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Planar Permutation Graphs

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Planar Permutation Graphs (1)

by

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INTRODUCTION

One of the best known graphs in all of graph theory is the Petersen graph, shown in Figure 1, named after the Swedish mathematician. Petersen [3] proved that every cubic bridgeless graph contains a 1-factor. He also showed that not every such graph is 1-factorable by exhibiting a counterexample which has become classic.

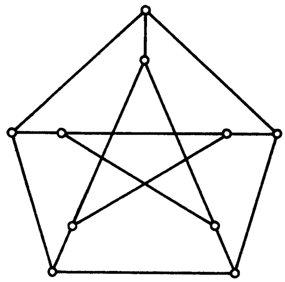

FIG. 1. — The Petersen graph.

This graph consists of two disjoint cycles of length 5 (a pentagon and a pentagram) joined by 5 additional lines. This is made clear in Figure 2 *a* as we see how the two cycles are linked. This graph is then redrawn in Figure 2 *b* to produce a labeling of the familiar Petersen graph shown in Figure 1.

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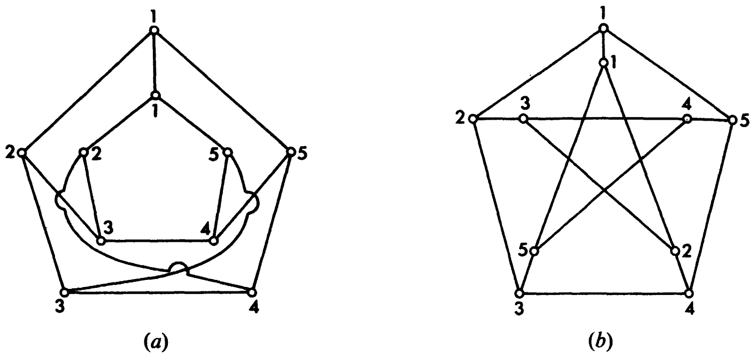

FIG. 2. — The first permutation graph.

The points of each of the two copies of C_5 (the cycle of length 5) are labeled cyclically 1 through 5, with the points of the exterior cycle joined to the points of the interior cycle according to the rule

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}.$$

Thus the numbers on the top row of the permutation α correspond to the exterior cycle and those in the second row to the interior cycle with a point i on the exterior cycle joined to a point j on the interior cycle if $\alpha(i) = j$. Therefore, the Petersen graph can be regarded as two disjoint copies of C_5 joined according to this permutation α. Looking at the Petersen graph from this viewpoint, we are led to the following, more general concept.

Consider two identical disjoint copies of a labeled graph G with p points. The *α-permutation graph* $P_\alpha(G)$ consists of these two copies of G along with p additional lines joining these graphs according to a given permutation α on $N_p = \{1, 2, \ldots, p\}$. A graph H is *a permutation graph* if there exists a labeled graph G, having p points, and a permutation α on the set N_p such that $H = P_\alpha(G)$. We note that the graph $P_\alpha(G)$ depends not only on the choice of the permutation α but on the particular labeling of G as well. In fact, there are four permutation graphs which can be obtained from C_5: the Petersen graph which is known to be nonplanar (see [*1*]), the pentagonal prism (Figure 3 *a*) which is planar, and the two nonplanar graphs in Figure 3 *b*. Certainly, more than one permutation may result in the same permutation graph; indeed, there are 10 permutations which produce the Petersen graph as there are for the pentagonal prism, and each of the graphs in Figure 3 *b* can be obtained from 50 permu-

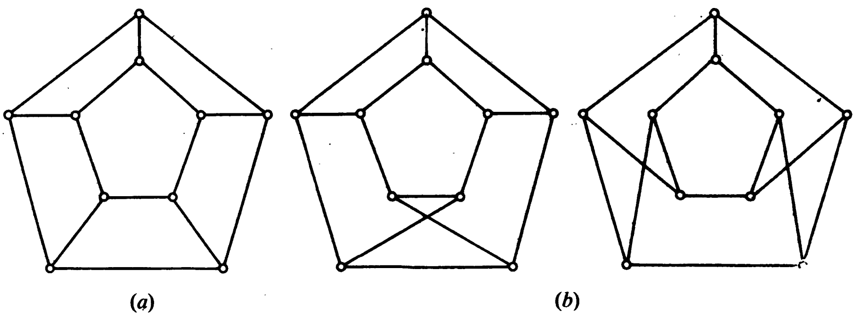

FIG. 3. — The other permutation graphs of C_5.

tations. For example, if the points of C_5 are labeled cyclically, then the prism results from either the identity permutation or the cyclic permutation (1 2 3 4 5).

PLANAR PERMUTATION GRAPHS

Although C_5 is obviously planar, we have seen that some permutation graphs of C_5 are planar and others nonplanar. We develop a criterion for a permutation graph of a cycle as well as any other 2-connected graph to be planar.

A graph G is *homeomorphic from* H if it is possible to insert points of degree two into the lines of H to produce G (A graph G_1 is *homeomorphic with* G_2 if there exists a graph G_3 which is homeomorphic from both G_1 and G_2). It is convenient to state in the following form the well-known theorem of Kuratowski [*2*]. A graph is planar if and only if it contains no subgraph homeomorphic from the complete graph K_5 or from the complete bigraph $K_{3,3}$.

Given that $P_\alpha(G)$ is planar, it is certainly clear that G is also planar since it is a subgraph of $P_\alpha(G)$. Furthermore, G must have the added property that it can be embedded in the plane so that all its points bound some region of G. Without loss of generality, we may assume this region to be exterior. If G did not have this property, then no matter how the points of the two copies of G are joined in the forming of a permutation graph, at least one of the added lines must cross some line in one of the copies of G so that $P_\alpha(G)$ would be nonplanar. A connected graph having at least 3 points which can be embedded in the plane so that all its points lie on the exterior region will be called *outerplanar*. A disconnected graph is considered outerplanar if all its components are. Of course every outerplanar nonseparable graph is hamiltonian. It is easy to see that all graphs with less than 6 lines are outerplanar. While all graphs with 6 lines are planar, there are two connected graphs among them which fail to be outerplanar, namely, the complete graph K_4 and the « theta-graph » $K_{2,3}$.

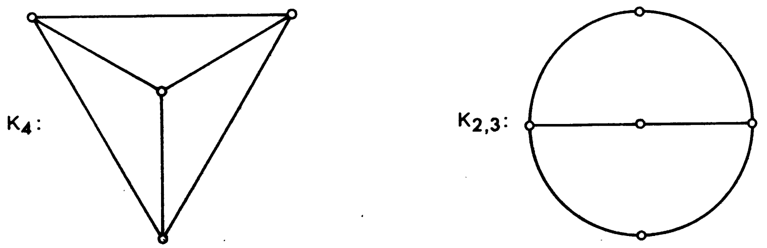

FIG. 4. — The prototypes of non-outerplanar graphs.

THEOREM 1. A graph G is outerplanar if and only if it contains no subgraph homeomorphic from K_4 or $K_{2,3}$.

Proof. It is obvious that G is not outerplanar if it contains a subgraph homeomorphic from K_4 or from $K_{2,3}$.

To prove the converse, let G contain no subgraph homeomorphic from K_4 or from $K_{2,3}$ but assume G is not outerplanar. If G is nonplanar, then, by Kuratowski's theorem, it contains a subgraph homeomorphic from K_5 or from $K_{3,3}$, so it certainly contains one homeomorphic from K_4 or from $K_{2,3}$. Hence, G is planar. Since G is not outerplanar, it must contain a block B with more than two points, which is not outerplanar. Embed B in the plane so that a maximum number of points lie on the exterior cycle Z. Since Z is not hamiltonian, there is at least one point which lies in the interior of Z. Let u be a point interior to Z which is adjacent to a point v_1 on Z. Since B is a block, deg $u \geq 2$. Hence, there is a path P from u to some other point v_2 on Z. There are two cases to consider.

Case 1. Points v_1 and v_2 are consecutive on Z.

In this case, some point of P different from v_2 must have degree at least 3; otherwise, the path could be transferred outside of Z to produce a planar embedding of B having a longer exterior cycle. Thus, there is a path from a point of P, say w, to a point v_3 of Z not containing any other point of P (See Figure 5 *a*). The lines of Z and the 3 paths from w to Z induce a subgraph of B homeomorphic from K_4.

Case 2. Points v_1 and v_2 are not consecutive on Z.

Clearly, the lines of Z and those of the path through u from v_1 to v_2 constructed in Case 1 induce a subgraph homeomorphic from $K_{2,3}$ (see Figure 5 *b*), completing the proof.

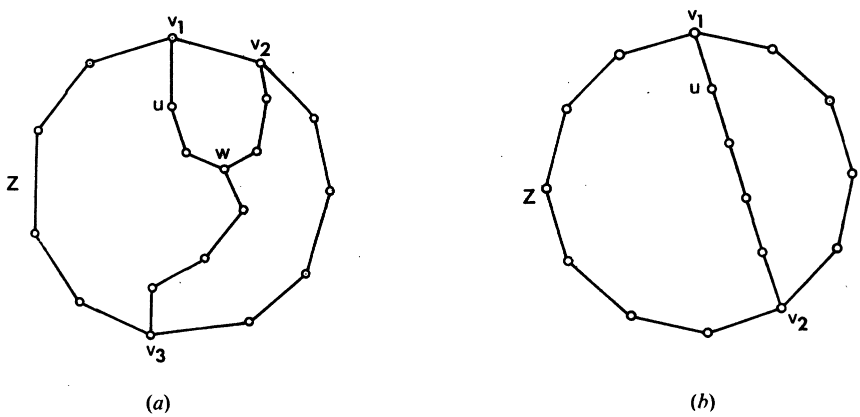

FIG. 5. — Homeomorphs of K_4 and $K_{2,3}$.

We now return to the Petersen graph and ask which permutations applied to a given cycle or, more generally, to a given nonseparable outerplanar graph G result in a planar permutation graph. This may depend on how G is labeled. Since G is outerplanar, it can be embedded in the plane so that its exterior cycle Z is hamiltonian. If we label the points of Z cyclically, 1 through p, then we say that G is « cyclically labeled ». It is convenient to assume that every nonseparable outerplanar graph is cyclically labeled. One sees that every nonseparable outerplanar graph has $2p$ cyclic labelings, p labelings of the points in cyclic clockwise order and p more counterclockwise.

Obviously, the number of ways of constructing a planar permutation graph from two disjoint copies of a nonseparable outerplanar graph with p points is the same as that of obtaining a planar permutation graph from two copies of C_p, namely $2p$, the number of permutations is the dihedral group D_p of degree p generated by the two permutations:

$$\alpha_1 = \begin{pmatrix} 1 & 2 \dots & p-1 & p \\ 2 & 3 \dots & p & 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 & 2 \dots & p \\ p & p-1 \dots & 1 \end{pmatrix}$$

When $\alpha \in D_p$, we say α is *dihedral.*

We now summarize these observations.

LEMMA. Given a nonseparable outerplanar graph G, the permutation graph $P_\alpha(G)$ is planar if and only if α is dihedral.

Combining Theorem 1 and the lemma, we arrive at a characterization of planar permutation graphs of nonseparable graphs.

THEOREM 2. The permutation graph $P_\alpha(G)$ of a nonseparable graph G is planar if and only if G is outerplanar and α is dihedral.

In general, the conclusion of Theorem 2 does not follow for connected outerplanar graphs with cutpoints, showing the necessity of the hypothesis that G is nonseparable. For example, consider the chain W_n with n points. It is easy to verify that all 24 permutation graphs of W_4 are planar, not just those obtained from the 8 permutations in D_4. This is not so for C_5 since $P_\alpha(C_5)$ is not planar when

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{pmatrix},$$

for it contains a subgraph homeomorphic from $K_{3,3}$.

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