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Renewal theorem and Markov chains

by

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RÉSUMÉ. — Une nouvelle démonstration des théorèmes de renouvellement de Orey-Feller-Blackwell est donnée; elle utilise les propriétés des fonctions harmoniques d'un processus markovien *ad hoc.*

We will explore a remark by Feller [*1*], a remark unfortunately not initially remarked by the author, concerning a Markov process associated with the renewal process. From considerations of the asymptotic properties of this Markov process we obtain the Orey-Feller-Blackwell renewal theorems in a unified « simple » way.

Consider a sequence of iid random variables $T_1, T_2, \ldots$ with distribution functions F concentrated on $(0, \infty)$, that is $F(0) = 0$. Following Feller [*2*] we may have a non-negative variable S_0 with a proper distribution G and we put

$$S_n = S_0 + T_1 + \ldots + T_n$$

The renewal process S_n is called pure if $S_0 = 0$ and delayed otherwise (for brevity a delayed process with starting distribution G will be called a G-process as opposed to a pure process). Also for any $t > 0$ there is a unique subscript N_t [*3*] such that $S_{N_t} < t \leq S_{N_t+1}$ (note this definition for the stopping time $N_t + 1$ differs slightly from Feller's). We define the excess (waiting) time at t as $S_{N_t+1} - t$ and we define

$Y^t(x)$ = the probability the excess time at t for the pure process $\leq x$,
$Y^t_G(x)$ = the probability the excess time at t for the G-process $\leq x$,
$u(t)$ = expected number of renewals for the pure process by t,

and

$V_G(t)$ = expected number of renewals for the G-process by t.

We now consider $R_+ = [0, \infty]$ as the state space of a Markov process where the transition probabilities are defined as follows. For $A \in \mathscr{B}(\mathscr{R}_+)$, the Borel sets on $\mathscr{R}_+$, $t > 0$ and $x \in \mathscr{R}_+$

$$
\begin{aligned}
P_t(x, A) &= \Pr(\text{the excess time at } t \text{ for a } \delta_x\text{-process} \in A) && \text{if } x < t \\
P_t(x, A) &= 1 \quad \text{if } x \in t + A && \text{if } x \geq t \\
P_t(x, A) &= 0 \quad \text{if } x \notin t + A && \text{if } x \geq t \\
P_0(x, A) &= 1 \quad \text{if } x \in A \\
P_0(x, A) &= 0 \quad \text{if } x \notin A.
\end{aligned}
$$

(The renewal δ_x-process gives S_0 distribution δ_x where δ_x means the distribution where all the probability is concentrated at x). In more picturesque language we say $P_t(x, A)$ is the probability of starting at x and with variable steps of length T finally jumping right over t and into the set $t+A$. Of course if $x \geq t$ we are already passed t and we only ask if we are already in $t + A$. Also we note that $P_t(x, A) = \Pr$ (the excess time for the pure process at time $t - A$) since these probabilities are invariant under translation.

It is quickly seen that for all $x \in \mathscr{R}_+$, $t \geq 0$, $A \to P_t(x, A)$ is a probability on $\mathscr{B}(\mathscr{R}_+)$. Also for all $A \in \mathscr{B}(\mathscr{R}_+)$, $x \to P_t(x, A)$ is measurable. Thus we need only establish the Chapmann-Kolmogorov relation to show that $(P_t)_{t \geq 0}$ is a semi-group of transition probabilities.

LEMMA 1.

$$P_{t_1+t_2}(x, A) = \int_0^\infty P_{t_1}(x, ds) P_{t_2}(s, A).$$

Proof. — For $x < t_1$.

$P_{t_1+t_2}(x, A) = \Pr$ (the excess time for the δ_x-process at $t_1 + t_2 \in A$). Conditioning on the excess time at t_1 we have

$$
\begin{aligned}
P_{t_1+t_2}(x, A) &= \int_0^\infty \Pr(\text{excess time for } \delta_x\text{-process at } t_1 + t_2 \in A \mid \text{excess time for } \delta_x\text{-process at } t_1 = s) dY^{t_1-x}(s) \\
&= \int_0^{t_2} \Pr(\text{excess time for } \delta_s\text{-process at } t_2 \in A) dY^{t_1-x}(s) \\
&\quad + \int_{t_2}^\infty \chi_{\{s \in t_2 + A\}} dY^{t_1-x}(s) \text{ by definition of } P_{t_2}(s, A).
\end{aligned}
$$

Finally we notice that if $I_y = [0, y]$ then $P_t(x, I_y) = Y^{t-x}(y)$. $x < t$. Thus for $x < t$

$$P_{t_1+t_2}(x, A) = \int_0^\infty P_{t_2}(s, A)P_{t_1}(x, ds).$$

For $t_1 \leq x < t_1 + t_2$.

$$\begin{aligned} P_{t_1+t_2}(x, A) &= \Pr (\text{the excess time for } \delta_{x-t_1}\text{-process at } t_2 \in A) \\ &= P_{t_2}(x - t_1, A) \\ &= \int_0^\infty \delta_{x-t_1}(ds)P_{t_2}(s, A). \end{aligned}$$

Now for $x \geq t_1$, $P_{t_1}(x, x - t_1) = 1$ by definition and we see that for $t_1 \leq x < t_1 + t_2$

$$P_{t_1+t_2}(x, A) = \int_0^\infty P_{t_2}(s, A)P_{t_1}(x, ds).$$

Lastly for $t_1 + t_2 \leq x$.

$$\begin{aligned} P_{t_1+t_2}(x, A) &= 1 \quad \text{if} \quad x - t_1 - t_2 \in A \\ &= 0 \quad \text{if} \quad x - t_1 - t_2 \notin A. \end{aligned}$$

Now $P_{t_1}(x, ds) = \delta_{x-t_1}(ds)$, which gives

$$\int_0^\infty P_{t_2}(s, A)P_{t_1}(x, ds) = P_{t_2}(x - t_1, A).$$

Since $x - t_1 \geq t_2$, we have

$$\begin{aligned} P_{t_2}(x - t_1, A) &= 1 \quad \text{if} \quad (x - t_1) - t_2 \in A \\ &= 0 \quad \text{if} \quad (x - t_1) - t_2 \notin A. \end{aligned}$$

This is precisely $P_{t_1+t_2}(x, A)$. Thus for $x \geq t_1 + t_2$

$$P_{t_1+t_2}(x, A) = \int_0^\infty P_{t_2}(s, A)P(x, ds),$$

and we have the result for all x.

Given the initial distribution on the state space (or delay distribution if you like) we may now construct a Markov process $(X_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathscr{F}, P_G)$, where G is the delay distribution, such that

$$P_G(X_{t_n} \in A/X_{t_0} = x_0, \ldots, X_{t_k} = x_k, t_0 <, \ldots, t_k < t_n) = P_{t_n - t_k}(x_{t_k}, A).$$

We note in passing that

$$P_G(X_t \in [0, y]) = Y_G^t(y).$$

We are interested in the limits of these distributions as t goes to infinity.

We approach this problem by looking at the properties of the harmonic functions on the related space-time process. We pick an arbitrary sequence $I = \{ t_n \}_{n=0}^{\infty}\ 0 = t_0 < t_1 < t_2 < \ldots < t_n \ldots t_n \to \infty$. The random variables X_{t_n} are by construction all measurable with respect to the measure space $(\Omega, \mathscr{F}, P_G)$ and we define $\mathscr{F}^n$ to be the σ-field generated by $\{ X_{t_m}, m \geq n \}$, and let $\mathscr{F}^{\infty} = \bigcap_{n=0}^{\infty} \mathscr{F}^n$. A random variable Y is (following [*4*]) a tail random variable if there exists a sequence (f_n) of $\mathscr{F}$-measurable functions on Ω such that $Y = f_n(X_{t_n}, X_{t_{n+1}}, \ldots)$. If f can be chosen independently of n so that

$$Y = f(X_{t_n}, X_{t_{n+1}}, \ldots)$$

then Y is called invariant. If $Y = \chi_A$ where Y is a tail random variable (respectively an invariant random variable) then A is called a tail event (respectively an invariant event). The class of all tail events (invariant events) form a σ-field called the tail σ-field (the invariant σ-field). We will call the pair $(X_{t_n}, t_n)_{n=0}^{\infty}$, the space-time chain associated with (X_{t_n}) and we see that we can construct a Markov chain on $\mathscr{R}_+ \times I$ with transition probabilities

$$\tilde{P}((x, t_n), \{ A, t_{n+1} \}) = P_{t_{n+1}-t_n}(x, A), x \in \mathscr{R}_+, t_n, t_{n+1} \in I, A \in \mathscr{B}(\mathscr{R}_+).$$

A real valued measurable function g on $\mathscr{R} \times I$ is called space-time harmonic if

$$g(x, t_n) = \int P_{t_{n+1}-t_n}(x, ds) g(s, t_{n+1}) x \in \mathscr{R}_+, t_n, t_{n+1} \in I.$$

We should note that the restriction of our space-time process to the times t is not obligatory and is only done to avoid certain measurability questions and to apply more easily the theorems in [*4*].

We now examine the space-time harmonic functions which are bounded.

LEMMA 2. — If h is a bounded space-time harmonic function then

$$h(x, t_n) = h(x - (t_{n+1} - t_n), t_{n+1}), \qquad x \geq t_{n+1} - t_n.$$

Proof. — A space-time harmonic function satisfies

$$h(x, t_n) = \int_0^{\infty} h(s, t_{n+1}) \tilde{P}((x, t_n), ds \times t_{n+1})$$

$$= \int_0^{\infty} h(s, t_{n+1}) P_{t_{n+1}-t_n}(x, ds).$$

Now for $t_{n+1} - t_n \leq x$, $P_{t_{n+1}-t_n}(x, ds) = \delta_{x-(t_{n+1}-t_n)}(ds)$. Therefore

$$h(x, t_n) = h(x - (t_{n+1} - t_n), t_{n+1}) \qquad \text{for} \qquad t_{n+1} - t_n \leq x.$$

We may extend this equality right along the diagonal.

We need now only consider $h(x, 0)$ to establish results on all space-time.

LEMMA 3. — (*a*) If F is periodic with period p $h(x, 0)$ is constant on the periods of F. That is $h(x, 0) = h(x + p, 0) \forall x \geq 0$.

(*b*) If F is aperiodic $h(x, 0)$ is almost surely constant with respect to Lebesgue measure.

(*c*) If F isaperiodic and there exists some convolution F^{*n} of F which is not singular with respect to Lebesgue measure then $h(x, 0)$ is constant.

Proof.

$$h(0, t_n) = \int_0^\infty \tilde{P}\{(0, t_n), ds \times t_{n+1}\} h(s, t_{n+1})$$
$$= \int_0^\infty P_{t_{n+1}-t_n}(0, ds) h(s, t_{n+1}).$$

From the equality along the diagonals we have

$$h(t_n, 0) = \int_0^\infty P_{t_{n+1}-t_n}(0, ds) h(s + t_{n+1}, 0).$$

Letting $q(x) = h(x, 0)$ we have

$$q(t_n) = \int_0^\infty P_{t_{n+1}-t_n}(0, ds) q(s + t_{n+1})$$

where $q(t_n)$ is measurable and bounded. Since the t_n were choosen arbitrarily we have in general that

$$q(x) = \int_0^\infty P_t(0, ds) q(x + t + s) \qquad \forall t, x \geq 0. \tag{1}$$

(*a*) Now if F has period p we may set $t = p$ to get $q(x) = \int_0^\infty q(x+s) dF(s)$ and by Choquet-Deny [5] we have $q(x)$ is constant on the periods of F.

(*b*) If F is aperiodic we remark that by regularization of (Eq. 1) we have functions $q^\varepsilon(x)$ which are bounded, uniformly continuous solutions of (Eq. 1) and which tend almost surely to $q(x)$ (w. r. t. Lebesgue measure) as $\varepsilon \to 0$. Also letting $t \to 0$ in $q^\varepsilon(x) = \int_0^\infty P_t(0, ds) q^\varepsilon(x + t + s)$ we have $q^\varepsilon(x) = \int_0^\infty q^\varepsilon(x + s) dF(s)$ and again by [5] $q^\varepsilon(x)$ is constant. Thus $q(x)$ is constant almost surely.

(*c*) We note that if F^{*n} is not singular w. r. t. Lebesgue measure we can pick a $\bar{t}$ sufficiently big so that $F_e^{*n}(\bar{t}) > 0$ (F_e^{*n} is the absolutely continuous part of F^{*n}). Hence $P_{\bar{t}}(0, ds)$ is not singular and we may employ the method in Note 1 to prove that $q(x)$ is constant.

We remark that the aperiodic distribution F giving mass $\frac{1}{2}$ to 1 and $\sqrt{2}$ and the function

$$
\begin{aligned}
h(x, 0) &= 1 \quad \text{if} \quad x = p + q\sqrt{2}\ p.q \text{ positive integers} \\
&= 0 \quad \text{if} \quad x \neq p + q\sqrt{2} \text{ for any } p.q.
\end{aligned}
$$

generate a counter example to an extension of Lemma 3 (*b*).

It is now useful to distinguish between the cases when F is arithmetic (without loss of generality having period 1) and when F is non-arithmetic. Henceforth if F is arithmetic the t_n and x's will be restricted to the positive integers. Moreover for the arithmetic case we will define the measure m to be the counting measure on the positive integers while in the non-arithmetic case m will be Lebesgue measure. With this in mind we note that the distribution,

$$E(r) = \frac{1}{\mu}\int_0^r (1 - F(y))m(dy)$$
$$\mu = \text{mean of F}$$

called the equilibrium distribution, gives a stationary delayed renewal process [*6*]. Hence $Y_E^t(r) = E(r)\forall t \geq 0$, and seewe X^t is stationary w. r. t. P_E.

The utility of the space-time chain is seen in the following.

PROPOSITION 1. — The following conditions are equivalent.

(*a*) For all probability measures μ and ν on $\mathscr{B}(\mathscr{R}_+)$

$$\lim_{n\to\infty} \| \mu P_{t_n} - \nu P_{t_n} \| \to 0. \quad \text{Where} \quad \mu P(A) = \int P(x, A)\mu(dx) \quad \text{for} \quad A \in \mathscr{B},$$

and $\| \nu \|$ is the total variation of ν.

(*b*) The only bounded space-time harmonic functions are constants.

Proof. — The proof is an adaptation of the proof for Proposition 4.3 in [*4*].

THEOREM 1. — If (*a*) F is arithmetic and α is any probability measure on the positive integers or if (*b*) F^{n*} is not singular w. r. t. Lebesgue measure for some n and α is a probability measure on $[0, \infty)$ then $\lim_{t\to\infty} \| \alpha P_t - e \| = 0$ (in the arithmetic case t is an integer), where e is the equilibrium measure having distribution $E(r)$.

Proof. — In the arithmetic case we restrict attention to the positive integers. Hence by Lemmas 3 (*a*) and 3 (*c*) respectively we see the bounded space-time harmonic functions are constant. Hence by Proposition 1, we have for probability measures α and e

$$\lim_{t\to\infty} \| \alpha P_t - eP_t \| = 0.$$

However

$$eP_t(A) = \int_0^\infty P_t(x, A)dE(x)$$
$$= e(A)$$

and we have the result. Q. E. D.

COROLLARY 1 (Feller). — If F is arithmetic with period 1 and $F(0) = 0$ then

$$\lim_{n\to\infty} \Pr \{ \text{renewal at } n \} = 1/\mu (\mu \text{ is mean of F})$$

Proof.

$$\Pr \{ \text{renewal at } n \} = P_n(0, 0).$$

Thus

$$\lim_{n\to\infty} \{ \text{renewal at } n \} = \frac{1}{\mu} e \{ 0 \} = 1/\mu, \text{ from Theorem 1.}$$

If F is aperiodic but not absolutely continuous w. r. t. Lebesgue measure we must be a little more subtle.

PROPOSITION 2. — If F is aperiodic and if G is a probability measure which is absolutely continuous with respect to Lebesgue measure ($G \ll m$) then the tail field of $\{ X_{t_n} \}$ is trivial w. r. t. P_G.

Proof. — Let A be a tail event of $\{ X_{t_n} \}$. Then by Proposition 4.1 of [*4*] A is an invariant event of $\{ X_{t_n}, t_n \}$. Consider the function defined on space-time by

$$h_A(Z) = \tilde{E}_Z(A) \qquad (Z \text{ of the form } Z = (x, t_n).$$

By Proposition 4.2 of [*4*] $h_A(Z)$ is bounded and space-time harmonic. Also

$$h_A(X_n, t_n) = \tilde{E}_{[X_n, t_n]}[A] = \tilde{E}_\eta[\theta^n A \mid \mathscr{F}_n] = \tilde{E}_\eta[A \mid \mathscr{F}_n]$$

(where again η is the initial distribution given by

$$\eta \{ Axt_0 \} = G(A).$$

Thus $h_A(X_n, t_n)$ converges a. s. w. r. t. $\tilde{P}_\eta$ to A. Moreover by Lemma 3 (*b*) we know $h_A(Z) = C$ (constant) a. s. w. r. t. Lebesgue measure.

$$\text{Let } B = \{ (x, t) \mid h(x, t) \neq C \}.$$

Since $G \ll m$ we have $\tilde{P}_\eta \{ (X_n, t_n) \in B \} = 0$. Thus a. s.-$\tilde{P}_\eta$ $h_A(X_n, t_n)$ converges to C and hence A is trivial w. r. t. $\tilde{P}_\eta$. Thus A is trivial w. r. t. P_G.

THEOREM 2. — If F is aperiodic and if α is a probability measure which is absolutely continuous w. r. t. Lebesgue measure then

$$\lim_{t_n \to \infty} \| \alpha P_{t_n} - e \| = 0.$$

Proof. — By Proposition 2, the tail field of $\{ X_{t_n} \}$ is trivial w. r. t. P_e (since the equilibrium distribution $e \ll m$). Hence as in Theorem 4.1 of [*4*].

$$\lim_{t_n \to \infty} \sup_{A \in F^n} | P_e(A \cap B) - P_e(A)P_e(B) | = 0 \quad \text{for every} \quad B \in \mathscr{F}. \qquad (2)$$

$$\text{Let } A = \{ X_{t_n} \in F \} ; \qquad B = \{ X_{t_0} \in G \}.$$

Thus we have

$$\lim_{t_n \to \infty} \sup_{F} \left| \int_G P_{t_n}(x, F) e(dx) - e(F)e(G) \right| = 0.$$

Now $\alpha \ll m$. Also $m \ll e$ on $\{ x \mid F(x) < 1 \}$ by the construction of e. Moreover by truncation there exists a probability measure $\tilde{\alpha} \ll m$ concentrated on $[0, T]$ such that $\| \tilde{\alpha} - \alpha \| < \varepsilon$ for any ε. Hence

$$\| \tilde{\alpha} P_{t_n} - \alpha P_{t_n} \| < \varepsilon.$$

Also by the construction of our chain we see $\tilde{\alpha} P_T$ is concentrated on $\{ x \mid F(x) < 1 \}$, and hence for t_n sufficiently large $\tilde{\alpha} P_{t_n} \ll e$. We may therefore consider the case $\alpha \ll e$. Let $\mathscr{F}(x) = \frac{d\alpha}{de}(x)$ and hence $\int \mathscr{F} \, de = 1$. Let $\mathscr{F}_k(x)$ be step functions such that $\mathscr{F}_k \uparrow \mathscr{F}$. Thus by Eq. 2 we have

$$\lim_{t_n \to \infty} \sup_{F} \left| \int P_{t_n}(x, F) \mathscr{F}_k(x) e(dx) - \int \mathscr{F}_k(x) e(dx) \cdot e(F) \right| = 0.$$

However we remark that

$$\left| \int P_{t_n}(x, F) \mathscr{F}_k(x) e(dx) - \int P_{t_n}(x, F) \mathscr{F}(x) e(dx) \right|$$
$$\leq \int P_{t_n}(x, F) \, | \mathscr{F}_k(x) - \mathscr{F}(x) | \, e(dx) \leq \int | \mathscr{F}_k(x) - \mathscr{F}(x) | \, e(dx).$$

This uniform bound implies

$$\lim_{t_n\to\infty} \sup_{F} \left| \int P_{t_n}(x, F)\mathscr{F}(x)e(dx) - \int \mathscr{F}(x)e(dx)\cdot e(F) \right| = 0.$$

and
$$\lim_{t_n\to\infty} \sup_{F} \left| \int P_{t_n}(x, F)\alpha(dx) - e(F) \right| = 0$$

and
$$\lim_{t_n\to\infty} \sup_{F} | \alpha P_{t_n}(F) - e(F) | = 0$$

and hence

$$\lim_{n\to\infty} \| \alpha P_{t_n} - e \| = 0.$$

COROLLARY 2. — For F aperiodic $Y^t(x)$ converges weakly to $E(x)$.

Proof. — For any interval $[x, y]$ we pick $0 < \varepsilon < x$ and by considering the possible trajectories of our process, as well as the translation invariance of our transition probabilities we see

$$P_{t_n}(s, [x + \varepsilon, y]) \leq P_{t_n}(0, [x, y]) \leq P_{t_n}(s, [x, y + \varepsilon]) + P_{t_n}(s, [0, \varepsilon])$$

for $0 \leq s \leq \varepsilon$. Consider the uniform probability measure u_ε on $[0, \varepsilon]$. Integrating our inequality we have

$$u_\varepsilon P_{t_n}[x + \varepsilon, y] \leq P_{t_n}(0, [x, y]) \leq u_\varepsilon P_{t_n}[x, y + \varepsilon] + u_\varepsilon P_{t_n}[0, \varepsilon].$$

Hence by Theorem 2. We have for all $\varepsilon < x$

$$e[x + \varepsilon, y] \leq \underline{\lim}\, P_{t_n}(0, [x, y]) \leq \overline{\lim}\, P_{t_n}(0, [x, y]) \leq e[x, y + \varepsilon] + e[0, \varepsilon].$$

Thus by the continuity of $E(x)$ we have $\lim_{t_n\to\infty} P_{t_n}(0, [x, y]) = e[x, y]$ which implies

$$\lim_{t_n\to\infty} Y^{t_n}(x) = E(x).$$

COROLLARY 3 (Blackwell). — If F is aperiodic then $\lim_{t\to\infty} u(t+h)-u(t)=h/\mu$.

Proof. — $u(t + h) - u(t)$ is the expected number of renewals in $[t, t + h]$. If we condition on the excess time at t we have

$$u(t + h) - u(t) = \int_0^{h-} \{1 + u(h - s)\}\, dY^t(s)$$

(if we step over t and land at $t + s < t + h$ then we restart the process at $t + s$ and we take on the average $u(h - s)$ more steps before $t + h$). Next we remark that $1 + u(h - s)$ is a decreasing function of s and hence has a countable number of discontinuities on $[0, h]$. Moreover by Corollary 2, Y^t converges weakly to E and $E \ll m$. Hence the set of discontinuities of $1 + u(h - s)$ has measure 0 w. r. t. E and we have

$$\lim_{t \to \infty} u(t + h) - u(t) = \int_0^h (1 + u(h - s)) dE(s)$$

(see Theorem 5.2 (*iii*) in [7] for example).

However since E is the equilibrium measure

$$\int_0^h (1 + u(h - s) dE(s) = V_E(h) = h/\mu \qquad \text{(see [6])}.$$

Remarks. — If F has infinite mean we have no equilibrium probability measure. We still have however an invariant measure e with distribution

$$E(x) = \int_0^x (1 - F(s)) m(ds).$$

We have for every $\gamma > 0$ and every $x \in R_+$

$$\frac{P_{t_n}(x, F)}{e(F) + \gamma} \nrightarrow 0 \quad \text{uniformly in} \quad F \in \mathscr{B}(R_+) \quad \text{as} \quad t_n \nrightarrow \infty.$$

(See Theorem 7.3 in [4]). We have all the obvious extensions of Corollaries 1, 2 and 3.

If our distribution F is not concentrated on the half line but has $\mu > 0$ we can still consider the excess time at $t \in R_+$ (since the walk drifts to the right) and we can prove our theorems based on the strict ladder distribution [8].

NOTE 1. — If h is measurable and bounded and satisfies

$$h(x) = \int_0^\infty h(x + y) dF(y) \tag{3}$$

and if F^{*n} is not singular (w. r. t. Lebesgue measure), then $h(x)$ is constant.

Proof. — By Choquet and Deny's lemma $h(x)$ is almost surely constant (say C). Subtracting C from both sides of (Eq. 3) we have

$$g(x) = \int_0^\infty g(x + y) dF(y) \tag{4}$$

where $g(x) = h(x) - C$ is almost surely 0.

By convolution of (Eq. 4) we have

$$g(x) = \int_0^\infty g(x + y) dF^{*n}(y).$$

Now $F^{*n} = G = pG_e + qG_d$ where G_e generates a measure absolutely continuous with respect to Lebesgue measure and G_d generates a measure singular with respect to Lebesgue measure [7] and $p > 0, q > 0, p + q = 1$. Now

$$g(x) = p\int_0^\infty g(x+y)dG_e(y) + q\int_0^\infty g(x+y)dG_d(y)$$
$$= q\int_0^\infty g(x+y)dG_d(y).$$

since $g(x) = 0$. a. e.

Again using

$$g(x+y) = \int_0^\infty g(x+y+y_1)dG(y_1)$$

we have

$$g(x) = q^2\int_0^\infty g(x+y)dG_d^{*2}(y)$$
$$= q^n\int_0^\infty g(x+y)dG_d^{*n}(y) \downarrow 0.$$

Thus $g(x) = 0$ everywhere and hence $h(x) = C$ everywhere.

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