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Ergodic properties of an operator obtained from a continuous representation

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SUMMARY. — Let S be a locally compact Abelian semi-group, and let $\{ T(t) : t \in S \}$ be a strongly continuous representation by linear contractions on a Banach space X. For a regular probability ν on the Borel sets of S, we define $U = \int T(t)\nu(dt)$. We study the fixed points of U and U*, and the convergence properties of $\{ U^n x \}$, using assumptions on the support of ν. Applications are made to random walks.

For a strongly continuous one-parameter semi-group $\{ T(t) : t \geqslant 0 \}$ we define $R = \int_0^\infty e^{-t}T(t)dt$, and show that R^n converges uniformly if (and only if) $\alpha^{-1}\int_0^\alpha T(t)dt$ converges uniformly as $\alpha \to \infty$.

1. INTRODUCTION

Let X be a Banach space, and let $\{ T(t) : t > 0 \}$ be a strongly continuous semi-group of linear operators, with $\| T(t) \| \leqslant 1$. It is well-known that many properties of $\{ T(t) \}$ can be derived from (or are equivalent to) properties of $R = \int_0^\infty e^{-t}T(t)dt$.

In this note we look at more general semi-groups, and study the ergodic properties of representation-averages. We start with a set S which is an Abelian semi-group (maybe without unit), and which has a locally compact (Hausdorff) topology. We call S a topological *semi-group* if the map $(t, s) \to t + s$ is continuous from $S \times S$ into S. An *operator representation* of S is a function T from S into the space L(X) of bounded linear operators on the Banach space X, such that $T(t + s) = T(t)T(s)$. The representation is *weakly continuous* if $\langle x^*, T(.)x \rangle$ is continuous, for $\forall x \in X$, $x^* \in X^*$, and *continuous* if $t \to T(t)x$ is continuous for $\forall x \in X$.

Let ν be a regular probability on the Borel sets of S. The operator $U_\nu x = \int T(t)x d\nu$ is defined in the strong operator topology (if T(t) is continuous), and we wish to study the ergodic properties of U_ν.

If G is a locally compact Abelian group with Haar measure λ, we have a continuous representation of G in the space of finite signed measures $\ll \lambda$, defined by $T(g)\mu(A) = \mu(A - g)$, and $U_\nu \mu = \nu * \mu$. We also have that $(T(t)f)(g) = f(g + t)$ is a continuous representation of G in the space $C_0(G)$ of bounded continuous functions vanishing at infinity (closure in sup norm of the functions with compact support). In this case $U_\nu f = \nu * f$. Our aim is to generalize also some of the results known for $\{ \nu^{*n} \}$, to our general set-up, *i. e.* to obtain results for iterates of a convolution on a semi-group.

2. FIXED POINTS AND ERGODICITY

In order to obtain convergence theorems, some assumptions on the support of ν are needed (this is well-known already for ν^{*n} in a group).

DEFINITION. — The probability ν is *ergodic* if for every open set $V \subset S$ there are natural numbers n_i, m_i, points s_i, $r_i \in \text{supp}(\nu)$, and a $t \in V$, such that $t + \sum_{i=1}^{j} n_i s_i = \sum_{i=1}^{k} m_i r_i$ (when S is group, ν is ergodic if and only if the group generated by its support is dense).

THEOREM 1. — *Let* T(t) *be a continuous operator representation of a locally compact topological Abelian semi-group* S, *with* $\| T(t) \| \leqslant 1$. *Let* ν *be an ergodic regular probability, and let* $U_\nu x = \int T(t)x d\nu$. *If* $U_\nu x_0 = x_0$, *then* $T(t)x_0 = x_0$ *for every* $t \in S$.

Proof. — Let μ and η be probabilities in S. For $x \in X$ and $x^* \in X^*$ we have

$$\langle x^*, U_\eta U_\mu x \rangle = \int_S \langle x^*, T(t)U_\mu x \rangle d\eta(t) = \int_S \langle T(t)^* x^*, U_\mu x \rangle d\eta(t)$$

$$= \int_S \left[\int_S \langle T(t)^* x^*, T(s)x \rangle d\mu(s) \right] d\eta(t) = \int_S \left[\int_S \langle x^*, T(t + s)x \rangle d\mu(s) \right] d\eta(t)$$

$$= \langle x^*, U_\mu U_\eta x \rangle$$

using Fubini's theorem. Hence $U_\mu U_\eta = U_\eta U_\mu$.

Let $A \subset S$ be a Borel set with $0 < \nu(A) < 1$. Then

$$x_0 = \int_A T(s)x_0 d\nu + \int_{S\setminus A} T(s)x_0 d\nu$$
$$= \nu(A)\nu(A)^{-1}\int_A T(s)x_0 d\nu + [1-\nu(A)][1-\nu(A)]^{-1}\int_{S\setminus A} T(s)x_0 d\nu,$$

and since the two operators are commuting contractions (by the first part), we apply Lemma 1 of Falkowitz [4], to conclude that $\nu(A)^{-1}\int_A T(s)x_0 d\nu = x_0$.

Hence, for $x^* \in X^*$ we have $\int_A \langle x^*, (I - T(s))x_0 \rangle d\nu = 0$, for every $A \subset S$; hence $\nu\{ s : \langle x^*, (I - T(s))x_0 \rangle \neq 0 \} = 0$. Since $\langle x^*, (I - T(s))x_0 \rangle$ is continuous, we have $\langle x^*, x_0 \rangle = \langle x^*, T(s)x_0 \rangle$ for s in the support of ν. This being true for every $x^* \in X^*$, $T(s)x_0 = x_0$ for $s \in \operatorname{supp}(\nu)$.

Let $t \in S$. By ergodicity of ν, given a neighborhood V of t there are positive integers m_i and points $s_i \in \operatorname{supp}(\nu)$ such that

$$t_\alpha + \sum_{i=j+1}^{k} m_i s_i = \sum_{i=1}^{j} m_i s_i$$

for some $t_\alpha \in V$. Hence

$$T(t_\alpha)x_0 = T(t_\alpha)\prod_{i=j+1}^{k} T(s_i)^{m_i}x_0 = T\left(t_\alpha + \sum_{i=j+1}^{k} m_i s_i\right)x_0$$
$$= T\left(\sum_{i=1}^{j} m_i s_i\right)x_0 = \prod_{i=1}^{j} T(s_i)^{m_i}x_0 = x_0$$

Taking $t_\alpha \to t$, continuity of the representation yields $T(t)x_0 = x_0$ for $t \in S$. Q. E. D.

REMARKS. 1. — If $\sup \{ \| T(t) \| : t \in S \} = M < \infty$, then $|||\, x \,||| = \max \{ \| x \|, \sup_{t\in S} \| T(t)x \| \}$ is an equivalent norm such that $|||\, T(t) \,||| \leqslant 1$, and the theorem can then be applied.

2. An inspection of the proof shows that we have used only weak continuity of $T(t)$ and Pettis integrability. Thus, for X non-separable and reflexive and S non-separable, only weak continuity is needed.

3. The methods of harmonic analysis used by Falkowitz [4] for $S = (0, \infty)$ cannot yield the result for ν *singular* with continuous distribution.

4. If ν is not ergodic, the theorem may fail: let $T(t)x = e^{it}x$ for $t \geqslant 0$. Then $T(2\Pi) = I$, and for ν supported on the point 2Π, $U_\nu = I$, while no common fixed-point exists.

THEOREM 2. — *Under the assumptions of theorem 1,* $x_0^* \in X^*$ *satisfies* $U_\nu^* x_0^* = x_0^*$ *if and only if* $T(t)^* x_0^* = x_0^*$ *for every* $t \in S$.

The proof is the same as the previous one, replacing $T(s)$ by $T(s)^*$, x_0 by x_0^*, $x^* \in X^*$ by $x \in X$, and continuity by weak-* continuity. Note that $\int_A T(s)^* d\nu = \left[\int_A T(s) d\nu\right]^*$ is defined in the weak-* topology.

COROLLARY 3. — *Under the assumptions of theorem 1,* $x \in X$ *satisfies* $\left\| N^{-1} \sum_{n=1}^{N} U_\nu^n x \right\| \underset{N \to \infty}{\longrightarrow} 0$ *if and only if* $\langle x^*, x \rangle = 0$ *for every* $x^* \in X^*$ *satisfying* $T^*(t) x^* = x^*$ *for* $\forall t \in S$.

Proof. — It follows from the Hahn-Banach theorem that

$$\left\| N^{-1} \sum_{n=1}^{N} U_\nu^n x \right\| \to 0 \Leftrightarrow x \in \overline{(I - U_\nu)X} \Leftrightarrow \langle x^*, x \rangle = 0$$

whenever $U_\nu^* x^* = x^*$. Apply now theorem 2.

REMARK. — Theorem 2 can be used to prove the result of Choquet and Deny [*1*] that for ν ergodic in a LCA group G, $\nu * f = f$ if and only if f is constant *a. e.* in G.

We can certainly restrict ourselves to the subspace

$$\left\{ x : \left\| N^{-1} \sum_{n=1}^{N} U_\nu^n x \right\| \to 0 \right\}$$

(and then $\{ T^*(t) \}$ have no common fixed-points, by corollary 3). We then ask for conditions to ensure that $U_\nu^n \to 0$ strongly. Note that if X is reflexive, $N^{-1} \sum_{n=1}^{N} U_\nu^n$ converges strongly.

DEFINITION. — The probability ν is *aperiodic* if for every open set $V \subset S$ there are natural numbers m_i, points r_i, $s_i \in \mathrm{Supp}(\nu) = \sigma$, and a $t \in V$, such that $t + \sum_{i=1}^{j} m_i s_i = \sum_{i=1}^{j} m_i r_i$. When S is a group, ν is aperiodic if and only if $\sigma - \sigma$ generates S. An equivalent condition is that σ is not contained in any coset of a closed subgroup of the group S.

THEOREM 4. — *Let* S *be a locally compact topological Abelian semi-group, and let* T(*t*) *be a continuous operator representation with* $\| T(t) \| \leqslant 1$. *If* ν *is an aperiodic regular probability on* S, *then* $\left\| N^{-1} \sum_{n=1}^{N} U_\nu^n x_0 \right\| \underset{N\to\infty}{\longrightarrow} 0$ *if and only if* $\| U_\nu^n x_0 \| \underset{n\to\infty}{\longrightarrow} 0$. (Hence if X is reflexive, U_ν^n converges strongly).

Proof. — For simplicity, we denote U_ν by U, and for any Borel set A with $0 < \nu(A)$ we define $U_A x = \nu(A)^{-1} \int_A T(t) x d\nu$. If $0 < \nu(A) < 1$, then U_A and $U_{A'}$ commute (where $A' = S\backslash A$), and $U = \nu(A)U_A + [1 - \nu(A)]U_{A'}$. By lemma 2.1 of [*8*], we have that $\| U^n(U_A - U_{A'}) \| \to 0$. Hence $\| U^n(U - U_A) \| \to 0$.

As noted above, we may and do assume that $\left\| N^{-1} \sum_{n=1}^{N} U^n x \right\| \to 0$ for every $x \in X$, so that $T^*(t)x^* = x^*$ for every $t \in S$ implies $x^* = 0$.

We have that, for $x \in X$ and $0 < \nu(A) < 1$, $\| U^n(U_A - U)x \| \underset{n\to\infty}{\longrightarrow} 0$. Let $x^* \in X^*$ satisfy $\langle x^*, (U_A - U)x \rangle = 0$ for each Borel set A with $0 < \nu(A) < 1$ and every $x \in X$. Hence $\langle x^*, U_A x \rangle = \langle x^*, Ux \rangle$ for $x \in X$, $0 < \nu(A) < 1$.

Let $t \in \operatorname{supp}(\nu)$. Fix $x \in X$. Given $\varepsilon > 0$, there is an open A containing t such that $| \langle x^*, T(s)x - T(t)x \rangle | < \varepsilon$ for $s \in A$. Hence (taking smaller A, if necessary, to obtain $\nu(A) < 1$, which is possible since ν is not supported on one point),

$$| \langle x^*, Ux \rangle - \langle x^*, T(t)x \rangle | = | \langle x^*, U_A x \rangle - \langle x^*, T(t)x \rangle | < \varepsilon,$$

showing that $U^*x^* = T(t)^*x^*$ for $t \in \operatorname{supp}(\nu)$.

We now want to show $B_\infty \equiv \bigcap_{n=1}^{\infty} U^{*n}B = \{ 0 \}$, where B is the unit ball of X. Let $x^* \in B_\infty$. Then there are $y_n^* \in B$ with $U^{*n}y_n^* = x^*$. Hence for $x \in X$ and A with $0 < \nu(A) < 1$, we have

$$| \langle x^*, (U_A - U)x \rangle = | \langle U^{*n}y_n^*, (U_A - U)x \rangle | \leq \| U^n(U_A - U)x \| \to 0.$$

Hence, by the preceeding paragraph, $T(t)^*x^* = U^*x^*$ for $t \in \operatorname{supp}(\nu) = \sigma$.

It is known [*9*] that U^* maps B_∞ onto B_∞, so that there are $x_n^* \in B_\infty$ with $U^{*n}x_n^* = x^* = x_0^*$. By the above, $T(t)^*x_n^* = U^*x_n^*$ for $n \geq 0$ and $t \in \sigma$.

Let $t \in S$. Take V open with $t \in V$. Then, by aperiodicity, there are integers $m_i > 0$, points r_i: $s_i \in \sigma$, and a $t_V \in V$ such that

$$t_V + \sum_{i=1}^{j} m_i r_i = \sum_{i=1}^{j} m_i s_i.$$

Let $m = \sum_{i=1}^{j} m_i$. For each x_n^* we have

$$T(t_V)^* U^{*m} x_n^* = T(t_V + \Sigma m_i r_i)^* x_n^* = U^{*m} x_n^*.$$

In particular, taking $n = m$, we get $T(t_V)^* x^* = x^*$. Letting $V \to \{ t \}$, we get $T(t)^* x^* = x^*$ for $t \in S$. Hence $x^* = 0$. Thus $B_\infty = \{ 0 \}$, and $\| U^n x \| \to 0$ for every $x \in X$, by [*9*] (see also [*2*]). Q. E. D.

REMARKS. — 1. It is known [*2*] that the condition that $\sigma - \sigma$ generate G, is equivalent to the convergence $\| \nu^{*n} * \mu \| \to 0$ for every $\mu \ll \lambda$ with $\mu(G) = 0$. Thus, there is no *general* weaker assumption on σ to imply theorem 4 for every representation.

2. The result of [*5*] is now a corollary of theorem 4.

COROLLARY 5. — *If* ν *is ergodic and* $0 \in \mathrm{supp}(\nu)$ *then the conclusion of Theorem 4 holds.*

COROLLARY 6 [*6*]. — *Let* G *be a LCA group and let* ν *be as above. Then* : (1) *for every* $f \in C(G)$ *wtth compact support* $\nu^{*n} * f$ *converges uniformly.* (2) *For every finite measure* $\mu \ll \lambda$ *with* $\mu(G) = 0$, $\| \nu^{*n} * \mu \| \to 0$.

Proof. — (1) Let $C_0(G)$ be the closure (in sup norm) of the functions in C(G) with compact support (if G is compact, $C_0(G) = C(G)$). $T(t)f(x) = f(x + t)$ is a continuous representation. For G compact the Haar measure λ is the only invariant functional for all $T(t)^*$, hence for U_ν^*. Thus, for $f \in C(G)$ with $\int f d\lambda = 0$, $\left\| N^{-1} \sum_{i=1}^{N} U_\nu^n f \right\| \to 0$ and, by theorem 4, $\| U_\nu^n f \| \to 0$. If G is not compact, there is no element in $C_0(G)_*$ which is invariant for every $T(t)^*$, hence $\left\| N^{-1} \sum_{i=1}^{N} U_\nu^n f \right\| \to 0$ for $f \in C_0(G)$, and $\| U_\nu^n f \| \to 0$. But $U_\nu f = \nu * f$. (2) follows from the continuity of the representation in $L_1(\lambda)$.

REMARK. — It follows that ν^{*n} converges weak-* in $C_0(G)^*$, since $\langle \nu^{*n}, f \rangle = \langle \nu, \nu^{*n-1} * f \rangle$. Results on the convergence of ν^{*n} in non-Abelian groups were obtained by Derriennic [*2*] and Foguel [*7*]. See also [*13*].

REMARK. — In general, one cannot relax the assumption on the support in theorem 4 (which is necessary for part (2) of corollary 6 [*2*]). However, the recent book of Mukherjea and Tserpes [*13*] contains a proof of part (1) of corollary 6, for non-compact LCA groups, which assumes only ergodicity.

COROLLARY 7. — *Let* S *and* ν *be as in theorem 4, and* $C_0(S)$ *the closure of the continuous functions with compact support. For* S *non-compact assume that, for compact sets* A *and* B, *the set* $A - B = \{ y : \exists x \in B, x + y \in A \}$ *is compact. Then for every* $f \in C(S)$ *with compact support* $\nu^{*n} * f$ *converges uniformly.*

LEMMA 8. — *Let* S *be non-compact as in the corollary. If* U *is open with compact complement and* $x \in S$, *there are open sets* W *and* $V \ni x$ *such that* W *has compact complement,* $\overline{V}$ *is compact, and* $W + V \subset U$.

Proof. — Let V be an open set containing x, with compact closure $\overline{V}$. Let $W = \bigcap_{x \in \overline{V}} \{ y : y + x \in U \}$. Then the complement of W is

$$W^c = \bigcup_{x \in \overline{V}} \{ y : y + x \in U^c \} = U^c - \overline{V},$$

which is compact by assumption, and $W + V \subset U$.

LEMMA 9. — *Under the assumptions of the corollary,* $T(t)f(x) = f(x + t)$ *is a continuous operator representation of* S *in* $C_0(S)$.

Proof. — Assume S is non-compact. Let f be continuous with compact support. Let A be compact with $\{ |f| > 0 \} \subset A$. Let $U = A^c$. Since $\{ x : f(x + t) > 0 \} \subset A - t$ which is compact, $T(t)$ maps $C_0(s)$ into itself.

Fix $t \in S$ and $\varepsilon > 0$. By lemma 8 there is an open set V_0 containing t, and an open set W with $B = W^c$ compact, such that $W + V_0 \subset U$.

For $x \in B$, there are open sets U_x and V_x with $x \in U_x$, $t \in V_x$, such that $|f(y + s) - f(x + t)| < \varepsilon/2$ for $y \in U_x$, $s \in V_x$ (by joint continuity of addition). Since B is compact, we take an open sub-cover $\{ U_{x_i} \}_{i=1}^{n}$ to cover B, and define $V = V_0 \bigcap \left(\bigcap_{i=1}^{n} V_{x_i} \right)$.

Let $s \in V$. If $x \in B$, $x \in U_{x_i}$ for some i, and since $s \in V_{x_i}$, we have

$$|f(x+s) - f(x+t)| \leqslant |f(x+s) - f(x_i+t)| + |f(x+t) - f(x_i+t)| < \varepsilon$$

If $x \notin B$, then $x \in W$, so that $x + s \in x + V_0 \subset W + V_0 \subset U$, implying $f(x+s) = 0$. Also $x + t \in x + V_0$, so $f(x+t) = f(x+s) = 0$. Thus $\| T(s)f - T(t)f \| \leqslant \varepsilon$.

The proof of the compact case is similar (and simpler).

PROOF OF COROLLARY 7. — We note that there is at most one probability μ on S with $\mu(A - x) = \mu(A)$ for every Baire set A. If $T(t)^*\mu = \mu$ and $T(t)^*\eta = \eta$ for every $t \in S$, we have $\eta * \mu = \mu$ and $\mu * \eta = \eta$ (both are probabilities), hence $\mu = \eta$.

If S is compact, such a probability μ exists, by the Markov-fixed point theorem [*3*, p. 456]. In that case $C_0(S) = C(S)$, and by corollary 3, for $f \in C(S)$ with $\int f d\mu = 0$, $\left\| N^{-1} \sum_{n=1}^{N} v^{*n} * f \right\| \to 0$. Apply now theorem 4, to obtain $\left\| v^{*n} * f - \int f d\mu \right\| \to 0$ for $f \in C(S)$.

If S is not compact, it is shown in [*11*, theorem 5] that (under our assumptions on S) there is no probability μ with $T(t)^*\mu = \mu$ for $\forall t \in S$. Hence, by corollary 3, $\left\| N^{-1} \sum_{n=1}^{N} v^{*n} * f \right\| \to 0$ for every $f \in C_0(S)$, and by theorem 4 $\| v^{*n} * f \| \to 0$ for $f \in C_0(S)$.

REMARK. — Corollary 7 relaxes some of the assumptions on S of [*11*, corollary 3], and yields a better convergence result. However, more is assumed here on v. Mukherjea [*11*] also obtains some results for v^{*n} in the non-Abelian case. More references are given in [*11*].

COROLLARY 10. — *Let v be as in theorem 4*. Then $v^{*n} * f$ *converges uniformly to a constant, for every* $f \in AP(S)$.

Proof. — We define $T(t)f(s) = f(t+s)$ for $f \in C(S)$ (bounded continuous). If $S(f) = \{ T(t)f \}$ has compact weak-closure, we have that also, for $t_0 \in S$, $\{ T(t)T(t_0)f \}$ has compact weak-closure, so that AP(S) is invariant under $\{ T(t) : t \in S)$ (and $\{ T(t)f \}$ has compact weak-closure in AP(S)). We show weak continuity of the representation in AP(S). Fix t; let $t_\alpha \to t$, and take a subnet t_β such that $T(t_\beta)f$ converges weakly, say to g. By joint continuity of addition, since $\delta_s \in AP(S)^*$, we get

$$g(s) = \langle g, \delta_s \rangle = \lim \langle T(t_\beta)f, \delta_s \rangle = \lim f(t_\beta + s) = f(t+s) = T(t)f(s).$$

Hence $T(t_\beta)f \to T(t)f$ weakly, so that $T(t_\alpha)f \to T(t)f$ weakly.

If $S(f)$ has compact weak-closure, $\overline{co}\ S(f)$ (closed convex hull) is weakly compact [3, p. 434]. Now $\langle x^*, Uf \rangle = \int \langle x^*, T(t)f \rangle\, dv(t)$ defines an element Uf in $AP(S)^{**}$. But $\overline{co}\ S(f)$ is compact in the w^*-topology of $AP(S)^{**}$, and the separation theorem [3, p. 417] shows that $Uf \in \overline{co}\ S(f)$, and $v * f = Uf \in AP(S)$. Also $U^n f \in \overline{co}\ S(f)$ for every n, hence $N^{-1} \sum_{n=1}^{N} U^n f$ is weakly sequentially compact, so converges strongly. Apply theorem 4 to obtain the norm convergence of $U^n f$, say to g. Hence $Ug = g$, and by theorem 1 $T(t)g = g$ for $\forall t \in S$. But for $t \neq s$,

$$g(s) = T(t)g(s) = g(s+t) = T(s)g(t) = g(t),$$

so g is constant.

REMARK. — A different proof of corollary 10 for groups is given in [6].

3. UNIFORM ERGODICITY

Given a contraction T on the Banach space X, the uniform convergence of its averages is the strong convergence when T is looked upon as an operator (by multiplication on the left) in the space L(X) of bounded linear operators on X. Thus, if $N^{-1} \sum_{n=1}^{N} T^n$ converge uniformly (or strongly), theorem 4 implies that for $U = \frac{1}{2}(T^2 + T^3)$, U^n converges uniformly (strongly, respectively).

However, T^2 and T^3 may both fail to be ergodic. A first example, based on the construction in [12], was observed by Prof. R. Sine, and applies to the strong topology. We produce an example in the uniform operator topology (which, by the above remarks, applies to the strong topology as well). This is a modification of the example in [10]. Let $0 < \lambda_n < 1$ with $\lambda_n \uparrow 1$, and let $w = \exp(2\Pi i/3)$. Let $X = l_2$ (complex) and define $y = Tx$ by $y_{2n-1} = -\lambda_n x_{2n-1}$, $y_{2n} = \lambda_n w x_{2n}$. Since $I - T$ is invertible, T is uniformly ergodic, but T^2 and T^3 are not uniformly ergodic. Note that $\{ T^n \}$ converges strongly, but not uniformly.

If we are given a strongly continuous one parameter semi-group of contractions $\{ T(t) : t > 0 \}$, we cannot apply theorem 4 in the uniform operator topology, since the representation into L(X) is not continuous. However, for the resolvent operators $\lambda \int_0^\infty e^{-\lambda t} T_t dt$, we can still obtain uniform convergence.

THEOREM 11. — *Let $\{ T(t) : t > 0 \}$ be a strongly continuous semi-group of linear contractions in X, such that $\alpha^{-1} \int_0^\alpha T(t)dt$ converge uniformly as $\alpha \to \infty$. For $\lambda > 0$ define $R_\lambda x = \int_0^\infty e^{-\lambda t} T(t)x dt$. Then $(\lambda R_\lambda)^n$ converges uniformly.*

Proof. — We reduce the theorem to the case that $\{ T(t) \}$ is continuous at 0, with $\lim_{t \to 0} T(t)x = x$. Let $Y = \text{span} \bigcup_{t > 0} T(t)X$. Since $R_\lambda x$ is defined in the strong operator topology, $R_\lambda x \in Y$. On Y the restriction of $\{ T(t) \}$ is uniformly ergodic, and strongly continuous. If $E = \lim_{\alpha \to \infty} \alpha^{-1} \int_0^\alpha T(t)dt$, $EX \subset Y$, and

$$\| (\lambda R_\lambda)^n_x - Ex \| = \| (\lambda R_\lambda)^{n-1} \lambda R_\lambda x - E \lambda R_\lambda x \| \leqslant \| (\lambda R_\lambda)^{n-1} - E \|_y \| x \|.$$

Hence, we may assume continuity at zero. In that case, the result is somewhat more general, as follows.

THEOREM 12. — *Let $\{ T(t) : t \geqslant 0 \}$ be a strongly continuous semi-group of linear operators on X, with $\| T(t) \| / t \xrightarrow[t \to \infty]{} 0$. Then $\alpha^{-1} \int_0^\alpha T(t)dt$ converges uniformly if and only if $(\lambda R_\lambda)^n$ converges uniformly for some (every) $\lambda > 0$.*

Proof. — We assume first that $\alpha^{-1} \int_0^\alpha T(t)dt$ converges uniformly, and we may assume (by decomposing the space) that the limit is zero. Then, by the uniform ergodic theorem [*10*], the infinitesimal generator A is onto X. Fix $\lambda > 0$. It is known that $(\lambda I - A)R_\lambda = I$ (see [*3*, p. 622]). The uniform ergodic theorem also implies that $N^{-1} \sum_{n=1}^{N} (\lambda R_\lambda)^n$ converges uniformly, and by our assumption the limit is zero. Hence, $I - \lambda R_\lambda$ is invertible. The generator A is one-to-one, so A^{-1} is a bounded operator on X and $A^{-1} = - R_\lambda (I - \lambda R_\lambda)^{-1}$.

Let $0 \neq \mu \in \sigma(\lambda R_\lambda)$ (note that $\mu \neq 1 \in \rho(\lambda R_\lambda)$). By the spectral mapping theorem, $\mu/\lambda(\mu - 1) \in \sigma(A^{-1})$. By [3, VII.9.2] (with $\alpha = 0$) we have that $\nu = \lambda(\mu - 1)/\mu \in \sigma(A)$. Hence, $\mathrm{Re}\ \nu \leqslant 0$ (since $\mathrm{Re}\ \nu > 0 \Rightarrow \nu \in \rho(A)$ by [3, p. 622]). But $\mu = \lambda/(\lambda - \nu)$ with $\lambda > 0$, so $|\mu| \leqslant \lambda/(\lambda - \mathrm{Re}\,\nu) \leqslant 1$. Thus $|\mu| = 1$ if and only if $\nu = 0$. But then $\mu = 1$, contradicting $1 \in \rho(\lambda R_\lambda)$. Hence $|\mu| < 1$ for $\mu \in \sigma(\lambda R_\lambda)$, so that $r(\lambda R_\lambda) < 1$, and $\| (\lambda R_\lambda)^n \| \to 0$.

The converse follows from the uniform ergodic theorem [10].

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