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Section B :
Calcul des Probabilités et Statistique.

Strong ratio limit theorems for mixing Markov operators

by

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ABSTRACT. — Let P be a Markov operator (positive linear contraction of L_1), which is assumed mixing (*i. e.*, $uP^n \to 0$ for $u \in L$, with $\int u = 0$). For fixed $f \in L_\infty$, we obtain conditions for the convergence of $\langle uP^n, f \rangle / \langle vP^n, f \rangle$ for every $0 \leqslant v, u \in L_1$, and for the convergence of $\langle uP^n, g \rangle / \langle vP^n, f \rangle$ for $0 \leqslant g \leqslant f$ and every $0 \leqslant v, u \in L_1$.

1. NOTATIONS

Let (X, Σ, m) be a σ-finite measure space, and let P be a positive linear contraction of $L_1(X, \Sigma, m)$. Its action will be written by uP, while the action of the adjoint on L_∞ will be written as Pf, so that $\langle uP, f \rangle = \langle u, Pf \rangle$.

Identifying L_1 with the space $M(X, \Sigma, m)$ of finite signed measures $\ll m$ (via the Radon-Nikodym theorem), P can be represented as an operator on $M(X, \Sigma, m)$, still denoted by P, so that $d(\nu P)/dm = uP$ when $u = d\nu/dm$.

A positive (finite or σ-finite) mesure $\lambda \ll m$ is *invariant* if $\lambda P = \lambda$, and many

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limit theorems require (or imply) this assumption. For existence theorems, see [*1*].

P is called *conservative* if $\sum_{n=0}^{\infty} u\mathrm{P}^n$ is 0 or ∞ *a. e.*, for $0 \leqslant u \in \mathrm{L}_1$. It is *ergodic* if $f \in \mathrm{L}_\infty, \mathrm{P}f = f \Rightarrow f \equiv 1$, (and by the Hahn-Banach theorem, $\int u = 0 \Rightarrow \left\| \frac{1}{\mathrm{N}} \Sigma\, u\mathrm{P}^n \right\| \to 0$), and *mixing* if $u\mathrm{P}^n$ converges weakly to 0 in L_1, for u with $\int udm = 0$.

2. STRONG RATIO LIMIT THEOREMS

Strong ratio limit theorems for a Markov operator deal with the convergence of the expressions $\langle \eta\mathrm{P}^n, g \rangle / \langle \nu\mathrm{P}^n, f \rangle$, where $\nu, \eta \ll m$ are probability measures, and $0 \leqslant f, g \in \mathrm{L}_\infty$. If P has a finite invariant measure and is ergodic, all these expressions converge if and only if P is mixing. We shall assume now that P is mixing, and irreducible (*i. e.*, if $\mathrm{L}_1(\mathrm{A})$ is invariant under P then $m(\mathrm{A}) = 0$ or $m(\mathrm{X} - \mathrm{A}) = 0$). In that case, either P has a finite invariant measure equivalent to m, or P has no finite invariant measures.

LEMMA 2.1. — *If* P *has no finite invariant measure and* $u\mathrm{P}^n$ *converges weakly, then* $\| u\mathrm{P}^n \|_1 \to 0$. This is theorem 5.1 of Krengel and Sucheston [*8*].

We first deal with particular cases of the convergence.

THEOREM 2.1. — *Let* P *be an irreducible mixing Markov operator and let* $0 \leqslant f \in \mathrm{L}_\infty$. *A necessary and sufficient condition for the convergence* $\lim \langle \nu\mathrm{P}^n, f \rangle / \langle \eta\mathrm{P}^n, f \rangle = 1$ *for every probability measures* $\nu, \eta \ll m$ *is the existence of a probability measure* $\mu \ll m$ *satisfying*

$$\text{(2.1)} \qquad \limsup \| \mathrm{P}^n f / \langle \mu, \mathrm{P}^n f \rangle \|_\infty < \infty$$

$$\text{(2.2)} \qquad \liminf_{n \to \infty} \langle \mu, \mathrm{P}^{n+1} f \rangle / \langle \mu, \mathrm{P}^n f \rangle \geqslant 1 .$$

Proof. — Let $f_n = \mathrm{P}^n f / \langle \mu, \mathrm{P}^n f \rangle$. If the convergence holds, $f_n \to 1$ in the weak-* topology of L_∞ and (2.1) follows, while (2.2) follows by putting $\nu = \mu\mathrm{P}$, $\eta = \mu$, and necessity is proved.

Sufficiency : If P has a finite invariant measure, mixing implies the convergence, so we may assume there is no finite invariant measure. Fix a probability measure ν. By lemma 2.1, $\|(\nu - \mu)P^n\| \to 0$. For $\varepsilon > 0$ fix k such that $\|(\nu - \mu)P^k\| < \varepsilon$. By (2.1) $\|f_n\|_\infty \leqslant M$ for $n > n_0$.

$$\begin{aligned}\limsup_{n\to\infty} |\langle \nu P^n, f\rangle/\langle \mu P^n, f\rangle - 1| &= \limsup_{n\to\infty} |\langle (\nu-\mu)P^k, P^{n-k} f\rangle/\langle \mu P^n, f\rangle| \\ &= \limsup_{n\to\infty} |\langle (\nu - \mu)P^k, f_{n-k}\rangle| \{\langle \mu P^{n-k}, f\rangle/\langle \mu P^n, f\rangle\} \\ &\leqslant \varepsilon M \limsup_{n\to\infty} \langle \mu P^{n-k}, f\rangle/\langle \mu P^n, f\rangle \leqslant \varepsilon M,\end{aligned}$$

the last inequality by (2.2). Let $\varepsilon \to 0$ to conclude the proof.

THEOREM 2.2. — *Let P be an irreducible mixing Markov operator and let $0 \leqslant g \leqslant f \in L_\infty$. Let μ and f satisfy* (2.1) *and* (2.2). *If $\langle \mu P^n, g\rangle/\langle \mu P^n, f\rangle$ converges to a nonzero limit α, then for every probability measures $\nu, \eta \ll m$ we have*

$$\lim_{n\to\infty} \langle \nu, P^n g\rangle/\langle \eta, P^n f\rangle = \alpha .$$

Proof. — We show that μ and g satisfy (2.1) and (2.2).

$$\begin{aligned}\langle \mu P^{n+1}, g\rangle/\langle \mu P^n, g\rangle &= \{\langle \mu P^{n+1}, g\rangle/\langle \mu P^{n+1}, f\rangle\} \\ &\quad \{\langle \mu P^{n+1}, f\rangle/\langle \mu P^n, f\rangle\}\{\langle \mu P^n, f\rangle/\langle \mu P^n, g\rangle\} \to 1 .\end{aligned}$$

If n is large enough,

$$\|P^n g/\langle \mu P^n, g\rangle\|_\infty \leqslant \|P^n f/\langle \mu P^n, f\rangle\|_\infty \{\langle \mu P^n, f\rangle/\langle \mu P^n, g\rangle\} \leqslant M2/\alpha .$$

Apply theorem 2.1 to g.

COROLLARY 2.1. — *Let P be a conservative mixing Markov operator and let $A \in \Sigma$ with $m(A) > 0$. If $\mu \ll m$ is a probability measure such that $\mu P^{n+1}(B)/\mu P^n(A)$ converges for $B \subset A$, and μ satisfies*

$$\limsup \|P^n 1_A/\mu P^n(A)\|_\infty < \infty ,$$

then there exists a σ-finite invariant measure $\lambda \sim m$ with $\lambda(A) < \infty$, and there is a sequence of sets $A = A_0 \subset A_1 \subset \ldots A_k \uparrow X$ with $\lambda(A_k) < \infty$ *such that for $0 \leqslant f, g \in L_\infty(A_k)$ and every probabilities $\nu, \eta \ll m$* we have

$$\lim_{n\to\infty} \langle \nu P^n, g\rangle/\langle \eta P^n, f\rangle = \int g d\lambda \Big/ \int f d\lambda . \tag{2.3}$$

Proof. — The existence of λ is known ([*11*], lemma 1.1), and

$$\lim \mu P^{n+1}(B)/\mu P^n(A) = \lambda(B)/\lambda(A) .$$

Putting $B = A$ we have (2.2) (with $f = 1_A$) and (2.3) holds for $0 \leqslant f, g \in L_\infty(A)$ by theorem 2.2. The existence of $\{A_k\}$ follows now from theorem 3.4 of [*11*].

REMARK 2.1. — The condition of mixing instead of ergodicity in the corollary is not very restrictive: it is satisfied in aperiodic Harris operators [7] [13], and in aperiodic random walks [2]. Point transformations (with $\lambda(X) = \infty$) do not satisfy (2.1) when $\int f d\lambda < \infty$, since (2.1) then implies $\| P^n f \|_\infty \to 0$.

2. An example of a random walk satisfying the conditions of the corollary was given in [12]. Hence the results are more general than those obtained by Levitan and Smolowitz [10], who assume Harris recurrence (and have longer proofs).

3. If P is conservative and ergodic, then a weaker version of theorem 1 can be proved by the methods of [11]. We need a condition stronger than (2.2) (in a certain sense): For every $\nu \ll m$ we have

$$\lim_{n\to\infty} \sup \langle \nu P^{n+1}, f \rangle / \langle \nu P^n, f \rangle = 1 .$$

The corresponding version of corollary 2.1 is given in [12].

The next proposition yields a sufficient criterion for corollary 2.1 to hold.

PROPOSITION. — *Let* P *be ergodic and conservative with σ-finite invariant measure* $\lambda \sim m$, *and let* $0 \leqslant f \in L_\infty$ *satisfy* $0 < \int f d\lambda < \infty$. *Let* $\mu \ll m$ *be a probability measure such that* $\limsup \| P^n f / \langle \mu P^n, f \rangle \|_\infty < \infty$. *If for* $0 \leqslant g \in L_\infty$ *with* $\int g d\lambda < \infty$ *the sequence* $P^n g(x)/P^n f(x)$ *converges* μ-*a. e., then* $\liminf_{n\to\infty} \langle \mu P^n, g \rangle / \langle \mu P^n, f \rangle \geqslant \int g d\lambda \Big/ \int f d\lambda$.

Proof. — By the Chacon-Ornstein theorem

$$\lim_{N\to\infty} \sum_{n=0}^{N} P^n g(x) \Big/ \sum_{n=0}^{N} P^n f(x) = \int g d\lambda \Big/ \int f d\lambda$$

a. e., and since $\sum_{n=0}^{\infty} P^n f(x) \equiv \infty$, $P^n g(x)/P^n f(x) \to \int g d\lambda \Big/ \int f d\lambda$ μ-a. e. Let S be the support of μ (i. e., $S = \{ d\mu/dm > 0 \}$). Fix $\alpha < \int g d\lambda \Big/ \int f d\lambda$. For $x \in S$ there is a first $N(x)$ such that for $n \geqslant N(x)$, $P^n g(x)/P^n f(x) > \alpha$. We have $\{ x \in S : N(x) \leqslant k \} = \bigcap_{n=k}^{\infty} \{ x : P^n g(x) - \alpha P^n f(x) > 0 \} \cap S$, showing the measurability of $N(x)$ on S. Putting $B_k = \{ x \in S : N(x) > k \}$, $\mu(B_k) \downarrow 0$.

Now

$$\langle \mu P^n, g \rangle - \alpha \langle \mu P^n, f \rangle = \int P^n(g - \alpha f)d\mu \geqslant \int_{B_n} P^n(g - \alpha f)d\mu \geqslant -\alpha \int_{B_n} P^n f\, d\mu .$$

Hence, with $M = \sup_{n \geqslant n_0} \| P^n f / \langle \mu P^n, f \rangle \|_\infty$, we have

$$\langle \mu P^n, g \rangle / \langle \mu P^n, f \rangle - \alpha \geqslant -\alpha \int_{B_n} \{ P^n f / \langle \mu P^n, f \rangle \} d\mu \geqslant -\alpha M \mu(B_n) \to 0 .$$

Hence $\liminf \langle \mu P^n, g \rangle / \langle \mu P^n, f \rangle \geqslant \alpha$. Now let $\alpha \to \int g d\lambda \Big/ \int f d\lambda$.

COROLLARY 2.2. — *Let* P *be a conservative mixing Markov operator with* σ*-finite invariant measure* $\lambda \sim m$, *and let* $0 < \lambda(A) < \infty$. *Let* μ *be a probability measure satisfying* (2.1) *and* (2.2) *(with* $f = 1_A$*). If for every* $B \subset A$ $P^n 1_B(x)/P^n 1_A(x)$ *converges on a nonnull set, then the results of corollary* 2.1 *hold.*

Proof. — Fix $B \subset A$. Let μ_0 be a probability supported in

$$\{ x : P^n 1_B(x)/P^n 1_A(x) \text{ converges} \} .$$

By theorem 2.1 μ_0 also satisfies (2.1) and (2.2) (with $f = 1_A$). Applying the proposition to 1_B and to $1_A - 1_B = 1_{A-B}$, with μ_0, we have

$$\lim \mu_0 P^n(B)/\mu_0 P^n(A) = \lambda(B)/\lambda(A) .$$

We now apply theorem 2.2 (with μ_0) to obtain $\mu P^n(B)/\mu P^n(A) \to \lambda(B)/\lambda(A)$. Since $\mu P^{n+1}(A)/\mu P^n(A) \to 1$, we can apply corollary 2.1.

Remarks. — 1. If μ and f satisfy (2.1), then a sufficient condition for (2.2) is that $P^{n+1}f(x)/P^n f(x)$ converges μ-a. e., when P is conservative and ergodic. $\left(\text{Then the limit is 1, since } \sum_{n=0}^{\infty} P^n(x) \equiv \infty\right)$. This is proved as in the proposition, by taking $g = Pf$, and $\alpha < 1$. (We do not require an invariant measure; e. g., $f \equiv 1$).

2. The proof of the proposition is a modification of the proof of the ratio limit theorem (for sums) in [*3*].

THEOREM 2.3. — *Let* P *be an irreducible mixing Markov operator and let* $\mu \ll m$ *be a probability measure.*

(*i*) *The set of all functions* $0 \leqslant f \in L_\infty$ *which satisfy* (2.1) *and* (2.2) *is* P*-invariant and closed under addition.*

(*ii*) *Let* f *satisfy* (2.1) *and* (2.2). *If* $0 \leqslant g \in L_\infty$ *satisfies* $P^K g \geqslant \beta f$ *and* $\sum_{n=0}^{N} P^n f \geqslant \alpha g$ *for some* K, N, α, β *positive, then* (2.1) *and* (2.2) *are satisfied with* g *replacing* f.

Proof. — (*i*) P-invariance is trivial. Let $f = f_1 + f_2$ with $0 \leqslant f_i$ satisfying (2.1) and (2.2).

$$\| P^n f \|_\infty / \langle \mu P^n, f \rangle \leqslant \| P^n f_1 \|_\infty / \langle \mu P^n, f \rangle + \| P^n f_2 \|_\infty / \langle \mu P^n, f \rangle$$
$$\leqslant \| P^n f_1 \| / \langle \mu P^n, f_1 \rangle + \| P^n f_2 \| / \langle \mu P^n, f_2 \rangle$$

and (2.1) follows. For $\varepsilon > 0$, if $n \geqslant n_0(\varepsilon)$,

$$\langle \mu P^{n+1}, f \rangle = \langle \mu P^{n+1}, f_1 \rangle + \langle \mu P^{n+1}, f_2 \rangle$$
$$\geqslant (1 - \varepsilon) \langle \mu P^n, f_1 \rangle + (1 - \varepsilon) \langle \mu P^n, f_2 \rangle = (1 - \varepsilon) \langle \mu P^n, f \rangle .$$

(*ii*) For $n > K$ *we have*

$$\langle \mu P^n, f \rangle / \langle \mu P^n, g \rangle = \{ \langle \mu P^{n-k}, f \rangle / \langle \mu P^n, g \rangle \} \langle \mu P^n, f \rangle / \langle \mu P^{n-k}, f \rangle$$
$$\leqslant \beta^{-1} \langle \mu P^n, f \rangle / \langle \mu P^{n-k}, f \rangle$$

and the right-hand side tends to β^{-1} by theorem 2.1. Hence

$$\langle \mu P^n, f \rangle / \langle \mu P^n, g \rangle \leqslant M$$

for $n \geqslant n_1$. For any $k \geqslant 0$,

$$\alpha \| P^{n-k} g / \langle \mu P^n, g \rangle \| \leqslant \left\| \sum_{j=0}^{N} P^{n-k+j} f \right\| / \langle \mu P^n, g \rangle \leqslant$$

$$\frac{\langle \mu P^n, f \rangle}{\langle \mu P^n, g \rangle} \sum_{j=0}^{N} \{ \| P^{n-k+j} f \| / \langle \mu P^{n-k+j}, f \rangle \} \{ \langle \mu P^{n-k+j}, f \rangle / \langle \mu P^n, f \rangle \} .$$

Hence, since the last factors tend to 1, we have

$$\limsup_{n \to \infty} \| P^{n-k} g / \langle \mu P^n, g \rangle \|$$
$$\leqslant \alpha^{-1} M(N + 1) \limsup_{i \to \infty} \| P^i f / \langle \mu P^i, f \rangle \| = \underset{i \to \infty}{M_1} .$$

M_1 does not depend on k.

Let $\nu \ll m$ be a probability. For $\varepsilon > 0$ take k such that $\| (\mu - \nu) P^k \| < \varepsilon$. Then

$$\limsup | \langle \nu P^n, g \rangle / \langle \mu P^n, g \rangle - 1 |$$
$$= \limsup_{n \to \infty} | \langle (\nu - \mu) P^k, P^{n-k} g \rangle | / \langle \mu P^n, g \rangle \leqslant \varepsilon M_1 .$$

The theorem shows that if $f = 1_B$ satisfies (2.1) and (2.2), so does 1_A with

$$A = \{ x : \sum_{n=0}^{N} P^n 1_B(x) \geqslant 1 \}$$

DÉFINITIONS. — A function $0 \leqslant f \in L_\infty$ is *small* if for every $0 \leqslant g \in L_\infty$ ($g \not\equiv 0$) there are K and $\beta > 0$ such that $\sum_{n=0}^{K} P^n g \geqslant \beta f$. f is *very small* if for every $0 \leqslant g \in L_\infty$ there are K and $\beta > 0$ such that $P^K g \geqslant \beta f$.

It was proved by Horowitz [4] that if P is conservative and ergodic and has a « small » function $f \not\equiv 0$, then it is Harris. If P is Harris, then there are « very small » functions [5]. Levitan [9] treated ratio limit theorems on sets A which satisfies $\inf_{x,y \in A} q_k(x, y) = \beta > 0$ for some k, where q_k is the kernel bounded by P^k. Clearly 1_A is then « very small ». If we define on $X = \{ 0, 1, 2, \dots \}$ $p_{ii+1} = 1/2$, $p_{i0} = 1/2$, $p_{ij} = 0$ otherwise, then $P^{j+1} 1_{\{j\}} \geqslant 1/2^{j+1}$ and 1 is a « very small » function; but

$$\sup_n \inf_{i,j} p_{ij}^{(n)} = 0 ,$$

so X itself does not satisfy Levitan's condition.

COROLLARY 2.3. — *Let* P *be an aperiodic Harris operator. It for some « very small » function* f *and a probability measure* $\mu \ll m$ *the relations* (2.1) *and* (2.2) *hold, then for every « small » function g and probabilities* η, $\nu \ll m$ *we have* $\lim \langle \nu P^n, g \rangle / \langle \eta P^n, g \rangle = 1$.

THEOREM 2.4. — *Let* X *be a locally compact* σ*-compact Abelian group, with Haar measure* λ*, and let* P *be an aperiodic random walk on* X. *If*

(*i*) *For some compact set* C, $\lim \sup \{ P^n(0, C) \}^{1/n} = 1$;

(*ii*) $\lim \sup \| P^n 1_B / \mu P^n(B) \|_\infty < \infty$ *for some relatively compact open set B with* $\lambda(\partial B) = 0$, *and* μ *with compact support*;

Then $\lim \langle \nu P^n, g \rangle / \langle \eta P^n, f \rangle = \int g d\lambda \Big/ \int f d\lambda$ *for* $0 \leqslant f, g \in L_\infty$ *with compact supports and any two probabilities* ν, $\eta \ll \lambda$.

Proof. — By Stone [15], condition (*i*) implies that $P^n 1_B(x) / P^n 1_B(y) \to 1$ uniformly for x, y in compact sets. We can therefore obtain

$$P^n 1_B(x) / \mu P^n(B) \to 1$$

uniformly on compact sets. Let $d\mu_1 = \lambda(B)^{-1} 1_B$. Then $\mu_1 P^n(B) / \mu P^n(B) \to 1$.

Then $\| P^n 1_B \|_\infty / \mu_1 P^n(B) = \{ \| P^n 1_B \|_\infty / \mu P^n(B) \} \mu P^n(B) / \mu_1 P^n(B)$ is also bounded. Hence we may and do assume $d\mu/d\lambda = \lambda(B)^{-1} 1_B$.

The adjoint Markov operator is the random walk induced by $p^*(A) = p(-A)$, and condition (*i*) applies to P^* as well (since we may assume that C is symmetric). Hence by Stone [*15*] we have

$$P^{*n+r} 1_B(x) / P^{*n} 1_B(y) \to 1$$

uniformly for x, y in compact sets. Let $0 \leqslant f, g \in L_\infty$ have compact supports. The above uniform convergence yields easily that

$$(*) \quad \langle \mu P^{n+r}, g \rangle / \langle \mu P^n, f \rangle = \langle g, P^{*n+r} 1_B \rangle / \langle f, P^{*n} 1_B \rangle \to \int g d\lambda \Big/ \int f d\lambda .$$

When $g = f = 1_B$, (2.1) and (2.2) are satisfied.

Let $A = B \cup \{ f > 0 \} \cup \{ g > 0 \}$. Then A is relatively compact, and its closure can be included in a relatively compact open set E with $\lambda(\partial E) = 0$ (using Urysohn's lemma).

Since the random walk is aperiodic, for a continuous $0 \leqslant h \leqslant 1$ supported in B (e. g., $h(a) = 1$ at a point of B, $h = 0$ outside B) we have $\sum_{n=0}^{\infty} P^n h(x) > 0$ on $\bar{E}$, hence the compactness of $\bar{E}$ yields N and α such that

$$\sum_{n=0}^{N} P^n 1_B \geqslant \sum_{n=0}^{N} P^n h \geqslant \alpha 1_{\bar{E}} .$$

By theorem 2.3, we can apply theorem 2.1 (P is mixing by Foguel [*2*]) to have (2.1) and (2.2) sarisfied by μ and $1_{\bar{E}}$. Relation (*) and theorem 2.2 conclude the proof.

Remarks. — 1. We see from the proof that (*ii*) can be replaced by (*ii*)′ $\limsup \| P^n f / \langle \mu P^n, f \rangle \|_\infty < \infty$ for some $0 \leqslant f \in L_\infty$ with compact support and μ with $d\mu(d\lambda = \lambda(B)^{-1} 1_B$, where B is as in (*ii*).

2. The conditions of the theorem are clearly necessary.

Levitan and Smolowitz [*10*] treated also the case that P is Harris and self-adjoint with respect to the invariant measure λ (i. e., $\langle u, Pf \rangle = \langle f, Pu \rangle$ for $f, u \in L_2(\lambda)$). We obtain results without the Harris assumption. The following lemma was observed by Pruitt [*14*] for Markov chains, and its arguments are used in [*10*].

LEMMA 2.2. — *Let* T *be a self-adjoint operator in a Hilbert space* H *satisfying* $\| T \| \leqslant 1$. *If* $Tf \neq 0$, *then* $T^n f \neq 0$ *for every* n, $\lim \| T^{n+1} f \| / \| T^n f \|$

exists, and does not exceed 1. *If* $\sum_{n=1}^{\infty} \| T^n f \| = \infty$, *then* $\| T^{n+1} f \| / \| T^n f \| \to 1$.

Proof. — The spectrum $\sigma(T)$ is contained in $[-1, 1] = I$. Let f satisfy $Tf \neq 0$. Let $E(.)$ be the resolution of the identity of T. Define, for Borel subsets of $I = [-1, 1]$, $\mu(A) = \langle E(A)f, f \rangle = \| E(A)f \|^2$. By the spectral theorem

$$\| T^n f \|^2 = \langle T^{2n} f, f \rangle = \int_I t^{2n} d\mu = \int_I t^{n+1} t^{n-1} d\mu$$
$$\leqslant \left\{ \int_I t^{2(n+1)} d\mu \right\}^{1/2} \left\{ \int_I t^{2(n-1)} d\mu \right\}^{1/2} = \| T^{n+1} f \| \, \| T^{n-1} f \| .$$

Thus $T^n f \neq 0 \Rightarrow T^{n+1} f \neq 0$, and $T^n f \neq 0$ for every n. Furthermore, $\| T^n f \| / \| T^{n-1} f \| \leqslant \| T^{n+1} f \| / \| T^n f \|$ and the sequence $\| T^{n+1} f \| / \| T^n f \|$ is increasing, and bounded by 1. The last statement is clear.

THEOREM 2.5. — *Let* P *be a conservative mixing Markov operator with* σ*-finite invariant measure* $\lambda \sim m$, *and assume that* P *is self-adjoint. Let* $0 \leqslant f \in L_\infty$ *satisfy* $0 < \int f d\lambda < \infty$. *A necessary and sufficient condition for the convergence* $\lim \langle \nu P^n, g \rangle / \langle \eta P^n, f \rangle = \int g d\lambda \Big/ \int f d\lambda$ *to hold for every* $0 \leqslant g \leqslant f$ *and probabilities* $\nu, \eta \ll m$ *is* $\limsup \| P^{2n} f / \langle P^{2n} f, f \rangle \|_\infty < \infty$.

Proof. — Define μ by $d\mu/d\lambda = \left(\int f d\lambda \right)^{-1} f$. Necessity is clear. $Q = P^2$ is also conservative and mixing. By lemma 2. 2, $\| P \|_2 \leqslant 1$ implies

$$\langle \mu Q^{n+1}, f \rangle / \langle \mu Q^n, f \rangle = \langle P^{2(n+1)} f, f \rangle / \langle P^{2n} f, f \rangle$$
$$= \| P^{n+1} f \|_2^2 / \| P^n f \|_2^2 \to 1 ,$$

since,

$$\| f \|_2 \sum_{n=0}^{\infty} \| P^n f \|_2 \geq \sum_{n=0}^{\infty} \| P^n f \|_2^2 = \sum_{n=0}^{\infty} \langle P^n f, P^n f \rangle = \sum_{n=0}^{\infty} \langle f, P^{2n} f \rangle = \infty .$$

Hence f, μ and Q satisfy (2.1) and (2.2) and for every ν, $\eta \ll m$ we have, by theorem 2.1,

$$\lim \langle \nu P^{2n}, f \rangle / \langle \eta P^{2n}, f \rangle = 1 .$$

Substituting νP for ν and ηP for η, we have $\langle \nu P^n, f \rangle / \langle \eta P^n, f \rangle \to 1$. If $0 \leqslant g \leqslant f$, then

$$\langle \mu P^n, g \rangle / \langle \mu P^n, f \rangle$$
$$= \langle P^n g, f \rangle / \langle P^n f, f \rangle = \langle P^n f, g \rangle / \langle P^n f, f \rangle \to \int g d\lambda \Big/ \int f d\lambda$$

when $d\nu/d\lambda = \left(\int g d\lambda \right)^{-1} g$. We now apply theorem 2.2.

COROLLARY 2.4. — *Let* P *be an aperiodic self-adjoint Harris operator. If for some « very small » function f we have* $\lim\sup \| P^{2n} f / \langle P^{2n} f, f \rangle \|_\infty < \infty$, *then for every « small » functions g, h and probabilities* $\nu, \eta \ll m$ *we have* $\lim \langle \nu P^n, g \rangle / \langle \eta P^n, h \rangle = \int g d\lambda \Big/ \int h d\lambda$.

Proof. — P has a σ-finite invariant measure, and for any « small » function h, $\int h d\lambda < \infty$ (Horowitz [5]).

Now apply theorems 2.3 and 2.5.

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