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Uniform and ratio limit theorems for Markov renewal and semi-regenerative processes on a general state space

by

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RÉSUMÉ. — Nous généralisons la « technique du fenderie » des chaînes markoviennes aux processus semi-markoviennes. En utilisant cette méthode, nous montrons les théorèmes de renouvellement du type uniforme et les théorèmes quotient pour les processus semi-markoviennes dans un espace mesurable quelconque.

ABSTRACT. — We prove uniform and ratio Markov renewal limit theorems for φ -recurrent Markov renewal processes on a general state space by using the so called splitting technique introduced recently for Markov chains.

0. INTRODUCTION

In a recent paper ([15]) the so called splitting method was introduced for φ -recurrent Markov chains on a general state space. It enables the full force of elementary discrete time renewal theory to be used in the analysis of general Markov chains. One of the main purposes of the present paper is to generalize this method to Markov renewal processes and to use it in the study of Markov renewal limit theorems. It reduces, similarly as in the case of Markov chains, the analysis of general Markov renewal

processes to the analysis of processes having an atom (that is a subset of the state space from the points of which all transitions are identical). If the original process is φ -recurrent (see Section 1), this atom will be recurrent, and we can use the more elementary results proved for Markov renewal processes possessing an atom.

Sections 1 and 2 contain the preliminaries. In Section 3 we introduce the splitting technique. At first a minorization condition called (M_α) is formulated, which we have to assume for the splitting technique to succeed. Theorem 3.1 and its Corollary give sufficient conditions for (M_α) . The rest of this section is devoted to the description of the splitting technique.

In Section 4 we shortly consider Markov renewal processes having an atom, and recall some results from [1].

Section 5 deals with uniform Markov renewal theorems. By using the splitting technique we extend the results of [1] dealing with Markov renewal processes possessing an atom to processes on a general state space. Specifically, in Theorem 5.1 we shall consider the limit of the expression $\lambda * R * f(t)$, ($t \rightarrow \infty$), uniformly over f in a suitable set of functions on $S \times \mathbb{R}_+$ ((S, \mathcal{F}) is the state space, $\mathbb{R}_+ = [0, \infty)$, \mathcal{R}_+ = the Borel σ -field of \mathbb{R}_+ , λ is a fixed probability measure on $\mathcal{F} \times \mathcal{R}_+$ and R is the Markov renewal kernel of the process). Theorem 5.1 complements the earlier renewal theorems (see e. g. Jacod [7], Kesten [9], McDonald [11]) allowing a more general starting distribution λ and function f , and providing a uniform convergence over the set of functions. We also give sufficient conditions for the boundedness of the signed measure $\lambda * R$ on $\mathcal{F} \times \mathcal{R}_+$, where λ is a finite measure on $\mathcal{F} \times \mathcal{R}_+$ with total mass zero (Theorem 5.2).

In Section 6 we study ratio limit theorems for Markov renewal processes. We extend a recent result proved for Markov chains ([15], Theorem 7.1) to Markov renewal processes. Our results complement the earlier results of Jacod [7], allowing more general starting distributions λ and μ , and functions f and g .

In Section 7 we formulate as a corollary of Theorem 5.1 a total variation convergence result for semi-regenerative processes.

1. NOTATION AND PRELIMINARIES

Let S be an arbitrary set and \mathcal{F} a σ -field of subsets of S . We denote by $b\mathcal{F}_+$ the set of bounded measurable functions from S into \mathbb{R}_+ . For the Lebesgue measure on \mathcal{R}_+ we write l , and dt for $l(dt)$. Any map

$$N : S \times \mathcal{F} \times \mathcal{R}_+ \rightarrow \mathbb{R}_+ \cup \{ + \infty \}$$

satisfying

(1.1a) for any fixed $E \in \mathcal{F} \times \mathcal{R}_+$, $N(\cdot, E)$ is a measurable function on S , and

(1.1b) for any fixed $x \in S$, $N(x, \cdot)$ is a (non-negative) measure on $\mathcal{F} \times \mathcal{R}_+$, is called a *kernel*. If in addition, N satisfies

$$(1.1c) \quad N(x, S \times \mathbb{R}_+) = 1 \quad \text{for all } x \in S,$$

then it is called a *semi-Markov kernel* (on (S, \mathcal{F})).

Let M and N be any two kernels, μ be a measure on $\mathcal{F} \times \mathcal{R}_+$ and f (resp. g) be a non-negative measurable function on $S \times \mathbb{R}_+$ (resp. S). The following notations are used throughout this paper:

$$(1.2) \quad \tilde{N}(x, A) = N(x, A \times \mathbb{R}_+), \quad \tilde{\mu}(A) = \mu(A \times \mathbb{R}_+), \quad \tilde{f}(x) = \int_{\mathbb{R}_+} f(x, t) dt;$$

$$(1.3a) \quad M * N(x, A \times \Gamma) = \int_S \int_{\mathbb{R}_+} M(x, dy \times dt) N(y, A \times (\Gamma - t)),$$

where $\Gamma - t = \{u - t; u \in \Gamma\}$ and by convention any measure or function on $(\mathbb{R}_+, \mathcal{R}_+)$ is extended to the whole real line by setting them zero on $(-\infty, 0)$;

$$(1.3b) \quad N^{*0}(x, A \times \Gamma) = \Delta(x, A \times \Gamma) = 1_A(x) \varepsilon_0(\Gamma), \\ N^{*n} = N^{*(n-1)} * N, \quad n \geq 1,$$

where 1_A denotes the indicator function of the set A , and for any point a , ε_a denotes the probability measure assigning unit mass to a ;

$$(1.4) \quad \mu * N(A \times \Gamma) = \int_S \int_{\mathbb{R}_+} \mu(dy \times dt) N(y, A \times (\Gamma - t));$$

$$(1.5) \quad N * f(x, t) = \int_S \int_{\mathbb{R}_+} N(x, dy \times du) f(y, t - u);$$

$$(1.6) \quad g \otimes \mu(x, A \times \Gamma) = g(x) \mu(A \times \Gamma);$$

$$(1.7) \quad I_g N(x, A \times \Gamma) = g(x) N(x, A \times \Gamma), \quad NI_g(x, A \times \Gamma) = \int_A N(x, dy \times \Gamma) g(y);$$

we write I_A for I_{1_A} , ($A \in \mathcal{F}$), and I for I_S . For any signed measure μ the corresponding absolute value measure is denoted by $|\mu|$, and the total variation of μ (i. e. the total mass of $|\mu|$) by $\|\mu\|$. For any transition kernels on (S, \mathcal{F}) and for their operations on measures and functions we use the standard notations of Markov chain theory (see [15] or [19]).

Let Q be a fixed semi-Markov kernel. We denote by

$$(X, T) = \{ (X_n, T_n); n \geq 0 \}$$

the associated *Markov renewal process* (MRP); that is a Markov chain on $(S \times \mathbb{R}_+, \mathcal{F} \times \mathcal{R}_+)$ with transition probabilities

$$(1.8) \quad P((x, t), A \times \Gamma) = Q(x, A \times (\Gamma - t)), \quad x \in S, \quad A \in \mathcal{F}, \quad \Gamma \in \mathcal{R}_+.$$

We denote by $\mathbb{P}_{x,t}$ the canonical probability measure on the sample space $(\Omega, \Sigma) = (S \times \mathbb{R}_+, \mathcal{F} \times \mathcal{R}_+)^{\infty}$ of the MRP (X, T) induced by the semi-Markov kernel Q and the start $X_0 = x, T_0 = t, (x \in S, t \in \mathbb{R}_+)$. For any measure λ on $\mathcal{F} \times \mathcal{R}_+$ we denote $\mathbb{P}_{\lambda} = \int_S \int_{\mathbb{R}_+} \lambda(dx \times dt) \mathbb{P}_{x,t} \{ \cdot \}$. We write \mathbb{P}_x for $\mathbb{P}_{x,0}$ and \mathbb{P}_{μ} for $\mathbb{P}_{\mu \times \varepsilon_0}$, (μ a measure on \mathcal{F}).

DEFINITION 1. — Let φ be a non-trivial σ -finite measure on \mathcal{F} . We say that the MRP (X, T) is φ -irreducible (φ -recurrent), if the embedded Markov chain $\{X_n\}$ with transition probabilities $\tilde{Q}(x, A) = Q(x, A \times \mathbb{R}_+)$ is φ -irreducible (φ -recurrent) (see e. g. [17], p. 4). When $\{X_n\}$ is φ -recurrent, we use the notation π for the unique (up to scalar multiplication) invariant measure of \tilde{Q} , which is known to exist ([17], p. 31), and we write \mathcal{F}^+ for the set $\{A \in \mathcal{F}; \pi(A) > 0\}$.

We define the kernels U_f ($f \in b\mathcal{F}_+, f \leq 1$) in a similar way as they are defined for Markov chains (see e. g. [19], p. 48)

$$(1.9) \quad U_f = \sum_{n \geq 0} Q * (I_{1-f}Q)^{*n} = \sum_{n \geq 0} (QI_{1-f})^{*n} * Q;$$

in particular $U_1 = Q$ and $U_0 = \sum_{n \geq 1} Q^{*n}$. We write R for the kernel $\Delta + U_0 = \sum_{n \geq 0} Q^{*n}$, and call it the *Markov renewal kernel* corresponding to Q . The following well known probabilistic interpretation is valid:

$$(1.10) \quad R(x, E) = \mathbb{E}_x \left[\sum_{n \geq 0} 1_E(X_n, T_n) \right], \quad x \in S, \quad E \in \mathcal{F} \times \mathcal{R}_+.$$

PROPOSITION 1.1 (*Resolvent equation*). — For $f, g \in b\mathcal{F}_+$ with $f \leq g \leq 1$,

$$U_f = \sum_{n \geq 0} (U_g I_{g-f})^{*n} * U_g = \sum_{n \geq 0} U_g * (I_{g-f} U_g)^{*n},$$

which implies

$$U_f = U_g + U_f * I_{g-f} U_g = U_g + U_g * I_{g-f} U_f.$$

Proof. — Similar as in the case of Markov chains (see [19], p. 49). \square
 Next we define the concept of an atom:

DEFINITION 2. — A set $B \subset S$ is called an *atom*, provided that $Q(x, \cdot) \equiv Q(y, \cdot)$ for all $x, y \in B$. For any function f on S , which is constant on B , we write $f(B)$ for $f(x)$, ($x \in B$); in particular $Q(B, \cdot) \equiv Q(x, \cdot)$, ($x \in B$). The atom B is called *recurrent*, if for all $x \in S$, $X_n \in B$ i. o. (\mathbb{P}_x -a. s.).

2. ON THE CONCEPT OF POSITIVE RECURRENCE

This section contains some preliminaries which are needed later in Sections 5 and 6 in the study of limit theorems. We denote for $x \in S$

$$(2.1) \quad m(x) = \mathbb{E}_x T_1 = \int_{\mathbb{R}_+} t Q(x, S \times dt).$$

DEFINITION 1. — The MRP (X, T) is called *positive recurrent*, if it is φ -recurrent for some φ , and if

$$(2.2) \quad \pi(m) = \int_S \int_{\mathbb{R}_+} t \pi(dx) Q(x, S \times dt) < \infty.$$

The following Proposition 2.2 is due to Jacod ([7], Proposition 11). However, we shall give a proof to it in order to illustrate the use of the Resolvent equation. We denote for all $A \in \mathcal{F}$

$$(2.3) \quad \tau_A = \inf \{ n \geq 1 ; X_n \in A \},$$

$$(2.4) \quad \begin{aligned} T_A &= T_{\tau_A} && \text{on } \{ \tau_A < \infty \}, \\ &= \infty && \text{on } \{ \tau_A = \infty \}. \end{aligned}$$

The following lemma is needed in the proof of Proposition 2.2 and also later in Sections 5 and 6.

LEMMA 2.1. — For any probability measure λ on $\mathcal{F} \times \mathcal{R}_+$ and $A \in \mathcal{F}^+$,

$$\begin{aligned} \mathbb{E}_\lambda T_A &= \mathbb{E}_\lambda T_0 + \mathbb{E}_\lambda \left[\sum_{n=0}^{\tau_A - 1} m(X_n) \right] \\ &= \int_{\mathbb{R}_+} t \lambda(S \times dt) + \tilde{\lambda}(I + \tilde{U}_A I_{A^c})m. \end{aligned}$$

Proof. — For any measure λ on $\mathcal{F} \times \mathcal{R}_+$ denote by λ' the measure $\int_{\mathbb{R}_+} t\lambda(\cdot \times dt)$ on \mathcal{F} , and for any kernel N , by N' the transition kernel $\int_{\mathbb{R}_+} tN(\cdot, \cdot \times dt)$ on (S, \mathcal{F}) . It is easy to see that

$$(2.5) \quad (\lambda * N)' = \tilde{\lambda}N' + \lambda'\tilde{N},$$

and

$$(2.6) \quad \left(\sum_{n \geq 0} N^{*n}\right)' = \left(\sum_{m \geq 0} \tilde{N}^m\right)N' \left(\sum_{n \geq 0} \tilde{N}^n\right).$$

In particular, for $f \in b\mathcal{F}_+$ ($f \leq 1$),

$$(2.7) \quad U'_f = (I + \tilde{U}_f I_{1-f})Q'(I + I_{1-f}\tilde{U}_f),$$

where for $\tilde{U}_f = U_f(\cdot, \cdot \times \mathbb{R}_+)$ we have

$$(2.8) \quad \tilde{U}_f = \sum_{n \geq 0} \tilde{Q}(I_{1-f}\tilde{Q})^n,$$

as is easily seen by the Resolvent equation and by monotone convergence theorem. Now we get (we write 1 for 1_S):

$$(2.9) \quad \begin{aligned} \mathbb{E}_\lambda T_A &= \int_{\mathbb{R}_+} t \sum_{n \geq 1} \mathbb{P}_\lambda \{ X_n \in A, \tau_A \geq n, T_n \in dt \} \\ &= (\lambda * U_A)'(A) \\ &= \lambda'\tilde{U}_A(A) + \tilde{\lambda}U'_A(A) \\ &= \lambda'\tilde{U}_A(A) + \tilde{\lambda}(I + \tilde{U}_A I_{A^c})Q'(I + I_{A^c}\tilde{U}_A)(A) \\ &= \lambda'1 + \tilde{\lambda}(I + \tilde{U}_A I_{A^c})m, \end{aligned}$$

since $Q'1 = m$, and by recurrence, $\tilde{U}_A(A) = 1$ (see [19], p. 74). □

PROPOSITION 2.2. — Suppose that the MRP (X, T) is φ -recurrent. Then for any $A \in \mathcal{F}^+$,

$$\int_A \pi(dx)\mathbb{E}_x T_A = \pi(m).$$

Thus, (X, T) is positive recurrent, if and only if for some $A \in \mathcal{F}^+$ (and hence for all $A \in \mathcal{F}^+$), $\mathbb{E}_{\pi|_A}[T_A] < \infty$.

Proof. — Applying the preceding lemma with $\lambda = \pi I_A \times \varepsilon_0$, we get

$$\begin{aligned} \mathbb{E}_{\pi|_A}[T_A] &= \pi I_A (I + \tilde{U}_A I_{A^c})m \\ &= \pi(m), \end{aligned}$$

since by (2.8) and ([19], p. 77), $\pi I_A \tilde{U}_A = \pi$. □

From the preceding proposition we immediately get that $\mathbb{E}_x T_A$ is finite for π -almost all $x \in A$. The following proposition shows that this holds for π -almost all $x \in S$. It is a semi-Markov generalization of Proposition 3.1 of Cogburn [4].

PROPOSITION 2.3. — Suppose that (X, T) is positive recurrent. Then $\mathbb{E}_x T_A$ is finite for π -almost all $x \in S$, all $A \in \mathcal{F}^+$.

Proof. — By Lemma 2.1

$$\mathbb{E}_x T_A = m(x) + \tilde{U}_A I_{A^c} m(x).$$

The assertion is a direct consequence of Proposition 5.15 of [15], since m is π -integrable. \square

Using similar arguments as in the proof of Lemma 5.11 of [15] we get the following useful inequality.

LEMMA 2.4. — For any probability measure λ on $\mathcal{F} \times \mathcal{R}_+$ and sets $E, F \in \mathcal{F}$:

$$\mathbb{E}_\lambda T_E \leq \mathbb{E}_\lambda T_F + \sup_{x \in F} \mathbb{E}_x T_E.$$

3. THE SPLITTING TECHNIQUE FOR MARKOV RENEWAL PROCESSES

Recall from (1.9) the definition of the kernels U_f . When $f \equiv \alpha$ for some $\alpha \in (0, 1]$, we have $U_\alpha = \sum_{n \geq 1} (1 - \alpha)^{n-1} Q^{*n}$. We call (M_α) , where $0 < \alpha \leq 1$,

the following minorization condition:

(M_α) : There exist $h \in b\mathcal{F}_+$ with $h < 1$, $\pi(h) > 0$, and a probability measure ν on $\mathcal{F} \times \mathcal{R}_+$ such that $\alpha U_\alpha \geq h \otimes \nu$.

In this section (see Theorem 3.1 and its Corollary) we shall at first seek suitable sufficient conditions for (M_α) to hold for some $\alpha \in (0, 1]$. After that we introduce the splitting technique, for which we have to assume (M_α) for some $\alpha \in (0, 1]$. Theorem 3.1 *i*) is a semi-Markov counterpart to the existence theorem of C-sets for Markov chains (cf. [17], Theorem 2.1).

THEOREM 3.1. — *i*) Suppose that \mathcal{F} is countably generated. Let φ be a non-trivial σ -finite measure on \mathcal{F} . Suppose that there exist $E \in \mathcal{F}$ with

$\varphi(E) > 0$ and an interval $\Gamma_0 \in \mathcal{R}_+$ with $l(\Gamma_0) > 0$ such that for all $x \in E$, $\varphi \times l$ -negligible $N \in \mathcal{F} \times \mathcal{R}_+$ and all $\Gamma \subset \Gamma_0$ with $l(\Gamma) > 0$:

$$(3.1) \quad \mathbb{P}_x \{ (X_n, T_n) \in E \times \Gamma \setminus N \text{ for some } n \geq 1 \} > 0,$$

(or equivalently $U_0(x, E \times \Gamma \setminus N) > 0$). Then there exist $k \geq 1, \beta > 0, C \subset E$ with $\varphi(C) > 0, D \subset E \times \mathbb{R}_+$ with $\varphi \times l(D) > 0$, such that for all $x \in S, A \in \mathcal{F} \times \mathcal{R}_+$

$$(3.2) \quad Q^{*k}(x, A) \geq \beta 1_C(x) \varphi \times l(A \cap D).$$

In particular, (M_2) holds with

$$h = \beta(1 - \alpha)^{k-1} \varphi \times l(D) 1_C, \quad v = \varphi \times l(\cdot \cap D) / \varphi \times l(D).$$

ii) Assume in addition that (X, T) is φ -irreducible. Then h and v can be chosen such that the measure $\int_S h(x)v(dx \times \cdot)$ on \mathcal{R}_+ is non-trivial and absolutely continuous w. r. t. Lebesgue measure.

Proof. — i) We first prove that for all $x \in E$, almost all $u \in \Gamma_0$

$$(3.3) \quad \int_E q(x, y, u) \varphi(dy) > 0,$$

where $q(x, \cdot, \cdot)$ denotes the density of $U_0(x, \cdot)$ (see (1.8)) w. r. t. the measure $\varphi \times l$. Assume the contrary, i. e. that for some $x \in E, \Gamma_3 \subset \Gamma_0$ with $l(\Gamma_3) > 0$

$$\int_E \int_{\Gamma_3} q(x, y, u) \varphi(dy) du = 0.$$

Then there exists $N_x \subset E \times \Gamma_3$ with $\varphi \times l(N_x) = 0$ such that, for all $(y, u) \in E \times \Gamma_3 \setminus N_x, q(x, y, u) = 0$. Denote by N'_x the support of the singular part of the measure $U_0(x, \cdot)$; then $\varphi \times l(N'_x) = 0$. Denote $N = N_x \cup N'_x$. We have

$$U_0(x, E \times \Gamma_3 \setminus N) = \int_E \int_{\Gamma_3} q(x, y, u) \varphi(dy) du = 0,$$

contradicting the assumption (3.1). Hence we have (3.3).

For any $V \in \mathcal{F} \times \mathcal{F} \times \mathcal{R}_+$ denote

$$(3.4) \quad V_1(y, t) = \{ x : (x, y, t) \in V \}, \quad V_2(x, t) = \{ y : (x, y, t) \in V \},$$

$$(3.5) \quad V^{(1)}(x) = \{ (y, t) : (x, y, t) \in V \}, \quad V^{(2)}(y) = \{ (x, t) : (x, y, t) \in V \}.$$

We write $q_m(x, \cdot, \cdot)$ for the density of $Q^{*m}(x, \cdot)$ w. r. t. $\varphi \times l$, and denote $q = \sum_{i \geq 1} q_i$ (q is a version of the density of U_0). The densities q_m ($m \geq 1$),

can be assumed to be jointly measurable w. r. t. $\mathcal{F} \times \mathcal{F} \times \mathcal{R}_+$ (cf. [17], p. 5). We write for $m, n \geq 1$

$$(3.6) \quad H^{(m,n)} = \{ (x, y, t) \in E \times E \times \Gamma_0 : q_m(x, y, t) \geq n^{-1} \},$$

$$H = \bigcup_{m,n \geq 1} H^{(m,n)} = \{ (x, y, t) \in E \times E \times \Gamma_0 : q(x, y, t) > 0 \}.$$

By (3.3) $\varphi(H_2(x, u)) > 0$ on $E \times \Gamma_0$ except on a $\varphi \times l$ -null set. Let $\Gamma_1, \Gamma_2 \in \mathcal{R}_+$ be l -positive sets such that for all $t \in \Gamma_1, \Gamma_2 \subset \Gamma_0 \cap (t - \Gamma_0)$. This is possible, since Γ_0 is an l -positive interval. We have

$$(3.7) \quad 0 < \int_E \int_{\Gamma_2} \varphi(H_2(x, u)) \varphi(dx) du \\ = \int_E \int_{\Gamma_2} \varphi(H_1(y, u)) \varphi(dy) du \quad \text{by Fubini's theorem.}$$

For any fixed $t \in \Gamma_1, t - u \in \Gamma_0$ for all $u \in \Gamma_2$; hence $\varphi(H_2(x, t - u)) > 0$ for all $(x, u) \in E \times \Gamma_2$ except on a $\varphi \times l$ -null set. This and (3.7) imply that

$$\varphi \times l \{ (y, u) \in E \times \Gamma_2 : \varphi(H_1(y, u)) > 0, \varphi(H_2(y, t - u)) > 0 \} > 0.$$

Hence there exist $m_1(t), n_1(t), m_2(t), n_2(t) \in \{ 1, 2, \dots \}$ such that for

$$F^t = H^{(m_1(t), n_1(t))}, \quad G^t = H^{(m_2(t), n_2(t))},$$

we have

$$(3.8) \quad \varphi \times l \{ (y, u) \in E \times \Gamma_2 : \varphi(F_1^t(y, u)) > 0, \\ \varphi(G_2^t(y, t - u)) > 0 \} > 0.$$

By the differentiation theorem of Doob ([5], p. 612, Theorem 2.5), there exists a $\varphi \times \varphi \times l$ -null set N^t such that for all $(x, y, u) \notin N^t$ (in the following we write $t - G^t = \{ (x, y, u) : (x, y, t - u) \in G^t \}$):

$$(3.9) \quad \lim_{n \rightarrow \infty} \frac{\varphi \times \varphi \times l(F^t \cap E_x^n \times E_y^n \times D_u^n)}{\varphi \times \varphi \times l(E_x^n \times E_y^n \times D_u^n)} = 1_{F^t}(x, y, u),$$

$$(3.10) \quad \lim_{n \rightarrow \infty} \frac{\varphi \times \varphi \times l((t - G^t) \cap E_x^n \times E_y^n \times D_u^n)}{\varphi \times \varphi \times l(E_x^n \times E_y^n \times D_u^n)} = 1_{G^t}(x, y, t - u),$$

where E_x^n, E_y^n are defined as in Revuz [19], p. 160, and D_u^n denotes that set in the n 'th partition of \mathbb{R}_+ , to which u belongs (similarly as for S , we construct a sequence $\{ \mathcal{P}_n \}$ of partitions of \mathbb{R}_+ such that \mathcal{P}_{n+1} is finer than \mathcal{P}_n and

$$\mathcal{R}_+ = \bigvee_n \sigma(\mathcal{P}_n).$$

We get from (3.8)

$$\varphi \times l \{ (y, u) \in E \times \Gamma_2 : \varphi((F^t \setminus N^t)_1(y, u)) > 0, \\ \varphi(((t - G^t) \setminus N^t)_2(y, u)) > 0 \} > 0.$$

Hence we can by (3.9), (3.10), choose $n^t \geq 1$, $(x^t, y^t, u^t) \in F^t \setminus N^t$ and $(y^t, z^t, u^t) \in (t - G^t) \setminus N^t$ such that

$$\varphi \times \varphi \times l(F^t \cap E_{x^t}^{n^t} \times E_{y^t}^{n^t} \times D_{u^t}^{n^t}) \geq \frac{3}{4} \varphi(E_{x^t}^{n^t}) \varphi(E_{y^t}^{n^t}) l(D_{u^t}^{n^t}) > 0,$$

and

$$\varphi \times \varphi \times l((t - G^t) \cap E_{y^t}^{n^t} \times E_{z^t}^{n^t} \times D_{u^t}^{n^t}) \geq \frac{3}{4} \varphi(E_{y^t}^{n^t}) \varphi(E_{z^t}^{n^t}) l(D_{u^t}^{n^t}) > 0.$$

Now, since the range of the map

$$\Gamma_1 \ni t \rightarrow (F^t, G^t, E_{x^t}^{n^t}, E_{y^t}^{n^t}, E_{z^t}^{n^t}, D_{u^t}^{n^t})$$

is countable and $l(\Gamma_1) > 0$, there exists $\Gamma_4 \subset \Gamma_1$ with $l(\Gamma_4) > 0$ such that for $t \in \Gamma_4$

$$F^t \equiv F = H^{(m_1, n_1)} \quad \text{for some } m_1, n_1,$$

$$G^t \equiv G = H^{(m_2, n_2)} \quad \text{for some } m_2, n_2,$$

$$E_{x^t}^{n^t} \equiv E_1, E_{y^t}^{n^t} \equiv E_2, E_{z^t}^{n^t} \equiv E_3, D_{u^t}^{n^t} \equiv D_0 \quad \text{for some } E_1, E_2, E_3 \in \mathcal{F}^+, D_0 \in \mathcal{D}_+.$$

Hence for all $t \in \Gamma_4$

$$\varphi \times \varphi \times l(F \cap E_1 \times E_2 \times D_0) \geq \frac{3}{4} \varphi(E_1) \varphi(E_2) l(D_0),$$

$$\varphi \times \varphi \times l((t - G) \cap E_2 \times E_3 \times D_0) \geq \frac{3}{4} \varphi(E_2) \varphi(E_3) l(D_0).$$

Denote

$$C = \left\{ x \in E_1 : \varphi \times l[F^{(1)}(x) \cap E_2 \times D_0] \geq \frac{3}{4} \varphi(E_2) l(D_0) \right\},$$

$$D = \left\{ (z, t) \in E_3 \times \Gamma_4 : \varphi \times l[(t - G^{(2)}(z)) \cap E_2 \times D_0] \geq \frac{3}{4} \varphi(E_2) l(D_0) \right\}.$$

Then $\varphi(C) > 0$ and $\varphi \times l(D) > 0$, and for all $x \in C, (z, t) \in D$

$$\varphi \times l[F^{(1)}(x) \cap (t - G^{(2)}(z))] \geq \frac{1}{2} \varphi(E_2) l(D_0),$$

and for $k = m_1 + m_2$ (cf. [17], p. 5, Proposition 1.2)

$$q_k(x, z, t) \geq \int_{F^{(1)}(x) \cap (t - G^{(2)}(z))} q_{m_1}(x, y, u) q_{m_2}(y, z, t - u) \varphi(dy) du \\ \geq n_1^{-1} n_2^{-1} \varphi \times l[F^{(1)}(x) \cap (t - G^{(2)}(z))] \\ \geq (2n_1 n_2)^{-1} \varphi(E_2) l(D_0) > 0. \quad \square$$

ii) Denote $\nu_0 = \varphi \times l(\cdot \cap D)$, and choose $m \geq 1$ such that

$$\tilde{\nu}_0 \tilde{Q}^m(C) > 0.$$

Then

$$Q^{*(k+m)} \geq \beta 1_C \otimes (\nu_0 * Q^{*m}),$$

and the measure $\int_S \beta 1_C(x)(\nu_0 * Q^{*m})(dx \times \cdot) = \beta \nu_0 * Q^{*m}(C \times \cdot)$ is non-trivial and absolutely continuous w. r. t. Lebesgue measure. \square

COROLLARY. — Let φ be a non-trivial σ -finite measure on \mathcal{F} . Suppose that there exist $E \in \mathcal{F}$ with $\varphi(E) > 0$ and an interval $\Gamma_0 \in \mathcal{R}_+$ with $l(\Gamma_0) > 0$ such that for all $x \in E$, $F \subset E$ with $\varphi(F) > 0$ and $\Gamma \subset \Gamma_0$ with $l(\Gamma) > 0$:

$$(3.11) \quad \mathbb{P}_x \{ X_n \in F, T_n \in \Gamma \text{ for some } n \geq 1 \} > 0.$$

Then the conclusions of Theorem 3.1 are valid. \square

Remark. — Conditions (3.1) and (3.11) can be compared with the concept of spread-outness for measures on \mathcal{R}_+ (see [19], p. 90). As a trivial corollary, part ii) of Theorem 3.1 gives a sufficient condition for the spread-outness of the measure $\int_S h(x)\nu(dx \times \cdot)$ (cf. the assumptions of Theorems 5.1 and 5.2). \square

Next we shall describe the splitting technique for Markov renewal processes. It is an obvious extension of the splitting technique for Markov chains, and the reader is referred to [15] for a detailed study of this method in the context of Markov chains.

The splitting method introduces an atom, in a way which we shall next describe, to a Markov renewal process satisfying condition (M_1) , i. e. $Q \geq h \otimes \nu$. The general case (M_α) , $\alpha \leq 1$, can then be treated by considering the MRP $\{ (X_n^{(\alpha)}, T_n^{(\alpha)}); n \geq 0 \}$ corresponding to the semi-Markov kernel

$$\alpha U_\alpha = \alpha \sum_{n \geq 0} (1 - \alpha)^{n-1} Q^{*n}.$$

Let us assume in the rest of this section that condition (M_1) holds. As in [15], Section 2, we denote for all $x \in S$, $E \in \mathcal{F}$:

$$(3.12) \quad \begin{aligned} x_0 &= (x, 0), & x_1 &= (x, 1); \\ E_0 &= E \times \{0\}, & E_1 &= E \times \{1\}, & E^* &= E \times \{0, 1\}; \end{aligned}$$

\mathcal{F}^* denotes the σ -algebra generated by the sets E_i , ($E \in \mathcal{F}$, $i = 0, 1$). We identify E and E^* for $E \in \mathcal{F}$, in particular we can then write $\mathcal{F} \subset \mathcal{F}^*$.

Any function f on S is extended to S^* by defining

$$(3.13) \quad f(x_1) = f(x_0) = f(x), \quad (x \in S);$$

any measure λ on \mathcal{F} is split onto \mathcal{F}^* by defining for all $E \in \mathcal{F}$

$$(3.14) \quad \lambda(E_0) = \lambda I_{1-h}(E), \quad \lambda(E_1) = \lambda I_h(E).$$

We define the transition kernel \bar{Q} from S^* into $\mathcal{F} \times \mathcal{R}_+$ as follows:

$$(3.15a) \quad \bar{Q}(x_0, A \times \Gamma) = [1 - h(x)]^{-1}(Q(x, A \times \Gamma) - h(x)v(A \times \Gamma)),$$

$$(3.15b) \quad \bar{Q}(x_1, A \times \Gamma) = v(A \times \Gamma), \quad x \in S, \quad A \in \mathcal{F}, \quad \Gamma \in \mathcal{R}_+;$$

and we define the semi-Markov kernel Q^* on (S^*, \mathcal{F}^*) as follows: for any fixed $z \in S^*$, $\Gamma \in \mathcal{R}_+$, $Q^*(z, \cdot \times \Gamma)$ is the splitting onto \mathcal{F}^* (see (3.14)), of the measure $\bar{Q}(z, \cdot \times \Gamma)$ on \mathcal{F} ; i. e. for all $A \in \mathcal{F}$

$$(3.16a) \quad Q^*(z, A_0 \times \Gamma) = \int_A \bar{Q}(z, dy \times \Gamma)(1 - h(y)),$$

$$(3.16b) \quad Q^*(z, A_1 \times \Gamma) = \int_A \bar{Q}(z, dy \times \Gamma)h(y),$$

(the superscript '*' in (3.16) should not be confused with the notation for convolution). We shall denote by $(X^*, T^*) = \{X_n^*, T_n^*; n \geq 0\}$ the MRP corresponding to Q^* and with state space (S^*, \mathcal{F}^*) . From (3.15b) we see that the set $S_1 \subset S^*$ is an atom for the MRP (X^*, T^*) . It is easily seen that the embedded Markov chain $\{X_n^*\}$ of (X^*, T^*) is the splitting, as defined in Section 2 of [15], of the embedded Markov chain $\{X_n\}$ of (X, T) ; in particular (X^*, T^*) is φ -recurrent, if (X, T) is φ -recurrent (use Theorem 2.3 of [15]). Since $\varphi(S_1) = \varphi(h) > 0$, we see that, in the case when (X, T) is φ -recurrent, S_1 is a recurrent atom for (X^*, T^*) .

We denote $X_n^* = (X_n, Y_n)$, (where $X_n \in S, Y_n \in \{0, 1\}$), and call X_n the first coordinate of X_n^* . The following proposition is easy to prove by using the definition of the splitting.

PROPOSITION 3.3. — *i)* (cf. Proposition 2.1 of [15]). For any measures μ on $\mathcal{F} \times \mathcal{R}_+$, η on $\mathcal{F}^* \times \mathcal{R}_+$

$$\begin{aligned} \mu * \bar{Q} &= \mu * Q && (\text{on } \mathcal{F} \times \mathcal{R}_+), \\ \mu * Q^* &= \mu * Q && (\text{on } \mathcal{F}^* \times \mathcal{R}_+), \\ \eta * Q^* &= \eta * \bar{Q} && (\text{on } \mathcal{F}^* \times \mathcal{R}_+); \end{aligned}$$

(the measure μ on $\mathcal{F} \times \mathcal{R}_+$ is split onto $\mathcal{F}^* \times \mathcal{R}_+$ by splitting the marginal measures $\mu(\cdot \times \Gamma)$ onto \mathcal{F}^* , ($\Gamma \in \mathcal{R}_+$)).

ii) (cf. Proposition 2.2 of [15]). For any probability measure μ on

$\mathcal{F} \times \mathcal{R}_+$ the marginal distribution w. r. t. \mathbb{P}_μ^* (\mathbb{P}_μ^* denotes the canonical probability measure on the sample space $(S^* \times \mathbb{R}_+, \mathcal{F}^* \times \mathcal{R}_+)^{\infty}$ of (X^*, T^*) , induced by the initial probability μ and the kernel Q^*), of the « first coordinate MRP » $(X, T^*) = \{(X_n, T_n^*)\}$ and the \mathbb{P}_μ -distribution of the original MRP $(X, T) = \{(X_n, T_n)\}$ are identical. \square

4. MARKOV RENEWAL PROCESSES POSSESSING AN ATOM

Throughout this section we shall assume the existence of a recurrent atom $B \subset S$, i. e. B is such that

$$(4.1) \quad Q(x, \cdot) \equiv Q(B, \cdot), \quad x \in B,$$

(recall the convention of notation in Section 1).

Note that for the split MRP (X^*, T^*) introduced in Section 3 the set $B = S_1 \subset S^*$ is an atom. The following notations for an MRP having an atom B will be used in the sequel:

$$(4.2) \quad F(x, \Gamma) = U_B(x, B \times \Gamma) = \mathbb{P}_x \{ T_B \in \Gamma \},$$

$$(4.3) \quad V(\Gamma) = R(B, B \times \Gamma) = \varepsilon_0(\Gamma) + U_0(B, B \times \Gamma),$$

$$(4.4) \quad F_B(\Gamma) = F(B, \Gamma) = \mathbb{P}_B \{ T_B \in \Gamma \},$$

$$(4.5) \quad \psi(A \times \Gamma) = U_B(B, A \times \Gamma) \\ = \mathbb{E}_B \left[\sum_{n=1}^{T_B} 1_{\{X_n \in A, T_n \in \Gamma\}} \right], \quad x \in S, \quad A \in \mathcal{F}, \quad \Gamma \in \mathcal{R}_+.$$

By using standard renewal arguments and the fact that B is an atom, we can prove:

LEMMA 4.1. — i) V is a renewal measure,

$$(4.6) \quad V = \sum_{n=0}^{\infty} F_B^{*n}.$$

ii) We have the following first-entrance-last-exit decomposition of U_0

$$(4.7) \quad U_0 = U_B + F * V * \psi,$$

$(F * V * \psi(x, A \times \Gamma))$ means $(F(x, \cdot) * V * \psi(A \times \cdot))(\Gamma)$. \square

Note that for all $A \in \mathcal{F}$

$$(4.8) \quad \tilde{\psi}(A) = \psi(A \times \mathbb{R}_+) = \mathbb{E}_B \left[\sum_{n=1}^{\tau_B} 1_{\{X_n \in A\}} \right] = \pi(A)/\pi(B),$$

where π denotes the invariant measure of the φ -recurrent Markov chain $\{X_n\}$, (φ any probability measure concentrated on B).

We recall from [I] the following theorems:

THEOREM 4.2. — ([I], Theorem 7) Assume that F_B is spread out (i. e. for some $n \geq 1$, F_B^{*n} has an absolutely continuous component w. r. t. Lebesgue measure). Then for any measurable $f: S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying

$$(4.10a) \quad \pi(\bar{f}) < \infty,$$

where $\bar{f}(\cdot)$ is any measurable version of $\sup_{t \geq 0} f(\cdot, t)$;

$$(4.10b) \quad \pi \times l(f) < \infty;$$

$$(4.10c) \quad \text{for all } x \in S, \quad \lim_{t \rightarrow \infty} f(x, t) = 0;$$

and for any probability measure λ on $\mathcal{F} \times \mathcal{R}_+$ such that

$$(4.11) \quad \mathbb{E}_{\tilde{\lambda}} \left[\sum_{n=0}^{\tau_B-1} \bar{f}(X_n) \right] = \tilde{\lambda}(I + \tilde{U}_B I_{B^c}) \bar{f} < \infty,$$

we have

$$(4.12) \quad \lim_{t \rightarrow \infty} \sup_{|g| \leq f} |\lambda * R * g(t) - [\pi(m)]^{-1} \pi \times l(g)| = 0. \quad \square$$

THEOREM 4.3. — ([I], Theorem 8). Assume that F_B is spread out, (X, T) is positive recurrent and $\pi(S) < \infty$. For any finite signed measure λ on $\mathcal{F} \times \mathcal{R}_+$ with $\lambda(S \times \mathbb{R}_+) = 0$, and such that

$$(4.13) \quad \mathbb{E}_{|\lambda|}[\tau_B] < \infty,$$

$$(4.14) \quad \mathbb{E}_{|\lambda|}[\tau_B] < \infty,$$

we have

$$(4.15) \quad \|\lambda * R\| = |\lambda * R|(S \times \mathbb{R}_+) < \infty. \quad \square$$

5. UNIFORM MARKOV RENEWAL LIMIT THEOREMS

In this section we extend Theorems 4.2 and 4.3 to the more general case, where instead of an atom we assume only the minorization condi-

tion (M_α) (Theorems 5.1 and 5.2). The proofs are based on the splitting technique described in Section 3: we can apply Theorems 4.2 and 4.3 to the MRP (X^*, T^*) having the atom S_1 . We shall compare our results with those of Jacod, Kesten and McDonald at the end of this section.

We recall from [15] the concept of a *g-regular measure* $(g : S \rightarrow \mathbb{R}_+)$. A probability measure μ on \mathcal{F} is called *g-regular* (w. r. t. the embedded Markov chain $\{X_n\}$), provided that

$$(5.1) \quad \mu \tilde{U}_A g = \mathbb{E}_\mu \left[\sum_{n=1}^{\tau_A} g(X_n) \right] < \infty \quad \text{for all } A \in \mathcal{F}^+.$$

(for a detailed study of this concept see Section 5 of [15] and Section 1 of [16]).

At first we prove the generalization of Theorem 4.2:

THEOREM 5.1. — Assume that the MRP (X, T) is φ -recurrent, satisfies condition (M_α) for some $\alpha \in (0, 1]$, and that the measure $\int_S h(x)v(dx \times \cdot)$ on \mathcal{R}_+ is spread-out (cf. Theorem 3.1 *i*) and *ii*). Then for any measurable $f : S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying (4.10) and for any probability measure λ on $\mathcal{F} \times \mathcal{R}_+$ such that $\tilde{\lambda}(f) < \infty$ and $\tilde{\lambda}$ is an *f-regular* probability measure, we have

$$(5.2) \quad \lim_{t \rightarrow \infty} \sup_{|g| \leq f} |\lambda * R * g(t) - [\pi(m)]^{-1} \pi \times l(g)| = 0.$$

Proof. — *i*) We consider first the special case $\alpha = 1$, i. e. we assume that $Q \geq h \otimes v$. We construct the MRP (X^*, T^*) as described in Section 3. It is easy to see that π (split onto \mathcal{F}^*) is invariant for the embedded Markov chain $\{X_n^*\}$, and that $\pi(m^*) = \pi(m)$. In particular, (X^*, T^*) is positive recurrent if and only if (X, T) is. The set $S_1 \subset S^*$ is a recurrent atom for (X^*, T^*) . For F_{S_1} we have (see (4.4))

$$(5.3) \quad \begin{aligned} F_{S_1}(\Gamma) &= \mathbb{P}_z^* \{ T_{S_1} \in \Gamma \} && (z \in S_1, \Gamma \in \mathcal{R}_+), \\ &\geq \mathbb{P}_z^* \{ X_1^* \in S_1, T_1^* \in \Gamma \} \\ &= Q^*(z, S_1 \times \Gamma) \\ &= v(S_1 \times \Gamma) && \text{by (3.16b),} \\ &= \int_S h(x)v(dx \times \Gamma) && \text{by (3.14).} \end{aligned}$$

Since, by assumption, the measure $\int_S h(x)v(dx \times \cdot)$ is spread-out, F_{S_1} is spread-out, too.

In order to apply Theorem 4.2 we have to check that (4.11) holds with

$B = S_1$. By Lemma 5.12 and Theorem 5.14 of [15] we know that $\tilde{\lambda}$ is \bar{f} -regular for $\{X_n\}$ if and only if

$$(5.4) \quad \tilde{\lambda} \tilde{U}_{S_1}^* \bar{f} < \infty.$$

Hence we have by assumption

$$(5.5) \quad \tilde{\lambda}(I + \tilde{U}_{S_1}^* I_{S_0}) \bar{f} < \infty.$$

Now Theorem 4.2 gives (recall the convention (3.13))

$$(5.6) \quad \lim_{t \rightarrow \infty} \sup_{|g| \leq f, g: S \rightarrow \mathbb{R}} |\lambda * R^* * g(t) - [\pi(m)]^{-1} \pi \times l(g)| \\ \leq \lim_{t \rightarrow \infty} \sup_{|g| \leq f, g: S^* \rightarrow \mathbb{R}} |\lambda * R^* * g(t) - [\pi(m)]^{-1} \pi \times l(g)| = 0,$$

where R^* denotes the Markov renewal kernel corresponding to Q^* . The assertion follows after observing that by Proposition 3.3 i)

$$(5.7) \quad \lambda * R^* = \sum_{n \geq 0} \lambda * (Q^*)^{*n} \quad ((Q^*)^{*n} \text{ denotes the } n\text{'th convolution} \\ \text{power of the kernel } Q^*), \\ = \sum_{n \geq 0} \lambda * Q^{*n} \\ = \lambda * R.$$

i) Assume now that $\alpha \in (0, 1)$. Then the MRP $(X^{(\alpha)}, T^{(\alpha)})$ corresponding to the kernel αU_α satisfies condition (M_1) . It is easy to see that π is invariant for the Markov chain $\{X_n^{(\alpha)}\}$ with transition probability

$$\alpha \tilde{U}_\alpha = \alpha \sum_{n \geq 1} (1 - \alpha)^{n-1} \tilde{Q}^n.$$

By (2.2)

$$(5.8) \quad m^{(\alpha)} = \alpha U_\alpha' 1 \quad (\text{cf. the proof of Lemma 2.1}), \\ = \alpha(I + (1 - \alpha)\tilde{U}_\alpha)Q'(I + (1 - \alpha)\tilde{U}_\alpha)1 \\ = (I + (1 - \alpha)\tilde{U}_\alpha)m \\ = \sum_{n \geq 0} (1 - \alpha)^n \tilde{Q}^n m$$

from which we get that $\pi(m^{(\alpha)}) = \alpha^{-1} \pi(m)$.

By Lemma 1.2 of [16], λ is \bar{f} -regular w. r. t. the Markov chain $\{X_n^{(\alpha)}\}$, too. Hence, by part i), we get

$$(5.9) \quad \lim_{t \rightarrow \infty} \sup_{|g| \leq f} |\lambda * R^{(\alpha)} * g(t) - \alpha[\pi(m)]^{-1} \pi \times l(g)| = 0,$$

where

$$\begin{aligned}
 (5.10) \quad \mathbf{R}^{(\alpha)} &= \sum_{n \geq 0} \alpha^n \mathbf{U}_\alpha^{*n} \\
 &= \Delta + \alpha \mathbf{U}_0 && \text{by the Resolvent equation,} \\
 &= (1 - \alpha)\Delta + \alpha \mathbf{R}.
 \end{aligned}$$

The final assertion follows from (5.9), (5.10) and from the remark that by Egoroff's theorem, by (4.10c), and since $\tilde{\lambda}(\bar{f}) < \infty$,

$$\lim_{t \rightarrow \infty} \lambda * f(t) = 0. \quad \square$$

COROLLARY. — Assume that (X, T) is ϕ -recurrent, satisfies condition (M_α) for some $\alpha \in (0, 1]$, and that the measure $\int_S h(x)v(dx \times \cdot)$ is spread-out.

i) For any $\pi \times l$ -integrable and bounded $f : S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\lim_{t \rightarrow \infty} f(x, t) = 0$ for all $x \in S$ and the function \bar{f} is special w. r. t. $\{X_n\}$ in the sense of Neveu [13] (i. e. $\sup_{x \in S} U_A \bar{f}(x) < \infty$ for all $A \in \mathcal{F}^+$; see also [15], Lemma 5.6), and for any probability measure λ on $\mathcal{F} \times \mathbb{R}_+$, we have (5.2).

ii) Let $f : S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfy (4.10). Then for π -almost all $x \in S$

$$(5.11) \quad \lim_{t \rightarrow \infty} \sup_{|g| \leq f} |\mathbf{R} * g(x, t) - [\pi(m)]^{-1} \pi \times l(g)| = 0.$$

Proof. — i) Note that f satisfies (4.10a), since \bar{f} is special ([13], Section 4). As a direct consequence of the definitions we get that any probability measure on \mathcal{F} is \bar{f} -regular. Since \bar{f} is bounded, we have $\tilde{\lambda}(\bar{f}) < \infty$. Theorem 5.1 now gives the assertion.

ii) Note that $\bar{f}(x) < \infty$ and ε_x is \bar{f} -regular for π -almost all $x \in S$ ([15], Theorem 5.14). Theorem 5.1 with $\lambda = \varepsilon_{(x,0)}$ gives (5.11). \square

Next we prove the generalization of Theorem 4.3:

THEOREM 5.2. — Assume that the MRP (X, T) is positive recurrent, satisfies condition (M_α) for some $\alpha \in (0, 1]$, the measure $\int_S h(x)v(dx \times \cdot)$ is spread-out, and that $\pi(S) < \infty$. Then for any finite signed measure λ on $\mathcal{F} \times \mathbb{R}_+$ with $\lambda(S \times \mathbb{R}_+) = 0$, such that

$$(5.12) \quad \mathbb{E}_{|\lambda|}[\tau_A] < \infty \quad \text{for all } A \in \mathcal{F}^+ \quad (\text{i. e. } |\lambda| \sim \text{is 1-regular}),$$

$$(5.13) \quad \mathbb{E}_{|\lambda|}[\mathbf{T}_A] < \infty \quad \text{for all } A \in \mathcal{F}^+,$$

we have

$$(5.14) \quad \|\lambda * \mathbf{R}\| < \infty.$$

Before proving the theorem, we consider a preliminary result, which has also some independent interest giving equivalent conditions for (5.13).

LEMMA 5.3. — Assume only that the MRP (X, T) is positive recurrent. For any $\alpha \in (0, 1]$ and any probability measure μ on $\mathcal{F} \times \mathcal{R}_+$ the following three conditions are equivalent :

- i) $\mu'1 < \infty$ and $\mathbb{E}_{\tilde{\mu}} \sum_{n=0}^{\tau_A-1} m(X_n) = \tilde{\mu}(I + \tilde{U}_A I_{A^c})m < \infty$ for all $A \in \mathcal{F}^+$;
 ii) $\mathbb{E}_{\mu} T_A < \infty$ for all $A \in \mathcal{F}^+$;
 iii) $\mathbb{E}_{\mu} T_A^{(\alpha)} \stackrel{\text{def}}{=} \mathbb{E}_{\tilde{\mu}} [\inf \{ T_n^{(\alpha)} ; n \geq 1, X_n^{(\alpha)} \in A \}] < \infty$ for all $A \in \mathcal{F}^+$.

The following condition iv) implies the equivalent conditions i)-iii):

- iv) $\mu'1 < \infty$, $\tilde{\mu}(m) < \infty$ and $\tilde{\mu}$ is m -regular .

Proof. — The equivalence of i) and ii) follows directly from Lemma 2. 1. By the same lemma

$$\begin{aligned}
 (5.15) \quad \mathbb{E}_{\mu} T_A^{(\alpha)} &= \mu'1 + \tilde{\mu}(I + \tilde{U}_A^{(\alpha)} I_{A^c})m^{(\alpha)} \\
 &= \mu'1 + \tilde{\mu}(I + \alpha \tilde{U}_{\alpha 1_A} I_{A^c})(I + (1 - \alpha)\tilde{U}_{\alpha})m \\
 &\quad \text{by Lemma 1.1 of [15] and by (5.8),} \\
 &= \mu'1 + \tilde{\mu}(I + \tilde{U}_{\alpha 1_A} I_{1-\alpha 1_A})m \quad \text{by [19], p. 49,} \\
 &\geq \mu'1 + \tilde{\mu}(I + \tilde{U}_A I_{A^c})m = \mathbb{E}_{\mu} T_A .
 \end{aligned}$$

The last inequality gives the implication iii) \Rightarrow ii).

Assume now that ii) holds. Let $A \in \mathcal{F}^+$ be arbitrary. By Theorem 5.14 iii) or [15], and since m is π -integrable, we can choose $G \in \mathcal{F}^+$ such that $G \subset A$, $\sup_{x \in G} m(x)$ and $\sup_{x \in G} U_G m(x)$ are bounded, say by $\gamma < \infty$. We have

$$\begin{aligned}
 (5.16) \quad \mathbb{E}_{\mu} T_A^{(\alpha)} &\leq \mathbb{E}_{\mu} T_G^{(\alpha)} \\
 &= \mu'1 + \tilde{\mu}(I + \tilde{U}_{\alpha 1_G} I_{1-\alpha 1_G})m \\
 &= \mu'1 + \tilde{\mu}(I + (1 - \alpha)\tilde{U}_{\alpha 1_G} I_G)(I + \tilde{U}_G I_{G^c})m \quad \text{by [19], p. 49,} \\
 &= \mu'1 + \tilde{\mu} \sum_{n \geq 0} (1 - \alpha)^n (\tilde{U}_G I_G)^n (I + \tilde{U}_G I_{G^c})m \quad \text{by [19], p. 48,} \\
 &= \mu'1 + \tilde{\mu}(I + \tilde{U}_G I_{G^c})m + \tilde{\mu} \sum_{n \geq 1} (1 - \alpha)^n (\tilde{U}_G I_G)^n (1 + \tilde{U}_G I_{G^c})m \\
 &\leq \mathbb{E}_{\mu} T_G + 2\gamma \tilde{\mu} \sum_{n \geq 1} (1 - \alpha)^n (\tilde{U}_G I_G)^n 1 \\
 &= \mathbb{E}_{\mu} T_G + 2\gamma(1 - \alpha^{-1}), \quad \text{since } \tilde{U}_G 1_G \equiv 1 \quad \text{by recurrence,} \\
 &< \infty \quad \text{by assumption.}
 \end{aligned}$$

That *iv)* implies *i)* follows directly from the definition of *m*-regularity. \square

We now turn to the proof of the theorem.

PROOF OF THEOREM 5.2. — *i)* Assume first that $\alpha = 1$. We again construct the MRP (X^*, T^*) . As in the proof of Theorem 4.2 we get that F_{S_1} is spread-out and (X^*, T^*) is positive recurrent. By Proposition 2.3, $\mathbb{E}_z^* T_{S_1}$ is finite for π -almost all $z \in S^*$. Using similar arguments as in the proof of Theorem 5.14 of [15] we can find $E \in \mathcal{F}^+$ such that $\sup_{z \in E^*} \mathbb{E}_z^* T_S$ is finite. Then by Lemma 2.4, Proposition 3.3 and (5.13)

$$(5.18) \quad \mathbb{E}_{[\lambda]}^* T_{S_1} \leq \mathbb{E}_{[\lambda]} T_E + \sup_{z \in E^*} \mathbb{E}_z^* T_{S_1} < \infty .$$

Theorem 4.3 now gives

$$\begin{aligned} \infty > \| \lambda * R^* \| &= 2 \sup_{E \in \mathcal{F}^* \times \mathcal{R}_+} | \lambda * R^*(E) | \\ &\geq 2 \sup_{E \in \mathcal{F} \times \mathcal{R}_+} | \lambda * R^*(E) |, \quad (\mathcal{F} \subset \mathcal{F}^*), \\ &= 2 \sup_{E \in \mathcal{F} \times \mathcal{R}_+} | \lambda * R(E) |, \quad (\text{cf. (5.7)}) \\ &= \| \lambda * R \| . \end{aligned}$$

ii) Assume now that $\alpha \in (0, 1)$. By Lemma 1.3 of [16] and by assumption, $|\lambda| \sim$ is 1-regular w. r. t. $\{X_n^{(\alpha)}\}$. By Lemma 5.3 and by assumption, $\mathbb{E}_{[\lambda]} T_A^{(\alpha)} < \infty$ for all $A \in \mathcal{F}^+$. Hence, by part *i)* and by (5.10)

$$(5.19) \quad \| \lambda * R \| \leq \alpha^{-1} \| \lambda * R^{(\alpha)} \| + (\alpha^{-1} - 1) \| \lambda \| < \infty . \quad \square$$

COROLLARY. — Assume that (X, T) is positive recurrent, satisfies condition (M_α) for some $\alpha \in (0, 1]$, the measure $\int_S h(x) \nu(dx \times \cdot)$ is spread-out, and that $\pi(S) < \infty$. Then for π -almost all $x, y \in S$

$$(5.20) \quad \| R(x, \cdot) - R(y, \cdot) \| < \infty .$$

Proof. — Note that for π -almost all $x \in S$, $\mathbb{E}_x \tau_A < \infty$ (since $\pi(S) < \infty$), and $\mathbb{E}_x T_A < \infty$ (Proposition 2.3). Theorem 5.2 with $\lambda = \varepsilon_{(x,0)} - \varepsilon_{(y,0)}$ gives (5.20). \square

Jacod (Théorème 3 of [8]) has given sufficient conditions for the *weak convergence* of the measures $R(x, A \times (t + \cdot))$, $x \in S$, $A \in \mathcal{F}$, $t \rightarrow \infty$. Jacod makes the following assumptions: 1) the embedded chain $\{X_n\}$ is φ -recurrent, 2) the closed group generated by the *support* of the process (see Definition on p. 87 of [7]) is $\mathbb{R} \times d\mathbb{Z}$, and 3) A is such that, for some $\varepsilon > 0$, $\sup_{x \in A} R(x, A \times [0, \varepsilon])$ is finite. Assumption 1 is also one of our basic assumptions. Assumption 2 is somewhat weaker than our condition (M_α) . We do not need assumption 3. Jacod is concerned only with the weak

convergence of the Markov renewal measure in contrast with our uniform convergence results.

Kesten [9] has two kinds of Markov renewal theorems. Theorems 1 and 2 of [9] involve some topological assumptions about the state space and the embedded Markov chain, and the results involve the continuity of the functions appearing in the theorems. Theorems 3 and 4 of Kesten have as basic assumptions for the embedded chain the total variation convergence to an invariant probability measure and a certain assumption on the support of the process (see Kesten's Condition II.3). The first assumption is somewhat weaker than the assumption of positive recurrence. The second assumption is somewhat weaker than our condition (M_α) . On the other hand, Kesten's results involve stronger assumptions on the function f (continuity w. r. t. the argument t or a directly Riemann integrability condition), and the convergence is not uniform over the set of functions.

McDonald [10], [11], gives sufficient conditions for the total variation convergence of the « age process », that is the continuous time Markov process $(X(t), U(t))$ defined by

$$X(t) = X_{n-1}, \quad U(t) = t - T_{n-1} \quad \text{for} \quad T_{n-1} \leq t \leq T_n,$$

and deduces from these results uniform Markov renewal theorems (uniform only over the space S ; cf. our Theorems 5.1 and 5.2 and McDonald's [11] Proposition 2, Corollary 2 and Theorem 3). McDonald makes two basic assumptions: a mixing condition (Definition 3 of [10]) and a condition, which states that, for some distribution function G with finite mean, the sojourn time distributions are bounded below uniformly by G ,

$$Q(x, S \times [0, t]) \geq G(t) \quad \text{for all } x \in S, \quad \text{all } t \geq 0$$

(cf. Definition 5 of [10] and Theorems 2, 3 and 4 of [11]). Again the first assumption is somewhat weaker than our condition (M_α) . The second assumption of McDonald is, on the other hand, more restrictive than ours. As mentioned above, the statements of our Theorems 5.1 and 5.2 are stronger than those of McDonald.

6. RATIO LIMIT THEOREMS

In this section we study the limit of the ratio

$$(6.1) \quad \frac{\lambda * U_0 * f(t)}{\mu * U_0 * g(t)} = \frac{\sum_{n \geq 1} \lambda * Q^{*n} * f(t)}{\sum_{n \geq 1} \mu * Q^{*n} * g(t)} \quad \text{as } t \rightarrow \infty,$$

where λ and μ are probability measures on $\mathcal{F} \times \mathcal{R}_+$, and f and g are non-negative measurable functions on $S \times \mathbb{R}_+$. For earlier works on this subject the reader is referred to Jacod [7], Section II. 3. Our results complement Jacod's results allowing more general starting distributions λ, μ and functions f, g (cf. our Corollary 3).

The corresponding problem for discrete time φ -recurrent Markov chains (ratio limit for sums of transition probabilities) has been investigated e. g. in [12], [14] and [15]. For the most general result for φ -recurrent Markov chains the reader is referred to [15], Section 7. There has been proved that for P a φ -recurrent transition probability on (S, \mathcal{F}) with invariant measure π , for λ and μ probability measures on \mathcal{F} , and for f and g non-negative π -integrable functions on S , such that λ is f -regular and μ is g -regular, the

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \lambda P^i f}{\sum_{i=1}^n \mu P^i g} \text{ exists and equals } \pi(f)/\pi(g).$$

Here we shall extend this result to Markov renewal processes. We assume throughout this section that the MRP (X, T) is φ -recurrent, and that the minorization condition (M_α) holds (recall that Theorem 3.1 gives a sufficient condition for (M_α)). In the following we call $f : S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ non-decreasing, provided that $f(x, \cdot)$ is non-decreasing for all $x \in S$, and we denote by $\bar{f}(x)$ the limit $\lim_{t \rightarrow \infty} f(x, t)$.

THEOREM 6.1. — For any probability measures λ and μ on $\mathcal{F} \times \mathcal{R}_+$ and any measurable non-decreasing functions $f, g : S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$(6.2) \quad \tilde{\lambda} \text{ is } f\text{-regular and } \tilde{\mu} \text{ is } g\text{-regular,}$$

$$(6.3) \quad \pi(\bar{f}) < \infty \quad \text{and} \quad 0 < \pi(\bar{g}) < \infty,$$

we have

$$(6.4) \quad \lim_{t \rightarrow \infty} \frac{\lambda * U_0 * f(t)}{\mu * U_0 * g(t)} = \frac{\pi(\bar{f})}{\pi(\bar{g})}.$$

Proof. — The proof is similar to the corresponding proof in the Markov chain case (see [15], Section 7):

i) Assume first that there exists an atom $B \subset S$ such that $\tilde{\lambda} \tilde{U}_B \bar{f} < \infty$ and $\tilde{\mu} \tilde{U}_B \bar{g} < \infty$. By Lemma 4.1 ii), and since f is non-decreasing, we get

$$(6.5) \quad \begin{aligned} \sup_{t \in \mathbb{R}_+} | \lambda * U_0 * f(t) - \lambda * F * V * \psi * f(t) | \\ \leq \sup_{t \in \mathbb{R}_+} \lambda * U_B * f(t) \\ \leq \tilde{\lambda} \tilde{U}_B \bar{f} < \infty, \end{aligned}$$

and similarly for $\mu * U_0 * g$.

Since by recurrence $\lim_{t \rightarrow \infty} \mu * U_0 * g(t) = \tilde{\mu}(\bar{g}) \cdot \infty = \infty$, we get

$$(6.6) \quad \lim_{t \rightarrow \infty} \frac{\lambda * U_0 * f(t)}{\mu * U_0 * g(t)} = \lim_{t \rightarrow \infty} \frac{\lambda * F * V * \psi * f(t)}{\mu * F * V * \psi * g(t)}.$$

By Lemma 3.1 of Pyke and Schaufele [18],

$$(6.7) \quad \begin{aligned} \lim_{t \rightarrow \infty} \frac{(\lambda * F) * V * (\psi * f)(t)}{V * 1(t)} &= \lim_{t \rightarrow \infty} \lambda * F * \psi * f(t) \\ &= \tilde{\psi}(\bar{f}) \\ &= \frac{\pi(\bar{f})}{\pi(\mathbf{B})} \end{aligned} \quad (\text{see (4.8)}),$$

from which the assertion follows.

ii) In the case when we have only (M_1) , we can apply part i) to the MRP (X^*, T^*) . Note that, since $\tilde{\lambda}$ is \bar{f} -regular, we have $\tilde{\lambda}U_{S_1}^* \bar{f} < \infty$, and similarly $\tilde{\mu}U_{S_1}^* \bar{g} < \infty$ (cf. Theorem 5.14 of [15]). By part i)

$$(6.8) \quad \lim_{t \rightarrow \infty} \frac{\lambda * U_0 * f(t)}{\mu * U_0 * g(t)} = \frac{\pi(\bar{f})}{\pi(\bar{g})},$$

(note that by Proposition 3.3 $\lambda * U_0^* * f = \lambda * U_0 * f$).

iii) The final assertion now follows by applying part ii) to the MRP $(X^{(\alpha)}, T^{(\alpha)})$ after observing that

$$(6.8) \quad U_0^{(\alpha)} = \sum_{n \geq 1} \alpha^n U_\alpha^n = \alpha U_0$$

and $\tilde{\lambda}$ is \bar{f} -regular w. r. t. $\{X_n\}$ if and only if $\tilde{\lambda}$ is \bar{f} -regular w. r. t. $\{X_n^{(\alpha)}\}$ ([16], Lemma 1.2). \square

COROLLARY 1. — For any non-decreasing functions $f, g : S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that f, g are π -integrable, $\pi(\bar{g}) > 0$, for π -almost all $x, y \in S$

$$(6.10) \quad \lim_{t \rightarrow \infty} \frac{U_0 * f(x, t)}{U_0 * g(x, t)} = \frac{\pi(\bar{f})}{\pi(\bar{g})}.$$

Proof. — The assertion follows from Theorem 5.14 of [15], according to which ε_x is \bar{f} -regular (resp. \bar{g} -regular) for π -almost all $x \in S$. \square

COROLLARY 2. — For any probability measures λ and μ on $\mathcal{F} \times \mathbb{R}_+$, and $F, G \in \mathcal{F}$ with $\pi(F) < \infty, 0 < \pi(G) < \infty$, such that

$$(6.11) \quad \tilde{\lambda} \text{ is } 1_F\text{-regular and } \tilde{\mu} \text{ is } 1_G\text{-regular w. r. t. } \{X_n\},$$

we have

$$(6.12) \quad \lim_{t \rightarrow \infty} \frac{\lambda * U_0(F \times [0, t])}{\mu * U_0(G \times [0, t])} = \lim_{t \rightarrow \infty} \frac{\mathbb{E}_\lambda \left[\sum_{n \geq 1} 1_{\{X_n \in F, T_n \leq t\}} \right]}{\mathbb{E}_\mu \left[\sum_{n \geq 1} 1_{\{X_n \in G, T_n \leq t\}} \right]} = \frac{\pi(F)}{\pi(G)}.$$

COROLLARY 3. — (cf. [7], II. 3, Proposition 6). For any $F, G \in \mathcal{F}$ with $\pi(F) < \infty, 0 < \pi(G) < \infty$, for π -almost all $x, y \in S$

$$(6.13) \quad \lim_{t \rightarrow \infty} \frac{U_0(x, F \times [0, t])}{U_0(y, G \times [0, t])} = \frac{\pi(F)}{\pi(G)}.$$

Proof. — For π -almost all $x \in S$, ε_x is 1_F -regular and 1_G -regular ([15], Theorem 5.14). \square

Jacod proves this result by assuming only that the embedded chain is φ -recurrent.

COROLLARY 4. — For any non-decreasing functions $f, g : S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, such that \bar{f} and \bar{g} are special (see the Corollary of Theorem 5.1) and for any probability measures λ, μ on $\mathcal{F} \times \mathcal{R}_+$

$$(6.14) \quad \lim_{t \rightarrow \infty} \frac{\lambda * U_0 * f(t)}{\mu * U_0 * g(t)} = \frac{\pi(\bar{f})}{\pi(\bar{g})}.$$

Proof. — Recall that the function $\bar{f} : S \rightarrow \mathbb{R}_+$ is special, if and only if for all probability measures η on \mathcal{F} , η is \bar{f} -regular. \square

7. A LIMIT THEOREM FOR ψ -RECURRENT SEMI-REGENERATIVE PROCESSES

Assume in this section that S is a locally compact space with a countable base and \mathcal{B} is the Borel σ -field of S . Denote by \mathcal{F} the σ -field of universally measurable sets over (S, \mathcal{B}) (see [2], p. 2). Let (Ω, Σ) be a measurable space on which is defined a semi-group of translations $\{\Theta_t; t \in \mathbb{R}_+\}$ and a stochastic process $X_t : \Omega \rightarrow S$ with right continuous trajectories such that $X_{t+s} = X_t \circ \Theta_s, t, s \in \mathbb{R}_+$. Denote $\Sigma_t = \sigma(X_s : 0 \leq s \leq t)$; we assume that Σ

is minimal in the sense that $\Sigma = \bigvee_{t \in \mathbb{R}_+} \Sigma_t$. Let $\mathbb{P}_{x,t}(d\omega)$ be a transition probability from $(S \times \mathbb{R}_+, \mathcal{F} \times \mathcal{R}_+)$ into Σ ; we write \mathbb{P}_x for $\mathbb{P}_{x,0}$ and for

any measure μ on $\mathcal{F} \times \mathcal{R}_+$ we denote $\mathbb{P}_\mu(d\omega) = \int_{\mathcal{S} \times \mathcal{R}_+} \mu(dx \times dt) \mathbb{P}_{x,t}(d\omega)$.

Next we shall define what we mean by a semi-regenerative process (cf. Definition 4.1 of Cinlar [3]).

DEFINITION 1. — Let τ be a stopping time w. r. t. $\{\Sigma_t\}$. The pair $(\{X_t; t \in \mathbb{R}_+\}, \{\tau_n; n = 0, 1, \dots\})$, where $\{\tau_n\}$ is a sequence of finite, non-negative (\mathbb{P}_x -a. s. for all $x \in \mathcal{S}$) stopping times w. r. t. $\{\Sigma_t\}$ such that $(X_{\tau_0}, \tau_0) = (x, t)$ $\mathbb{P}_{x,t}$ -a. s. for all $x \in \mathcal{S}, t \in \mathbb{R}_+$ and iteratively for all n

$$(7.1) \quad \tau_{n+1} = \tau_n + \tau \circ \Theta_{\tau_n},$$

is a *semi-regenerative process*, provided that for all probability measures μ on $\mathcal{F} \times \mathcal{R}_+$ and all real valued bounded measurable functions Y on \mathcal{R}

$$(7.2) \quad \mathbb{E}_\mu[Y \circ \Theta_{\tau_n} | \Sigma_{\tau_n}] = \mathbb{E}_{X_{\tau_n}}[Y].$$

Let $V_t = \sup \{ \tau_n; n \geq 0, \tau_n \leq t \}$, $U_t = t - V_t$, $X_t^- = X_{V_t}$, and for any initial distribution μ of (X_{τ_0}, τ_0) ,

$$P_t(\mu, A) = \mathbb{P}_\mu \{ (X_t, X_t^-, U_t) \in A \}, \quad t \in \mathbb{R}_+, \quad A \in \mathcal{F} \times \mathcal{F} \times \mathcal{R}_+.$$

As a corollary of Theorem 5.1 we easily get

THEOREM 7.1. — Assume that the embedded MRP $\{(X_{\tau_n}, \tau_n)\}$ is φ -recurrent, satisfies condition (M_α) for some $\alpha \in (0, 1]$ (see Section 3), the measure $\int_{\mathcal{S}} h(x)v(dx \times \cdot)$ is spread out, and that $\pi(\mathcal{S}) < \infty$. Then for any probability measure λ on $\mathcal{F} \times \mathcal{R}_+$ such that $\tilde{\lambda}$ is 1-regular (w. r. t. the embedded Markov chain $\{X_{\tau_n}; n \geq 0\}$; see (5.1)), we have the total variation norm (on $\mathcal{F} \times \mathcal{F} \times \mathcal{R}_+$) convergence:

$$\lim_{t \rightarrow \infty} \| P_t(\lambda, \cdot) - \pi_\infty \| = 0,$$

where π_∞ is the following probability measure on $\mathcal{F} \times \mathcal{F} \times \mathcal{R}_+$:

$$\pi_\infty(E \times F \times \Gamma) = [\pi(m)]^{-1} \int_F \int_\Gamma \pi(dx) du \mathbb{P}_x \{ X_u \in E, \tau > u \}. \quad \square$$

In an obvious way we would also obtain a ratio limit theorem for semi-regenerative processes. We leave the details to the reader.

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