $r \to 1$ a. e. Denoting $\mathrm{Im}\left(\frac{e^{i\theta}+z}{e^{i\theta}+z}\right)$ by $q_z(\theta)$ and $q_z(\theta)\,d\theta$ by $d\pi_z(\theta)$, we see that $\pi_z \circ \emptyset^{-1} = \pi_\emptyset \mathrm{P}_{\emptyset(z)}$ and this combined with the fact that $\emptyset^{-1}\mathrm{T}\emptyset = \tau$ gives us that:

$$\pi_z \circ \tau^{-1} = \pi_{\mathrm{M}(z)}.$$

So τ is a non-singular transformation of $(\mathbb{T}, \lambda)$, and is conservative iff T is conservative.

Let $\hat{\tau}$ be the operator dual to τ, acting on L^1. Then $\hat{\tau}q_z(t) = q_{\mathrm{M}(z)}(t)$ and τ is conservative iff

$$(3.1) \qquad \sum_{n=1}^{\infty} q_{\mathrm{M}^n(z)}(t) = \infty \qquad \text{a. e. } \forall z \in \mathrm{U}.$$

We next show that $\mathrm{M}^n(z) \to 1$ as $n \to \infty$ $\forall z \in \mathrm{U}$. This will follow from the fact that $f^n(\omega) \to \infty$ as $n \to \infty$ $\forall \omega \in \mathrm{R}^{2+}$ which we now demonstrate. From 0.1:

$$v_{n+1}(\omega) = v_n(\omega) + v_n(\omega) \int_{\infty}^{\infty} \frac{(1+t^2)d\mu(t)}{(t-\mathrm{U}_n)^2 + v_n^2} \geq v_n(\omega).$$

Hence $v_n \uparrow v_\infty$. It is not hard to see that if $v_\infty < \infty$, we must have $|U_n| \to \infty$. Hence $M^n(z) \to 1$.

Now choose $z \in U$ and let $M^n(z) = r_n e^{i\theta_n}$. We have $r_n \to 1$ and $\theta_n \to 0$. Also:

$$q_{M^n(z)}(t) = \frac{1 - r_n^2}{1 - 2r_n \cos(\theta_n - t) + r_n^2} \sim \frac{1 - r_n}{1 - \cos t} \quad \text{as} \quad n \to \infty .$$

For $t \neq 0$. Thus:

(3.2) T is conservative iff $\displaystyle\sum_{n=1}^{\infty} 1 - |M^n(z)| = \infty \qquad \forall z \in U$.

The second condition is the same as

$$\sum_{n=1}^{\infty} 1 - |M^n(z)|^2 = \infty \qquad \forall z \in U .$$

Now if $\omega = a + ib \in \mathbb{R}^{2+}$, then

$$1 - \left|\frac{\omega - i}{\omega + i}\right|^2 = \frac{4b}{a^2 + (b+1)^2}$$

From the definition of M, we have

$$1 - \left|M^n\left(\frac{\omega - i}{\omega + i}\right)\right|^2 = \frac{4v_n(\omega)}{U_n(\omega) + (v_n + 1)^2} \sim \frac{4v_n(\omega)}{|f^n(\omega)|^2} \quad \text{as} \quad n \to \infty \qquad \square$$

THEOREM 3.2. — Let $f \in I(\mathbb{R}^{2+})$ be non-linear, $T = T(f)$ and $\alpha(f) = 1$. If T is conservative then T is rationally ergodic, and

$$\mathscr{A}(T) = \left\{ \sum_{k=1}^{n} \frac{v_k(\omega)}{|f^k(\omega)|^2} \right\} \qquad \text{for every} \quad \omega \in \mathbb{R}^{2+} .$$

Proof. — We first prove ergodicity, and here again, it is more comfortable to work on U. We prove the ergodicity of τ (as defined in Lemma 3.1). If T is conservative then by (3.2):—

$$\sum_{n=1}^{\infty} 1 - |M^n(z)| = \infty \qquad \forall z \in U .$$

Since $M^n(z) \to 1$, we must have that the points $\{M^n(z)\}_{n \geq 1}$ are distinct. Now, let $h \in N(U)$ (defined on p. 303 of [9]). If $h(\bar{M}(z)) = h(z)$ for all $z \in U$ then by theorem 15-23 of [9], h must be constant. The ergodicity of τ is deduced from this as follows:

Let $A \subseteq T$ be a τ-invariant measurable set and let

$$v(z) = \int_{\pi}^{\pi} q_z(\theta) 1_A(\theta) \frac{d\theta}{2\pi} .$$

Then $v(\mathrm{M}(z)) = v(z)$, $\| v \circ \mathrm{M}^n \|_\infty \leq 1$ $\forall n \geq 1$, and $v(re^{i\theta}) \to 1_{\mathrm{A}}(\theta)$ a. e. as $r \to 1$. Now v can be regarded as the imaginary part of an analytic function $\mathrm{F} \in \mathrm{H}(\mathrm{U})$. By theorem 17-26 of [9] $\mathrm{F} \in \mathrm{H}^1(\mathrm{U}) \subseteq \mathrm{N}(\mathrm{U})$ and $\| \mathrm{F} \circ \mathrm{M}^n \|_1 \leq \mathrm{A}$ $\forall n \geq 1$.

Moreover: $\mathrm{F}(\mathrm{M}(z)) = \mathrm{F}(z) + c$ where $c \in \mathbb{R}$.

Let $\mathrm{F}^*(e^{i\theta}) = \lim_{r \uparrow 1} \mathrm{F}(re^{i\theta})$, then $\mathrm{F}^*(\tau e^{i\theta}) = \mathrm{F}^*(e^{i\theta}) + c$. The conservativity of τ yields that $c = 0$ (since the set $[| \mathrm{F}^* | \leq \mathrm{M}]$ has positive measure for some M, and so every point of this set returns infinitely often to it under iterations of τ — an impossibility if $c \neq 0$). Thus, F is constant and hence also $1_{\mathrm{A}}(0)$.

We now turn to rational ergodicity. Let

$$b_n(\omega) = \frac{| f^n(\omega) |^2}{v_n(\omega)}$$

Since $f^n(\omega) \to \infty$, it is clear that:

$$\pi b_n(\omega) \hat{\mathrm{T}}^n \phi_\omega(t) \to 1 \tag{3.3}$$

uniformly on compact subsets of $\mathbb{R}$. Let

$$a_n(\omega) = \sum_{k=1}^{n} \frac{1}{\pi b_k(\omega)}.$$

From (3.3) we have that

$$\frac{1}{a_n(\omega)} \sum_{k=0}^{n-1} \hat{\mathrm{T}}^k \phi_\omega \to 1 \tag{3.4}$$

uniformly on compact subset of $\mathbb{R}$.

Now, since T is a conservative ergodic transformation, it follows that $\hat{\mathrm{T}}$ is a conservative ergodic Markov operator, and we have from (3.4), by the Chacon-Ornstein theorem (see [3]) that:

$$\frac{1}{a_n(\omega)} \sum_{k=0}^{n-1} \hat{\mathrm{T}}^k f \to \int_{\mathbb{R}} f d\lambda \quad \text{a. e.} \quad \forall f \in \mathrm{L}^1. \tag{3.5}$$

Hence

$$\exists\, a_n \to \infty \text{ s. t. } \quad \frac{a_n(\omega)}{a_n} \to 1 \quad \text{for every } \omega \in \mathbb{R}^{2+}.$$

We will prove rational ergodicity of T by showing that bounded intervals are in B(T).

Let $\mathrm{A} = [a, b]$ where $-\infty < a < b < \infty$.

Then $1_A \leq c\phi_i$
Hence, by (3.4), there is a $C_1 < \infty$ s. t.

$$\frac{1}{a_n}\sum_{k=0}^{n-1} \hat{T}^k 1_A(x) \leq C_1 \qquad \text{for} \quad n \geq 1,\ x \in A. \tag{3.6}$$

This, combined with (3.5), gives (by dominated convergence)

$$\frac{1}{a_n}\sum_{k=0}^{n-1} \lambda(A \cap T^{-k}A) \to \lambda(A)^2 \tag{3.7}$$

To complete the proof that T is rationally ergodic, we show that:

$$\int_A \left(\sum_{k=0}^{n-1} 1_A \circ T^k\right)^2 d\mu \leq 2C_1\, a_n^2 \qquad \text{for} \quad n \geq 1. \tag{3.8}$$

$$\int_A \left(\sum_{k=0}^{n-1} 1_A \circ T^k\right)^2 d\mu \leq 2\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \lambda(A \cap T^{-k}(A \cap T^{-l}A))$$
$$= 2\sum_{l=0}^{n-1} \int_{A \cap T^{-l}A} \sum_{k=0}^{n-1} \hat{T}^k 1_A d\lambda \leq 2C_1\, a_n^2 \qquad \square$$

We now turn to exactness. The following elementary lemma plays a similar role to that of lemma 1.2.

LEMMA 3.3. — If $b_n \to \infty$, $B_n \sim b_n$ and

$$\frac{a_n}{b_n} \to 0 \qquad \text{as} \quad n \to \infty$$

then

$$\| \phi_{a_n + ib_n} - \phi_{iB_n} \|_1 \to 0 \qquad \text{as} \quad n \to \infty.$$

THEOREM 3.4. — Let $f \in I(\mathbb{R}^{2+})$, $T = T(f)$ and assume

$$f(\omega) = \omega + \int_{-K}^{K} \frac{dv(t)}{t - \omega}$$

then: T is exact, rationally ergodic and $\mathscr{A}(T) = \{ \sqrt{n} \}$.

Proof. — Let $L = \max \{ v(\mathbb{R}), v(\mathbb{R})^2 \}$ and assume that $K \geq \frac{1}{4}$. We write $f^n(\omega) = u_n(\omega) + iv_n(\omega)$. The assumption of the theorem means that

$$u_{n+1} = u_n + \int_{-K}^{K} \frac{t - u_n}{(t - u_n)^2 + v_n^2} d\nu(t) \tag{3.9}$$

$$v_{n+1} = v_n + v_n \int_{-K}^{k} \frac{d\nu(t)}{(t - u_n)^2 + v_n^2}$$

The first part of the proof of this result consists of deducing the asymptotic behaviour of u_n and v_n. For this, we assume that $\omega = a + iL$ where $a \in \mathbb{R}$. The recurrence relations (3.9) show us that

$$v_n(\omega) \geq L \qquad \text{for every} \quad n \geq 1 .$$

And this enables us to deduce the boundless of $|u_n(\omega)|$ as follows: Not-ing that:

$$\left| \int_{-K}^{K} \frac{t - u_n}{(t - u_n)^2 + v_n^2} d\nu(t) \right| \leq \frac{\nu(\mathbb{R})}{2v_n} \leq \frac{1}{2}$$

we see that:

If $\quad u_n \geq K \qquad$ then $\qquad -K \leq K - \frac{1}{2} \leq u_{n+1} \leq u_n .$

If $\quad u_n \leq -K \qquad$ then $\qquad u_n \leq u_{n+1} \leq -K + \frac{1}{2} \leq K$

If $\quad u_n \leq K \qquad$ then

$$u_{n+1} \leq u_n + (K - u_n) \int_{-K}^{K} \frac{d\nu}{(t - u_n)^2 + v_n^2} \leq u_n + \frac{(K - u_n)}{v_n^2} \nu(R) \leq K$$

If $\quad u_n \geq -K \qquad$ then $\qquad u_{n+1} \geq -K .$

Hence $|u_n(a + iL)| \leq |a| V^K$ for $n \geq 1$.

The recurrence relations (3.9) now imply that $v_n \to \infty$ as $n \to \infty$ and hence

$$v_{n+1}^2 - v_n^2 = 2v_n^2 \int_{-K}^{K} \frac{d\nu(t)}{(t - u_n)^2 + v_n^2}$$

$$+ v_n^2 \left(\int_{-K}^{K} \frac{d\nu(t)}{(t - u_n)^2 + v_n^2} \right)^2 \to 2\nu(\mathbb{R}) \qquad \text{as} \quad n \to \infty$$

Hence $v_n(a + iL) \sim \sqrt{2\nu n}$ as $n \to \infty$.

Lemma 3.3 now shows us that for every $a \in \mathbb{R}$:

$$|| \hat{T}^n \phi_{a+iL} - \phi_{i\sqrt{2\nu n}} || \to 0 \qquad \text{as} \quad n \to \infty . \tag{3.19}$$

We now obtain exactness by Lin's criterion by an argument similar to that of theorem 1.1 (the rational ergodicity of T has already been established, and its asymptotic type characterised, by theorem 3.2).

Let $u \in L^1$, $\int_R u d\lambda = 0$, and $\varepsilon > 0$:

By Wiener's Tauberian theorem, there are $\alpha_1 \dots \alpha_N$, $a_1 \dots a_N \in R$ such that

$$\left\| u - \sum_{k=1}^{N} \alpha_k \phi_{a_k + iL} \right\|_1 < \varepsilon/2$$

Whence:

$$\| \hat{T}^n u \|_1 \leq \left\| \hat{T}^n \left(u - \sum_{k=1}^{N} \alpha_k \phi_{a_k + iL} \right) \right\|_1 + \left\| \hat{T}^n \sum_{k=1}^{N} \alpha_k \phi_{a_k + iL} - \sum_{k=1}^{N} \alpha_k \phi_{i\sqrt{2vn}} \right\|_1 + \left\| \sum_{k=1}^{N} \alpha_k \phi_{i\sqrt{2vn}} \right\|_1$$

$$\| \hat{T}^n u \|_1 \leq \left\| u - \sum_{k=1}^{N} \alpha_k \phi_{a_k + iL} \right\|_1 + \sum_{k=1}^{N} \alpha_k \| \hat{T}^n \phi_{a_k + iL} - \phi_{i\sqrt{2vn}} \|_1 + \left| \sum_{k=1}^{N} \alpha_k \right| < \varepsilon + o(1) \quad \square$$

We note that the « generalized Boole transformation » (proven ergodic in [7]) falls within the scope of this last theorem.

If we added $\beta \neq 0$ to f in theorem 3.4, we would obtain that for Im ω large enough $| u_n(\omega) | \geq c_1 n$ and $v_n(\omega) \leq c_2 \log n$ (where $f^n(\omega) = u_n(\omega) + iv_n(\omega)$). The methods of lemma 3.1 would yield that $T(f)$ is dissipative.

The following corollary follows immediately from lemma 3.1 and theorem 3.2.

COROLLARY 3.5. — Let $f \in I(\mathbb{R}^{2+})$ and let $T = T(f)$, $f^n(i) = iv_n(i)$. If $\alpha(f) = 1$ then:

$$\text{T is conservative iff} \quad \sum_{n=1}^{\infty} \frac{1}{v_n(i)} = \infty$$

and in this case, T is rationally ergodic with

$$\mathscr{A}(T) = \left\{ \sum_{k=1}^{n} \frac{1}{\pi v_k(i)} \right\}.$$

Moreover, in case $f \in I_0$ and $\alpha(f) = 1$: we have that $v_n \to \infty$ and so:

$$v_{n+1}^2 - v_n^2 = 2v_n^2 \int_{-\infty}^{\infty} \frac{1 + t^2}{t^2 + v_n^2} d\mu(t) + v_n^2\left(\int_{-\infty}^{\infty} \frac{1 + t^2}{t^2 + v_n^2} d\mu(t)\right)^2 \to 2 \int_{-\infty}^{\infty} (1 + t^2) d\mu(t) \leq \infty$$

Hence:

$$\frac{v_n(i)}{\sqrt{n}} \to \sqrt{2 \int_{-\infty}^{\infty} (1 + t^2) d\mu(t)} \leq \infty$$

which means:

a) $\mathrm{T} \times \mathrm{T} \times \mathrm{T}$ is dissipative

b) $\dfrac{a_n(\mathrm{T})}{\sqrt{n}} \to c \in [0, \infty)$ as $n \to \infty$ (in case T is r. e.).

These last two properties are held in common with the restrictions of theorem 3.4, and with the Markov shifts of random walks on $\mathbb{Z}$.

The following example does not fall within the scope of theorem 3.4, (though theorem 3.2 does apply).

EXAMPLE 3.6. — $\mathrm{T}x = x + \alpha \tan x$ is exact, rationally ergodic with $a_n(\mathrm{T}) \sim \dfrac{\operatorname{Log} n}{\alpha}$ for $\alpha > 0$.

Proof. — Let $f(\omega) = \omega + \alpha \tan \omega$ and $f^n(\omega) = u_n(\omega) + iv_n(\omega)$. Then:

$$u_{n+1} = u_n + \frac{2\alpha \sin 2u_n e^{2v_n}}{e^{4v_n} - 2 \cos 2u_n e^{2v_n} + 1}$$

and

$$v_{n+1} = v_n + \alpha \frac{e^{4v_n}}{e^{4v_n} - 2 \cos 2u_n e^{v_n} + 1}$$

Whence:

$$v_{n+1} - v_n \geq \alpha \tanh v_n \geq \alpha \tanh v_0 > 0$$

so

$$v_n \sim \alpha n \quad \text{as} \quad n \to \infty .$$

On the other hand:

$$| u_{n+1} - u_n | \leq \frac{2\alpha e^{2v_n}}{(e^{2v_n} - 1)^2} \leq 4\alpha e^{-2v_n} \leq 4\alpha e^{-\alpha n} \quad \text{for} \quad n \text{ large}.$$

Hence $u_n \to u_\infty$, and the argument that T is exact now proceeds identically to the last argument of theorem 3.4. □

The following lemma will give examples of $f \in I_0(R^{2+})$ with $\alpha(f) = 1$ and $T = T(f)$ dissipative, and also uncountably many dissimilar Γ. *e.* (see [*1*]) restrictions $T(f)$ with $f \in I_0(\mathbb{R}^{2+})$, $\alpha(f) = 1$.

LEMMA 3.7. — Let $\mu \in S(\mathbb{R})$ be symetric with

$$c(x) = \mu(|t| \geq x) \sim \frac{1}{x^\alpha} \quad \text{where} \quad 0 < \alpha < 2.$$

Let

$$f_\alpha(\omega) = \omega + \int_{-\infty}^{\infty} \frac{1 + t\omega}{t - \omega} d\mu(t) \quad \text{and} \quad f^n(i) = iv_n.$$

Then: $v_n \sim cn^{1/\alpha}$ where c depends only on α.

Proof. — We have

$$v_{n+1} = v(1 + F(v_n))$$

where

$$F(b) = \int_{-\infty}^{\infty} \frac{1 + t^2}{t^2 + b^2} d\mu(t).$$

It is not difficult to see that

$$F(b) = \frac{\mu(\mathbb{R})}{b^2} + 2(b^2 - 1) \int_0^\infty \frac{xc(x)}{(x^2 + b^2)^2} dx$$

We first show that $F(b) \sim \frac{c_1}{b^\alpha}$ as $\alpha \to \infty$

Let $\varepsilon > 0$, and M be such that

$$\frac{1 - \varepsilon}{x^\alpha} \leq c(x) \leq \frac{1 + \varepsilon}{x^\alpha} \qquad \forall x \geq M$$

Writing

$$L_M(b) = \int_M^\infty \frac{x^{1-\alpha}}{(x^2 + b^2)^2} dx$$

we have that:

$$(1 - \varepsilon)L_M(b) = \int_M^\infty \frac{xc(x)dx}{(x^2 + b^2)^2} \leq (1 + \varepsilon)L_M(b).$$

Now

$$L_M(b) = \int_M^\infty \frac{x^{1-\alpha}}{(x^2 + b^2)^2} dx = \frac{1}{b^{2+\alpha}} \int_{M/b}^\infty \frac{x^{1-\alpha}dx}{(x^2 + 1)^2} \sim \frac{c}{b^{2+\alpha}} \qquad \text{as} \quad b \to \infty$$

where

$$c = \int_0^\infty \frac{x^{1-\alpha}dx}{(x^2 + 1)^2}$$

Since $\varepsilon > 0$ was arbitrary and $\alpha < 2$, we have that

$$F(b) \sim \frac{c}{b^\alpha} \quad \text{as} \quad b \to \infty .$$

Clearly, $v_n \to \infty$, hence:

$$\begin{aligned} v_{n+1}^\alpha - v_n^\alpha &= v_n^\alpha[(1 + F(v_n))^\alpha - 1] \\ &\sim \alpha v_n^\alpha F(v_n) \quad \text{as} \quad n \to \infty \\ &\to \alpha c \quad \text{as} \quad n \to \infty \end{aligned}$$

Thus $v_n \sim (\alpha c n)^{1/\alpha}$ as $n \to \infty$ □

We now let $T_\alpha = T(f_\alpha)$.

By corollary 3.5:

If $0 < \alpha < 1$ then T_α is dissipative.
If $1 \leq \alpha < 2$ then T_α is rationally ergodic and

$$\mathscr{A}(T_\alpha) = \begin{cases} \{ \log n \} & \text{if} \quad \alpha = 1 \\ \{ n^{1-1/\alpha} \} & \text{if} \quad 1 < \alpha < 2 . \end{cases}$$

If follows from theorem 2.4 of [1] that if $1 \leq \alpha_1 < \alpha_2 < 2$ then T_{α_1} and T_{α_2} are not factors of the same measure preserving transformation.

THEOREM 3.8. — Let $f \in I(\mathbb{R}^{2+})$ and $T = T(f)$.

Suppose $x_0 \in \mathbb{R}$ and f is analytic in a neighbourhood around x_0.

If $Tx_0 = x_0$, $T'(x_0) = 1$ and $T''(x_0) = 0$ then T preserves the measure ν_{x_0} where $d\nu_{x_0}(x) = \dfrac{dx}{(x - x_0)^2}$, and is exact, rationally ergodic with asymptotic type $\{ \sqrt{n} \}$

Remarks. — The conditions $Tx_0 = x_0$ and $T'(x_0) = 1$ correspond to: $\alpha(\phi_{x_0} f \phi_{x_0}^{-1}) = 1$. If, in this situation, $T''(x_0) \neq 0$; then T is dissipative. By possibly considering $g(\omega) = f(\omega + x_0) - x_0$ we may (and do) assume $x_0 = 0$.

Proof. — Let

$$f(\omega) = \omega + \sum_{n=3}^{\infty} a_n \omega^n \quad \text{for} \quad |\omega| \text{ small}.$$

Then

$$\begin{aligned} \frac{1}{f(\omega)} - \frac{1}{\omega} = \frac{\omega - f(\omega)}{f(\omega)} &= \frac{\omega}{f(\omega)} \sum_{n=3}^{\infty} a_n \omega^n \\ &\to 0 \quad \text{as} \quad \omega \to 0 . \end{aligned}$$

Hence

$$\frac{1}{f(\omega)} = \frac{1}{\omega} + \sum_{n=1}^{\infty} b_n \omega^n \qquad \text{for} \quad |\omega| \text{ small}.$$

Let $\tilde{f}(\omega) = -1/f\left(-\frac{1}{\omega}\right)$.

Then:

$$(3.11) \qquad \tilde{f}(\omega) = \omega + \sum_{n=1}^{\infty} b_n \omega^{-n} \qquad \text{for} \quad |\omega| \text{ large},$$

say $|\omega| \geq K$ and, since $\tilde{f} \in I(\mathbb{R}^{2+})$, $\alpha(\tilde{f}) = 1$:

$$(3.12) \quad \tilde{f}(\omega) = \omega + \beta + \int_{-\infty}^{\infty} \frac{1 + t\omega}{t - \omega} d\mu(t) \qquad \text{where} \quad \mu \in S(\mathbb{R}),\ \beta \in \mathbb{R}$$

In order to prove the theorem by applying theorem 3.4, we will show that

$$(3.13) \qquad \tilde{f}(\omega) = \omega + \int_{-K}^{K} \frac{d\nu(t)}{t - \omega} \qquad \text{where} \quad \nu \in S(\mathbb{R}).$$

Firstly, let $g(\omega) = \tilde{f}(\omega) - \omega$. By (3.11):

$$-ibg(ib) \to b_1 \qquad \text{as} \quad b \to \infty$$

But by (3.12):

$$-ibg(ib) = -ib\left(\beta - b^2 \int_{-\infty}^{\infty} \frac{td\mu(t)}{t^2 + b^2}\right) + ib \int_{-\infty}^{\infty} \frac{td\mu(t)}{t^2 + b^2} + b^2 \int_{-\infty}^{\infty} \frac{1 + t^2}{t^2 + b^2} d\mu(t).$$

Hence, we obtain, from the convergence of the real part, that

$$\int_{-\infty}^{\infty} (1 + t^2) d\mu(t) < \infty$$

and from the convergence of the imaginary part that:

$$b^2 \int_{-\infty}^{\infty} \frac{td\mu(t)}{t^2 + b^2} \to \beta \qquad \text{as} \quad b \to \infty.$$

which convergence, when combined with the previous one, gives

$$\int_{-\infty}^{\infty} td\mu(t) = \beta.$$

Now, let $d\nu(t) = (1 + t^2)d\mu(t)$, then $\nu \in S(\mathbb{R})$ and it follows easily that

$$\tilde{f}(\omega) = \omega + \int_{-\infty}^{\infty} \frac{d\nu(t)}{t - \omega} \tag{3.14}$$

Now, let $h_b(a) = \operatorname{Im} g(a + ib) = b \int_{-\infty}^{\infty} \frac{d\nu(t)}{(t + a)^2 + b^2}$. By (3.11) g is uniformly continuous on compact subsets of $[\,|\,\omega\,| \geq K]$, and so $h_b(a) \to 0$ as $b \to 0$ uniformly on compact subsets of $[\,|\,a\,| > K]$.

Let $dQ_b(x) = h_b(x)dx$, then $Q_b = P_{ib} * \nu$, and so $Q_b(A) \to \nu(A)$ for A a compact set. If A is a compact subset of $[\,|\,x\,| > K]$, then

$$\nu(A) = \lim_{b \downarrow 0} Q_b(A) = \lim_{b \downarrow 0} \int_A h_b(x)dx = 0\,.$$

Thus ν is concentrated on $[-K, K]$ and (3.13) is established. □

The transformations $T_\alpha x = \alpha x + (1 - \alpha)\tan x$ for $0 \leq \alpha < 1$ fall within the scope of theorem 3.9 (it was shown in [*11*] that T_0 is ergodic). It follows from asymptotic type considerations that the above transformations are dissimilar to $Tx = x + \alpha \tan x$.

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