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On Rudolph's representation of aperiodic flows

by

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RÉSUMÉ. — Récemment Daniel Rudolph a amélioré le théorème classique d'Ambrose sur la représentation des flots ergodiques préservant la mesure par des flots sous une fonction en prouvant que pour des flots apériodiques la fonction plafond f peut être choisie pour ne prendre que des valeurs p, q données, p/q devant être irrationnel. Il a montré aussi que la mesure de $\{ f = p \}$ peut être choisie à l'avance à ε près. Nous montrons ici que la construction de Rudolph peut être affinée de manière à donner à la mesure de $\{ f = p \}$ une valeur exacte donnée à l'avance. Ceci implique que tous les flots apériodiques peuvent être représentés sur le même espace et ne diffèrent que par « l'automorphisme de base » nécessaire à cette représentation.

A titre d'application, nous obtenons une version continue d'un théorème de Dye, dans une forme très précise. Nous traitons aussi le cas de flots non singuliers.

ABSTRACT. — Recently Daniel Rudolph sharpened the classical theorem of Ambrose on the representation of ergodic measure preserving flows

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by flows under a function f by proving that for aperiodic flows the ceiling function f can be chosen to take only the values $p, q > 0$, which can be preassigned with p/q irrational. He also proved that the measure of $\{ f = p \}$ can be preassigned up to an $\varepsilon > 0$. Here we show that Rudolph's construction admits a refinement which allows to preassign the measure of $\{ f = p \}$ exactly. This implies that all aperiodic flows can be represented on the same space and differ only by the « basis-automorphism » needed in this representation. As an application we get a continuous time version of Dye's theorem, of a very sharp form. We also treat the case of nonsingular flows.

1. INTRODUCTION

We begin by recalling the basic theorem of Ambrose [*1*] which establishes a link between the investigation of flows and that of measure preserving transformations.

By a measurable measure preserving flow on a space Σ with measure μ we mean a group $\{ T_t, t \in \mathbb{R} \}$ of μ-preserving transformations T_t such that $T_{t+s} = T_t \circ T_s$ $(t, s \in \mathbb{R})$ and such that the map $(\sigma, t) \rightarrow T_t\sigma$ is measurable $\Sigma \times \mathbb{R} \rightarrow \Sigma$ with respect to the completion of the product $\mu \times \lambda$ in $\Sigma \times \mathbb{R}$, where λ denotes Lebesgue measure.

A flow built under a function is given by a quadrupel $(B, T, \underline{m}, f)$ where B is a space carrying a finite measure $\underline{m}$, T is an invertible $\underline{m}$-preserving transformation of B onto B, T and T^{-1} are measurable, f is an $\underline{m}$-measurable map from B to $\{ t \in \mathbb{R} : t > 0 \}$ with $\sum_{i=0}^{\infty} f(T^i b) = \sum_{i=0}^{\infty} f(T^{-i} b) = \infty$ for all $b \in B$, and $\int f d\underline{m} = 1$. On

$$\Omega = \Omega(B, f) = \{ (b, x) : b \in B, 0 \leqq x < f(b) \}$$

a measure m is given by the restriction of the completion of $\underline{m} \times \lambda$ to Ω, and a measurable m-preserving flow $\{ S_t, t \in \mathbb{R} \}$, called the flow under f, is given by

$$S_t(b, x) = \left(T^i b, x + t - \sum_{j=0}^{i-1} f(T^j b) \right)$$

where i is the unique integer with

$$\sum_{j=0}^{i-1} f(\mathrm{T}^j b) \leqq x + t < \sum_{j=0}^{i} f(\mathrm{T}^j b).$$

Ambrose has shown that each ergodic measure preserving measurable flow $\{ \mathrm{T}_t \}$ on a non-atomic complete probability space is isomorphic mod nullsets to a flow under a function f which can be assumed bounded below by some $c > 0$.

Recently, in a break-through paper, Rudolph [*17*] has shown that one can, in fact, find a representation with an f taking only two values. If $\{ \mathrm{T}_t \}$ is ergodic and aperiodic the values $p, q > 0$ can be preassigned in advance subject to the condition that p/q be irrational. For $\varepsilon > 0$ given, Rudolph constructs a representation with $| m \{ f = p \} - m \{ f = g \} | < \varepsilon$. Rudolph does everything in Lebesgue-spaces since they are the spaces of interest in ergodic theory, but if one replaces the pointwise definition of aperiodicity by the more technical « setwise » definition given below, his proof goes through in general abstract probability spaces.

The condition of ergodicity is not important: Ambrose and Kakutani [*2*] have called $\{ \mathrm{T}_t \}$ proper if there is no A with $\mu(\mathrm{A}) > 0$ such that for all measurable $\mathrm{A}' \subset \mathrm{A}$ and all t the symmetric difference $\mathrm{A}' \triangle \mathrm{T}_t \mathrm{A}'$ has μ-measure zero. They have proved that proper flows still have a representation by a flow under a function (not bounded below), if one admits $\underline{m}$ σ-finite. Rudolph mentions that his result remains valid for aperiodic nonergodic $\{ \mathrm{T}_t \}$.

In this paper we describe a refinement of the construction of Rudolph which permits to eliminate the above $\varepsilon > 0$. For $0 < \tilde{\rho} < \infty$ given, we find a representation with $\underline{m} \{ f = p \} = \tilde{\rho} \underline{m} \{ f = q \}$. A direct consequence of this is the result that all aperiodic measurable measure preserving flows on Lebesgue spaces of total measure 1 admit modulo nullsets a representation on the same Ω and differ only by the automorphism T needed in this representation.

Another application is concerned with the theorem of Dye [*4*], which says that any two ergodic measure preserving invertible transformations T, T′ of nonatomic Lebesgue spaces Ω, Ω' of total measure 1 are weakly equivalent; i. e. there exists an invertible measure preserving $\varphi : \Omega \to \Omega'$ so that for a. e. $\omega \in \Omega$ the sequence $\varphi \{ \mathrm{T}^n \omega, n \in \mathbb{Z} \}$ is a permutation of $\{ \mathrm{T}'^k \varphi(\omega), k \in \mathbb{Z} \}$. In the case of continuous time T and T′ are replaced by aperiodic flows $\{ \mathrm{T}_t \}$ and $\{ \mathrm{T}'_t \}$ and φ is again a transformation mapping orbits $\{ \mathrm{T}_t \omega, t \in \mathbb{R} \}$ onto orbits $\{ \mathrm{T}'_t \omega', t \in \mathbb{R} \}$. In fact, φ not only preserves the length of orbit sections (which is automatic for discrete T, since a permutation preserves the counting measure), but it is again a « permutation » as follows: each orbit $\{ T_t\omega, t \in \mathbb{R} \}$ and $\{ T'_s\omega', s \in \mathbb{R} \}$ is cut into intervals $\{ T_t\omega, t_i \leqq t < t_{i+1} \}$ of length p or q, resp. $\{ T'_s\omega', s_k \leqq s < s_{k+1} \}$, $(i, k \in \mathbb{Z})$. The image under φ of the intervals of length p (resp. q) of the orbit of ω is a permutation of the intervals of length p (resp. q) of the orbit of $\varphi(\omega)$.

The fact that φ preserves the length of orbit sections is of importance since without this requirement the existence of a finite equivalent invariant measure would not be an invariant of weak equivalence.

Some years ago, I. Kubo [*13*] and the author [*10*] proved independently and at the same time a theorem giving a representation of nonsingular flows by flows under a function. In this theorem, too, the ceiling function can be constructed so as to assume only the values p, q, using the ideas of Rudolph together with some new estimates. The problem of preassigning the measures seems more difficult in this case, but it does not seem of particular interest since in the study of nonsingular measures one can usually replace the original measure by an equivalent measure.

I would like to thank F. Blanchard for stimulating conversations on the subject of this paper.

2. THE REPRESENTATION THEOREM

Following [*10*, I] we call a flow $\{ T_t \}$ in (Σ, μ) aperiodic if there is no $0 < \alpha_1 < \alpha_2$ and Σ_0 with $\mu(\Sigma_0) > 0$ such that for all μ-measurable $A \subset \Sigma_0$ A is contained up to nullsets in $\cup \{ T_t A : \alpha_1 < t < \alpha_2, t \text{ rational} \}$. If (Σ, μ) is a Lebesgue space this is equivalent to the obvious pointwise definition of aperiodicity. By lemma 2.4 in [*10*, I] a flow $\{ S_t \}$ under a function is aperiodic in this sense if and only if the basis automorphism T is aperiodic, i. e. there is no $B_0 \subset B$ with $\underline{m}(B_0) > 0$ such that for some $n \geqq 1$ and all measurable $E \subset B_0$ $\underline{m}(E \,\triangle\, T^n E) = 0$.

The principal result of this paper is

THEOREM 2.1. — Let $\{ T_t, t \in \mathbb{R}^1 \}$ be a measurable measure preserving aperiodic flow on a complete probability space (Σ, μ) and let $p, q > 0$ with p/q irrational and $0 < \tilde{\rho} < \infty$ be given. Then there exists a quadrupel $(B', T', \underline{m}', f')$ as above such that f' takes only the values p and q, $\underline{m}'(f' = p) = \tilde{\rho}\underline{m}'(f' = q)$, and such that $\{ T_t, t \in \mathbb{R}^1 \}$ is mod nullsets isomorphic to the flow $\{ S'_t, t \in \mathbb{R} \}$ under f' defined on $\Omega' = \Omega(B', f')$.

Proof. — Before beginning the formal proof we give a description of the argument, which leaves out some technical points and the determination of various constants.

It is clear that any aperiodic flow is proper. By the theorem of Ambrose-Kakutani we may therefore assume that $\{ T_t \}$ is a flow under a function given by a quadrupel $(B'', T'', \underline{m}'', f'')$ with the ceiling function f'' not bounded below and the measure on B'' possibly σ-finite. The transformation T'' is aperiodic and this permits to pass to a representation $(B, T, \underline{m}, f)$ for which f is bounded below and hence $\underline{m}$ finite. Simply observe that the proof of Ambrose and Kakutani shows that B'' is an at most countable union of T''-invariant sets B_i'' such that f'' is bounded below on each B_i'' separately, and apply Rohlin's lemma to T'' on these sets B_i'' with sufficiently fast increasing $n_i's$; see [*10*, I]. Here we need Rohlin's lemma in the following form: if T is measure preserving and aperiodic in a space Σ with finite measure μ then there exists for every $\varepsilon > 0$ and $n \in \mathbb{N}$ a measurable set F such that $F, T^{-1}F, \ldots, T^{-(n-1)}F$ are disjoint, $\mu\left(\Sigma \setminus \bigcup_{i=0}^{n-1} T^{-i}F\right) < \varepsilon$ and $\bigcup_{i=0}^{\infty} T^{-i}F = \Sigma$. We note for later use that we even may assume $\Sigma = \bigcup_{i=0}^{2n-1} T^{-i}F$, as the proof of the lemma shows.

Clearly we may also assume f to be bounded above, say $0 < c \leqq f \leqq K$, since a representation with an f unbounded from above is easy to change into one with a bounded f. Actually, the passage from B'' to B is all that is needed to make Rudolph's proof of his theorem work in the nonergodic aperiodic case.

Rudolph's proof consists of an inductive construction, where at each stage he changes the « names » of points in two different ways. We shall also need these changes but introduce at each step an additional change of names in order to correct the frequencies of orbit sections of length p and q. We shall therefore suppose that the reader is familiar with part II of the paper of Rudolph, which brings out the ideas quite lucidly. In particular we do not discuss some subtle questions of measurability because it is clear from Rudolph's proof how they can be treated.

Roughly speaking our argument shall run as follows: we may assume that $\{ T_t \}$ is the flow $\{ S_t \}$ in $\Omega(B, f)$. Pick a set $F_1 \subset B$ such that for a very large n_1 the sets $F_1, TF_1, \ldots, T^{n_1-1}F_1$ are disjoint and $\bigcup_{i=0}^{2n_1-1} T^iF_1 = B$. Let $\tau_1(b) = \inf \{ k \geqq 1 : T^k b \in F_1 \}$ $(b \in F_1)$ be the first return time under T to F_1 and $f_1(b) = \sum_{\nu=0}^{\tau_1(b)-1} f(T^\nu b)$. We may regard $\{ S_t \}$ as a flow in $\Omega(F_1, f_1)$ with basis automorphism T_1 = induced transformation on F_1, i. e. $T_1 b = T^{\tau_1(b)} b$ $(b \in F_1)$.

For each fixed $b \in F_1$ let $l = f_1(b)$ and divide the interval $[0, l[$ into subintervals $I_\nu = [x_\nu, x_{\nu+1}[$ with $0 = x_1 < x_2 < \ldots < x_{N+1} \leqq x_{N+2} = l$. The intervals I_ν $(\nu = 1, \ldots, N)$ shall have length p or q and the « rest » I_{N+1} shall have length $\leqq q$. We may assume $p < q$. We call intervals of length p p-intervals and intervals of length q q-intervals. $H^1 = 2n_1 K$ is an upper bound for f_1 and $H_1 = n_1 c$ is a large lower bound for f_1. Therefore we can choose the number N_p of p-intervals and the number N_q of q-intervals in such a way that $N_p(N_p + N_q)^{-1}$ is close to $\rho = \tilde{\rho}(1 + \tilde{\rho})^{-1}$ (this means that N_p is of the order $\tilde{\rho} N_q$).

It is convenient to mark the beginnings of the intervals I_ν. Let $\alpha = 2^{-1} \operatorname{Min}(p, q - p)$ and let $J_\nu = [x_\nu, x_\nu + \alpha] \subset I_\nu$ $(1 \leqq \nu \leqq N)$. Do this for every $b \in F_1$ and let $\mathscr{B}_1$ be the set of all (b, x) with $0 \leqq x < f_1(b)$ for which x belongs to some J_ν constructed with $l = f_1(b)$.

Let $\mathscr{F}$ be the collection of all measurable A in $\Omega(B, f)$ with the property that for each measurable $h : B \to \mathbb{R}$ the set $\{ b \in B : (b, h(b)) \in A \}$ is a measurable subset of B. Just like Rudolph we have to do all the constructions within this sub-σ-algebra of the family of m-measurable sets, see in particular his lemma 2. It is easy to pick the I_ν's for different b's so that $\mathscr{B}_1$ belongs to $\mathscr{F}$.

The set $\mathscr{B}_1$ is of course a first approximation for the desired set which shall correspond to $\{ (b', x') : 0 \leqq x' \leqq \alpha \}$ in $\Omega(B', f')$. The proof now consists of an inductive improvement of this approximation.

After the i-th step F_i, T_i, f_i with $H_i \leqq f_i \leqq H^i$, and $\mathscr{B}_i$ have been determined. B_i comes from a partition of the intervals $[0, l[$ with $l = f_i(b)$ $(b \in F_i)$ into N_p p-intervals, N_q q-intervals and a rest of length $\leqq q$ (Of course, l, N_p, etc. depend on i and b, but we do not want to carry too many indices). The approximation is already so good that $| N_p(N_p + N_q)^{-1} - \rho | < \eta_i$ for some small $\eta_i > 0$.

In step $i + 1$ we find a large n_{i+1} and an $F_{i+1} \subset F_i$ so that $F_{i+1}, T_i F_{i+1}, T^2 F_{i+1}, \ldots, T^{n_{i+1}-1} F_{i+1}$ are disjoint and

$$F_i = \bigcup_{\nu=0}^{2n_{i+1}-1} T_i^\nu F_{i+1}\,.$$

Put

$$\tau_{i+1}(b) = \inf \{ t \geqq 1 : T_i^t b \in F_{i+1} \} \qquad (b \in F_{i+1}),$$

$T_{i+1}b = T_i^{\tau_{i+1}(b)}b$, and $f_{i+1}(b) = \sum_{\nu=0}^{\tau_{i+1}(b)-1} f_i(T_i^\nu b)$. We may regard $\{ S_t \}$ as the flow under f_{i+1} in $\Omega(F_{i+1}, f_{i+1})$ with basis automorphism T_{i+1}.

Now consider $[0, l[$ with $l = f_{i+1}(b)$ for some $b \in F_{i+1}$. The letter l serves as a general symbol for the endpoint of the intervals which we want to subdivide even though it really depends on b and the step of the construction. The interval $[0, l[$ carries already p-intervals and q-intervals from the previous i-th step of the construction.

These « old » intervals now lie in the subintervals $[0, f_i(b)[$, $[f_i(b), f_i(b) + f_i(T_i b)[$, ... of the new interval $[0, l[$. If we call the strings of p-intervals and q-intervals in these subintervals « blocks », there exist between the blocks intervals which correspond to the rest left over in the i-th step. Such a « rest » can be empty but in general it has positive length.

To remove these gaps we use Rudolph's trick. Let $\varepsilon_i = 2^{-i}$. Since p/q is irrational, the function

$$g(\varepsilon) = \inf \{ t > 0 : \forall x \geqq t \ \ \exists m, n \in \mathbb{N} \text{ with } 0 < x - np - mq < \varepsilon \}$$

is finite for all $\varepsilon > 0$. If the n_i in the last step was large enough we can be sure that H_i is much larger than $g(\varepsilon_{i+1})$. Let $[z_1, z_1'[$ be the leftmost rest in $[0, l[$. Let r_1 be the largest righthand endpoint of an interval (of length p or q) with $r_1 \leqq z_1' - g(\varepsilon_{i+1})$ (Clearly $z_1' = f_i(b)$). We shall have ε_{i+1} so small that $\varepsilon_{i+1} \leq p$ and $g(\varepsilon_{i+1}) \geqq q$. Since $z_1' \geqq H_i$ and $z_1 \geqq H_i - q$, we can be sure that $r_1 \geqq H_i - q - g(\varepsilon_{i+1})$ can be found. By the definition of $g(\varepsilon)$ there exist m, n with $0 \leqq z_1' - np - mq - r_1 < \varepsilon_{i+1}$. Now redefine the intervals in $[r_1, z_1'[$ putting there m q-intervals and n p-intervals starting at r_1 and leaving a gap of length g_1 at most ε_{i+1} between the last of these intervals and z_1'.

Now consider the next block of connected p-intervals and q-intervals, i. e. if $[z_2, z_2'[$ is the second « rest » from the previous construction, we mean the interval $[z_1', z_2[$ forming a block of p-intervals and q-intervals. Note that $z_2' - z_1' = f_i(T_i b)$ so that this block has length $\geqq H_i - q$. Move the block $[z_1', z_2[$ by the length g_1 to the left so that the gap disappears and there is now a longer connected string of p-intervals and q-intervals. Let r_2 be the largest righthand endpoint of an interval of this longer string with $r_2 \leqq z_2' - g(\varepsilon_{i+1})$. We can again redefine the intervals on $[r_2, z_2'[$ putting there p-intervals and q-intervals and leaving a gap of length $g_2 \leqq \varepsilon_{i+1}$ between the last of these intervals and z_2'. Now shift the next block so the left by g_2 and continue this way until the string is within a distance $\leqq 2^{-i}l$ of l, but stop the procedure before it is within a distance $\leqq 2^{-i-1}l$ of l.

Most of the intervals stemming from the previous step of the construction have only been shifted by less than ε_{i+1} and only a small proportion has been radically changed.

This may deteriorate our approximation of the desired frequencies, but no too much. The last portion of length $\approx 2^{-i-1}l$ of $[0, l[$ is reserved to improve the frequencies again. For this we must have many p-intervals and many q-intervals in this last section so that by replacing some p-intervals by q-intervals we could increase the frequencies of q-intervals if necessary or the other way around. Now for large i this last section is relatively short compared with l so that we cannot improve much. Therefore, η_i in the previous step must already be quite small. The way to make sure that there are enough p-intervals and q-intervals of the previous step in the last section is to pick the n_{i+1} so large that $3\mathrm{H}^i \leqq 2^{-i-1}\mathrm{H}_{i+1}$ so that at least one third of the last section is composed of blocks. Moreover, n_{i+1} must be so large that changing one interval after another the frequencies change so slowly that one can get within η_{i+1} of ρ.

Now assume that the new partition of $[0, l[$ into p-intervals, q-intervals and a rest $\leqq q$ has been found for every $b \in \mathrm{F}_{i+1}$ and $l = f_{i+1}(b)$. For simplicity we write again $0 = x_1 < x_2 < \ldots < x_{\mathrm{N}+1} \leqq x_{\mathrm{N}+2} = l$ for this partition. Then let

$$\mathscr{B}_{i+1} = \{ (b, x) : b \in \mathrm{F}_{i+1} \text{ and } x_\nu \leqq x \leqq x_\nu + \alpha \text{ for some } 1 \leqq \nu \leqq \mathrm{N} \}.$$

Formally this set is of course defined on $\Omega(\mathrm{F}_{i+1}, f_{i+1})$ which is different from $\Omega(\mathrm{F}_i, f_i)$ but there is an obvious isomorphism between these spaces so that one can regard all sets $\mathscr{B}_i$ as subsets of $\Omega(\mathrm{B}, f)$. Since the change of the old partition into the new partition as carried out above consists only in small shifts except on a small part of $[0, l[$ the sets $\mathscr{B}_i$ shall converge and the limit $\mathscr{B}$ shall be such that a. e. orbit has the following property: it always stays in $\mathscr{B}$ during (not necessarily closed) intervals of length α and then stays in $\mathscr{B}^c$ during a time interval of length $p - \alpha$ or $q - \alpha$ before re-entering $\mathscr{B}$. Since in the limit the p-intervals have exactly the right frequency one can define

$$\mathrm{B}' = \{ (b, x) \in \Omega(\mathrm{B}, f) : \mathrm{S}_t(b, x) \in \mathscr{B} \text{ for all rational } t \text{ with } 0 < t < \alpha \}.$$

B' is a cross-section with the property that the time between two visits to B' always equals p or q.

We do not want to go into details about the question of measurability of the sets $\mathscr{B}_i$. The main argument goes like this. Using lemma 2 of Rudolph [17] it follows from the construction of $\mathscr{B}_1$ that $\mathscr{B}_1 \in \mathscr{F}$. When we know $\mathscr{B}_i \in \mathscr{F}$ we can partition B_{i+1} into measurable subsets E such that the intervals $[0, l[$ and $[0, l'[$ with $l = f_{i+1}(b)$, $l' = f_{i+1}(b')$ for any two elements b', $b \in \mathrm{E}$ have nearly the same length and for b and b' the old partition is characterized by the same sequence $(p, p, q, q, q, p, r, p, q, \ldots)$, saying that the first interval of the old partition has length p, the second has length p, the third length q, etc., r indicating a rest. Also the lengths of the sequence of rests shall be nearly the same for b, b' belonging to the same E. Then the new partition of $[0, l[$ can be chosen identical for all $b \in \mathrm{E}$, except for the length of the rest. This then implies $\mathscr{B}_{i+1} \in \mathscr{F}$, compare [*17*]. Hence $\mathscr{B} = \bigcup_{n=1}^{\infty} \left(\bigcap_{i=n}^{\infty} \right) \mathscr{B}_i \in \mathscr{F}$ and as in [*17*] the lemma 1 of [*17*], due to Ambrose, yields the theorem.

The main work now is that of making all precise estimates fit together. We start with some simple lemmas; recall that we had assumed $p < q$.

Lemma 2.2. — Assume an interval of length $l \geqq 2(p + q)$ is subdivided in two different ways into N_p (resp. N'_p) p-intervals and N_q (resp. N'_q) q-intervals + some rest of length $\leqq q$ in both subdivisions. Then we have

$$(2.1) \quad \Delta =_{\mathrm{def}} | \mathrm{N}_p(\mathrm{N}_p + \mathrm{N}_q)^{-1} - \mathrm{N}'_p(\mathrm{N}'_p + \mathrm{N}'_q)^{-1} | \leqq 2ql^{-1}(|\mathrm{N}_p - \mathrm{N}'_p| + |\mathrm{N}_q - \mathrm{N}'_q|).$$

Proof. —

$$\begin{aligned}\Delta &= |(\mathrm{N}_p\mathrm{N}'_q - \mathrm{N}'_p\mathrm{N}_q)(\mathrm{N}_p + \mathrm{N}_q)^{-1}(\mathrm{N}'_p + \mathrm{N}'_q)^{-1}| \\ &\leqq (\mathrm{N}_p|\mathrm{N}_q - \mathrm{N}'_q| + \mathrm{N}_q|\mathrm{N}_p - \mathrm{N}'_p|)(\mathrm{N}_p + \mathrm{N}_q)^{-1}(\mathrm{N}'_p + \mathrm{N}'_q) \\ &\leqq |\mathrm{N}_q - \mathrm{N}'_q|(\mathrm{N}'_p + \mathrm{N}'_q)^{-1} + |\mathrm{N}_p - \mathrm{N}'_p|(\mathrm{N}'_p + \mathrm{N}'_q)^{-1}\end{aligned}$$

Now apply $q(\mathrm{N}'_p + \mathrm{N}'_q) \geqq p\mathrm{N}'_p + q\mathrm{N}'_q \geqq 1/2$ to get (2.1). ■

An important special case is obtained when the second subdivision is obtained from the first either by replacing one q-interval by so many p-intervals that the rest is again $\leqq q$ or by giving up one p-interval and replacing it by zero or one q-intervals with rest $\leqq q$. If w is the integer for which $(w - 1)p < q < wp$ this particular change results in a change of the frequencies with

$$(2.2) \qquad \Delta \leqq 4wql^{-1}.$$

Lemma 2.3. — Let $0 < \gamma$, $0 < \rho < 1$, and $0 < \delta < 1$ be given. Assume $[0, l[$ with $l \geqq 2(p + q)$ is divided into N_p p-intervals, N_q q-intervals and a rest $\leqq q$. Assume further that the interval $[l - \delta l, l[$ contains at least $[\gamma\delta l] + 1$ p-intervals and also $[\gamma\delta l] + 1$ q-intervals. If $\eta_0 = 2^{-1}\rho(1 - \rho)$ and

$$(2.3) \qquad | \mathrm{N}_p(\mathrm{N}_p + \mathrm{N}_q)^{-1} - \rho | \leqq \mathrm{Min}\,\{ \eta_0, \eta_0\gamma\delta p \}$$

then one can pass to a new subdivision with N'_p, resp. N'_q intervals and a rest $\leqq q$ which differs from the original subdivision only in $[l - \delta l, l[$ and for which

$$(2.4) \qquad | N'_p(N'_p + N'_q)^{-1} - \rho | \leqq 4wql^{-1}.$$

Proof. — By (2.2) one change of an interval changes the relative frequency by at most $4wql$. Now assume the p-intervals are too frequent and that $[\gamma\delta l] + 1$ p-intervals in the last section $[l - \delta l, l[$ are changed into the corresponding number of q-intervals. Then the new frequencies, say N^*_p, N^*_q satisfy $N_p \geqq N^*_p$, $N_q \leqq N^*_q$ and

$$\begin{aligned} N_p(N_p + N_q)^{-1} - N^*_p(N^*_p + N^*_q)^{-1} &\geqq N_p(N_p + N_q)^{-1} - N^*_p(N^*_p + N_q)^{-1} \\ = N_q(N_p + N_q)^{-1} \cdot (N_p - N^*_p)(N^*_p + N_q) &\geqq \eta_0([\gamma\delta l] + 1)(N^*_p + N_q)^{-1} \end{aligned}$$

since $N_p(N_p + N_q)^{-1} \geqq \rho - \eta_0 \geqq \eta_0$ and $[\gamma\delta l] + 1$ intervals are changed. We can continue

$$\geqq \eta_0\gamma\delta l(N_p + N_q)^{-1} \geqq \eta_0\gamma\delta p$$

since $(N_p + N_q)p \leqq l$. Because of (2.3) we can, therefore, gradually change the intervals until we get within a distance $\leqq 4wql^{-1}$ of ρ. If there are too many q-intervals a symmetric argument works. ■

LEMMA 2.4. — If $[0, l[$ is subdivided in two ways as in lemma 2.2 and the two subdivisions differ only on a set of Lebesgue measure $\leqq l_0$ (except possibly for translations allowed on the complement of this set), then

$$(2.5) \qquad \Delta \leqq 4wl_0l^{-1}.$$

Proof. — Apply lemma 2.2 and use that $| N_p - N'_p | \leqq l_0 p^{-1}$, $| N_q - N'_q | \leqq l_0 q^{-1}$. Hence

$$\Delta \leqq 2ql^{-1}l_0 \{ p^{-1} + q^{-1} \} \leqq 2ql^{-1}l_0 \{ wq^{-1} + q^{-1} \} \leqq 4wl_0l^{-1}. \qquad ■$$

Now we can determine the parameters of the construction inductively. Recall that we started with $0 < c \leqq f \leqq K$, $p < q$, $\rho = \tilde{\rho}(1 + \tilde{\rho})^{-1}$, $(w - 1)p < q < wp$, $\eta_0 = 2^{-1}\rho(1 - \rho)$. We chose $0 < \varepsilon < 1$ so small that $\varepsilon < p$ and $g(\varepsilon) > q$ and pose

$$(2.6) \qquad \begin{aligned} \varepsilon_i &= 2^{-i}\varepsilon && (i \geqq 1) \\ \gamma &= 12^{-1}\eta_0 q^{-1} \\ \eta_i &= \operatorname{Min} \{ 2^{-1}\eta_0, 2^{-i-2}\eta_0\gamma p \} && (i \geqq 1) \end{aligned}$$

To define n_1 first pick a rational number $r = u_1(u_1 + u_2)^{-1}$ $(u_i \in \mathbb{N})$ with $| r - \rho | < 2^{-3}\eta_1$. We say that an interval $[0, \lambda_0[$ is subdivided nearly r-periodically if the subdivision starts with u_1 p-intervals, continues with u_2

q-intervals, then with u_1 p-intervals, then again with u_2 q-intervals, etc. until the right endpoint of the so defined string is $\geqq \lambda_0 - u_1 p - u_2 q$. The interval right of this endpoint may be subdivided in an arbitrary way with rest $\leqq q$. If J is sufficiently large any interval $\mathrm{I} = [0, \lambda_0[$ with $\lambda_0 \geqq \mathrm{J}\cdot(u_1 p + u_2 q)$ has the property:

(2.7) For all nearly r-periodical subdivisions the numbers N_p, N_q of p-intervals resp. q-intervals satisfy $|\,\mathrm{N}_p(\mathrm{N}_p + \mathrm{N}_q)^{-1} - \rho\,| < \eta_1$.

Fix such a $\mathrm{J} \geqq 2$ and let n_1 be so large that $\mathrm{H}_1 = n_1 c$ satisfies:

$$\mathrm{H}_1 \geqq \mathrm{J}\cdot(u_1 p + u_2 q) \geqq 2(p + q) \tag{2.8}$$

$$\mathrm{H}_1 > 4w(g(\varepsilon_2) + 2p + 2q)\eta_1^{-1} \tag{2.9}$$

Now $\mathrm{H}^1 = 2n_1\mathrm{K}$ is an upper bound for f_1. When n_i has been determined we know that $2^i n_1 n_2 \ldots n_i\mathrm{K} = \mathrm{H}^i$ is an upper bound for f_i and $\mathrm{H}_i = n_1\cdot n_2 \ldots n_i c$ is a lower bound. Let n_{i+1} be so large that

$$\mathrm{H}_{i+1} > 2^{i+1}\cdot 3\mathrm{H}^i \tag{2.8}$$

and

$$\mathrm{H}_{i+1} > 4w(g(\varepsilon_{i+2}) + q)\eta_{i+1}^{-1} \tag{2.9}$$

and

$$\mathrm{H}_{i+1} \geqq 3\cdot 2^{i+3} q\eta_0^{-1}\,. \tag{2.10}$$

Together with (2.6) this determines all constants needed for the construction. Of course in the first step the intervals $[0, l[$ with $l = f_1(b) \geqq \mathrm{H}_1$ are subdivided nearly r-periodically. Therefore, (2.7) holds for this subdivision.

We have to check inductively that the subdivision of $[0, f_i(b)[$ into N_p p-intervals and N_q q-intervals and a rest can be constructed with

$$|\,\mathrm{N}_p(\mathrm{N}_p + \mathrm{N}_q)^{-1} - \rho\,| < \eta_i \tag{2.11}$$

in step i. Since this has already been established for $i = 1$ we now assume it proved for i and consider $[0, f_{i+1}(b)[$ for some $b \in \mathrm{B}_{i+1}$. First regard the changes that come from the radical modifications of the old subdivision which were necessary to remove the rests. A sequence of such radical changes was carried out but each involved an interval of length $\leqq g(\varepsilon_{i+1}) + q$ and enabled us to move on a piece of length $\geqq \mathrm{H}_i$. Thus if l_0 is the total length of the part of $[0, l[$ where we have had to perform these changes, then $l_0/l \leqq (g(\varepsilon_{i+1}) + q)/\mathrm{H}_i$. By lemma 2.4 the new relative frequency $\mathrm{N}'_p(\mathrm{N}'_p + \mathrm{N}'_q)^{-1}$ differs by $\Delta \leqq 4wl_0/l$ from the old. By (2.9)

$\Delta \leqq 4w(g(\varepsilon_{i+1}) + q)\mathrm{H}_i^{-1} \leqq \eta_i$. Together with (2.11) this implies $|\mathrm{N}'_p(\mathrm{N}'_p + \mathrm{N}'_q)^{-1} - \rho| < 2\eta_i = \mathrm{Min}\{\eta_0, 2^{-(i+1)}\eta_0\gamma p\}$.

Put $\delta = 2^{-(i+1)}$ and assume for a moment that lemma 2.3 is applicable. Then we can pass to a new subdivision with N''_p, resp. N''_q intervals which differs from the subdivision just obtained only on $[l - \delta l, l[$ and for which by (2.4) the relative frequency can be estimated:

$$|\mathrm{N}''_p(\mathrm{N}''_p + \mathrm{N}''_q)^{-1} - \rho| \leqq 4wql^{-1} \leqq 4wq\mathrm{H}_{i+1}^{-1} < \eta_{i+1}.$$

The last inequality follows from (2.9) again. This means that we have then verified (2.11) for step $i + 1$.

It remains to check the hypothesis of lemma 2.3. By (2.8) at least a third of the interval $[l - 2^{-(i+1)}l, l[$ is composed of complete intervals of the form $\left[\sum_{\nu=0}^{k} f_i(\mathrm{T}_i^\nu b), \sum_{\nu=0}^{k+1} f_i(\mathrm{T}_i^\nu b)\right[$, i. e. intervals which were subdivided with good frequencies in the previous step. Therefore, if M_p is the number of p-intervals in the union of these « complete » subintervals, and M_q the number of q-intervals, we have

$$\mathrm{M}_p(\mathrm{M}_p + \mathrm{M}_q)^{-1} \geqq \eta_0, \tag{2.12}$$

since the frequencies in step i were good.

As the total length of these « complete » subintervals is $\geqq 3^{-1}2^{-(i+1)}l$ and the rest in each of them occupies less than half of it, the estimate

$$(\mathrm{M}_p + \mathrm{M}_q)q \geqq 3^{-1}2^{-(i+2)}l \tag{2.13}$$

must hold. Together with (2.10) and (2.12) this yields

$$\mathrm{M}_p \geqq \eta_0 3^{-1}2^{-(i+2)}lq^{-1} = 2\gamma\delta l \geqq [\gamma\delta l] + 1.$$

A symmetric argument proves $\mathrm{M}_q \geqq [\gamma\delta l] + 1$. Thus, the application of lemma 2.3 was legitimate and the inductive proof of (2.11) is complete.

Next we want to check that the sets $\mathscr{B}_i$ converge. Let

$$\mathrm{D}'_i = \left\{(b, x) \in \Omega(\mathrm{B}_{i+1}, f_{i+1}) : \exists k < \tau_{i+1}(b) \text{ with } \sum_{\nu=0}^{k} f_i(\mathrm{T}_i^\nu b) - g(\varepsilon_{i+1}) - 2q \leqq x \leqq \sum_{\nu=0}^{k} f_i(\mathrm{T}_i^\nu b) + q\right\},$$

$$\mathrm{D}''_i = \{(b, x) \in \Omega(\mathrm{B}_{i+1}, f_{i+1}) : 0 \leqq x \leqq q \text{ or } x \geqq f_{i+1}(b) - 2^{-i}f_{i+1}(b) - q\}.$$

If $(x, b) \notin D_i = D_i' \cup D_i''$, then all of the translates $S_t(x, b)$ ($|t| < q$) belong to a part of $\Omega(B_{i+1}, f_{i+1})$ where the new subdivision differs from the old only by a translation of magnitude $\leqq \varepsilon_{i+1}$.

Considering each fiber $\{ (b, x) \in \Omega(B_{i+1}, f_{i+1}) : 0 \leqq x < f_{i+1}(b) \}$ for fixed b separately and applying the Fubini theorem we find

$$m(D_i') \leqq (g(\varepsilon_{i+1}) + 3q)/H_i .$$

By (2.8) $H_i \geqq 2^i c$ and by (2.9) $(g(\varepsilon_{i+1}) + q)/H_i < 4^{-1}w^{-1}\eta_i$. As $\sum_{i=1}^{\infty} \eta_i < \infty$ we have shown $\sum_{i=1}^{\infty} m(D_i') < \infty$. Similarly $\sum_{i=1}^{\infty} m(D_i'') < \infty$ follows from $m(D_i'') \leqq 2q\underline{m}(B_{i+1}) + 2^{-i} \leqq 2qH_{i+1}^{-1} + 2^{-i}$. As also $\sum_{i=1}^{\infty} \varepsilon_{i+1} < \infty$, the convergence of the sets $\mathscr{B}_i$ to a set $\mathscr{B}$ with the property that each orbit stays in $\mathscr{B}$ always for a duration α and then stays outside for a duration $p - \alpha$ or $q - \alpha$ is now straightforward. As $\mathscr{B}_i$ produces the desired relative frequency of p-intervals up to η_i and η_i converges to zero. $\mathscr{B}$ produces exactly the desired relative frequency. ■

The next theorem is an immediate consequence of theorem 2.1:

THEOREM 2.5. — There exists a measure space $(B, \underline{m})$ and a (two-valued) function $f : B \to \mathbb{R}^+$ such that for every aperiodic measurable measure preserving flow $\{ T_t, t \in \mathbb{R} \}$ on a Lebesgue-space (Σ, μ) with $\mu(\Sigma) = 1$ there exists an invertible $\underline{m}$-preserving transformation $T : B \to B$ for which the flow $\{ S_t, t \in \mathbb{R} \}$ on $\Omega(B, f)$ constructed with the basis-automorphism T is isomorphic mod 0 to $\{ T_t, t \in \mathbb{R} \}$.

Proof. — We may preassign $p, q > 0$ with p/q irrational and take $B = [0, 2(p + q)^{-1}]$, $\underline{m}$ = Lebesgue-measure on B, $f(b) = p$ on $[0, (p+q)^{-1}[$ and $= q$ on $[(p + q)^{-1}, 2(p + q)^{-1}]$. If (Σ, μ) is a Lebesgue-space and $\{ T_t \}$ is represented on $\Omega(B', f')$ then $(B', \underline{m}')$ is a Lebesgue space (with not necessarily normalized measure). This was already pointed out by Rohlin [*16*]. By theorem 2.1 we can represent $\{ T_t \}$ on $\Omega(B', f')$ where $\underline{m}' \{ f' = p \} = \underline{m}' \{ f' = q \}$. As $\underline{m}' \{ f' = p \} = \underline{m} \{ f = p \}$ and $\underline{m}' \{ f' = q \} = \underline{m} \{ f = q \}$ and nonatomic Lebesgue spaces with the same total measure are isomorphic mod 0 there also exists a representation on $\Omega(B, f)$. The measure $\underline{m}'$ is nonatomic since $\{ T_t \}$ is aperiodic. ■

Remarks. — *a*) The proof of theorem 2.1 also establishes the following.

Let $P = \{ b \in B' : f'(b) = p \}$. The representation can be constructed in such a way that $\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} 1_P(T'^k b) = \rho \underline{m}'$ – a. e. on B'. This is equivalent to $\underline{m}' \{ f' = p \} = \tilde{\rho}\underline{m}' \{ f' = q \}$ in the ergodic case but in the non-ergodic aperiodic case it is stronger.

b) The second part of Rudolph's paper is devoted to a coding argument which shows that one can find a representation such that $\{ \{ f'=p \}, \{ f' = q \} \}$ is a generator for T' provided the entropy of the flow $h(T_1)$ is so small that $2^{-1}h(T_1)(p + q) < 1$. This argument needs no change. Thus it is possible to find a representation with $\underline{m}' \{ f' = p \} = m' \{ f' = q \}$ for which $\{ f' = p \}$ generates in B' under the same entropy condition.

c) If there is a way to define (B', f', T', $\underline{m}'$) in such a way that the points of B' visit $\{ f' = p \}$ very regularly, then the representation under 2-valued functions might open up a new approach to the theorem of Jacobs, Denker and Eberlein [*7*] [*3*] on the prevalence of uniquely ergodic flows.

d) There has been great interest in the last couple of years in « filtered flows » studied by P. A. Meyer, Sam Lazaro and others for their probabilistic interest [*14*] [*15*]. Some years earlier the author [*11*] had already given a representation theorem for filtered flows with some different probabilistic applications. Unfortunately, the fact that the representation of « filtered flows » had already been started—they were originally called flows with increasing σ-algebras—was overlooked.

We call a flow $\{ T_t, t \in \mathbb{R}^1 \}$ in (Σ, μ) filtered if the σ-algebra $\mathcal{G}$ on which μ is defined contains a sub-σ-algebra $\mathcal{G}_0$ which is increasing (i. e. $t \geqq 0 \Rightarrow T_t^{-1}\mathcal{G}_0 \subset \mathcal{G}_0$). This can for example correspond to the « future » or « past » of a process, depending on the direction of the shift. Usually the case where $\mathcal{G}_0$ is exhaustive, i. e. the σ-algebra $\mathcal{G}_\infty$ generated by all $T_t\mathcal{G}_0 (t \geqq 0)$ is $\mathcal{G}$, is of interest. It was shown by the author in [*11*] that measure preserving measurable flows that are proper on $\mathcal{G}_\infty$ admit a representation as flows under a function *adapted to* $\mathcal{G}_0$. By this we mean the following. There is a $\mathcal{G}_0^*$ such that $\mathcal{G}_0 \supset \mathcal{G}_0^* \supset T_{t_0}^{-1}\mathcal{G}_0$ for some $t_0 > 0$ and under the isomorphism not only $\mathcal{G}$ but also $\mathcal{G}_0^*$ is a product σ-algebra. If $\mathcal{C}$ is the σ-algebra on which $\underline{m}$ on B is defined there is a $\mathcal{C}_0 \subset \mathcal{C}$ with $T^{-1}\mathcal{C}_0 \subset \mathcal{C}_0$ such that $\mathcal{G}_0^*$ is the product of $\mathcal{C}_0$ with the Lebesgue-measurable sets. What is more important, the ceiling function f can be chosen $\mathcal{C}_0$-measurable, see [*11*, Theorem 2]. It would now be very interesting to know if even in this refined representation theorem f can be chosen 2-valued. This question seems very hard to me and the answer may well be negative (Of course $\{ T_t \}$ on $\mathcal{G}_\infty$ has to be aperiodic).

3. NONSINGULAR FLOWS

We now discuss the representation of nonsingular flows $\{ T_t, t \in \mathbb{R} \}$ by flows under a two-valued function.

A nonsingular transformation in a finite measure space $(\Sigma, \mathscr{A}, \mu)$ is a bijective transformation $T : \Sigma \to \Sigma$ for which T and T^{-1} are measurable and the measure $T\mu$ defined by $T\mu(A) = \mu(T^{-1}A)$ is equivalent to μ, i. e. $T\mu(A) = 0$ iff $\mu(A) = 0$. A group $\{ T_t, t \in \mathbb{R} \}$ of nonsingular transformations is called a nonsingular flow.

By a theorem of Kubo and the author [*13*], [*10*, II] proper measurable nonsingular flows on a complete probability space can be represented as flows $\{ S_t, t \in \mathbb{R}^1 \}$ under a function as follows. T in the finite measure space $(B, \underline{m})$ is now nonsingular. The measure m on $\Omega(B, f)$ is no longer a product-measure. If $\underline{m} \otimes \lambda$ is the completed product of $\underline{m}$ and the Lebesgue measure λ then m is equivalent to $\underline{m} \otimes \lambda$ restricted to $\Omega(B, f)$ and given by a strictly positive density $h(b, x)$ on $\Omega(B, f)$ with regard to $m \otimes \lambda$. (Actually, I did not formulate the representation in this simple way in [*10*] since I discussed the more general case of semiflows $\{ T_t, t \geqq 0 \}$ (the T_t are then not invertible). But the above formulation is easy to deduce from theorem 4.1 and 4.2 in [*10*, II], see p. 19 there).

If $\{ T_t \}$ is aperiodic one may again assume $0 < c \leqq f \leqq K < \infty$. This can be shown as in the measure preserving case [*10*, I, p. 187] since Rohlin's lemma remains true for nonsingular transformations [*8*, p. 282]. But more is true; we shall see that the argument of Rudolph goes through with only little extra work and thus we get:

THEOREM 3.1. — If $\{ T_t, t \in \mathbb{R} \}$ is an aperiodic measurable nonsingular flow on a complete probability space $(\Sigma, \mathscr{A}, \mu)$ and $p, q > 0$ with p/q irrational are given, there exists a finite measure space $(B', \underline{m}')$, a measurable $f' : B' \to \{ p, q \}$, a nonsingular transformation T' in B' and a strictly positive density $h'(b, x)$ in $\Omega(B', f')$ such that $\{ T_t, t \in \mathbb{R} \}$ is isomorphic mod 0 to the flow $\{ S_t, t \in \mathbb{R} \}$ under f', and the measure m' in $\Omega(B', f')$ is given by the density $h'(b, x)$ with respect to $\underline{m}' \otimes \lambda$.

Sketch of proof. — The main argument is the same as in the measure preserving case. A new difficulty, however, comes in when the question of convergence of the sets $\mathscr{B}_i$ is studied. The reason is that we can no longer be sure that a set with short orbit sections has small measure.

We shall be able to handle the small translations of blocks by passing to the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(A) = 2p^{-1} \int_0^{p/2} \mu(T_s^{-1}A)ds \qquad (A \in \mathscr{A}).$$

before even starting the construction. The radical changes between the good blocks also cause trouble since they happen on longer and longer intervals. The measure of these sets will be made small by a more careful choice of the sets F_i.

By the representation theorem of Kubo and the author we may start with a representation on some $\Omega(B, f)$ with $0 < c \leqq f \leqq K < \infty$. On B there is given $\underline{m}$. Considering both μ and the equivalent measure $\tilde{\mu}$ we get on $\Omega(B, f)$ two densities h and $\tilde{h}$. We may assume $c > 2p$. For $2^{-1}p \leqq x < f(b)$ $\tilde{h}$ can be chosen in its equivalence class with

$$\tilde{h}(b, x) = 2p^{-1} \int_0^{p/2} h(b, x - s)ds. \tag{3.1}$$

In $\Omega(F_i, f_i)$ we shall of course still use the restriction of $\underline{m}$ to F_i as the measure on the basis, but the density will now depend on i, call it h_i and $\tilde{h}_i$. Then (3.1) is replaced by

$$\tilde{h}_i(b, x) = 2p^{-1} \int_0^{p/2} h_i(b, x - s)ds \qquad (2^{-1}p \leqq x < f_i(x)). \tag{3.2}$$

Consider an interval $[0, l[$ with $l = f_{i+1}(b)$, $b \in F_{i+1}$. Let

$$0 = x_1 < x_2 < x_3 < \ldots < x_N \leqq l - q$$

be the left endpoints of all p-intervals and q-intervals from the subdivision coming from step i, i. e. the entrance points into $\mathscr{B}_i$ (The left endpoints of the « rests » are not yet considered here).

For the present theorem one does not need the correction of the frequencies in $[2^{-i}l, l[$. The removal of gaps and the small shifts of intervals are therefore, as in Rudolph's paper, continued through the entire interval $[0, l[$ (except for a rest $\leqq q$). Therefore, $\mathscr{B}_{i+1} \triangle \mathscr{B}_i$ can be written as a union $D_i^1 \cup D_i^2$ where D_i^1 is due to a translation of size $\leqq \varepsilon_{i+1}$ of blocks and D_i^2 comes from the radical changes at the end of a block.

For $(b, x) \in D_i^1$ we have

$$x \in \bigcup_{i=3}^{N} [x_i - \varepsilon_{i+1}, x_i] \cup \bigcup_{i=3}^{N} [x_i + \alpha - \varepsilon_{i+1}, x_i + \alpha] = : U_1 \cup U_2 .$$

(The first block with at least 2 intervals is not moved).

The conditional measure on the fiber above b is governed by h_i, resp. $\tilde{h}_i$. We may assume $\varepsilon < p$ so that $\varepsilon_{i+1} < p2^{-(i+1)}$. Applying Fubini we can estimate the conditional measure of U_1 as follows:

$$\int_{x_i-\varepsilon_{i+1}}^{x_i} \tilde{h}_i(b, x)dx = 2p^{-1}\int_0^{p/2}\left(\int_{x_i-\varepsilon_{i+1}}^{x_i} h_i(b, x-s)dx\right)ds$$
$$\leqq 2p^{-1}\varepsilon_{i+1}\int_{x_i-\varepsilon_{i+1}-p/2}^{x_i} h(b, t)dt$$

Hence

$$\int_{U_1} \tilde{h}_i(b, x)dx \leqq 2p^{-1}\varepsilon_{i+1}\int_p^{l-p} h(b, t)dt$$
$$= 4p^{-2}\varepsilon_{i+1}\int_0^{p/2}\left(\int_p^{l-p} h(b, t)dt\right)ds$$
$$\leqq 4p^{-2}\varepsilon_{i+1}\int_0^{p/2}\left(\int_p^{l-p/2} h(b, x-s)dx\right)ds$$
$$\leqq 2p^{-1}\varepsilon_{i+1}\int_0^{l} \tilde{h}(b, x)dx$$

In the same way it follows that $\int_{U_2} \tilde{h}_i(b, x)dx \leqq 2p^{-1}\varepsilon_{i+1}\int_0^l \tilde{h}(b, x)dx$. Now integrating the conditional measures over b we have proved that

$$(3.3) \qquad \tilde{m}(D_i^1) \leqq 4p^{-1}\varepsilon_{i+1}\,.$$

Next we estimate $\tilde{m}(D_i^2)$. We may consider $\tilde{m}$ defined on each $\Omega(F_i, f_i)$ by the natural isomorphism between these spaces. We need Rohlin's lemma in the following form. For $i \geqq 2$ we can find $F_i' \subset F_{i-1}$ such that $F_i', T_{i-1}F_i', \ldots, T_{i-1}^{n_i-1}F_i'$ are disjoint and $F_{i-1} = \bigcup_{v=0}^{n_i} T_{i-1}^{v}F_i'$. This is not hard to derive from the usual formulation of Rohlin's lemma.

We may choose the n_i increasing so fast that $H_{i-1} > g(\varepsilon_{i+1}) + q$ and $n_i \geqq 2^i$. For $0 \leqq v \leqq n_i$ consider

$$X_v = T_{i-1}^{v+(n_i-1)}F_i' \cup T_{i-1}^{v+n_i}F_i'$$

and

$$\theta_v = \tilde{m}(\{(b, x) : b \in X_v, 0 \leqq x < f_{i-1}(b)\})$$

Let v_0 be the index which minimizes θ_v, then it is easy to check that $\theta_{v_0} \leqq 6n_i^{-1}$. Let $F_i = T_{i-1}^{v_0}F_i'$. By the choice of v_0 the set

$$W_i = \{(b, x) : b \in F_i, f_i(b) - H_{i-1} \leqq x < f_i(b)\}$$

has measure $\tilde{m}(W_i) \leqq \theta_{v_0} \leqq 6n_i^{-1}$. As $g(\varepsilon_{i+1}) + q < H_{i-1}$ the set D_i^2

where the radical changes are performed when passing from $\mathscr{B}_i$ to $\mathscr{B}_{i+1}$ is contained in the image of W_i under the natural isomorphism $\Omega(F_i, f_i) \to \Omega(F_{i+1}, f_{i+1})$. Hence

$$\tilde{m}(D_i^2) \leqq 6n_i^{-1} \leqq 6 \cdot 2^{-i}.$$

Together with (3.3) it follows that the sets $\mathscr{B}_i$ converge since

$$\mathscr{B}_{i+1} \triangle \mathscr{B}_i = D_i^1 \cup D_i^2 .$$

The set $\mathscr{B} = \bigcup_{n=1}^{\infty} \left(\bigcap_{i=n}^{\infty} \mathscr{B}_i \right)$ has again the desired property that any orbit stays in $\mathscr{B}$ always for intervals of length α and then outside during intervals of length $p - \alpha$ or $q - \alpha$. The measurability considerations are the same as before.

We can now forget $\tilde{\mu}$ and $\tilde{m}$ since the convergence of the sequence $\mathscr{B}_i$ in the metric $\tilde{d}(A, B) = \tilde{m}(A \triangle B)$ implies that in the corresponding metric for the equivalent measure m.

Now it suffices to point out that lemma 3.1 in [*10*, II] can play exactly the same role as lemma 1 in [*17*] in order to get from the cross-section to the representation (Actually, a simpler lemma would suffice since lemma 3.1 was designed even for semiflows).

4. DYE'S THEOREM FOR FLOWS

We now return to the study of measure preserving transformations and flows. In this final section we shall obtain a continuous time version of Dye's theorem [*4*] on the weak equivalence of ergodic measure preserving transformations as an application of theorem 2.1 with $\tilde{\rho} = 1$.

We refer to the paper of Hajian, Ito, and Kakutani [*5*] for a very elegant proof and presentation of Dye's theorem for discrete time. In the case of discrete time the condition of aperiodicity, which we shall need, is not explicitly stated in the formulation of the theorem since it follows from ergodicity in (nonatomic) Lebesgue spaces.

The notion of weak equivalence for groups of transformations has been introduced by W. Krieger [*12*] and it has turned out to be very effective for the study of countable groups. However, for flows we feel that a stronger notion is desirable for the reason indicated in the introduction.

THEOREM 4.1. — Let $\{ T_t, t \in \mathbb{R} \}$, $\{ T'_t, t \in \mathbb{R} \}$ be measurable measure preserving ergodic aperiodic flows of Lebesgue-spaces (Σ, μ), resp. (Σ', μ')

of total measure 1. Then there exists a measure-preserving invertible transformation φ of Σ onto Σ' (mod 0) such that for all $\omega \in \Sigma$ φ maps the orbit Orb $(\omega) = \{ T_t\omega, t \in \mathbb{R}^1 \}$ onto the orbit Orb $(\omega') = \{ T_t'\omega', t \in \mathbb{R} \}$ of $\omega' = \varphi(\omega)$ in such a way that φ on orbits preserves the Lebesgue measure. In fact, given $p, q > 0$ with p/q irrational one can find φ with the following property. There exist measurable partitions $\{ P, P^c \}$ of Σ, resp. $\{ P', P'^c \}$ of Σ', such that $P' = \varphi(P)$ and the orbit of each $\varphi \in \Sigma$ consists of intervals of length p in P and intervals of length q in P^c, that of $\omega' \in \Sigma'$ of intervals of length p in P' and of length q in P'^c.

Proof. — Pass to representations with $(B, \underline{m}, T, f)$ and $(B', \underline{m}', T', f')$ such that f and f' assume only the values p and q and

$$\underline{m}\{ f = p \} = \underline{m}'\{ f' = p \} = \underline{m}\{ f = q \} = \underline{m}'\{ f' = q \}.$$

We may assume that $\Sigma = \Omega(B, f)$, $\Sigma' = \Omega(B', f')$ and that the given flows are those under f, resp. f'. Let T_0, T_0' be the transformations on $B_p = \{ f = p \}$, resp. $B_p' = \{ f' = p \}$ induced by T, resp. T'. They are ergodic and $\{ f = p \}$, $\{ f' = p \}$ are nonatomic Lebesgue spaces with the same total measure. By the theorem of Dye there exists a measure preserving invertible transformation $\psi : B_p \to B_p'$ which maps the orbit of each $b \in B_p$ under T_0 *onto* the orbit of $\psi(b)$ under T_0'. Of course the measure on B_p, resp. B_p' is the restriction of m, resp. m'. By lemma 2 in the quoted paper of Hajian, Ito and Kakutani there exists a partition of B_p say $\{ B_1, B_2, B_3, \dots \}$, (some B_i may be empty), such that

$$\{ T^1B_1, T^2B_2, T^3B_3, \dots \}$$

is a partition of $\{ f = q \}$. An analogous partition exists for B_p'. Let ρ be the measure preserving invertible transformation $\{ f = p \} \to \{ f = q \}$ defined by $\rho(b) = T^ib$ $(b \in B_i)$. Let ρ' denote the corresponding transformation $\{ f' = p \} \to \{ f' = q \}$. We can now define $\varphi : \Omega(B, f) \to \Omega(B', f')$ as follows:

$$\varphi((b, x)) = \begin{cases} (\psi(b), x) & b \in \{ f = p \}, \quad 0 \leqq x < p \\ (\rho'\psi\rho^{-1}(b), x) & b \in \{ f = q \}, \quad 0 \leqq x < q \end{cases}$$

Checking the stated properties of φ is straightforward. ■

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