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Minimal moment conditions in the limit theory for general Markov branching processes ⁽¹⁾

by

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0. INTRODUCTION

After revising the underlying positivity concept, several standard limit theorems for Markov branching processes have recently been proved with an infinite or general set of types. The theory as presented in [6, 7, 10] is somewhat inhomogeneous, however. The degree of generality varies from paper to paper. In particular [6, 10] and the application to branching diffusions in [7] assume a local branching law, thus excluding for example multitype branching diffusions. Besides, the moment conditions in [6] are not quite minimal. In this note we formulate a coherent theory in a completely general setting and discuss its conditions for processes constructed from a transition function on the type space, a bounded termination density, and a not necessarily local branching kernel.

As an introduction let us look in intuitive terms at a simple branching diffusion. Let a particle undergo standard Brownian motion restricted to $X = \left(0, \frac{\pi}{2}\right]$ with absorption at 0 and instantaneous reflection at $\frac{\pi}{2}$. The differential generator of the transition semigroup is then given by $L\eta = \frac{1}{2}\eta''$ and the boundary conditions $\eta(0) = \eta'\left(\frac{\pi}{2}\right) = 0$, with η' , η'' denoting the

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first and second derivatives. Curtail the lifetime of this process through application of the multiplicative functional $\exp \{ -kt \}$, where k is a positive constant. If the diffusion is stopped at a point in X , replace the particle at the point and instant of termination by $n \geq 0$ new particles, subject to the distribution $\pi = \{ p_n \}$. Let the new particles move and reproduce independently, but according to the same probability laws as their ancestor. The resulting Markov process $\{ \hat{x}_t, \hat{P}^x \}$ on the space \hat{X} of unordered finite populations of particles positioned in X is an example of a branching diffusion.

A key role in the investigation of Markov branching processes is played by the moment semigroup. For any bounded measurable function ξ on X define $\hat{x}_t[\xi]$ as 0. if the population \hat{x}_t is empty, and as $\xi(x_1) + \dots + \xi(x_n)$, if it consists of n particles at points $x_1, \dots, x_n \in X$, respectively. Note that $\hat{x}_t[1_Y]$ is simply the number of particles in $Y \subset X$. Let $\langle x \rangle$ denote the population consisting of just one particle situated at $x \in X$. In terms of our example, suppose that

$$m := \sum_n n p_n < \infty$$

Then $M^t \xi(x) := E^{(x)} \hat{x}_t[\xi]$ defines a semigroup M^t , whose differential generator is $L + k(m - 1)$. That is,

$$\begin{aligned} M^t \xi(x) &= \sum_{\nu=0}^{\infty} \rho_{\nu}^t \varphi_{\nu}^*[\xi] \varphi_{\nu}(x), & t > 0, \\ \rho_{\nu} &= \exp \{ - (2\nu + 1)^2 + k(m - 1) \}, \\ \varphi_{\nu}^*[\xi] &= \int_0^{\frac{\pi}{2}} \xi(y) \sin [(2\nu + 1)y] dy, & \varphi_{\nu}(x) = \frac{4}{\pi} \sin [(2\nu + 1)x]. \end{aligned}$$

From this, defining $\rho := \rho_0$, $\varphi := \varphi_0$, and $\varphi^* := \varphi_0^*$,

$$\begin{aligned} M^t \xi &= \rho^t \varphi \varphi^*[\xi] \{ 1 + \Delta_{t,\xi} \}, & t > 0, \\ M^t \varphi &= \rho^t \varphi, & \varphi^*[M^t \xi] &= \rho^t \varphi^*[\xi], & \varphi^*[\varphi] &= 1, \\ \limsup_{t \rightarrow \infty} \sup_{x, \xi} | \Delta_{t,\xi}(x) | &= 0. \end{aligned}$$

This property, which is stronger than what can be inferred from the general positivity theory of Krein and Rutman [0], has been used decisively in the proofs of limit theorems with otherwise minimal moment conditions. It can also be verified in cases where a full spectral representation is not available [9].

Let $\widehat{\mathfrak{A}}$ be the σ -algebra induced on \widehat{X} by the Borel algebra on X . Define the extinction probability

$$q(x) := \mathbf{P}^{(x)}(\lim_{t \rightarrow \infty} \widehat{x}_t[\mathbf{1}] = 0), \quad x \in X.$$

Specialized to our example, the limit results we shall be concerned with are the following:

Suppose $\rho > 1$. Then $q(x) < 1$ for all $x \in X$, but $q(x) \rightarrow 1$ as $x \rightarrow 0$. There exists a random variable W such that for any almost everywhere continuous, bounded ξ

$$\rho^{-t} \widehat{x}_t[\xi] \rightarrow \varphi^*[\xi]W \quad \text{a. s.},$$

as $t \rightarrow \infty$. We have $\mathbf{E}^{\widehat{x}}W = \widehat{x}[\varphi]$ if and only if

$$(0.1) \quad \sum_n p_n n \log n < \infty,$$

and $W = 0$ a. s. otherwise. The proof is dominated by martingale techniques [7].

Now suppose $\rho < 1$. Then $q = \mathbf{1}$. More precisely, there exists a constant $\gamma \geq 0$ such that for large t

$$\mathbf{P}^{\widehat{x}}\{\widehat{x}_t[\mathbf{1}] > 0\} \sim \gamma \rho^t \widehat{x}[\varphi],$$

where $\gamma > 0$ if and only if (0.1) is satisfied. There exists a probability measure \mathbf{P} on $(\widehat{X}, \widehat{\mathfrak{A}})$ such that for any measurable decomposition $\{A_v\}_{0 \leq v \leq j}$ and any $\widehat{x} \in \widehat{X}$ with $\widehat{x}[\mathbf{1}] > 0$

$$\mathbf{P}^{\widehat{x}}(\widehat{x}_t[1_{A_v}] = n_v; v = 1, \dots, j \mid \widehat{x}_t[\mathbf{1}] > 0) \rightarrow \mathbf{P}(\widehat{y}[1_{A_v}] = n_v; v = 1, \dots, j),$$

as $t \rightarrow \infty$. The limit distribution \mathbf{P} has a bounded first moment, namely $\gamma^{-1} \varphi^*$ if and only if $\gamma > 0$. The proof is based on the generating functional method [10].

Finally, suppose $\rho = 1$. Then again $q = \mathbf{1}$. If we assume in addition that

$$\mu := \frac{1}{2} \varphi^*[\varphi^2] \sum_n p_n n(n-1) < \infty$$

— in our example automatically $\mu > 0$ —, then for large t

$$\mathbf{P}^{\widehat{x}}(\widehat{x}_t[\mathbf{1}] > 0) \sim (\mu t)^{-1} \widehat{x}[\varphi],$$

and for any measurable decomposition $\{A_v\}_{1 \leq v \leq j}$ of X and any $\widehat{x} \in \widehat{X}$ with $\widehat{x}[\mathbf{1}] > 0$ the conditional distribution function of the vector

$t^{-1}(\widehat{x}_t[1_{A_1}], \dots, \widehat{x}_t[1_{A_j}])$, given $\widehat{x}_t[1] > 0$, converges to the d. f. of a vector of the form

$$(\varphi^*[1_{A_1}], \dots, \varphi^*[1_{A_j}])w, \quad \mathbf{P}^{\widehat{x}}(w \geq \lambda) = \exp \{ -\mu^{-1}\lambda \}, \quad \lambda \geq 0.$$

The proof again uses generating functionals [6].

As one would expect, results of this type hold also in more general settings, where X is an orientable, connected, bounded manifold and L a uniformly elliptic differential operator with mixed boundary conditions, whose coefficients may vary with x , as may k and the p_n .

A further generalization which is of more than formal interest concerns the character of the branching law. In our example we have assumed a « local » branching law, *i. e.*, in a branching event all new particles emerge at the point x of death of their immediate ancestor. Consider, however, a multitype branching diffusion, *i. e.* a branching diffusion where particles are distinguished not only by their position in space but also by an additional property varying on a finite set. Any such process is equivalent to an ordinary branching diffusion on a disconnected domain, and unless the latter process can be decomposed into processes on connected domains, the branching law cannot be local in the above sense. We will have to admit more general probability kernels $\pi(\cdot, \cdot) | X \otimes \widehat{\mathfrak{U}}$ as branching laws.

Finally, it is of interest to formulate the theory without reference to explicit models, but in terms of an abstract process or transition function, which satisfies a condition reflecting the basic independence assumption. Such a formulation is possible, but it involves technical conditions whose examination requires again a more detailed setting.

Proofs will be spelled out only where they deviate significantly enough from the proofs in [6, 7, 10].

1. SET-UP

Let (X, \mathfrak{A}) be a measurable space, \mathcal{B} the Banach algebra of all bounded, complex-valued, \mathfrak{A} -measurable functions ξ on X with supremum-norm $\|\xi\|$, \mathcal{B}_+ the non-negative cone in \mathcal{B} , and $\mathcal{F} := \{ \xi \in \mathcal{B} : \|\xi\| \leq 1 \}$, $\mathcal{F}_+ := \mathcal{B}_+ \cap \mathcal{F}$. Define

$$\widehat{X} := \bigcup_{n=0}^{\infty} X^{(n)}$$

where $X^{(n)}$, $n \geq 1$, is the symmetrization of the direct product of n disjoint copies of X and $X^{(0)} := \{ \theta \}$ with some extra point θ . Let $\widehat{\mathfrak{A}}$ be the σ -algebra on \widehat{X} induced by \mathfrak{A} .

By definition a transition function $\mathbf{P}_t(\widehat{X}, \widehat{\mathfrak{A}})$ on $(\widehat{X}, \widehat{\mathfrak{A}})$ with parameter set $T = \mathbb{Z}_+$, or $T = \mathbb{R}_+$, is a branching transition function if its generating functional,

$$F_t(\widehat{x}, \xi) := \mathbf{P}_t(\widehat{x}, \{\theta\}) + \sum_{n=1}^{\infty} \int_{X^{(n)}} \mathbf{P}_t(\widehat{x}, d\langle x_1, \dots, x_n \rangle) \prod_{v=1}^n \xi(x_v); \quad \xi \in \overline{\mathcal{P}},$$

satisfies

$$(1.1) \quad \begin{aligned} F_t(\theta, \xi) &= 1, \\ F_t(\langle x_1, \dots, x_n \rangle, \xi) &= \prod_{v=1}^n F_t(\langle x_v \rangle, \xi) \end{aligned}$$

for all $t \in T$, $\xi \in \overline{\mathcal{P}}$, and $\langle x_1, \dots, x_n \rangle \in X^{(n)}$, $n > 0$. Correspondingly, a Markov process $\{\widehat{x}_t, \mathbf{P}^{\widehat{x}}\}$ on $(\widehat{X}, \widehat{\mathfrak{A}})$ is a Markov branching process if it has a branching transition function.

In particular, we shall refer to the following more explicit setting:

Suppose $T = \mathbb{R}_+$, let X be a locally compact Hausdorff space with countable open base, and let \mathfrak{A} be the topological Borel algebra on X . If X is non-compact, let $X \cup \{\partial\}$ be the one-point compactification of X . Define \mathcal{C}_0 as the subalgebra of all continuous $\xi \in \mathcal{B}$ such that $\lim_{x \rightarrow \partial} \xi(x) = 0$ if X is non-compact. Suppose to be given

(A.1) a transition semigroup $\{T_t\}_{t \geq 0}$ on \mathcal{B} , which is strongly continuous on \mathcal{C}_0 with $T_t \mathcal{C}_0 \subseteq \mathcal{C}_0$ for $t \geq 0$,

(A.2) a termination density $k \in \mathcal{B}_+$ and a branching kernel π on $X \otimes \widehat{\mathfrak{A}}$. As is well-known, these data uniquely determine a right-continuous strong Markov branching process on $(\widehat{X}, \widehat{\mathfrak{A}})$, cf. [3, 4]. If $\{T_t\}$ is the transition semigroup of a diffusion, this process is called a branching diffusion.

For every \mathfrak{A} -measurable function ξ on X define

$$\begin{aligned} \widehat{x}[\xi] &:= 0; & \widehat{x} &= \theta, \\ &:= \sum_{v=1}^n \xi(x_v); & \widehat{x} &= \langle x_1, \dots, x_n \rangle \in X^{(n)}; \quad n > 0. \end{aligned}$$

Let $\mathbf{0}(x) := 0$ and $\mathbf{1}(x) := 1 \forall x \in X$. If \mathbf{P}_t is a branching transition function on $(\widehat{X}, \widehat{\mathfrak{A}})$ such that

$$M^t[\xi](\cdot) := \int_{\widehat{X}} \widehat{x}[\xi] \mathbf{P}_t(\langle \cdot \rangle, d\widehat{x}) \in \mathcal{B}; \quad t \geq 0,$$

for $\xi = \mathbf{1}$ and thus all $\xi \in \mathcal{B}$, then $\{M^t\}_{t \geq 0}$ is a semigroup of linear-bounded operators on \mathcal{B} . In the (A.1-2) framework the assumption

$$(A.3) \quad \int_{\widehat{X}} \widehat{x}[\mathbf{1}] \pi(\cdot, d\widehat{x}) \in \mathcal{B}$$

assures that

$$m[.](y) := \int_{\widehat{X}} \widehat{x}[.] \pi(y, d\widehat{x}); \quad y \in X,$$

defines a linear-bounded operator on \mathcal{B} , which in conjunction with $k \in \mathcal{B}_+$ implies $M^t : \mathcal{B} \rightarrow \mathcal{B}$ for all $t \geq 0$, cf. [3].

We assume throughout that the following condition is satisfied:

(M) *The moment semigroup $\{M^t\}_{t \geq 0}$ can be represented in the form*

$$M^t = \rho^t \varphi \varphi^* + Q_t; \quad t > 0,$$

where $\rho \in]0, \infty[$, $\varphi \in \mathcal{B}_+$, φ^* is a non-negative, linear-bounded functional on \mathcal{B} , and $Q_t : \mathcal{B} \rightarrow \mathcal{B}$ such that

$$\begin{aligned} \varphi^*[\varphi] = 1, \quad \varphi^*[Q_t[.]] = 0, \quad Q_t[\varphi] = 0; \quad t > 0, \\ |Q_t[\xi]| \leq \alpha_t \varphi^*[\xi] \varphi; \quad \xi \in \mathcal{B}_+, \quad t > 0, \end{aligned}$$

with some $\alpha_t : \mathbb{T} \rightarrow [0, \infty[$ satisfying

$$\rho^{-t} \alpha_t \rightarrow 0; \quad t \rightarrow \infty.$$

We propose to call a Markov branching process positively regular if it satisfies (M). For finite X this definition is equivalent to the historic one. Verifications of (M) for large classes of branching diffusions and related processes are to be found in [6, 7] and particularly in [9]. While admitting $\inf_{x \in X} \varphi(x) = 0$, we can assume without loss of generality that $\varphi(x) > 0 \forall x \in X$. In case of a restricted branching diffusion this merely means that any totally absorbing barrier is by definition not included in X . Note also that $\varphi^*[1_A]$ is automatically σ -additive in $A \in \mathfrak{A}$. Here 1_A is the indicator function of A .

EXAMPLE 1. — *Branching processes with a finite set of types.* Suppose $X = \{1, \dots, K\}$, $T_t \xi \equiv \xi$, and

$$\pi(x, \{ \widehat{x}[1_{\{y\}}] = n_y; y = 1, \dots, K \}) = p_x(n_1, \dots, n_K),$$

where p_x is a probability distribution on \mathbb{Z}_+^K for every $x \in X$. Then m is given as a $K \times K$ matrix with elements

$$m_{xy} = \sum_{n_1=0}^{\infty} \dots \sum_{n_K=0}^{\infty} n_y p_x(n_1, \dots, n_K).$$

If $(k(x)m_{xy})$ is finite and irreducible, then M^t is finite and strictly positive, and it follows by Perron's theorem on positive matrices that (M) is satisfied with $\varphi(x) > 0$, $x \in X$, and $\varphi^*[\xi] > 0$, $0 \neq \xi \in \mathcal{B}_+$.

EXAMPLE 2. — *Branching diffusions on simply connected, bounded domains.* Suppose $\Omega \subset X \subset \bar{\Omega}$, where Ω is a simply connected bounded open set of class C^3 in \mathbb{R}^N and $\bar{\Omega}$ the closure of Ω . Let T_t be given by its differential generator, a uniformly elliptic operator

$$L = \sum_{i,j=1}^N a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^N b_j \frac{\partial}{\partial x_j}$$

defined on the set of restrictions to X of all twice continuously differentiable functions ξ on $\bar{\Omega}$ satisfying the boundary condition

$$\left(\alpha \xi + \beta \frac{\partial \xi}{\partial n} \right) \Big|_{\partial \Omega} = 0.$$

Here $\frac{\partial}{\partial n}$ denotes the derivative in the direction of the exterior normal and $\partial \Omega$ the boundary of Ω . Let a_{ij}, b_j be restrictions of functions that have Hölder continuous second, respectively first derivatives on $\bar{\Omega}$, and let α, β be non-negative, $\alpha(x) + \beta(x) > 0$ for all $x \in \partial \Omega$, with Hölder continuous second derivatives on $\partial \Omega$. By convention

$$\bar{\Omega} \setminus X = \{ \beta = 0 \}.$$

a) The kernel π is called a *local branching law*, if it is of the form

$$\pi(x, \hat{A}) = p_0(x) 1_{\hat{A}}(\theta) + \sum_{n>0} 1_{\hat{A}}(\langle \overset{n}{x}, \dots, x \rangle) p_n(x), \quad \hat{A} \in \hat{\mathcal{A}},$$

where $\{ p_n(x) \}$ is a probability distribution on \mathbb{Z}_+ for every $x \in X$. If

$$m := \sum_n n p_n < \infty$$

then $m[\xi] = m\xi$ and (M) is satisfied with

$$\varphi^*[\xi] = \int_X \varphi^*(x) \xi(x) dx.$$

Moreover, $\varphi(x)$ and $\varphi^*(x)$ possess continuously differentiable extensions to $\bar{\Omega}$, which are strictly positive on X and vanish with strictly negative exterior normal first derivatives on $\bar{\Omega} \setminus X$, see [9].

b) An example of a *non-local branching law* is given by

$$\pi(x, \{ \hat{x}[1_{A_\nu}] = n_\nu; \nu = 1, \dots, j \}) = p_n(x) n! \sum_{\nu=1}^j \frac{1}{n_\nu!} \left(\int_{A_\nu} f(x, y) dy \right)^{n_\nu},$$

$$n = \sum_{\nu=1}^j n_\nu,$$

where $\{A_\nu\}_{0 \leq \nu \leq j}$ varies over the measurable decompositions of X , $\{p_n(x)\}$ is a probability distribution on \mathbb{Z}_+ , and $f(x, \cdot)$ a distribution density on X for every $x \in X$.

The intuitive interpretation is simply that if a branching event occurs at x , it results with probability p_n in n new particles, and these emerge at locations distributed on X independently of each other but each according to $f(x, \cdot)$.

If the function

$$m(x, y) := f(x, y) \sum_n n p_n(x)$$

is bounded on $X \otimes X$, then

$$m[\xi](x) = \int_X m(x, y) \xi(y) dy, \quad \xi \in \mathcal{B}, x \in X,$$

and (M) is satisfied with the same general properties of φ and \mathcal{S}^* as stated in part (a) of this example, cf. [9].

EXAMPLE 3. — *Branching diffusions on disconnected, bounded domains.* Let X and L be as in the preceding example, except that X is now the union of $K < \infty$ simply connected components X_ν , the closures \bar{X}_ν being disjoint. Then the branching law cannot be local, unless the branching process can be decomposed into K processes on the simply connected components. A special case are *multitype branching diffusions*. Here the \bar{X}_ν are all congruent. Suppose that even the X_ν are congruent, and let $\kappa_\nu x$ be the picture of $x \in X$ produced in X_ν by the given congruence. The natural analogue of a local branching law would then be a *quasi-local branching law*,

$$\begin{aligned} \pi(x, \hat{A}) = p_{0\dots 0}(x) 1_{\hat{A}}(\theta) + \sum_{\substack{n_1 \geq 0 \\ \dots \\ n_K \geq 0 \\ n_1 + \dots + n_K > 0}} \dots \sum_{n_K \geq 0} p_{n_1 \dots n_K}(x) \\ \times 1_{\hat{A}}(\langle \kappa_1 x, \dots, \kappa_1 x, \dots, \kappa_K x, \dots, \kappa_K x \rangle), \end{aligned}$$

where $\{p_{n_1 \dots n_K}(x)\}$ is a probability distribution on \mathbb{Z}_+^K for every $x \in X$. If

$$m_\nu := \sum_{n_1 \geq 0} \dots \sum_{n_K \geq 0} n_\nu p_{n_1 \dots n_K} \in \mathcal{B}; \quad \nu = 1, \dots, K,$$

then

$$m[\xi](x) = \sum_{\nu_K=1}^K m_\nu(x) \xi(\kappa_\nu x),$$

and if in addition the $K \times K$ -matrix with elements

$$m_{\nu\mu} = \int_{X_\mu} k(x) m_\nu(x) dx$$

is irreducible, then (M) is satisfied with the same general properties of φ and φ^* as stated in the preceding example, cf. [9].

Simple explicit cases of these and other examples along with some applications will be discussed in [11].

2. LIMIT THEOREMS

Let us first recall two results on supercritical processes ($\rho > 1$).

THEOREM 1 [7]. — *If $\{\widehat{x}_t, \widehat{\mathbf{P}}^x\}$ is a Markov branching process satisfying (M) with $\rho > 1$, then there exists a random variable W such that*

$$\lim_{N \ni n \rightarrow \infty} \rho^{-n} \widehat{x}_n[\xi] = \varphi^*[\xi]W \text{ a. s. } [\widehat{\mathbf{P}}^x]$$

for every ξ absolutely integrable with respect to φ^* . If

$$(2.1) \quad \varphi^*[\mathbf{E}^{\langle \cdot \rangle} \widehat{x}_t[\varphi] \log \widehat{x}_t[\varphi]] < \infty$$

for some $t \in T \setminus \{0\}$, then this inequality holds for all $t \in T$, and $\mathbf{E}^{\langle x \rangle} W = \varphi(x) \forall x \in X$. Otherwise $W = 0$ a. s. $[\widehat{\mathbf{P}}^x]$.

In order to handle $t \rightarrow \infty$, $t \in \mathbb{R}_+$, some additional structure is needed:

(C.1) *There exists a set of non-negative random variables $\{\Gamma_t; t > 0\}$ such that $\widehat{x}_s[\mathbf{1}] \leq \Gamma_t \forall s \in [0, t]$ and $\|\mathbf{E}^{\langle x \rangle} \Gamma_t\| \downarrow 1$ as $t \downarrow 0$.*

If $\{\widehat{x}_t, \widehat{\mathbf{P}}^x\}$ can be constructed from a system $[T, k, \pi]$ satisfying (A.1-3), define $y_t := \widehat{x}_t[\mathbf{1}] + n_t$ with $n_t := \# \{ \tau : \widehat{x}_{t-\tau}[\mathbf{1}] > \widehat{x}_t[\mathbf{1}]; 0 < \tau \leq t \}$. Then $\widehat{x}_s[\mathbf{1}] \leq y_t \forall s \in [0, t]$ a. s. $[\widehat{\mathbf{P}}^x]$ and

$$1 \leq \|\mathbf{E}^{\langle \cdot \rangle} y_t\| \leq \exp \{ \|k\| (\|m[\mathbf{1}]\| + 1)t \},$$

so that (C.1) is satisfied, cf. [7].

THEOREM 1* [7, 8]. — *Let X be a separable metric space, \mathfrak{A} the topological Borel algebra, and $\{\widehat{x}_t, \widehat{\mathbf{P}}^x\}$ a right-continuous Markov branching process satisfying (M) with $\rho > 1$ and (C.1). Then*

$$\lim_{t \rightarrow \infty} \rho^{-t} \widehat{x}_t[\eta] = \varphi^*[\eta]W \text{ a. s. } [\widehat{\mathbf{P}}^x]$$

for all $\eta \in \mathfrak{B}$ which are continuous a. e. $[\varphi^*]$.

Given (A.1-3), condition (2.1) can be expressed in terms of k and π . For this we need the following property:

(B.1) *There exists a $c^* \in \mathbb{R}_+$ such that $\varphi^*[km[\xi]] \leq c^* \varphi^*[\xi]$ for all $\xi \in \mathfrak{B}_+$.*

Let us examine (B. 1) in terms of the examples given in section 1 :

E. 1. If X is finite, (B. 1) is satisfied with

$$c^* = \max_x \frac{\varphi^*[km[1_{\{x\}}]]}{\varphi^*[1_{\{x\}}]} .$$

E. 2. Given a branching diffusion with a local π as in example 2a, (B. 1) is satisfied with $c^* = \| km \|$. However, if π is non-local, (B. 1) need not be satisfied. Take a π of the form given in example 2b, and suppose $k(x)m(x, y)$ can be extended with continuous first partial derivatives to $\bar{X} \otimes \bar{X}$. Then (B. 1) holds if and only if $k(x)m(x, y) \rightarrow 0$ whenever $y \rightarrow z \in \bar{X} \setminus X$.

E. 3. In case of a multitype branching diffusion with quasi-local π as in example 3, (B. 1) is satisfied with

$$c^* = \max_v \| km_v \| \sup_x \frac{\varphi^*(x)}{\varphi^*(\kappa_v x)} .$$

which is finite due to the stated properties of $\varphi^*(x)$.

Concerning the involvement of φ^* see Remark 1 at the end of this section.

PROPOSITION 1. — *Given (A.1-3), suppose (M) and (B.1) are satisfied. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be concave with $f(0) = 0$. Then for any $t > 0$*

$$(2.2) \quad \varphi^*[\mathbf{E}^{\langle \cdot \rangle} \hat{x}_t[\varphi] f(\hat{x}_t[\varphi])] < \infty$$

if and only if

$$(2.3) \quad \varphi^* \left[k \int_{\hat{X}} \hat{x}[\varphi] f(\hat{x}[\varphi]) \pi(\cdot, d\hat{x}) \right] < \infty .$$

The proof is a routine extension of the proof given in a more special setting in [7], and there is no need to repeat the details. Note that, while $f(x) = \log x$ does not satisfy the assumptions of Proposition 1, (2.2) and (2.3) with

$$f(x) = 1_{[0, e)}(x) \frac{x}{e} + 1_{[e, \infty)}(x) \log x$$

are equivalent to (2.1) and

$$(2.4) \quad \varphi^* \left[k \int_{\hat{X}} \hat{x}[\varphi] \log \hat{x}[\varphi] \pi(\cdot, dx) \right] < \infty ,$$

respectively. Concerning extensions of Proposition 1 see Remark 2 below.

Turning now to $\rho \leq 1$, we introduce the mappings $F_t[\cdot] : \mathcal{F} \rightarrow \mathcal{F}$, $t \in \mathbb{T}$, defined by $F_t[\cdot](x) = F_t(\langle x \rangle, \cdot)$; $x \in X$. If $M^t : \mathcal{B} \rightarrow \mathcal{B}$, there exists a mapping $R^t(\cdot)[\cdot] : \mathcal{F} \otimes \mathcal{B} \rightarrow \mathcal{B}$, sequentially continuous with respect to the product topology on bounded regions, non-increasing in the first and linear-bounded in the second variable, such that

$$(2.5) \quad \mathbf{0} = R^t(\mathbf{1})[\eta] \leq R^t(\zeta)[\eta] \leq M^t[\eta]; \quad (\zeta, \eta) \in \bar{\mathcal{F}}_+ \otimes \mathcal{B}_+,$$

$$(2.6) \quad \mathbf{1} - F_t[\xi] = M^t[\mathbf{1} - \xi] - R^t(\xi)[\mathbf{1} - \xi]; \quad \xi \in \overline{\mathcal{P}},$$

cf. [2, 6]. We shall need the following property:

(R) For every $t \in T \setminus \{0\}$ there exists a mapping $g_t: \overline{\mathcal{P}}_+ \rightarrow \mathcal{B}$ such that

$$R^t(\xi)[\mathbf{1} - \xi] = g_t[\xi]\rho^t\varphi^*[\mathbf{1} - \xi]\varphi; \quad \xi \in \overline{\mathcal{P}}_+,$$

$$\lim_{\|\mathbf{1} - \xi\| \rightarrow 0} \|g_t[\xi]\| = 0.$$

If X is finite, (R) is automatically satisfied, since in that case $\inf \varphi > 0$ and for $\mathbf{0} \leq \xi \leq \mathbf{1}$, $\xi \neq \mathbf{1}$,

$$0 < \frac{\|\mathbf{1} - \xi\|}{\varphi^*[\mathbf{1} - \xi]} \leq (\inf_x \varphi^*[1_{\{x\}}])^{-1} < \infty.$$

To prove (R) in the $[T, k, \pi]$ setting for more general X , we need another consistency condition:

(B.2) There exists a $c \in \mathbb{R}_+$ such that $km[\varphi] \leq c\varphi$.

The discussion of (B.2) parallels that of (B.1):

In example 2a condition (B.2) is satisfied with $c = \|km\|$. In example 2b, assuming the same smoothness as above, (B.2) is satisfied if and only if $k(x)m(x, y) \rightarrow 0$ whenever $x \rightarrow z \in \overline{X} \setminus X$. In example 3 take

$$c = \max_v \|km_v\| \sup_x \frac{\varphi(\kappa_v x)}{\varphi(x)}.$$

Concerning the role of φ we again refer to Remark 1.

PROPOSITION 2. — Given (A.1-3), suppose (M) and (B.1-2) are satisfied. Then (R) holds.

A proof is to be found in section 3. It extends the argument given in [10].

In accordance with the respective remark in section 1 we tacitly assume from now on that $\varphi(x) > 0$ for all $x \in X$. In connection with the subcritical case ($\rho < 1$) we shall need the following continuity property:

(C.2) The space (X, \mathfrak{M}) is a topological measurable space, and there exists a compactification \overline{X} of X such that $(\mathbf{1} - F_t[\xi])/\varphi$ has a continuous extension on \overline{X} for every $t \in T \setminus \{0\}$ and $\xi \in \overline{\mathcal{P}}_+$.

A verification of (C.2) for a large class of branching diffusions has been given in [10]. The proof does not depend on whether or not π is local.

If P_t is a branching transition function satisfying (M) with $\rho < 1$, then by (2.5), (2.6), $\lim_{t \rightarrow \infty} P_t(\widehat{x}, \{\theta\}) = 1$ uniformly in $\widehat{x} \in X^{(n)}$ for every $n > 0$.

THEOREM 2. — Let \mathbf{P}_t be a branching transition function satisfying (M) with $\rho < 1$ and (R). Then there exists a $\gamma \in \mathbb{R}_+$ such that

$$(2.7) \quad \lim_{t \rightarrow \infty} \rho^{-t} \mathbf{P}_t(\widehat{x}, \{y \neq \theta\}) = \gamma \widehat{x}[\varphi]$$

uniformly in $\widehat{x} \in \mathbf{X}^{(n)}$ for every $n > 0$. Moreover, $\gamma > 0$ if and only if

$$(2.8) \quad \varphi^*[\mathbf{E}^{\langle \cdot \rangle} \widehat{x}_t[\varphi] \log \widehat{x}_t[\varphi]] < \infty$$

for some (and thus all) $t \in \mathbb{T} \setminus \{0\}$. If $\gamma = 0$, suppose (C.1) is satisfied. Then

for any $A_v \in \mathfrak{A}$ and $n_v \in \mathbb{Z}_+$, $v = 1, \dots, j$, with $\bigcup_{v=1}^j A_v = \mathbf{X}$, $j > 0$, the limit

$$p(A_1, \dots, A_j; n_1, \dots, n_j) := \lim_{t \rightarrow \infty} \frac{\mathbf{P}_t(\widehat{x}, \{\widehat{y}[1_{A_v}] = n_v; v = 1, \dots, j\} \cap \{\widehat{y} \neq \theta\})}{\mathbf{P}_t(\widehat{x}, \{\widehat{y} \neq \theta\})}$$

exists uniformly in $\widehat{x} \in \mathbf{X}^{(n)}$ for every $n > 0$ and is independent of \widehat{x} . The limits form a consistent set of probabilities, and if \mathbf{X} is a locally compact Hausdorff space with countable open base and \mathfrak{A} the topological Borel algebra, this set determines a probability measure \mathbf{P} on $(\widehat{\mathbf{X}}, \widehat{\mathfrak{A}})$ such that

$$\mathbf{P}(\widehat{y}[1_{A_v}] = n_v; v = 1, \dots, j) = p(A_1, \dots, A_j; n_1, \dots, n_j).$$

If $\gamma > 0$, then

$$\int_{\widehat{\mathbf{X}}} \widehat{x}[\cdot] \mathbf{P}(d\widehat{x}) = \frac{1}{\gamma} \varphi^*[\cdot].$$

but if $\gamma = 0$, then \mathbf{P} does not have a bounded first moment functional.

For the proof we refer to section 3. Details are given only where the argument deviates from the more special proof in [10].

We now turn to the critical case ($\rho = 1$).

LEMMA 3. — Let \mathbf{P}_t be a branching transition function satisfying (M) with $\rho = 1$. Then the value of

$$\mu := \frac{1}{2t} \varphi^* \left[\int_{\widehat{\mathbf{X}}} \{ \widehat{x}[\varphi]^2 - \widehat{x}[\varphi^2] \} \mathbf{P}_t(\langle \cdot \rangle, d\widehat{x}) \right],$$

which is non-negative, possibly infinite, does not depend on $t \in \mathbb{T} \setminus \{0\}$.

PROPOSITION 3. — Given (A.1-3) such that (M) with $\rho = 1$ and (B.1) are satisfied.

$$(2.9) \quad \mu = \frac{1}{2} \varphi^* \left[k \int_{\widehat{\mathbf{X}}} \{ \widehat{x}[\varphi]^2 - \widehat{x}[\varphi^2] \} \pi(\cdot, d\widehat{x}) \right].$$

Again, the proofs are deferred to section 3.

Let P_t be a branching transition function satisfying (M) with $\rho = 1$. Clearly, $\mu = 0$ if and only if $P_t(\langle \cdot \rangle, X^{(1)}) = 1$ a. s. $[\varphi^*] \forall t \in T$. If $\mu > 0$, then $\varphi^*[P_t(\langle \cdot \rangle, \{\theta\})] > 0 \forall t \in T \setminus \{0\}$. Assuming $\mu > 0$, define

$$N(t) := \{x \in X : P_t(\langle x \rangle, \{\theta\}) = 0\}; \quad t \in T \setminus \{0\},$$

$$q(x) := \lim_{t \rightarrow \infty} P_t(\langle x \rangle, \{\theta\}); \quad x \in X.$$

If $\varphi^*[1_{N(t)}] = 0$ for some $t > 0$, then $q = 1$ a. s. $[\varphi^*]$ as in [1; III, no. 11, 12]. If $\varphi^*[1_{N(t)}] > 0 \forall t > 0$, fix $s > 0$ such that $\alpha_s < 1$ and define

$$N := \bigcup_{n \in \mathbb{N}} N(ns).$$

A routine extension of [1; II, no. 6] shows that $P_{2s}(\langle x \rangle, \{\widehat{y}[1_N] > 1\}) > 0 \forall x \in X$ and, if

$$(2.10) \quad \inf_{x \in N} P_{2s}(\langle x \rangle, \{\widehat{y}[1_N] > 1\}) > 0,$$

that $\{0 < \widehat{y}[1] \leq d\}$, $d > 0$, is a transient event of the process $\{\widehat{x}_{2ns}, P^{\widehat{x}}; n \in \mathbb{Z}_+\}$ determined by P_{2s} , which implies again that $q = 1$.

If X is finite, (2.10) is automatic. If more generally (X, \mathfrak{A}) is a topological measurable space and N compact, then continuity of $P_{2s}(\langle x \rangle, \{\widehat{y}[1_N] \leq 1\})$ in $x \in N$ is sufficient for (2.10). Given (A.1-2), this continuity is guaranteed, if $T_t \xi(x)$ is continuous in $x \in X$ for all $t > 0$ and $\xi \in \mathcal{B}$, and that is the case for many diffusions, cf. [9].

From (1.1) and the Chapman-Kolmogorov equation

$$(2.11) \quad F_{t+s}[\xi] = F_t[F_s[\xi]]; \quad t, s \in T, \xi \in \overline{\mathcal{F}}.$$

By use of (2.11), (2.5), (2.6), and (M) it follows from $\varphi^*[1 - q] = 0$ that $\lim_{t \rightarrow \infty} P_t(\widehat{x}, \{\theta\}) = 1$ uniformly in $\widehat{x} \in X^{(n)}$ for every $n > 0$.

We shall need the following continuity property:

(C.3) *If $T = \mathbb{R}_+$, then for every $x \in X$ and every decomposition $\{A_1, \dots, A_j\}$ of X with $A_v \in \mathfrak{A}$, $v = 1, \dots, j$, $j > 0$, the function $P_t(\langle x \rangle, \{\widehat{y}[1_{A_v}] = n_v; v = 1, \dots, j\})$ is continuous in $t \in T$.*

Assuming (A.1-2), it is shown in section 3 that (C.3) is satisfied if $T_t : \mathcal{B} \rightarrow \mathcal{C}_0$ for $t > 0$. The latter is the case in all examples of section 1, cf. [9].

THEOREM 3. — *Let P_t be a branching transition function satisfying (M) with $\rho = 1$ and (R). If $\varphi^*[1 - q] = 0$ and $\mu < \infty$, then*

$$\lim_{t \rightarrow \infty} tP_t(\widehat{x}, \widehat{X} \setminus \{\theta\}) = \frac{1}{\mu} \widehat{x}[\varphi]$$

uniformly in $\widehat{x} \in X^{(n)}$ for every $n > 0$. If in addition (C.3) is satisfied, then for every decomposition $\{A_1, \dots, A_j\}$ of X with $A_v \in \mathfrak{A}$, $v = 1, \dots, j, j > 0$, and every $\widehat{x} \in \widehat{X} \setminus \{\theta\}$

$$\lim_{t \rightarrow \infty} \frac{\mathbf{P}_t \left(\widehat{x}, \left\{ \frac{1}{t} \widehat{y}[1_{A_v}] \leq \lambda_v; v = 1, \dots, j \right\} \cap \{ \widehat{y} \neq \theta \} \right)}{\mathbf{P}_t(\widehat{x}, \{ \widehat{y} \neq \theta \})} = \begin{cases} 0; & \min \lambda_v \leq 0, \\ 1 - \exp \left\{ - \min_v [(\mu \varphi^*[1_{A_v}])^{-1} \lambda_v] \right\}; & \min \lambda_v > 0 \end{cases}$$

uniformly in $(\lambda_1, \dots, \lambda_j) \in \mathbb{R}^j$.

REMARK 1. — The conditions (2.1), (2.4), (B.1-2), and $\mu < \infty$ are less implicit than they may appear to be. As illustrated by the examples, there is often enough general information about φ and φ^* to allow more explicit expressions. In the finite case, or if $\{\beta = 0\}$ is empty, we may replace $\varphi(x)$ and $\varphi^*(x)$ in the conditions simply by $\mathbf{1}$ and obtain conditions which are equivalent to the original ones. As an example for the case that $\{\beta = 0\}$ is not empty consider a branching diffusion on the finite interval $X = (a, b)$ with absorption at both endpoints. We may then replace both $\varphi(x)$ and $\varphi^*(x)$ by $(x - a)(b - x)$ and arrive at equivalent conditions.

REMARK 2. — Although Proposition 1 is already more general than is needed here, the full scope of the method of proof in [7] is of interest:

a) In order to prove that

$$(2.12) \quad \varphi^* \left[k \int_{\widehat{X}} \widehat{x}[\varphi]^n f(\widehat{x}[\varphi]) \pi(\cdot, d\widehat{x}) \right] < \infty$$

is sufficient for

$$\varphi^* [\mathbf{E}^{\langle \cdot \rangle} \widehat{x}_t[\varphi]^n f(\widehat{x}_t[\varphi])] < \infty$$

with f as in the proposition and $n = 2, 3, 4, \dots$, the corresponding higher order analogue of (A.3),

$$\int_{\widehat{X}} \widehat{x}[\mathbf{1}]^n \pi(\cdot, d\widehat{x}) \in \mathcal{B},$$

is needed. For finite X this is, of course, already contained in (2.12), but in general it is not. The necessity part of the proof goes through as before-

b) When replacing φ^* , or φ , the sensitive details of the proof are the following. The sufficiency part relies on (B.2) and (3.11), the necessity part on (3.12) and the submartingale property of $\{ \widehat{x}_t[\varphi]/\rho^t \}$. In fact, (3.11), (3.12), and the submartingale property are needed only with some positive continuous function in place of ρ^t .

3. PROOFS

PROOF OF PROPOSITION 2. — Given (A.1-2), let $\{x_t, P^x\}$ be the Markov process determined by $\{T_t\}$ and E^x the expectation with respect to P^x . Define

$$T_t^0 \xi(x) := E^x \left(\xi(x_t) \exp \left\{ - \int_0^t k(x_s) ds \right\} \right); \quad \xi \in \mathcal{B}, x \in X,$$

and let $f[\cdot](x)$ be the generating functional of $\pi(x, \cdot)$, $x \in X$. Then for every $\xi \in \overline{\mathcal{P}}$ the function $F_t[\xi](x); t \geq 0, x \in X$, is the unique solution of

$$u_t(x) = T_t^0 \xi(x) + H_t(x) + \int_0^t T_s^0 \{ kf[u_{t-s}] \} (x) ds,$$

$$H_t(x) := 1 - T_t^0 \mathbf{1}(x) - \int_0^t T_s^0 k(x) ds.$$

If we also assume (A.3), then for every $\xi \in \mathcal{B}$ the function $M^t[\xi](x); t \geq 0, x \in X$, is the unique solution of

$$(3.1) \quad v_t(x) = T_t^0 \xi(x) + \int_0^t T_s^0 \{ km[v_{t-s}] \} (x) ds.$$

cf. [3], [4]. It follows by use of (2.6) and the corresponding expansion for f ,

$$1 - f[\xi] = m[1 - \xi] - r(\xi)[1 - \xi]; \quad \xi \in \overline{\mathcal{P}},$$

that for every $\varepsilon > 0$ and $\xi \in \overline{\mathcal{P}}_+$ the function $R^t(\xi)[1 - \xi](x); t \geq \varepsilon, x \in X$, solves

$$(3.2) \quad w_t(x) = A_t(x) + B_t^\varepsilon(x) + \int_0^{t-\varepsilon} T_s^0 \{ km[w_{t-s}] \} (x) dx,$$

$$A_t(x) := \int_0^t T_s^0 \{ kr(F_{t-s}[\xi])[1 - F_{t-s}[\xi]] \} (x) ds,$$

$$B_t^\varepsilon(x) := \int_0^\varepsilon T_{t-s}^0 \{ km[R^s(\xi)[1 - \xi]] \} (x) ds.$$

In fact, $R^t(\xi)[1 - \xi](x)$ is the only bounded solution in $[\varepsilon, \varepsilon + \tau]$ for any $\tau > 0$, and thus equals the limit of the (non-decreasing) iteration sequence $\{w_t^{(v)}(x)\}_{v \in \mathbb{Z}_+}$, $w_t^{(0)} \equiv 0$. We estimate this sequence, modifying the argument given in [10].

Suppose $0 < \delta < \varepsilon/2$ and $\xi \in \overline{\mathcal{P}}_+$. By (2.5) and (2.6) there exists a $c_1 \geq 0$ such that $F_{t-s}[\xi] \geq \mathbf{1} - c_1 \|\mathbf{1} - \xi\| \mathbf{1}$ for $\delta \leq s \leq t - \delta, t \leq \varepsilon + \tau$. Equation (3.1) implies $T_t^0 \leq M^t$ on \mathcal{B}_+ . Finally, we have

$$0 = r(\mathbf{1})[\xi] \leq r(\xi)[\xi] \leq m[\xi] \forall (\zeta, \xi) \in \overline{\mathcal{P}}_+ \otimes \mathcal{B}_+.$$

Hence, making use of (M) and (B.1-2), for $t \geq \varepsilon$

$$\begin{aligned}
 A_t(x) &\leq \int_0^\delta + \int_{t-\delta}^t M^s[km[M^{t-s}[\mathbf{1} - \xi]]](x)ds \\
 &\quad + \int_\delta^{t-\delta} M^s[kr(\mathbf{1} - c_1 \|\mathbf{1} - \xi\| \mathbf{1})][M^{t-s}[\mathbf{1} - \xi]](x)ds \\
 &\leq \delta(c + c^*)(1 + \rho^{-\varepsilon/2}\alpha_{\varepsilon/2})\rho^t\varphi^*[\mathbf{1} - \xi]\varphi(x) \\
 &\quad + t(1 + \rho^{-\delta}\alpha_\delta)(1 + \rho^{-\varepsilon/2}\alpha_{\varepsilon/2})\|k\varphi\| \\
 &\quad \times \varphi^*[r(\mathbf{1} - c_1 \|\mathbf{1} - \xi\| \mathbf{1})[\varphi]]\rho^t\varphi^*[\mathbf{1} - \xi]\varphi(x).
 \end{aligned}$$

Since $\varphi^*[r(\mathbf{1} - c_1 \|\mathbf{1} - \xi\| \mathbf{1})[\varphi]] \rightarrow 0$ as $\|\mathbf{1} - \xi\| \rightarrow 0$, and since δ can be chosen arbitrarily small, this shows that

$$\begin{aligned}
 (3.3) \quad A_t &\leq t\Delta_{\varepsilon,\tau}[\xi]\rho^t\varphi^*[\mathbf{1} - \xi]\varphi; & \varepsilon \leq t \leq \varepsilon + \tau, \\
 &\lim_{\|\mathbf{1} - \xi\| \rightarrow 0} \Delta_{\varepsilon,\tau}[\xi] = 0; & \varepsilon > 0, \tau > 0.
 \end{aligned}$$

Using (2.5), (3.1), and the fact that $T_{t-s}^0 \leq M^{t-s}M^{\varepsilon-s}$ on \mathcal{B}_+ ,

$$\begin{aligned}
 (3.4) \quad B_t^\varepsilon(x) &\leq M^{t-\varepsilon} \left[\int_0^\varepsilon M^{\varepsilon-s}[km[M^s[\mathbf{1} - \xi]]](x)ds = : \bar{B}_t^\varepsilon(x); \quad t \geq \varepsilon, \right. \\
 &\quad \left. \int_0^{t-\varepsilon} T_s^0 \{ km[\bar{B}_{t-s}^\varepsilon] \} (x)ds \leq \bar{B}_t^\varepsilon(x); \quad t \geq \varepsilon. \right.
 \end{aligned}$$

Again by use of (M) and (B.1-2) it follows from (3.2-4) that

$$\lim_{v \rightarrow \infty} w_t^{(v)} \leq \{ e^{ct}t\Delta_{\varepsilon,\tau}[\xi] + \varepsilon c^*(1 + \rho^{-(t-\varepsilon)}\alpha_{t-\varepsilon}) \} \rho^t\varphi^*[\mathbf{1} - \xi]\varphi; \quad \varepsilon < t \leq \varepsilon + \tau.$$

Since $\varepsilon, \tau > 0$ were arbitrary, this implies (R). \square

The following lemma is used in the proofs of Theorems 2 and 3.

LEMMA 1. — *If P_t is a branching transition function such that (M) and (R) are satisfied and $\lim_{t \rightarrow \infty} P_t(\langle x \rangle, \theta) = 1 \forall x \in X$, then there exists for every $t \in T \setminus \{0\}$ a mapping $h_t : \bar{\mathcal{P}}_+ \rightarrow \mathcal{B}$ such that*

$$\begin{aligned}
 \mathbf{1} - F_t[\xi] &= (\mathbf{1} + h_t[\xi])\varphi^*[\mathbf{1} - F_t[\xi]]\varphi; \quad \xi \in \bar{\mathcal{P}}_+, \\
 \lim_{t \rightarrow \infty} \|h_t[\xi]\| &= 0 \text{ uniformly on } \bar{\mathcal{P}}_+,
 \end{aligned}$$

where $\varphi^*[\mathbf{1} - F_t[\xi]] > 0 \forall t > 0, \xi \in \bar{\mathcal{P}}_+ \cap \{\varphi^*[\mathbf{1} - \xi] > 0\}$.

The proof of this lemma is the same as in [10] except for the last statement, which we verify as follows. Suppose $\xi \in \bar{\mathcal{P}}_+ \cap \{\varphi^*[\mathbf{1} - \xi] > 0\}$ and $t > 0$. If $0 < \delta < 1$, then by (M) and (R)

$$\begin{aligned}
 \varphi^*[\mathbf{1} - F_t[\xi]] &\geq \varphi^*[\mathbf{1} - F_t[\mathbf{1} - \delta(\mathbf{1} - \xi)]] \\
 &\geq \rho^t\delta\varphi^*[\mathbf{1} - \xi] \{ 1 - \|g_t[\mathbf{1} - \delta(\mathbf{1} - \xi)]\| \},
 \end{aligned}$$

and there is a $\delta = \delta(t)$ such that $\|g_t[\mathbf{1} - \delta(\mathbf{1} - \xi)]\| < 1$.

PROOF OF THEOREM 2. — Given (M) with $\rho < 1$, there exists a $\gamma \in \mathbb{R}_+$ such that

$$(3.5) \quad \rho^{-t} \varphi^* [\mathbf{1} - F_t[\mathbf{0}]] \downarrow \gamma \quad \text{as } t \uparrow \infty.$$

Moreover, $\gamma > 0$ if and only if for some $\varepsilon < \|\varphi\|^{-1}$

$$(3.6) \quad \sum_{v=1}^{\infty} \varphi^* [\mathbf{R}^t(-\varepsilon \varphi \rho^{vt})[\varphi]] < \infty,$$

where $t \in \mathbb{T} \setminus \{0\}$ is arbitrary. The proof of these two statements is the same as in [10]. It is a routine extension of the argument given in [2].

Lemma 1 and (3.5) imply (2.7). The equivalence of (3.6) and (2.8) follows from the next lemma.

LEMMA 2. — Let $\mathbf{P}(\cdot, \cdot)$ be a stochastic kernel on $X \otimes \mathfrak{X}$ such that

$$\mathbf{M}[\cdot](x) := \int_{\widehat{X}} \widehat{y}[\cdot] \mathbf{P}(x, \widehat{dy}) ; \quad x \in X,$$

defines a bounded operator \mathbf{M} on \mathcal{B} . Let $F[\cdot](x)$ be the generating functional of $\mathbf{P}(x, \cdot)$, and expand

$$\mathbf{1} - F[\xi] = \mathbf{M}[\mathbf{1} - \xi] - \mathbf{R}(\xi)[\mathbf{1} - \xi]; \quad \xi \in \overline{\mathcal{F}},$$

as in (2.6). Finally, let ξ^* be a non-negative, linear-bounded functional on \mathcal{B} , sequentially continuous with respect to the product topology on bounded regions, let $\xi \in \overline{\mathcal{F}}_+$ such that $\xi(x) > 0 \forall x \in X$, and let $\lambda \in (0, 1)$. Then

$$(3.7) \quad \sum_{v=1}^{\infty} \xi^* [\mathbf{R}(\mathbf{1} - \lambda^v \xi)[\xi]] < \infty$$

if and only if

$$(3.8) \quad \xi^* \left[\int_{\widehat{X}} \widehat{x}[\xi] \log \widehat{x}[\xi] \mathbf{P}(\cdot, \widehat{dx}) \right] < \infty.$$

Proof. — We extend the proof of [10 : Lemma 4]. Notice the relation to the argument used in [5]. Clearly,

$$\begin{aligned} & \int_0^\infty \xi^* [\mathbf{R}(\mathbf{1} - \lambda^t \xi)[\xi]] dt - \xi^* [\mathbf{M}[\xi]] \\ & \leq \sum_{v=1}^{\infty} \xi^* [\mathbf{R}(\mathbf{1} - \lambda^v \xi)[\xi]] \leq \int_0^\infty \xi^* [\mathbf{R}(\mathbf{1} - \lambda^t \xi)[\xi]] dt. \end{aligned}$$

With the substitution $s = s(\widehat{x}, t) := -\widehat{x}[\log(\mathbf{1} - \lambda^t \xi)]/\widehat{x}[\xi]$

$$\begin{aligned} & \int_0^\infty \xi^* [\mathbf{R}(\mathbf{1} - \lambda^t \xi)[\xi]] \\ &= \xi^* \left[\int_{\widehat{x}} \int_0^\infty (\exp \{ \widehat{x}[\log(\mathbf{1} - \lambda^t \xi)] \} - 1 + \lambda^t \widehat{x}[\xi]) \lambda^{-t} dt \mathbf{P}(\cdot, d\widehat{x}) \right] \\ &= \xi^* \left[\int_{\widehat{x}} \int_0^{s(\widehat{x}, 0)} \{ s^{-2} (\exp \{ -\widehat{x}[\xi]s \} - 1 + \widehat{x}[\xi]s) \right. \\ & \qquad \qquad \qquad \left. + a(\widehat{x}, s) \} b(\widehat{x}, s) ds \mathbf{P}(\cdot, d\widehat{x}) \right], \end{aligned}$$

$$a(\widehat{x}, s(\widehat{x}, t)) := s^{-2}(\lambda^t - s)\widehat{x}[\xi] = \frac{\widehat{x}[\lambda^t \xi] - \widehat{x}[|\log(\mathbf{1} - \lambda^t \xi)|]}{(\widehat{x}[\log(\mathbf{1} - \lambda^t \xi)]/\widehat{x}[\xi])^2},$$

$$b(\widehat{x}, s(\widehat{x}, t)) := -\lambda^{-t} s^2 \left(\frac{\partial s}{\partial t} \right)^{-1} = \frac{1}{|\log \lambda|} \frac{(\widehat{x}[\log(\mathbf{1} - \lambda^t \xi)])^2}{\widehat{x}[\lambda^t \xi] \widehat{x}[\lambda^t \xi / (\mathbf{1} - \lambda^t \xi)]}.$$

Observing that $a(\widehat{x}, s(\widehat{x}, t))$ and $b(\widehat{x}, s(\widehat{x}, t))$ are bounded as functions of (\widehat{x}, t) on $\widehat{X} \otimes \mathbb{R}_+$, even if $\inf \xi = 0$, and substituting $\sigma = \widehat{x}[\xi]s$, we obtain the equivalence of (3.7) and

$$(3.9) \quad \xi^* \left[\int_{\widehat{x}} \int_0^{\widehat{x}[|\log(\mathbf{1} - \xi)|]} \sigma^{-2} (e^{-\sigma} - 1 + \sigma) d\sigma \mathbf{P}(\cdot, d\widehat{x}) \right] < \infty.$$

Since there exist real constants C_1 and C_2 such that

$$0 < C_1 \leq [\log(1 + \omega)]^{-1} \int_0^\omega \sigma^{-2} (e^{-\sigma} - 1 + \sigma) d\sigma \leq C_2 < \infty$$

for all $\omega > 0$, (3.9) is equivalent to

$$\xi^* \left[\int_{\widehat{x}} \widehat{x}[\xi] \log(1 + \widehat{x}[|\log(\mathbf{1} - \xi)|]) \mathbf{P}(\cdot, d\widehat{x}) \right] < \infty,$$

which is clearly equivalent to (3.8). \square

The remaining parts of the proof of Theorem 2 are the same as in [10].

PROOF OF LEMMA 3. — Let \mathcal{P} be the set of all non-negative, not necessarily finite-valued, \mathfrak{A} -measurable functions on X . Then

$$M_{(2)}^t[\xi](\cdot) := \int_{\widehat{X}} \{ \widehat{y}[\xi]^2 - \widehat{y}[\xi^2] \} \mathbf{P}_t(\langle \cdot \rangle, d\widehat{y})$$

defines a quadratic mapping $M_{(2)}^t : \mathcal{B}_+ \rightarrow \mathcal{P}$ for every $t \in T$. Extending $M^t[\cdot](x)$ to \mathcal{P} , we deduce from (2.11) that

$$M_{(2)}^{t+s}[\xi] = M_{(2)}^t[M^s[\xi]] + M^t[M_{(2)}^s[\xi]] \quad \forall s, t \in T, \xi \in \mathcal{P},$$

If we have (M) with $\rho = 1$ and extend φ^* to \mathcal{P} , it follows that $\varphi^*[M_{(2)}^t[\varphi]]$ is non-decreasing in $t \in T$ and

$$(3.10) \quad \varphi^*[M_{(2)}^t[\varphi]] = t\varphi^*[M_{(2)}^1[\varphi]]$$

for all rational $t \in T \setminus \{0\}$. Consequently (3.10) holds for all $t \in T \setminus \{0\}$. \square

PROOF OF PROPOSITION 3. — Let $\xi \in \mathcal{B}_+$, define

$$m_{(2)}[\xi](x) := \int_{\widehat{X}} \{ \widehat{y}[\xi]^2 - \widehat{y}[\xi^2] \} \pi(x, \widehat{dy}),$$

and extend $T_t^0[\cdot](x)$ and $m[\cdot](x)$ to \mathcal{P} . Then the function $M_{(2)}^t[\xi](x)$, $t \geq 0$, $x \in X$, is the minimal non-negative solution of

$$z_t(x) = \int_0^t T_s^0 \{ km[z_{t-s}] + km_{(2)}[M^{t-s}[\xi]] \} (x) ds,$$

cf. [3]. Given (M), it follows from (3.1) that

$$(3.11) \quad \varphi^*[T_t^0\xi] \leq \rho^t \varphi^*[\xi]; \quad t \geq 0, \xi \in \mathcal{P},$$

and, using (B.1), that

$$(3.12) \quad \varphi^*[T_t^0\xi] \geq (1 - c^*t)\rho^t \varphi^*[\xi]; \quad t \geq 0, \xi \in \mathcal{P},$$

Hence, if $\rho = 1$,

$$\begin{aligned} 0 \leq \varphi^* \left[\int_0^t T_s^0 \{ km[M_{(2)}^{t-s}[\varphi]] \} ds \right] &\leq tc^* \sup_{s \in [0, t]} \varphi^*[M_{(2)}^s[\varphi]] = 2c^*t^2\mu, \\ t(1 - c^*t)\varphi^*[km_{(2)}[\varphi]] &\leq \varphi^* \left[\int_0^t T_s^0 \{ km_{(2)}[\xi] \} ds \right] \\ &\leq t\varphi^*[km_{(2)}[\varphi]]; \quad t \in T. \end{aligned}$$

Letting $0 < t \downarrow 0$, we have (2.9). \square

PROOF OF THEOREM 3. — Given Lemmata 1, 3, the property (C.3), and Lemma 4 below, the proof is the same as in [6].

Verification of (C.3). If $T_t : \mathcal{B} \rightarrow \mathcal{C}_0$ for $t > 0$, then by

$$T_t^0 = T_t - \int_0^t T_{t-s} k T_s^0 ds$$

also $T_t^0 : \mathcal{B} \rightarrow \mathcal{C}_0$ for $t > 0$. Set $\widehat{A} = \{ \widehat{x}[1_{A_v}] = n_v; v = 1, \dots, j \}$. From the integral equation for F_t given in the proof of Proposition 2

$$P_t(\langle x \rangle, \widehat{A}) = \sum_{v=1}^j 1_{n_v=1} T_t^0 1_{A_v}(x) + 1_{\widehat{A}=\{\emptyset\}} H_t(x) + I_t(x)$$

$$I_t(x) := \int_0^t T_{t-s}^0 \left\{ k \int_{\widehat{X}} \pi(\cdot, d\widehat{x}) P_s(\widehat{x}, \widehat{A}) \right\} (x) ds.$$

Appealing to $T_t^0 : \mathcal{B} \rightarrow \mathcal{C}_0$, $t > 0$, and the continuity of T_t^0 on \mathcal{C}_0 , we immediately have continuity of the first two terms. As to the third term, note that

$$\begin{aligned} \| I_{t+\delta} - I_t \| &\leq \| T_\delta^0(T_\varepsilon^0 I_{t-\varepsilon}) - T_\varepsilon^0 I_{t-\varepsilon} \| + 3 \| k \| \varepsilon, \\ \| I_{t-\delta} - I_t \| &\leq \| T_{\varepsilon-\delta}^0(T_\varepsilon^0 I_{t-2\varepsilon}) - T_\varepsilon^0(T_\varepsilon^0 I_{t-2\varepsilon}) \| + 4 \| k \| \varepsilon, \end{aligned}$$

whenever $0 < 2\delta < 2\varepsilon < t$. \square

LEMMA 4. — *Assuming (M) with $\rho = 1$, (R), and $\mu < \infty$, we have*

$$\lim_{N \ni n \rightarrow \infty} \frac{1}{n\delta} \{ \varphi^*[\mathbf{1} - F_{n\delta}[\xi]]^{-1} - \varphi^*[\mathbf{1} - \xi]^{-1} \} = \mu$$

uniformly on $\overline{\mathcal{P}}_+ \cap \{ \varphi^*[\mathbf{1} - \xi] > 0 \}$ for every $\delta \in T \setminus \{ 0 \}$.

Proof. — Let $\xi \in \overline{\mathcal{P}}_+ \cap \{ \varphi^*[\mathbf{1} - \xi] > 0 \}$. Then by Lemma 1 also $\varphi^*[\mathbf{1} - F_t[\xi]] > 0 \forall t \in T$. Using (2.11),

$$\begin{aligned} \frac{1}{n\delta} \{ \varphi^*[\mathbf{1} - F_{n\delta}[\xi]]^{-1} - \varphi^*[\mathbf{1} - \xi]^{-1} \} \\ &= \frac{1}{n} \sum_{v=0}^{n-1} \frac{1}{\delta} \{ \varphi^*[\mathbf{1} - F_\delta[F_{v\delta}[\xi]]]^{-1} - \varphi^*[\mathbf{1} - F_{v\delta}[\xi]]^{-1} \} \\ &= \frac{1}{n} \sum_{v=0}^{n-1} \frac{1}{\delta} (1 - \varphi^*[\mathbf{1} - F_{v\delta}[\xi]] \Lambda_\delta[F_{v\delta}[\xi]])^{-1} \Lambda_\delta[F_{v\delta}[\xi]], \end{aligned}$$

$$\Lambda_\delta[\zeta] := \varphi^*[\mathbf{1} - \zeta]^{-2} \{ \varphi^*[\mathbf{1} - \zeta] - \varphi^*[\mathbf{1} - F_\delta[\zeta]] \}.$$

Given $\mu < \infty$, there exists for every $t \in T \setminus \{ 0 \}$ a functional $\varphi^*[\mathbf{R}_{(2)}^t(\cdot)]$ on $\overline{\mathcal{P}}_+ \otimes \mathcal{B}_+$, sequentially continuous on bounded regions in $\overline{\mathcal{P}}_+ \otimes \{ \xi = \eta\varphi : \eta \in \mathcal{B}_+ \}$, such that

$$0 = \varphi^*[\mathbf{R}_{(2)}^t(\mathbf{1})[\eta\varphi]] \leq \varphi^*[\mathbf{R}_{(2)}^t(\xi)[\eta\varphi]] \leq \varphi^*[\mathbf{M}_{(2)}^t[\eta\varphi]] \leq 2t\mu \| \eta \|^2$$

for $t \geq 0$, $(\xi, \eta) \in \overline{\mathcal{P}}_+ \otimes \mathcal{B}_+$, and

$$\varphi^*[\mathbf{1} - F_t[\zeta]] = \varphi^*[\mathbf{M}^t[\mathbf{1} - \zeta]] - \frac{1}{2} \varphi^*[\mathbf{M}_{(2)}^t[\mathbf{1} - \zeta]] + \frac{1}{2} \varphi^*[\mathbf{R}_{(2)}^t(\zeta)[\mathbf{1} - \zeta]]$$

for $t \geq 0$, $\zeta = \mathbf{1} - \eta\varphi \in \overline{\mathcal{P}}_+$, $\eta \in \mathcal{B}_+$. In view of $\rho = 1$ and Lemma 1,

$$\Lambda_\delta[F_t[\xi]] = \frac{1}{2} \varphi^*[\mathbf{M}_{(2)}^\delta[(\mathbf{1} + h_t[\xi])\varphi]] - \frac{1}{2} \varphi^*[\mathbf{R}_{(2)}^\delta(F_t[\xi])(\mathbf{1} + h_t[\xi])\varphi].$$

Since $1 \geq F_t[\xi](x) \geq F_t[\mathbf{0}](x) = \mathbf{P}_t(\langle x \rangle, \{ \theta \})$, we have

$$\lim_{t \rightarrow \infty} \Lambda_\delta[F_t[\xi]] = \delta\mu$$

uniformly in ξ . This completes the proof. \square

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