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Section B :
Calcul des Probabilités et Statistique.

Banach lattices valued amarts

by

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ABSTRACT. — Un espace de Banach réticulé est réflexif, si et seulement si tout amart fort à valeurs dans E, de classe (B), converge faiblement p. s. Si E possède la propriété de Radon-Nikodym, et si son dual contient un point quasi intérieur, toute sous-martingale à valeurs dans E, de classe (B), converge faiblement p. s.

Si E possède une base inconditionnelle, alors E possède la propriété de Radon-Nikodym, si et seulement si, toute sous-martingale positive bornée dans L^1, converge fortement p. s.

Si E est « minimal », toute sous-martingale à valeurs dans un intervalle d'ordre, converge fortement p. s.

I. INTRODUCTION

Let (Ω, F, P) be a probability space, and $(F_n)_n$ an increasing sequence of σ-algebras contained in F. The collection of bounded stopping times is denoted by T.

Let E be a Banach lattice, and consider a sequence $(X_n)_n$ of E-valued random variables adapted to $(F_n)_n$ (strongly measurable).

The sequence $(X_n)_n$ is called *strong amart* (resp. *weak amart*) iff each $(X_n)_n$ is Pettis integrable and $\int X_\tau$ converges in norm (resp. weakly).

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$(X_n)_n$ is called a *weak sequential amart* iff each X_n is Pettis-integrable and for each increasing sequence τ_n in T, there exists $z \in E$ (z can depend on τ_n) such that the weak limit of $\int X_{\tau_n}$ is z.

$(X_n)_n$ is called a *submartingale* (resp. a *supermartingale*) if we have for all $n \in \mathbb{N}$: $E^{F_n}[X_{n+1}] \geq X_n$ a. e. (resp. $E^{F_n}[X_{n+1}] \leq X_n$).

We recall that a Banach lattice E has an *order continuous norm* iff each order convergent filter is norm convergent. E is a *KB space* iff each norm bounded increasing sequence is norm convergent, which is equivalent to each of the following conditions:

a) E is weakly sequentially complete.

b) No Banach sublattice of E is vector lattice isomorphic to c_0.

For a proof, we refer to [*12*], and we can deduce immediately that every Banach lattice with the Radon-Nikodym property, is weakly sequentially complete.

II. WEAK CONVERGENCE OF AMARTS IN BANACH LATTICES

In [*3*], Chacon and Sucheston proved that a strong amart of class (B) $\left(\sup_{\sigma\in T} \int \| x_\sigma \| < \infty\right)$, valued in a Banach space with the Radon-Nikodym property and with a separable dual, converges weakly almost everywhere. The necessity of the separability of the dual is still an open question. In the case of Banach lattices, we can prove the following:

THEOREM (1). — For a Banach lattice E, the following condition are equivalent:

1) E is reflexive;

2) Each E-valued strong amart of class (B) converges weakly almost everywhere;

3) Each E-valued weak sequential amart of class (B) converges weakly a. e.;

4) Each E-valued weak amart of class (B) converges weakly a. e.;

LEMMA (1). — For a Banach lattice E, the following are equivalent:

a) E is reflexive;

b) E and E′ have the R. N. P.;

c) E has the R. N. P. and l^1 does not imbed in E;

Proof of the lemma. — *a*) $\Rightarrow$ *b*) By the theorem of Johnson and Stegall [*6*] this implication is true in any Banach space.

b) $\Rightarrow$ *a*) If *b*) holds, then c_0 does not imbed in E nor in E′, hence E is reflexive by (3.1) of [*9*].

a) $\Rightarrow$ *c*) By Theorem (5.16) in [*12*].

c) $\Rightarrow$ *a*) Since E has the R. N. P. it is weakly sequentially complete, and *a*) follows by Rosenthal's theorem in [*11*].

Proof of the theorem. — The equivalence 1) $\Leftrightarrow$ 4) is true in any Banach space, and it is a result of Brunel-Sucheston in [*1*].

1) $\Rightarrow$ 2) Since E and E′ have the R. N. P., then, by Stegall's theorem [*6*], we can reduce the problem to the convergence theorem of Chacon-Sucheston in [*3*].

2) $\Rightarrow$ 1) Since every L^1-bounded martingale, is a strong amart of class (B), and since weak convergence and strong convergence of a martingale are equivalent by [*3*], then, if (2) holds, E has the R. N. P. by Chatterji's theorem [*5*].

Now, if l^1 does imbed in E, let $(d_n)_n$ be it's canonical bases. Consider $(Y_n)_n$ a sequence of independent random variables:

$$Y_n : \Omega \to \{-1 ; +1\} \qquad \text{with} \qquad P[Y_n = 1] = P[Y_n = -1] = 1/2$$

let's define

$$X_n = \frac{1}{n} \sum_{i=1}^{n} Y_i d_i .$$

For all w, there exists $f \in E'$, such that $\lim_n f(X_n(w)) = 1$.

But, by the strong law of large numbers, we have $\lim_n f(X_n) = 0$ almost everywhere. Thus, $\lim_{\tau \to \infty} f(X_\tau) = 0$ a. e., and taking the integrals, $\int X_\tau$ converges to 0 weakly and then strongly since l^1 has the Shurr property. We conclude that $(X_n)_n$ is a strong amart which diverges weakly almost everywhere; hence l^1 does not imbed in E and E is reflexive.

1) $\Rightarrow$ 3) Is true in any Banach space by [*1*].

3) $\Rightarrow$ 2) It follows by this lemma:

LEMMA (2). — (Vector optional sampling theorem).

Let $(X_n)_n$ be a strong amart for $(F_n)_n$, and let $(\tau_k ; k \in \mathbb{N})$ be an incrasing sequence of bounded stopping times. Then: $(X_{\tau_k})_k$ is a strong amart for $(F_{\tau_k})_k$.

The proof of this lemma, is identical to the proof in the real case. For that, we refer to [*7*]. Now, we can deduce immediately, that every strong amart is a strong sequential amart and hence a weak sequential amart as observed in [*2*].

III. WEAK CONVERGENCE OF SUBMARTINGALES

The reason we are studying amarts in Banach lattices is that we can now introduce vector-valued submartingales and supermartingales. We can remark immediately, that if E is a weakly sequentially complete Banach lattice (for instance if E has the R. N. P.) then, a submartingale is a strong amart if and only if it satisfies one of these two equivalent conditions:

1) $\int X_n$ is order bounded from above.

2) $\int X_n$ is norm bounded.

To see that, it is enough to notice that if $\sigma, \tau \in T$ and $\sigma \leq \tau$, then $\int X_\sigma \leq \int X_\tau$, and $\sup_n \int X_n = \sup_\sigma \int X_\sigma$. The property follows by using the fact that E is w. s. c. *A fortiori*, if E has the R. N. P., then every L^1-bounded submartingale is a strong amart, but not necessarily of class (B), since generally $\| X_n \|$ is not a real submartingale. The following counterexample (oral communication of Y. Benyamini), shows that L^1-boundeness of a submartingale, is not enough for the weak convergence, even in reflexive spaces.

COUNTEREXAMPLE (1). — Let $\Omega_i = [0, 1]$, λ_i the Lebesgue measure, Σ_i the Borel sets. $\Omega = \prod_{i \in \mathbb{N}} \Omega_i$, $F = \prod_{i \in \mathbb{N}} \Sigma_i$ and $P = \prod_{i \in \mathbb{N}} \lambda_i$. Consider the disjoint blocks of integers $P_n = \{ 2^n + 1, \ldots, 2^{n+1} \}$. We define the random variables $(X_n)_n$, valued in l_2 by:

$$[X_m(w)]_{2^n+k} = \begin{cases} -1/2^n & \text{if } m < n \\ -m & \text{if } m = n \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \frac{k-1}{m2^m} \leq w_m \leq k/m2^m$$

$$(1 \leq k \leq 2^n)$$

a) $(X_n)_n$ is L^1-bounded:

$$\| X_m(w) \| = \begin{cases} \left[m^2 + \sum_{n>m} 2^n (1/2^n)^2 \right]^{1/2} \leq m + 1 & \text{if } \ w_m \leq 1/m \\ \left[\sum_{n>m} 2^n (1/2^n)^2 \right]^{1/2} \leq 1 & \text{if } \ w_m > 1/m \end{cases}$$

and

$$\int \| X_n(w) \| \, dP = \int_{\left\{ w_n > \frac{1}{m} \right\}} \| X_n(w) \| \, dP + \int_{\left\{ w_m \leq \frac{1}{n} \right\}} \| X_n(w) \| \, dP \leq 1 + \frac{m+1}{m} \leq 3 .$$

b) $(X_n)_n$ is a submartingale:

Since the X_n's are independent, it is enough to prove that: $X_m \leq E[X_{m+1}]$.
But

$$E[X_{m+1}]_j = \begin{cases} -1/2^n & \text{if } \ m+1 < n \\ -1/2^{m+1} & \text{if } \ m+1 = n . \\ 0 & \text{if } \ m+1 > n \end{cases}$$

c) $(X_n)_n$ does not converge weakly a. e.:

Let $A_m = \{ w \, ; \, 0 \leq w_m \leq 1/m \}$. $(A_m)_m$ are independent and $P(A_m) = 1/m$, hence $\sum_m P(A_m) = \infty$. By Borel-Cantelli, almost all w are in infinitely many A_m's. But, for $w \in A_m$, $\| X_m(w) \| \geq m$. It follows that $\| X_m(w) \|$ is not bounded for almost all w, hence, the weak convergence fails.

However, we will prove a weak convergence theorem for submartingales of class (B), valued in Banach lattices which have the Radon-Nikodym property, and a quasi-interior point in their dual, without assuming the separability of that dual (l^1 for instance).

THEOREM (2). — If E is a Banach lattice with the Radon-Nikodym property, such that E′ has a quasi interior point, then:

Each E-valued submartingale of class (B) converges weakly a. e.

LEMMA (3). — If E has the R. N. P. and E′ has a quasi interior point then: every positive potential of class (B) converges weakly to zero a. e.

Proof. — Let $(Z_n)_n$ be a positive potential of class (B). We can reduce the problem to that of the convergence of a positive potential such that $\sup_n \| X_n \| \in L^1$. A similar device was used in the case of real martingales and in the case of vectorial amarts in [3]. Since E has the R. N. P., we can prove like in [3], that for every $A \in F$, $\lim_{\tau} \int_A Z_\tau = 0$.

Now, let u be a quasi interior point of E'. $u(Z_n)$ is an L^1-bounded real potential. So, there exists a set N_u of measure zero, and $u(Z_n) \to 0$ outside N_u.

But, for every $f \in E'$, $f = \sup_m f \wedge mu$ and the convergence is in norm so, for all w outside N_u,

$$0 \leq f \wedge mu(Z_n(w)) \leq mu(Z_n(w))$$

and when $n \to \infty$, $f \wedge mu(Z_n(w)) \to 0$.

Now, $\sup_n \| Z_n(w) \| < \infty$ outside a set Ω with $P(\Omega) = 0$. Thus, for all $w \notin \Omega \cup N_u$, $f(Z_n(w)) = (f - f \wedge mu)(Z_n(w)) + (f \wedge mu)(Z_n(w))$ when $m \to \infty$, $(f - f \wedge mu)(Z_n(w)) \to 0$ uniformly in n, since $(Z_n(w))_n$ is norm bounded, and when $n \to \infty$, $f \wedge mu(Z_n(w))$ goes to zero. Finally $f(Z_n) \to 0$ outside $\Omega \cup N_u$.

Proof of the theorem. — For each $m \in \mathbb{N}$, $\{ E^{F_m}[X_m] ; n \geq m \}$ is an increasing sequence of random variables, because:

$$E^{F_m}[X_{n+1}] = E^{F_m}[E^{F_n}[X_{n+1}]] \geq E^{F_m}[X_n].$$

Since E^{F_m} is an L^1-contraction, and (X_n) of class (B), $(E^{F_m}[X_n])_n$ is of class (B); using lemma (2) in [8], $\sup_n \| E^{F_m}[X_n] \| \in L^1$, hence $\sup_n \| E^{F_m}[X_n] \| < \infty$ a. e.

Since E is weakly sequentially complete, $E^{F_m}[X_n]$ converges strongly a. e. to Y_m.

Y_m is an L^1-bounded martingale, so it converges in norm a. e. by Chatterji's theorem [5].

Let $Z_n = X_n - Y_n$. Z_n is a potential of class (B), since $(X_n)_n$ and $(Y_n)_n$ are of class (B). $(Z_n)_n$ is a negative submartingale since:

$$X_m \leq E^{F_m}[X_{m+1}] \leq \lim_{n \geq m} \uparrow E^{F_m}[X_n] = Y_m .$$

$(- Z_n)_n$ is a positive potential, so it converges weakly a. e. to zero, by the precedent lemma.

COROLLARY (1). — If E has the R. N. P. and a separable dual, or if E is reflexive, then each E-valued submartingale of class (B) converges weakly a. e.

Proof. — *a*) If E' is separable, then it has a quasi interior point by (6.2) in [12].

b) Since the $(X_n)_n$ are strongly measurable, they are almost separably valued. Let $F \subseteq E$ be a separable closed sublattice of E such that $P[X_n \in F] = 1$ for all n. F is reflexive by (5.16) of [*12*], and has a quasi interior point since it is separable. Thus F′ has a quasi interior point by (6.6) of [*12*]. Therefore, the theorem applies in F, and X_n converges a. e. in the weak topology of F, which is the relative topology from the weak topology on E.

Remark (1). — Is the existence of a quasi interior point in the dual necessary

IV. STRONG CONVERGENCE OF SUBMARTINGALES

According to the following counterexample, we cannot hope, generally, for a strong convergence of a submartingale.

COUNTEREXAMPLE (2). — A weakly convergent positive supermartingale, l_2-valued, which does not converge in norm.

Let (Ω, F, P) the same probability space, and $(P_n)_n$ the same disjoints blocks of integers, as in the first counterexample. We define the random variables $(X_n)_n$, valued in l_2 by:

$$[X_m(w)]_{2^n+k} = \begin{cases} -1/2^n & \text{if} \quad m < n \\ -1 & \text{if} \quad m = n \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \frac{k-1}{2^m} \leq w_m \leq \frac{k}{2^m}$$
$$1 \leq k \leq 2^n .$$

a) $(X_n)_n$ is a submartingale (negative).

Since they are independent, it's enough to prove that: $X_m \leq E[X_{m+1}]$. But

$$E[X_{m+1}]_j = \begin{cases} -1/2^n & \text{if} \quad m+1 \leqslant n \\ 0 & \text{otherwise} \end{cases} .$$

b) $X_n \to 0$ weakly a. e., because it's uniformly bounded and by the precedent theorem. Indeed:

$$\| X_n(w) \|_2^2 = 1 + \sum_{m>n} \left(\frac{1}{2^m}\right)^2 . 2^m = 1 + \sum_{m>n} 1/2^m \leq 2 .$$

c) X_n does not converge in norm to 0, since $\| X_n(w) \|_2 \geq 1$.

Since every submartingale can be decomposed into a martingale and a positive supermartingale, the strong convergence fails generally, according to the precedent counterexample and Chatterji's theorem for martingales.

However, for the case of positive submartingales, we can have the strong convergence in some « good spaces ».

PROPOSITION (1). — Let E a separable Banach lattice with the R. N. P., and $(X_n)_n$ an L^1-bounded positive submartingale taking values in E. Then:

a) $\| X_n \|$ is a real submartingale.

b) There exists a Bochner integrable random variable X_∞, such that:

$$\lim_n \| X_n \| = \| X_\infty \| \qquad \text{a. e.}$$

We need these two lemmas:

LEMMA (4). — If E is a separable Banach lattice, then there exists a denumberable set D in $B(E') \cap E'_+$, such that for all x in E_+,

$$\| x \| = \sup_{f \in D} f(x) .$$

Proof. — Since E is a Banach lattice, for every $x \in E_+$, $\| x \| = \{ \sup f(x) ; f \in B(E') \cap E_+ \}$. But $B(E') \cap E_+$ is w^*-compact metrizable since it's closed in $B(E')$ and the lemma follows.

LEMMA (5). — [*10*] Let I be a denumberable set, and for each $i \in I$, let $(X_n^i)_n$ be a real submartingale.

If $\sup_n \int \sup_i (X_n^i)^+ < \infty$. Then:

i) $(\forall i \in I) X_n^i \to X_\infty^i$ a. e.

ii) $\sup_i X_n^i$ is a submartingale.

iii) $\sup_i X_n^i \to \sup_i X_\infty^i$ a. e.

Proof of the proposition. — Let D be the denumberable set of lemma (4), and $(X_n)_n$ an L^1-bounded positive submartingale. By lemma (5), and since $(\forall f \in D) f(X_n)$ is a real submartingale, $\| X_n \| - \sup_{f \in D} f(X_n)$ is a real submartingale. So, $(X_n)_n$ is of class (B), cause if σ, $\tau \in T$ and $\sigma \leq \tau$ we have:

$$\int \| X_\sigma \| \leq \int \| X_\tau \| \qquad \text{and} \qquad \sup_n \int \| X_n \| = \sup_n \int \| X_\sigma \| .$$

Now, using the same device of theorem (2), we can reduce the case to a positive submartingale with $\sup_n \| X_n \| \in L^1$.

Following [3], since E has the R. N. P., there exists a Bochner integrable random variable X_∞, such that for every $A \in F$,

$$\lim_A \int_A X_\tau = \int_A X_\infty .$$

But, for every $f \in D$, $f(X_n)$ is an L^1-bounded real submartingale and it converges necessarily to $f(X_\infty)$ a. e.

Finally, $\|X_n\|$ converges to $\sup_{f \in D} f(X_\infty) = \|X_\infty\|$ a. e. by lemma (5).

COROLLARY (1). — If E is a weakly locally uniformly convex separable dual Banach lattice then:

a) Every positive L^1-bounded, E-valued submartingale converges weakly a. e.

b) Every L^1-bounded, E-valued submartingale, minorized by an L^1-bounded martingale (e. g. order bounded from below) converges weakly a. e.

c) Every L^1-bounded, E-valued supermartingale, majorized by an L^1-bounded martingale (e. g. order bounded from above) converges weakly a. e.

Proof. — *a)* E is separable and has the R. N. P., then there exists X_∞ Bochner integrable such that $\|X_n\| \to \|X_\infty\|$ a. e. But if $E = F'$, then F is separable. Let $(f_m)_m$ be a dense set of F. For every f_m, $f_m^+(X_n)$ and $f_m^-(X_n)$ are real submartingales, hence they converge, and necessarily to $f_m^+(X_\infty)$ (resp. $f_m^-(X_\infty)$) outside a negligeable set, since $\lim_n \int_A X_n = \int_A X_\infty$ for every $A \in F$. But $(f_m)_m$ is a dense family, and we can reduce the case one more time to $\sup_n \|X_n\| \in L^1$, hence X_n converges to X_∞ in the weak-star topology and almost everywhere. Since E is weakly locally uniformly convex, the weak convergence follows.

b) It follows immediately by applying the first part to $(X_n - Y_n)$ where Y_n is the martingale.

COROLLARY (2). — If E is a separable locally uniformly convex dual Banach lattice. Then:

a), *b)*, *c)* hold with strong convergence instead of weak convergence.

COROLLARY (3). — If E is a uniformly convex Banach lattice. Then:

a), *b)* and *c)* hold with strong convergence.

Proof. — E is reflexive:

Remark (2). — By Kadec's renorming theorem, every separable Banach space can be renormed in a locally uniformly convex space. But, if the Banach space is lattice, it is not known, whether the new norm can be compatible with the order. If the answer is positive, the precedent results will be true in more general spaces.

However, in spaces with unconditional bases, which are particular Banach lattices, under their Canonical order, we can state the following:

THEOREM (3). — If E is a Banach space with an unconditional bases, then the following are equivalent:

a) E has the R. N. P.

b) Each L^1-bounded positive, E-valued submartingale converges strongly a. e.

Proof. — *b*) $\Rightarrow$ *a*) Is true in any Banach lattice, since if $(X_n)_n$ is a martingale, $(X_n^+)_n$ and $(X_n^-)_n$ are positive submartingales, and the result follows from Chatterji's theorem.

a) $\Rightarrow$ *b*) Let $(e_k)_k$ be the unconditional bases, and $(f_k)_k$ the « coefficient functionals ». Put, for every $x \in E$,

$$R_q(x) = \sup_{q \leq k < m < \infty} \left\| \sum_{i=k+1}^{m} f_i(x) e_i \right\| .$$

According to Kadec's theorem, the norm:

$$||| x ||| = || x || + \sum_q 1/2q R_q(x) + \left(\sum_k 1/2^k f_k(x) \right)^{1/2} ,$$

verifies the following condition:

If $\lim_n ||| x_n ||| = ||| x_0 |||$ and $\lim_n f_k(x_n) = f_k(x_0)$ for all k, then

$$\lim_n || x_n - x_0 || = 0 .$$

But, for every k, f_k is a bounded linear functional, and $f_k(X_n) \to f_k(X_\infty)$ a. e. where X_∞ is the random variable satisfying $\lim_n \int_A X_n = \int_A X_\infty$ for every $A \in F$. The existence of X_∞ is assured by the R. N. P.

On the other hand, $||| \cdot |||$ is a Banach lattice norm, since $|| \cdot ||$ is and since the order is defined by the coefficients. Using the argument of proposition (1), $|||X_n||| \to |||X_\infty|||$ a. e. Finally, by the Kadec property of the new norm, we have that X_n converges strongly to X_∞ a. e.

COROLLARY (1). — If E has an order continuous norm. Then every order bounded submartingale (or supermartingale) converges strongly a. e.

Proof. — Since order intervals are weakly compact, the weak convergence follows from Brunel-Sucheston theorem in [2]. The strong convergence is obtained by applying the precedent device to $X_n - a$ (if X_n is a submartingale) and to $b - X_n$, if X_n is a supermartingale.

So, we obtained a convergence theorem for spaces which do not have the R. N. P. (c_0 !). Therefore, it will be interesting to see in which conditions, one can have Chacon's inequality [4]:

$$\limsup_{\sigma,\tau} \int (X_\sigma - X_\tau) \geq \int (\limsup X_n - \text{lim in } X_n)$$

and consequently, the strong convergence (order convergence).

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