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Distribution of crossings of level K in a busy cycle of the M/G/1 queue

by

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SUMMARY. — For the M/G/1 queuing system the distribution of the number of crossings from above of a level K by the virtual delay time during a busy cycle is derived; the busy cycle may be finite or infinite. Also the Laplace-Stieltjes transform of the distribution of the time of the first such crossing (if there is such a crossing) is obtained; similarly for the distribution of the time between two successive crossings from above during a busy cycle, and for the distribution of the time between the last crossing from above and the end of the busy cycle, if there is such a last crossing.

1. INTRODUCTION

Consider a single server queue M/G/1 with traffic intensity a. The average interarrival time is denoted by α, so that the average service time is αa; $B(t)$ will represent the distribution function of the service times, with $B(0+) = 0$; further

$$\beta(\rho) \stackrel{\text{def}}{=} \int_0^\infty e^{-\rho t} dB(t), \qquad \text{Re } \rho \geqq 0.$$

The virtual delay time at time t of the queueing process is denoted by $\underline{v}_t$. In the figure below a realisation of $\underline{v}_t$ during a busy cycle $\underline{c}$ is shown. This realisation of $\underline{v}_t$ has two crossing from above of level K during the busy cycle $\underline{c}$; here K is a positive constant.

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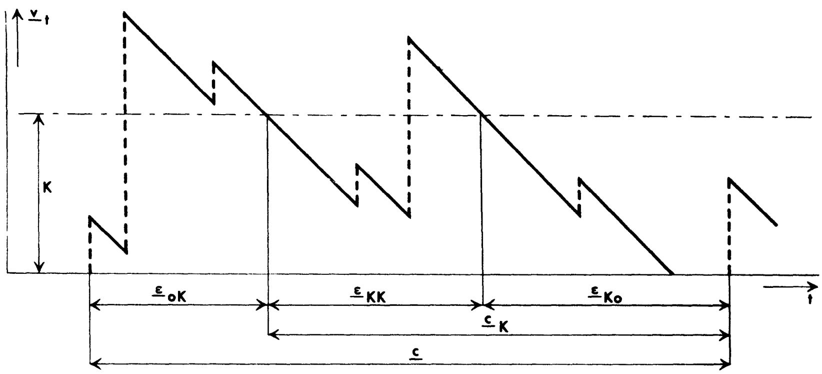

FIG. 1.

In the figure a number of other variables are indicated: $\underline{c}_{\mathrm{K}}$ denotes the length of a busy cycle starting with an initial virtual delay time equal to K; $\underline{\varepsilon}_{0\mathrm{K}}$ is the time from the start of the busy cycle until the moment of the first crossing from above of level K, if during the busy cycle $\underline{v}_t$ exceeds K; $\underline{\varepsilon}_{\mathrm{KK}}$ is the time between the moments of two successive crossings from above of level K during a busy cycle; finally, $\underline{\varepsilon}_{\mathrm{K}0}$ is the time between the moment of the last crossing from above and the end of the busy cycle, if there is a last crossing from above.

The main purpose of this paper is to study the distribution of $\underline{\Pi}_{\mathrm{K}}$, the number of crossings of level K from above by $\underline{v}_t$ during a busy cycle.

2. THE DISTRIBUTIONS OF $\underline{\varepsilon}_{0\mathrm{K}}$, $\underline{\varepsilon}_{\mathrm{KK}}$ AND $\underline{\varepsilon}_{\mathrm{K}0}$

From the queueing theory of the M/G/1 system it is well known (cf. [*1*]) that the function of η

$$(2.1) \qquad \beta(\eta) + (\eta - \rho)\alpha - 1, \qquad \mathrm{Re}\ \rho \geqq 0,$$

has one zero $\delta(\rho)$ with Re $\delta(\rho) > 0$ if Re $\rho > 0$, or if $a > 1$ and Re $\rho \geqq 0$; if $\rho = 0$ and $a < 1$ it has a single zero at $\eta = 0$, while for $\rho = 0$, $a = 1$ it has a zero of multiplicity two at $\eta = 0$. The zero $\delta(\rho)$ is a continuous function of ρ for Re $\rho \geqq 0$. Moreover, for Re $\rho \geqq 0$

$$(2.2) \qquad \mathrm{E}\left\{ e^{-\rho \underline{c}} \right\} = \frac{1 + \alpha\rho - \alpha\delta(\rho)}{1 + \alpha\rho},$$

$$(2.3) \qquad \mathrm{E}\left\{ e^{-\rho \underline{c}_{\mathrm{K}}} \right\} = \frac{1}{1 + \alpha\rho}\, e^{-\delta(\rho)\mathrm{K}}.$$

The stochastic variables $\underline{\varepsilon}_{0K}$, $\underline{\varepsilon}_{KK}$ and $\underline{\varepsilon}_{K0}$ are defined by

(2.4) $\underline{\varepsilon}_{0K} \overset{\text{def}}{=} \inf_{0<t<\underline{c}} \{ t : \underline{v}_{t-} > K > \underline{v}_{t+} \mid \sup_{0<\tau<\underline{c}} \underline{v}_\tau > K, \underline{v}_{0-} = 0, \underline{v}_{0+} > 0 \},$

$\overset{\text{def}}{=} \infty$, if no such finite t exists for the given conditions;

(2.5) $\underline{\varepsilon}_{KK} \overset{\text{def}}{=} \inf_{0<t<\underline{c}_K} \{ t : \underline{v}_{t-} > K > \underline{v}_{t+} \mid \sup_{0<\tau<\underline{c}_K} \underline{v}_\tau > K, \underline{v}_{0+} = K \},$

$\overset{\text{def}}{=} \infty$, if no such finite t exists for the given conditions;

(2.6) $\underline{\varepsilon}_{K0} \overset{\text{def}}{=} \inf_{0<t<\underline{c}_K} \{ t : \underline{v}_{t-} = 0 < \underline{v}_{t+} \mid \sup_{0<\tau<\underline{c}_K} \underline{v}_\tau < K, \underline{v}_{0+} = K \}.$

Define for $z \geqq 0$,

(2.7) $$\mathrm{F}(z) \overset{\text{def}}{=} \Pr\{ \underline{c} < z, \sup_{0<t<\underline{c}} \underline{v}_t < K \mid \underline{v}_{0-} = 0, \underline{v}_{0+} > 0 \},$$

(2.8) $$\mathrm{H}(z) \overset{\text{def}}{=} \Pr\{ \underline{c}_K < z, \sup_{0<\tau<\underline{c}_K} \underline{v}_\tau < K \mid \underline{v}_{0+} = K \},$$

(2.9) $$f(\rho) = \int_0^\infty e^{-\rho z} d\mathrm{F}(z) \quad , \quad h(\rho) \overset{\text{def}}{=} \int_0^\infty e^{-\rho z} d\mathrm{H}(z), \ \mathrm{Re}\,\rho \geqq 0.$$

Expressions for $f(\rho)$ and $h(\rho)$ have been derived in [2] (cf. (5.20) and (5.21) of [2]). These relations are : for Re $\rho \geqq 0$, Re $\eta >$ Re $\delta(\rho)$

(2.10) $$f(\rho) = \frac{1}{1+\alpha\rho} \frac{\dfrac{1}{2\pi i}\displaystyle\int_{C_\eta} e^{\eta K} \frac{\beta(\eta)}{\beta(\eta) + (\eta - \rho)\alpha - 1} d\eta}{\dfrac{1}{2\pi i}\displaystyle\int_{C_\eta} e^{\eta K} \frac{1}{\beta(\eta) + (\eta - \rho)\alpha - 1} d\eta},$$

(2.11) $$h(\rho) = \frac{1}{1+\alpha\rho} \left[\frac{1}{2\pi i} \int_{C_\eta} e^{\eta K} \frac{\alpha d\eta}{\beta(\eta) + (\eta - \rho)\alpha - 1} \right]^{-1};$$

here the integrals are to be read as

$$\frac{1}{2\pi i} \int_{C_\eta} \ldots d\eta = \lim_{b\to\infty} \int_{R-ib}^{R+ib} \ldots d\eta \quad , \quad R = \mathrm{Re}\,\eta.$$

In [2] it has been shown that, if $\underline{v}_t = u$ at some moment t with $0 \leqq u \leqq K$, then with probability one the system reaches the empty state in a finite time or passes level K (from below) in a finite time.
Consequently,

(2.12) $$f(0) = \Pr\{ \underline{c} < \infty, \sup_{0<\tau<\underline{c}} \underline{v}_\tau < K \mid \underline{v}_{0-} = 0, \underline{v}_{0+} > 0 \},$$

$$1 - f(0) = \Pr\{ \sup_{0<\tau<\underline{c}} \underline{v}_\tau \geqq K \mid \underline{v}_{0-} = 0, \underline{v}_{0+} > 0 \},$$

(2.13) $$h(0) = \Pr\{\underline{c}_K < \infty, \sup_{0<\tau<\underline{c}_K} \underline{v}_t < K \mid \underline{v}_{0+} = K\},$$

$$1 - h(0) = \Pr\{\sup_{0<\tau<\underline{c}_K} \underline{v}_\tau \geqq K \mid \underline{v}_{0+} = K\}.$$

A finite busy cycle which has at least one crossing of level K is the sum of one $\underline{\varepsilon}_{OK}$, of one $\underline{\varepsilon}_{KO}$ and of a random number of variables $\underline{\varepsilon}_{KK}$. Since for the M/G/1 system the interarrival times are all independent, and negative exponentially distributed with the same parameter it follows that every crossing of level K from above by $\underline{v}_t$ is a regeneration point; consequently the variables $\underline{\varepsilon}_{OK}$, $\underline{\varepsilon}_{KK}$ and $\underline{\varepsilon}_{KO}$ defined in (2.4), ..., (2.6) are independent variables. This conclusion leads to the following relations: for Re $\rho \geqq 0$,

(2.14) $$\mathrm{E}\{e^{-\rho \underline{c}}\} = f(\rho) + (1 - f(0))\mathrm{E}\{e^{-\rho \underline{\varepsilon}_{OK}}\}\mathrm{E}\{e^{-\rho \underline{c}_K}\},$$

(2.15) $$\mathrm{E}\{e^{-\rho \underline{c}_K}\} = h(\rho)\sum_{m=0}^{\infty} [(1 - h(0))\mathrm{E}\{e^{-\rho \underline{\varepsilon}_{KK}}\}]^m = \frac{h(\rho)}{1 - (1 - h(0))\mathrm{E}\{e^{-\rho \underline{\varepsilon}_{KK}}\}}.$$

From (2.14) and (2.15) and from (2.2) and (2.3) we have for Re $\rho \geqq 0$,

(2.16) $$\mathrm{E}\{e^{-\rho \underline{\varepsilon}_{OK}}\} = \{1 + \alpha\rho - \alpha\delta(\rho) - (1 + \alpha\rho)f(\rho)\}\frac{e^{K\delta(\rho)}}{1 - h(0)},$$

(2.17) $$\mathrm{E}\{e^{-\rho \underline{\varepsilon}_{KK}}\} = \frac{1}{1 - h(0)}\{1 - (1 + \alpha\rho)h(\rho)e^{K\delta(\rho)}\},$$

whereas from (2.6), (2.8) and (2.13)

(2.18) $$\mathrm{E}\{e^{-\rho \underline{\varepsilon}_{KO}}\} = \frac{h(\rho)}{h(0)}.$$

These relations describe the distributions of the variables $\underline{\varepsilon}_{OK}$, $\underline{\varepsilon}_{KK}$ and $\underline{\varepsilon}_{KO}$. It follows

(2.19) $$\Psi_{OK} \stackrel{\text{def}}{=} \Pr\{\underline{\varepsilon}_{OK} < \infty\} = \{1 - \alpha\delta(0) - f(0)\}\frac{e^{K\delta(0)}}{1 - f(0)},$$

(2.20) $$\Psi_{KK} \stackrel{\text{def}}{=} \Pr\{\underline{\varepsilon}_{KK} < \infty\} = \frac{1}{1 - h(0)}\{1 - h(0)e^{K\delta(0)}\}.$$

It is of some interest to consider the relations obtained above for the case $a \leqq 1$.

If $a \leqq 1$ then $\delta(0) = 0$ so that

(2.21) $$\Psi_{OK} = 1, \qquad \Psi_{KK} = 1.$$

If $a < 1$ then it is well known that the actual waiting time of the queueing process M/G/1 has a unique stationary distribution $W(t)$ of which the Laplace-Stieltjes transform is given by the Polaczek-Khinchin formula

$$(2.22)\qquad \int_{0-}^{\infty} e^{-\rho t} dW(t) = \frac{(1-a)\alpha\rho}{\beta(\rho)+\alpha\rho-1}\ ,\quad \operatorname{Re}\rho \geqq 0\ ,\quad a<1.$$

Define two nonnegative stochastic variables w and τ with joint distribution

$$(2.23)\qquad \Pr\{\underline{w} < t_1, \underline{\tau} < t_2\} = W(t_1)B(t_2), \qquad 0 \leqq t_1, \qquad 0 \leqq t_2,$$

so that $\underline{w}$ and $\underline{\tau}$ are independent by definition.

Using the inversion formula for the Laplace-Stieltjes transform it follows from (2.7), (2.8), (2.12), (2.13), (2.22) and (2.24)

$$(2.24)\qquad f(0) = \frac{\Pr\{\underline{w}+\underline{\tau} < K\}}{\Pr\{\underline{w} < K\}}, \qquad 1-f(0) = \frac{\Pr\{K-\underline{\tau} \leqq \underline{w} < K\}}{\Pr\{\underline{w} < K\}},$$

$$(2.25)\qquad h(0) = \frac{\Pr\{\underline{w} = 0\}}{\Pr\{\underline{w} < K\}}, \qquad 1-h(0) = \frac{\Pr\{0 < \underline{w} < K\}}{\Pr\{\underline{w} < K\}}.$$

The relations for $f(0)$ and $h(0)$ for all $a > 0$ have been found also by Takacs [3], who uses combinatorial methods.

3. DISTRIBUTION OF CROSSING OF LEVEL K

The distribution of $\underline{\Pi}_K$, the number of crossings from above of level K by $\underline{v}_t$, can be obtained by making use of the renewal property of these crossing points. It is necessary, however, to distinguish between the finite and infinite busy cycles if $a > 1$.

It follows

$$(3.1)\quad \Pr\{\underline{\Pi}_K = m, \underline{c} < \infty\} = f(0), \qquad m=0,$$
$$= (1-f(0))\Psi_{OK}\{(1-h(0))\Psi_{KK}\}^{m-1}h(0), \quad m=1,2,\ldots,$$

$$(3.2)\quad \Pr\{\underline{\Pi}_K = m, \underline{c} = \infty\} = (1-f(0))(1-\Psi_{OK}), \qquad m=0,$$
$$= (1-f(0))\Psi_{OK}\{(1-h(0))\Psi_{KK}\}^{m-1}(1-h(0))(1-\Psi_{KK}), \quad m=1,2,\ldots.$$

From (3.1) and (3.2) it is easily verified that

$$\sum_{m=0}^{\infty} \Pr\{\underline{\Pi}_K = m\} = 1,$$

so that the number of crossings of level K by $\underline{v}_t$ during a busy period is finite with probability one.

From (3.1) and (3.2) it is found

$$(3.3)\quad \mathrm{E}\{\underline{\Pi}_{\mathrm{K}}\,|\,\underline{c}<\infty\}=\frac{1-\alpha\delta(0)-f(0)}{h(0)}\,\frac{e^{-\mathrm{K}\delta(0)}}{1-\alpha\delta(0)},$$

$$\mathrm{Var}\{\underline{\Pi}_{\mathrm{K}}\,|\,\underline{c}<\infty\}=\frac{1-\alpha\delta(0)-f(0)}{h^2(0)}\,\frac{e^{-2\mathrm{K}\delta(0)}}{1-\alpha\delta(0)}\left\{1-h(0)e^{\mathrm{K}\delta(0)}+\frac{f(0)}{1-\alpha\delta(0)}\right\},$$

and if $a>1$

$$(3.4)\quad \mathrm{E}\{\underline{\Pi}_{\mathrm{K}}\,|\,\underline{c}=\infty\}=\frac{1-\alpha\delta(0)-f(0)}{h(0)}\,\frac{1-e^{-\mathrm{K}\delta(0)}}{\alpha\delta(0)},$$

$$\mathrm{Var}\{\underline{\Pi}_{\mathrm{K}}\,|\,\underline{c}=\infty\}=\frac{1-\alpha\delta(0)-f(0)}{h^2(0)}\,\frac{\{1-e^{-\mathrm{K}\delta(0)}\}e^{-\mathrm{K}\delta(0)}}{\alpha\delta(0)}$$
$$\times\left\{1-h(0)e^{\mathrm{K}\delta(0)}+\frac{1-f(0)}{\alpha\delta(0)}-\frac{1-\alpha\delta(0)-f(0)}{\alpha\delta(0)}e^{\mathrm{K}\delta(0)}\right\}.$$

If $a<1$ then $\underline{c}$, $\underline{\varepsilon}_{\mathrm{OK}}$ and $\underline{\varepsilon}_{\mathrm{KK}}$ are finite with probability one; in this case it follows from (3.1), (3.3), (2.24) and (2.25)

$$(3.5)\quad \Pr\{\underline{\Pi}_{\mathrm{K}}=m\}=\frac{\Pr\{\underline{w}+\underline{\tau}<\mathrm{K}\}}{\Pr\{\underline{w}<\mathrm{K}\}},\qquad m=0,$$

$$=\frac{\Pr\{\mathrm{K}-\underline{\tau}\leqq\underline{w}<\mathrm{K}\}}{\Pr\{\underline{w}<\mathrm{K}\}}\left[\frac{\Pr\{0<\underline{w}<\mathrm{K}\}}{\Pr\{\underline{w}<\mathrm{K}\}}\right]^{m-1}\frac{\Pr\{\underline{w}=0\}}{\Pr\{\underline{w}<\mathrm{K}\}},\qquad m=1,2,\ldots,$$

$$\mathrm{E}\{\underline{\Pi}_{\mathrm{K}}\}=\frac{1}{1-a}\Pr\{\mathrm{K}-\underline{\tau}\leqq\underline{w}<\mathrm{K}\},$$

$$\mathrm{Var}\{\underline{\Pi}_{\mathrm{K}}\}=\frac{1}{(1-a)^2}\Pr\{\mathrm{K}-\underline{\tau}\leqq\underline{w}<\mathrm{K}\}[\Pr\{0<\underline{w}<\mathrm{K}\}$$
$$+\Pr\{\underline{w}+\underline{\tau}<\mathrm{K}\}].$$

In [*2*] the distribution of $\underline{\Phi}_{\mathrm{K}}$, the number of overflows during a wet period (or busy cycle) of an M/G/1 dam with finite capacity K has been derived. It appears that for $a<1$, $\underline{\Phi}_{\mathrm{K}}$ and $\underline{\Pi}_{\mathrm{K}}$ have the same distribution.

Putting

$$\beta=a\alpha,$$

so that β is the mean service time then for the M/M/1 queueing system the results of this section specialize as follows

$$f(0)=\frac{1-e^{-(1-a)\mathrm{K}/\beta}}{1-ae^{-(1-a)\mathrm{K}/\beta}},\qquad h(0)=\frac{1-a}{1-ae^{-(1-a)\mathrm{K}/\beta}},\qquad a\neq 1,$$

$$=\frac{\mathrm{K}/\beta}{1+\mathrm{K}/\beta},\qquad =\frac{1}{1+\mathrm{K}/\beta},\qquad a=1,$$

$$\Psi_{\mathrm{OK}}=\Psi_{\mathrm{KK}}=1-\alpha\delta(0)=a^{-1},\qquad a>1,$$
$$=1,\qquad a\leqq 1;$$

for $a < 1$,

$$\mathrm{E}\{\underline{\Pi}_{\mathrm{K}}\} = e^{-(1-a)\mathrm{K}/\beta}, \qquad \mathrm{Var}\{\underline{\Pi}_{\mathrm{K}}\} = \frac{1+a}{1-\mathrm{a}} e^{-(1-a)\mathrm{K}/\beta}\{1 - e^{-(1-a)\mathrm{K}/\beta}\};$$

for $a = 1$,

$$\mathrm{E}\{\underline{\Pi}_{\mathrm{K}}\} = 1, \qquad \mathrm{Var}\{\underline{\Pi}_{\mathrm{K}}\} = 2\mathrm{K}/\beta;$$

for $a > 1$,

$$\mathrm{E}\{\underline{\Pi}_{\mathrm{K}} \mid \underline{c} < \infty\} = e^{(1-a)\mathrm{K}/\beta},$$

$$\mathrm{Var}\{\underline{\Pi}_{\mathrm{K}} \mid \underline{c} < \infty\} = \frac{1+a}{1-a} e^{(1-a)\mathrm{K}/\beta}\{e^{(1-a)\mathrm{K}/\beta} - 1\},$$

$$\mathrm{E}\{\underline{\Pi}_{\mathrm{K}} \mid \underline{c} = \infty\} = \frac{1 - e^{(1-a)\mathrm{K}/\beta}}{a-1},$$

$$\mathrm{Var}\{\underline{\Pi}_{\mathrm{K}} \mid \underline{c} = \infty\} = \frac{\{e^{(1-a)\mathrm{K}/\beta} - a\}\{e^{(1-a)\mathrm{K}/\beta} - 1\}}{(1-a)^2}.$$

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