ANNALES DE L'I. H. P., SECTION B

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Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's brownian motion of several parameters

Annales de l'I. H. P., section B, tome 3, nº 2 (1967), p. 121-226

<http://www.numdam.org/item?id=AIHPB_1967__3_2_121_0>

Ann. Inst. Henri Poincaré, Vol. III, n° 2, 1967, p. 121-225.

Section B : *Calcul des Probabilités et Statistique.*

Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters

by

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Dedicated to Paul Levy

CHAPTER I

LÉVY-SCHOENBERG KERNELS ON CERTAIN HOMOGENEOUS SPACES

§ 1. — Introduction [2].

Paul Lévy's studies of recent years have been much concerned with what he calls Brownian motion of several parameters. Specifically, he studies a Gaussian process $\{ \xi(a); a \in \mathrm{R}^d \}$ with parameter a running over Euclidean d-space R^d, which is centered, i. e.

$$\mathrm{E}(\xi(a)) = 0, \quad a \in \mathrm{R}^d \text{ [3]}, \tag{1.1}$$

Ce mémoire a reçu le prix fondé à l'occasion de l'élection de M. Paul Lévy à l'Académie des Sciences.

(1) Research supported in part by the National Science Foundation, grant No GP-3978, and a Sloan fellowship.

(2) A résumé of some of the results of this paper was presented at the fifth Berkeley symposium on mathematical statistics and probability theory, held at the University of California, Berkeley, in June, 1965. The résumé will appear in the proceedings of the symposium.

(3) E stands for the expectation.

and whose covariance $\mathbf{E}(\xi(a)\xi(b))$ is given by the kernel f on $\mathrm{R}^d \times \mathrm{R}^d$ defined by

$$(1.2) \qquad f(a, b) = \frac{1}{2}(|a| + |b| - |a - b|), \quad a, b \in \mathrm{R}^d,$$

where $|a|$ is the Euclidean length of $a \in \mathrm{R}^d$.

Lévy has also devoted some attention to a Gaussian process $\{\xi(a), a \in \mathrm{S}^d\}$ with the parameter a running over the d-sphere S^d, such that the process is again centered,

$$(1.3) \qquad \mathbf{E}(\xi(a)) = 0, \; a \in \mathrm{S}^d,$$

and has its covariance $\mathbf{E}(\xi(a)\xi(b))$ described by the kernel f on $\mathrm{S}^d \times \mathrm{S}^d$

$$(1.4) \qquad f(a, b) = \frac{1}{2}(d(a, o) + d(b, o) - d(a, b)), \; a, b \in \mathrm{S}^d,$$

where o is an arbitrary point of S^d and $d(x, y)$ is the distance between $x, y \in \mathrm{S}^d$ computed in the intrinsic geometry of S^d. See e. g. Lévy [*1*]-[*4*] (4).

The kernels on $\mathrm{R}^d \times \mathrm{R}^d$ (or $\mathrm{S}^d \times \mathrm{S}^d$) given by (1.2), (1.4) are both real valued, symmetric and positive definite; namely, given $a_1, \ldots, a_n \in \mathrm{R}^d$ (or S^d) and real numbers $\alpha_1, \ldots, \alpha_n$, one has

$$(1.5) \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j f(a_i, a_j) \geq 0.$$

As is well known, this last property is necessary and sufficient for the existence of a process whose covariance is the kernel f. For (1.2), this property follows from a result of Schoenberg [*1*], and indeed, in Lévy [*1*] it is used to establish the existence of the process $\{\xi(a), a \in \mathrm{R}^d\}$. On the other hand, in the case of (1.4), Lévy constructed the process $\{\xi(a), a \in \mathrm{S}^d\}$ by means of « white noise » integrals and then checked explicitly that its covariance was (1.4), proving thereby that (1.4) must be positive definite. As far as I know no analytical proof of this fact has appeared.

The processes mentioned above have many interesting properties, and there seem to be intimate connections, not yet transparent, between their study and various problems in harmonic analysis and differential equations.

Some indications may be found in McKean [*1*]. It is therefore natural to ask for a more comprehensive description of kernels on general spaces, which embody the main features of (1.2), (1.4), and to seek to develop a

(4) Square brackets [ ] refer to the bibliography at the end of this paper.

theory for the corresponding Gaussian processes. In Lévy [4, p. 309] some tentative remarks were made about the desirability of doing this, but there matters have stood, probably largely due to the ad hoc nature of the proofs showing that (1.2), (1.4) are positive definite.

The present paper grew out of an attempt to remedy this state of affairs and to abstract for study the relevant features of (1.2), (1.4). I study, on a fair variety of spaces, a class of kernels which possess these features, and of which (1.2), (1.4) are special cases. It will be seen that their study is amenable to methods of harmonic analysis and that a more or less complete description of the kernels in this class can be obtained by these methods.

The spaces on which this class of kernels is most efficiently studied are those that are germane to harmonic analysis. Thus one may treat, on the one hand, locally compact abelian groups, Hilbert and nuclear spaces; on the other hand the methods extend also to homogeneous spaces of compact groups, including as special cases all the compact symmetric spaces of É. Cartan's list, as well as to Riemannian symmetric spaces of non-compact type, whether exceptional or not.

Quite apart from yielding results of such generality, the abstract formulation seems to bring into focus the basic similarity underlying the problem in all these different situations.

However, while the earlier and relatively more complete part of the theory may be developed for all of the above spaces, the later (and more interesting) part of the theory may be attempted only when the underlying space has some differential structure. For this reason, general (non-Lie) abelian groups will not be considered except in the first part of this paper, the later preoccupation being with Riemannian symmetric spaces.

In § 2, a class of kernels, termed Lévy-Schoenberg kernels, is defined on the homogeneous space of a separable topological group. The definition is suggested by (1.2), (1.4). The problem of describing kernels of this class is then quickly seen to lead to the problem of describing a class of functions on the group, which are positive definite in the usual sense of harmonic analysis, and which, moreover, are infinitely divisible in a sense to be defined. In § 3, this problem is solved completely when the space is either a homogeneous space of a connected locally compact abelian separable group, a homogeneous space of an arbitrary arcwise connected compact group or an arbitrary connected Riemannian symmetric space of non-compact type. This permits the complete description of Lévy-Schoenberg kernels in each of these cases.

In § 4 examples are presented illustrating the theory. Incidentally, an analytical proof, free of the construction of white noise integrals, that (1.4) is positive definite will emerge from one class of examples.

In § 5 a general procedure is indicated whereby new Lévy-Schoenberg kernels may be constructed from a given one by what Bochner has called subordination. This enables one to point out analogues of (1.2), (1.4) in all these above situations.

In § 6 we permit ourselves a digression, and point out the connection between a Lévy-Schoenberg kernel and a certain semigroup of probability measures on an object which in each of the above cases is a Fourier-analytic dual to the space in question.

The succeeding sections of this paper are devoted to studying the Gaussian processes of which a given Lévy-Schoenberg kernel is the covariance. These processes have the given homogeneous space as their parameter set. Here our results are not as general as could be wished, since some assumption about the underlying homogeneous space seems to be necessary to get a clean theory. The basic assumption will be that the space carries a differential structure, and other assumptions will be made where appropriate. In § 7 we have a fairly general result on the continuity of sample functions of a process defined by a Lévy-Schoenberg kernel, which enables us in § 8 to obtain an orthogonal decomposition of some processes. § 9 is devoted to some remarks about the application of this decomposition to the study of the Markov property. Finally, in § 10, we conclude with some remarks about various questions to which this work seems to lead naturally.

§ 2. — Lévy-Schoenberg kernels.

Throughout this paper G will denote a separable topological group, K a closed subgroup of G. Further assumptions regarding G and K will be made in the appropriate context. G/K will denote the homogeneous space of cosets of the form xK, $x \in$ G. G/K is endowed with the quotient topology. x, y, z... will denote elements of G; a, b, c... will denote elements of G/K whenever their nature as cosets is not relevant, but when it is relevant, elements of G/K will be denoted by xK, yK, ..., etc. G acts on G/K in the usual way, $x(y\mathrm{K}) = (xy)\mathrm{K}$, $x, y \in$ G.

Definition 2.1. — By a kernel on a topological space S is meant a continuous complex valued function on S $\times$ S (5).

(5) All topological spaces are assumed Hausdorff.

DEFINITION 2.2. — A kernel on G/K will be said to be *positive definite* if for any $a_1, \ldots, a_n \in G/K$ and complex numbers $\alpha_1, \ldots, \alpha_n$, one has

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \bar{\alpha}_j f(a_i, a_j) \geq 0. \tag{2.1}$$

As is well known, if f is positive definite then $f(a, b) = \overline{f(b, a)}$, $a, b \in G/K$, i. e. f is Hermitian. Further, if a kernel f is a *real* valued kernel on G/K, then f is positive definite if and only if f satisfies (2.1) for any $a_1, \ldots, a_n \in G/K$ and *real* numbers $\alpha_1, \ldots, \alpha_n$. Note also that if φ is a continuous complex valued function on G then one gets a kernel f on G by letting $f(x, y) = \varphi(x^{-1}y)$, and f will be positive definite if and only if φ is a positive definite function on G in the usual sense of harmonic analysis, i. e., given $x_1, \ldots, x_n \in G$, and complex numbers $\alpha_1, \ldots, \alpha_n$, φ satisfies

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \bar{\alpha}_j \varphi(x_i^{-1} x_j) \geq 0. \tag{2.2}$$

These facts will be used without much comment below.

DEFINITION 2.3. — A kernel f on G/K is said to be a *Lévy-Schoenberg kernel* if it has the following properties.

(2.3) $\qquad f(a, b) = f(b, a) \qquad a, b \in G/K.$

(2.4) There exists a point $o \in G/K$ such that $f(a, o) = 0$ for all $a \in G/K$.

(2.5) The kernel r on G/K given by $r(a, b) = f(a, a) + f(b, b) - 2f(a, b)$ is invariant under G, i. e. $r(xa, xb) = r(a, b)$ for all $x \in G$, $a, b \in G/K$.

(2.6) f is positive definite.

Because of (2.3) and (2.6), a Lévy-Schoenberg kernel is automatically *real* valued. Note that both (1.2), (1.4) are Lévy-Schoenberg kernels. In the case of (1.2), R^d is to be viewed as the homogeneous space of the group G of all proper rigid motions of R^d, modulo the subgroup K consisting of proper rotations about 0. $K \cong SO(d)$. Thus the kernel of (1.2) lives on G/K. f clearly fulfills (2.3) and (2.4), the origin of R^d serving as the point o required by (2.4). The kernel $r(a, b)$ is just $| a - b |$ in this case, and this is surely invariant when a, b are subjected to the same rigid motion $x \in G$. Finally (2.6) is just Schoenberg's theorem quoted above. As for the kernel f of (1.4), one views S^d as the homogeneous space of

$G = SO(d+1)$ modulo $K \cong SO(d) =$ the subgroup of G which leaves fixed the point $o \in S^d$. (2.3), (2.4) are easily checked. In this case $r(a, b)$ is just $d(a, b)$ and this certainsly satisfies (2.5). Finally (2.6) is a consequence of Lévy's construction quoted above.

The reader must have noticed that the kernel r is the « polarization » of the kernel f and, when f is a Lévy-Schoenberg kernel, f can be recovered from r. Indeed one has in that case $r(a, o) = f(a, a)$, $r(b, o) = r(o, b) = f(b, b)$ so,

$$f(a, b) = \frac{1}{2}(r(a, o) + r(b, o) - r(a, b)) \tag{2.7}$$

which is reminiscent of (1.2), (1.4). It follows that any information about a Lévy-Schoenberg kernel is contained in its polarized kernel r. The classification of Lévy-Schoenberg kernels proceeds in the present paper via the classification of the corresponding polarized kernels.

The following simple and well-known facts will be used repeatedly in this paper and are therefore elevated to the status of a lemma.

LEMMA 2.4. — If f, g are positive definite kernels on a topological space S and t is a positive real number, then the kernels $tf, fg, f+g$ are all positive definite. If $\{h_n\}$ is a sequence of positive definite kernels converging pointwise to a continuous h on $S \times S$ then h is also a positive definite kernel. In particular, if f is positive definite then so is $\exp f$.

The fact that fg is positive definite goes back to a result of Schur to the effect that the tensor product of two nonnegative Hermitian operators on a finite dimensional complex vector space is again a nonnegative Hermitian operator. The rest of the assertions of the lemma are trivial and the proof is omitted.

The following observation is the key to the considerations of the first part of this paper.

LEMMA 2.5. — Suppose r is a real valued kernel on a topological space S such that $r(a, b) = r(b, a)$, $a, b \in S$, and suppose there is a point $o \in S$ such that $r(o, o) = 0$. Let f be defined in terms of r by

$$f(a, b) = \frac{1}{2}(r(a, o) + r(b, o) - r(a, b)). \tag{2.8}$$

Then f is positive definite if and only if for each $t \geq 0$, the kernel θ_t defined by $\theta_t(a, b) = \exp - tr(a, b)$ is positive definite.

Proof. — The first half of the proof goes as in Lévy [1, p. 276]. Suppose first that θ_t is positive definite. Since f is real valued, the positive definiteness of f will follow if one can show that for any finite signed Borel measure μ with compact support on S, one has

$$\int_S\int_S f(a, b)d\mu(a)d\mu(b) \geq 0. \tag{2.9}$$

But now, in view of the fact that $f(a, o) = 0 = f(b, o)$, one may modify the total mass of μ at will by placing point masses at o without changing the left side of (2.9). Thus (2.9) need only be proved under the additional hypothesis that the total mass of μ is 0. But then,

$$\begin{aligned} \int_S\int_S f(a, b)d\mu(a)d\mu(b) &= \frac{1}{2}\int_S\int_S (r(a, o) + r(b, o) - r(a, b))d\mu(a)d\mu(b) \\ &= -\frac{1}{2}\int_S\int_S r(a, b)d\mu(a)d\mu(b). \end{aligned} \tag{2.10}$$

So (2.9) is equivalent to showing that for each signed Borel measure of total mass 0 and compact support on S, one has

$$\int_S\int_S r(a, b)d\mu(a)d\mu(b) \leq 0. \tag{2.11}$$

Now, because θ_t is positive definite, there results

$$\begin{aligned} 0 &\leq \int_S\int_S \exp - tr(a, b)d\mu(b) \\ &= \int_S\int_S (1 - tr(a, b) + o(t^2))d\mu(a)d\mu(b) \\ &= -t\int_S\int_S r(a, b)d\mu(a)d\mu(b) + o(t^2) \qquad t \downarrow 0 \end{aligned} \tag{2.12}$$

where the fact that $o(t^2)$ is uniformly small on the compact support of μ was used. It is clear that (2.12) implies (2.11).

Conversely, suppose f defined by (2.8) is positive definite. Then for $t \geq 0$, tf and hence $\exp tf$ is positive definite. Now, if $\alpha_1, \ldots, \alpha_n$ are complex numbers, and $a_1, \ldots, a_n \in S$, then

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\bar{\alpha}_j \exp - tr(a_i, a_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\bar{\alpha}_j \exp - t(r(a_i, o) + r(a_j, o) - 2f(a_i a_j)). \end{aligned} \tag{2.13}$$

settingBut $\beta_i = \alpha_i \exp - tr(a_i, o), i = 1, \ldots, n$, this becomes

$$(2.14) \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_i\bar{\beta}_j \exp 2tf(a_i, a_j)$$

which is ≥ 0 because $\exp 2tf$ is positive definite for each $t \geq 0$. This concludes the proof.

COROLLARY 2.6. — If f is a Lévy-Schoenberg kernel on G/K then its polarized kernel r

$$(2.15) \qquad r(a, b) = f(a, a) + f(b, b) - 2f(a, b)$$

has the following properties

$$(2.16) \qquad r(a, b) = r(b, a) \qquad a, b \in G/K$$

$$(2.17) \qquad r(a, a) = 0 \qquad a \in G/K$$

$$(2.18) \qquad r(xa, xb) = r(a, b) \qquad a, b \in G/K, x \in G$$

(2.19) For each $t \geq 0$, the kernel θ_t defined by $\theta_t(a, b) = \exp - tr(a, b)$ is positive definite.

Conversely, if r is any real valued kernel on G/K satisfying (2.16)-(2.19), and for any point $o \in G/K$ the kernel f is defined by

$$(2.20) \qquad f(a, b) = \frac{1}{2}(r(a, o) + r(b, o) - r(a, b)).$$

Then f is a Lévy-Schoenberg kernel.

The point o might as well be taken as the identity coset eK of G/K. This will always be done below.

Suppose r is a kernel on G/K, satisfying (2.16)-(2.19) and let $\theta(a, b) = \exp - r(a, b)$. Then $\theta(xa, xb) = \theta(a, b), x \in G, a, b \in G/K$. This makes it possible to « lift » θ to a function on G. Namely if Φ is the function on G defined by $\Phi(x) = \theta(xK, eK)$, then $\theta(yK, zK) = \Phi(z^{-1}y), y, z \in G$, Φ is continuous and it is trivial to verify that the properties (2.16)-(2.19) of r imply

$$(2.21) \qquad \Phi(x^{-1}) = \Phi(x) \qquad x \in G$$

$$(2.22) \qquad \Phi(e) = 1$$

Where e is the identity of G.

(2.23) $$\Phi(k_1 x k_2) = \Phi(x); \qquad x \in G; \qquad k_1, k_2 \in K.$$

(2.24) For each $t \geq 0$, Φ^t is a positive definite function on G. Also,

(2.25) $$\Phi^t(x) \to 1 \text{ as } t \downarrow 0.$$

Note that (2.21), (2.24) imply that Φ is real valued.

Conversely, if a continuous function Φ on G satisfies (2.21)-(2.25), then, *provided it admits a continuous logarithm*, it is easily shown that the kernel r on G/K defined by

(2.26) $$r(a, b) = -\log \Phi(z^{-1}y); \qquad a = yK, \qquad b = zK$$

enjoys all the properties (2.16)-(2.19).

All this makes the following definitions pertinent.

DEFINITION 2.7. — A complex valued function Φ on G is said to be K-*spherical* if for all $x \in G$, $k_1, k_2 \in K$, one has $\Phi(k_1 x k_2) = \Phi(x)$.

DEFINITION 2.8. — Φ is said to be *normalized* if $\Phi(e) = 1$.

DEFINITION 2.9. — A continuous complex valued function Φ on G is said to be *imbeddable* if for each $t \geq 0$ Φ^t is positive definite and $\Phi^t(x) \to 1$ for each $x \in G$ as $t \downarrow 0$.

Note that if Φ is imbeddable then it is positive definite, and can be imbedded in a continuous one parameter semigroup (under pointwise multiplication), of continuous positive definite functions on G; namely, the semigroup $\{ \Phi^t, t \geq 0 \}$.

Thus, apart from the question of existence of a continuous logarithm for Φ, *the problem of describing all Lévy-Schoenberg kernels on* G/K *has been reduced to the problem of finding all real valued, normalized,* K-*spherical, continuous imbeddable positive definite functions on* G, or equivalently of finding continuous one parameter semigroups (under pointwise multiplication), of real valued normalized K-spherical positive definite functions on G. For the present, it is expedient to ignore the requirement that Φ be real valued.

Just as in classical probability theory, the following definitions are pertinent.

DEFINITION 2.10. — The class of all K-spherical continuous normalized complex-valued imbeddable functions on G will be called *the class* I *for the pair* (G, K).

DEFINITION 2.11. — A continuous positive definite function Φ on G is said to be *infinitely divisible* if for each positive integer n, there exists a *continuous* positive definite function Φ_n on G such that $\Phi(x) = (\Phi^n(x))^n$, $x \in G$.

A continuous imbeddable function Φ is clearly infinitely divisible.

DEFINITION 2.12. — By the *class* $\mathfrak{D}$ *for the pair* (G, K) we mean the class of all complex valued continuous K-spherical, normalized infinitely divisible positive definite functions on G.

One of the main points of the first part of this paper will be that in all the cases of concern in the present instance, each function Φ in $\mathfrak{D}$ is automatically imbeddable and has a continuous logarithm. This will accomplish a description of Lévy-Schoenberg kernels.

The next section will be concerned with characterizing the class $\mathfrak{D}$ in the following cases.

Case I.

G = A connected locally compact separable abelian group
K = any closed subgroup of G.

Case II.

G = the group of all proper rigid motions of Euclidean space R^d
K = the subgroup of G consisting of rotations about 0
= $\mathbf{SO}(d)$.

Case III.

G = a compact arcwise connected group
K = a closed subgroup of G.

Case IV.

G = a non-compact connected semisimple Lie group with a finite center
K = a maximal compact subgroup of G.
In this case G/K is a Riemannian symmetric space of non-compact type.

As mentioned above, in each of these cases, it turns out that if $\Phi \in \mathfrak{D}$ then Φ never vanishes, and is imbeddable. Further, one can get a more or less explicit formula for Φ as $\Phi(x) = \exp - \Psi(x)$ where Ψ can be described quite precisely. It is then possible to isolate those Ψ for which Φ is real valued. If now one sets $r(y\mathrm{K}, z\mathrm{K}) = \Psi(z^{-1}y)$ for such Ψ, then r gives rise to a Lévy-Schoenberg kernel. One thus gets a complete description of Lévy-Schoenberg kernels in all the situations above.

Imitating a definition originally due to Schoenberg [2] (or see Herz [1]), the functions Ψ might be called (restrictedly) negative definite K-spherical functions on G. For, Ψ is real, K-spherical, $\Psi(x^{-1}) = \Psi(x)$, $\Psi(e) = 0$ and finally, given $x_1, \ldots, x_n \in G$ and real numbers $\alpha_1, \ldots, \alpha_n$ such that $\sum_{i=1}^{n} \alpha_i = 0$, one has $\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j\Psi(x_i^{-1}x_j) \leq 0$. Thus one may equivalently regard the present problem as that of characterizing such functions.

It must be clear to the reader that all this is intimately connected with problems analogous to the classical theory of infinitely divisible probability measures on R, the culmination of which is the Lévy-Khinchine formula of probability theory. Indeed, in the present paper, the results of Case I follow rather readily from the analogues of the Lévy-Khinchine formula for this situation as derived by Parthasarathy, Ranga Rao and Varadhan [1]. The results in Case II are obtained simply by « radializing » the classical Lévy-Khinchine formula for R^d. The results of Cases III, IV, require fresh work, and in each case, a sort of Lévy-Khinchine formula results. Of course this means that with each function in $\mathfrak{D}$ there is associated a semi-group of probability measures (as in the classical case), on an appropriate dual object for G/K. This will be pointed out below.

It will appear below (1.2) is a Lévy-Schoenberg kernel for case II, and (1.4) is such a kernel for a specialization of Case III. One gets in this way an analytical proof that (1.4) is positive definite. On the other hand, in Case IV no instances of Lévy-Schoenberg kernels have hitherto been known.

In each case, a Lévy-Schoenberg kernel f gives rise to a centered Gaussian process $\{ \xi(a), a \in G/K \}$ with covariance f. The kernels (1.2) have a somewhat distinguished place among Lévy-Schoenberg kernels in Case II and similarly distinctive instances will be pointed out below in Case IV.

While we have limited ourselves in this paper to the discussion of situations where G/K is locally compact, we would like to state that for some situations when G/K is not locally compact, one can obtain similar results. For example if G is a separable Hilbert space and K is a closed subspace, the class $\mathfrak{D}$ for (G, K) can be characterized. Similarly, if G is a nuclear space and K a closed subspace such that G/K is complete (and therefore nuclear), a characterization can be given for the class $\mathfrak{D}$. (If G/K is not complete, it is more natural for the description of the class $\mathfrak{D}$ to work with its completion). Some remarks about these two cases are made in § 3 at the end.

In all these cases, analogues of (1.2) can be singled out by means of subordination. These kernels then give rise to what one may decide to call Brownian motions on G/K, thus perhaps providing a general basis for Lévy's idea of defining Brownian motions whose parameter runs over a general finite or infinite dimensional space.

§ 3. — The class $\mathfrak{D}$.

CASE I. — Here let G be a connected locally compact separable abelian group and K any closed subgroup. Obviously, a function Φ on G is K-spherical if and only if it is constant on K-cosets, and can therefore be lifted to a function Φ^* on the quotient group $G^* = G/K$, by setting $\Phi^*(xK) = \Phi(x)$. Conversely, given any function on G^*, one may compose it with the projection $\pi : G \to G^*$ and get a K-spherical function on G. It follows that Φ is in the class $\mathfrak{D}$ for (G, K) if and only if Φ^* is in the class $\mathfrak{D}$ for $(G^*, \{ e^* \})$ where e^* is the identity for G^*.

The description of the class $\mathfrak{D}$ for $(G^*, \{ e^* \})$ in this situation is due to Parthasarathy, Ranga Rao and Varadhan [*1*]. Their results give us the following theorem. Its proof, being a simple permutation of their results, is omitted.

THEOREM 3.1. — A real valued function Φ^* on G^* is in the class $\mathfrak{D}$ for $(G^*, \{ e^* \})$ if and only if it admits the representation

$$\Phi^*(a) = \exp - \left\{ g^*(a) + \int_{\widehat{G}^* - \{1\}} (1 - \chi(a)) dL^*(\chi) \right\} \tag{3.1}$$

Where L^*, g^* satisfy the following requirements:

(*a*) L^* is a nonnegative measure on the character group $\widehat{G}^*$ of G^* such that L^* gives finite mass to the complement of any neighborhood of the identity 1 of $\widehat{G}^*$, and $L^*(\{ 1 \}) = 0$.

(*b*) For any Borel set $A \subset \widehat{G}^*$, and $a \in G^*$, we have

$$L^*(- A) = L^*(A) \tag{3.2}$$

$$\int_{\widehat{G}^* - \{1\}} (1 - \chi(a)) dL^*(\chi) < \infty. \tag{3.3}$$

(*c*) g^* is a nonnegative continuous function on G^* which satisfies the functional equation

$$\frac{1}{2}(g^*(a + b) + g^*(a - b)) = g^*(a) + g^*(b). \tag{3.4}$$

Further, the correspondance (3.1) between real valued functions Φ^* in class $\mathfrak{D}$ for $(G^*, \{ e^* \})$ and pairs (g^*, L^*) satisfying the conditions (*a*), (*b*), (*c*) is one-to-one.

Let us note that the last part of the theorem makes essential use of the hypothesis that G, and therefore G^*, is connected.

COROLLARY 3.2. — A continuous K-spherical normalized real valued, positive definite function Φ on G is infinitely divisible if and only if it is imbeddable.

There is now an immediate consequence.

THEOREM 3.3. — A kernel f on G/K is a Lévy-Schoenberg kernel if and only if

$$f(a, b) = \frac{1}{2}(r(a, o) + r(b, o) - r(a, b))\,;\ a, b \in G/K \tag{3.5}$$

where $r(xK, yK) = \Psi(y^{-1}x)$ and Ψ is a K-spherical function on G such that the corresponding function Ψ^* on $G^* = G/K$ is given by

$$\Psi^*(a) = g^*(a) + \int_{\hat{G}^* - \{1\}} (1 - \chi(a)) d\mathbf{L}^*(\chi) \tag{3.6}$$

for a pair $(g^*, \mathbf{L}^*)$ satisfying the conditions of Theorem 3.1.

Proof. — If f is a Lévy-Schoenberg kernel and r is defined in terms of f by

$$r(a, b) = f(a, a) + f(b, b) - 2f(a, b), \tag{3.7}$$

then, since r is then invariant under G, it is of the form $r(xK, yK) = \Psi(y^{-1}x)$, where Ψ is such that the function $\Phi = \exp - \Psi$ is in the class $\mathfrak{D}$ for the pair (G, K) (cf. § 2). This however implies that Ψ^* is in the class I for the pair $(G^*, \{ e^* \})$, $G^* = G/K$, and this, together with Theorem 3.1 and its corollary, determines the form of Ψ^*. The converse is obtained by retracing these steps backwards.

Q. E. D.

When G is the vector group R^d and K is trivial $= \{ 0 \}$, the solutions to (3.4) are just the nonnegative definite quadratic forms $g^*(a) = \sum_{ij} g_{ij} a_i a_j$, as may be verified fairly simply. When K is larger, it is a subspace of G, and the forms which arise are constant on subspaces parallel to K. This merely means that if a basis of G is chosen so that it is a basis of K augmented by a basis of a complementary subspace of K in G, the forms $g^*(a)$ will not have any dependence on the variables in K when expressed with respect to this basis. The measure $\mathbf{L}^*$ will similarly have symmetry properties which reflect the size of K.

Those Lévy-Schoenberg kernels which correspond to a pair $(g^*, \mathbf{L}^*)$ with $\mathbf{L}^* \equiv 0$ may be called Gaussian. The question arises naturally of describing the analogues of (1.1) in this situation. We shall postpone this to a later section.

CASE II. — Here we let G be the (connected) group of all proper rigid motions of a Euclidean space R^d and let K be the subgroup of all proper rotations about $0 \in \mathrm{R}^d$. Then $\mathrm{K} = \mathbf{SO}(d)$. K is closed and normal in G and the quotient group G/K is precisely R^d as a topological group. The coset $o = e\mathrm{K}$ is precisely the point $0 \in \mathrm{R}^d$. Now, if Φ is a function on G such that $\Phi(xk) = \Phi(x)$ for all $x \in \mathrm{G}$, $k \in \mathrm{K}$, then Φ may be lifted to a function Φ^* on $\mathrm{G/K} = \mathrm{R}^d$, by setting $\Phi^* \circ \pi = \Phi$ where $\pi : \mathrm{G} \to \mathrm{G/K}$ is the natural projection.

Then Φ is positive definite on G if and only if Φ^* is positive definite on the topological group R^d. Further Φ is K-spherical if and only if Φ^* is a *radial* function on R^d. It follows that Φ is in the class $\mathfrak{D}$ (or I) for the pair (G, K) if and only if Φ^* is a *radial* function belonging to the class $\mathfrak{D}$ (or I) for the pair $(\mathrm{R}^d, \{0\})$.

The classical Lévy-Khinchine formula of probability theory, which describes the Fourier transforms of probability measures on R^d which are infinitely divisible under convolution, is nothing but a description of the class $\mathfrak{D}$ for the pair $(\mathrm{R}^d, \{0\})$. If Φ^* is to be in this class, and is to be a *radial* function on R^d, then we may « radialize » the classical Lévy-Khinchine formula, and get the following result. We again omit the proof which involves nothing which is not routine.

THEOREM 3.4. — A function Φ^* on $\mathrm{G/K} = \mathrm{R}^d$ with $d \geq 2$, is a radial function in the class $\mathfrak{D}$ for the pair $(\mathrm{R}^d, \{0\})$ if and only if it admits the representation

$$(3.8) \qquad \Phi^*(\alpha) = \exp - \left\{ g^*(a) + \int_{\lambda>0} (1 - \mathrm{Y}_d(\lambda \,|\, a\,|)) d\mathbf{L}^*(\lambda) \right\}$$

where Y_d is the Bessel function

$$(3.9) \qquad \mathrm{Y}_d(t) = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\cdot\Gamma\left(\frac{d-1}{2}\right)} \cdot \int_0^{\pi} e^{it\cos\theta} \sin^{d-2}\theta\, d\theta$$

$$= \Gamma\left(\frac{d}{2}\right)(2t^{-1})^{(d-2)/2}\cdot \mathrm{J}_{(d-2)/2}(t), \qquad t \geq 0$$

and g^*, $\mathbf{L}^*$ are respectively a function and a measure satisfying the following requirements (*a*), (*b*), (*c*).

(*a*) $\mathbf{L}^*$ is a nonnegative measure on $(0, \infty)$

(*b*) $\int_{\lambda>0} \lambda^2/1+\lambda^2 . d\mathbf{L}^*(\lambda) < \infty$

(*c*) g^* is a function on R^d, such that $g^*(a) = c \mid a \mid^2$, where c is a constant ≥ 0 and $\mid a \mid$ is the length of a.

Further, the correspondence (3.8) between radial functions in the class $\mathfrak{D}$ for $(\mathrm{R}^d, \{0\})$ and pairs $(g^*, \mathbf{L}^*)$ satisfying (*a*), (*b*), (*c*) is one-to-one.

COROLLARY 3.5. — A function Φ on G is in the class $\mathfrak{D}$ for (G, K) if and only if it is in the class I for (G, K).

THEOREM 3.6. — A kernel f on $\mathrm{G/K} = \mathrm{R}^d \, (d \geq 2)$ is a Lévy-Schoenberg kernel if and only if

$$(3.10) \qquad f(a, b) = \frac{1}{2}(r(a, o) + r(b, o) - r(a, b)) \, ; \qquad a, b \in \mathrm{R}^d$$

where $r(a, b) = \Psi^*(a-b)$ and Ψ^* is a function on R^d of the form

$$(3.11) \qquad \Psi^*(a) = g^*(a) + \int_{\lambda>0} (1 - \mathrm{Y}_d(\lambda \mid a \mid)) d\mathbf{L}^*(\lambda)$$

where g^*, $\mathbf{L}^*$ have the meanings described in Theorem 3.4.

Note that if $a = x\mathrm{K}$, $b = y\mathrm{K}$, $x, y \in \mathrm{G}$ then $r(x\mathrm{K}, y\mathrm{K}) = \Psi(y^{-1}x)$, where $\Psi = \Psi^* \circ \pi$, and π is the natural map $\mathrm{G} \to \mathrm{G/K}$.

By making various choices for the pairs $(g^*, \mathbf{L}^*)$ one gets various Lévy-Schoenberg kernels. For example, if we let $g^* = 0$ and $d\mathbf{L}^*(\lambda) = d\lambda/\lambda^{1+\alpha}$ for some $0 < \alpha < 2$, we get after some computation that $\Psi^*(a) = \mid a \mid^\alpha$. Thus, the kernels

$$(3.15) \qquad f(a, b) = \frac{1}{2}(\mid a \mid^\alpha + \mid b \mid^\alpha - \mid a - b \mid^\alpha) \qquad 0 \leq \alpha \leq 2$$

are all Lévy-Schoenberg kernels. For $\alpha = 1$ we have just the kernel (1.2). That the kernels (3.15) are positive-definite must surely have been known to many people, but I have not seen an explicit reference to this fact in the literature, and it seems worthwhile to point it out.

For $d = 1$, the case not treated above, a formula similar to (3.8) results, but Y_d is now replaced by the cosine function.

CASE III. — Here we let G be an *arcwise connected* compact group and let K be any closed subgroup of G. We want to describe the class $\mathfrak{D}$ for the pair (G, K). The necessary tools pertain to harmonic analysis on compact groups, and especially the Peter-Weyl theorem. See e. g. Loomis [*1*] or Weil [*1*]. Later, we shall see that special choices of G, K will lead to interesting results.

By a representation T of G we shall always mean a continuous unitary representation on a complex Hilbert space H(T). We denote by $\mathfrak{R}(G)$ the set of all (unitary) equivalence classes of irreducible representations of G. α, β, γ, ... will denote elements of $\mathfrak{R}(G)$. $\mathfrak{k}$ will denote the class of the trivial representation $x \to 1$ of G. dx will denote the Haar measure of G normalized to give unit mass to G.

Given a representation T of G on the Hilbert space H(T), one may, as is known, decompose T into its irreducible components.

DEFINITION 3.7. — A representation T of G is said to be K-*spherical* (or merely, *spherical*) if the decomposition of the restriction of T to K contains the trivial representation $k \to 1$ of K. This is the same thing as saying that there is a unit vector $v \in H(T)$ such that $T(k)v = v$ for all $k \in K$.

It is clear that given a class $\alpha \in \mathfrak{R}(G)$, either all members of α are spherical or none is spherical. We denote by $\mathfrak{R}_K(G)$ the set of all those classes α in $\mathfrak{R}(G)$ such that every member of α is spherical.

Given a representation T of G, the function $x \to (T(x)u, v)$ where $u, v \in H(T)$ and $(\ldots)$ is the inner product in H(T), will be called a function associated with T. If T happens to be irreducible, then a function associated with T is called an *elementary* function on G. One version of the Peter-Weyl Theorem asserts that the set of all finite linear combinations (with complex coefficients) of elementary functions is *uniformly* dense in the set of all continuous complex valued functions on G.

Here is a list of several well-known facts about positive definite functions that we shall have to use below. For details see Naimark [*1*] or Godement [*1*].

i) Given a representation T of G on H(T) and a unit vector $u \in H(T)$, the function $\varphi(x) = (T(x)u, u)$ is a normalized continuous positive definite function on G. If T is spherical and u is a vector fixed under all $T(k)$, $k \in K$, then φ is spherical. Conversely, if φ is a normalized positive definite continuous function on G there is then a representation T of G and a unit vector $u \in H(T)$ such that $\varphi(x) = (T(x)u, u)$. Further, if φ is spherical then T is spherical, and $T(k)u = u$ for all $k \in K$.

ii) Suppose that T and U are irreducible representations of G and φ, ψ are positive definite functions associated with T and U respectively.

If $\varphi = \psi$ then T is equivalent to U. In particular, since the function identically equal to 1 is associated with the class $\mathfrak{1}$, it follows that if T is any representation of class $\alpha \neq \mathfrak{1}$, then a nonzero positive definite function associated with T cannot be constant. We shall use this crucially below.

iii) A positive definite function is uniformly continuous on G if it is continuous in a neighborhood of $e \in G$.

iv) Let φ_n be a sequence of continuous positive definite functions on G and φ a continuous positive definite function on G such that $\varphi_n \to \varphi$ in the weak-* topology of $L_\infty(G)$ (regarded as the dual of $L_1(G)$). Then in fact $\varphi_n \to \varphi$ uniformly on G (Gelfand's lemma).

v) Let φ be a continuous positive definite function on G. Then exp φ is again a continuous positive definite function on G. Cf. Lemma 2.4. Thus, $x \to \exp(\varphi(x) - \varphi(e))$ is a normalized positive definite continuous function on G. In particular, if φ is normalized, then the function exp $(\varphi(x) - 1)$ is positive definite, normalized and continuous.

With these preparations we may begin our characterization of the class $\mathfrak{D}$. The reader should note that the arguments are more or less classical, and are dual to those of a previous paper of the author. See Gangolli [*1*].

LEMMA 3.8. — Let $\Phi \in \mathfrak{D}$. Then Φ does not vanish on G.

Proof. — If $\Phi \in \mathfrak{D}$ then so is $\bar{\Phi}$ and hence $|\Phi|^2$. Since $\Phi(x) = 0$ if and only if $|\Phi(x)|^2 = 0$, we may assume to begin with that Φ is real valued. Now, for each integer n let Φ_n be a continuous positive definite function such that $(\Phi_n(x))^n = \Phi(x)$, $x \in G$. Then $\Phi_n(x) = \Phi(x)^{1/n}$. As $n \to \infty$, $\Phi(x)^{1/n} \to \chi(x)$ where $\chi(x)$ is 0 if $\Phi(x) = 0$ and $\chi(x) = 1$ if $\Phi(x) \neq 0$. Since $\Phi_n(x) \to \chi(x)$ pointwise, $\chi(x)$ is positive definite. Now since $\Phi(e) = 1$, it follows that $\Phi(x) \neq 0$ in a neighborhood of e, hence that $\chi(x) \equiv 1$ in a neighborhood of e. Therefore $\chi(x)$ is continuous in a neighborhood of e, hence $\chi(x)$, being positive definite, is continuous on G. Since G is connected and χ can take only the values 0 or 1, χ must be identically 1 on G, proving that $\Phi(x)$ is never zero on G.

Q. E. D.

Since G is arcwise connected, it follows immediately that for a proper determination of the logarithm we have (6)

$$(3.16) \qquad \begin{aligned} -\log \Phi(x) &= \lim_{j\to\infty} j(1 - \Phi(x)^{1/j}) \\ &= \lim_{j\to\infty} j(1 - \Phi_j(x)), \end{aligned}$$

(6) Regarding this point see Rogalski [*1*]. His proof works for our situation as well.

the limit holding uniformly on every compact neighborhood of $e \in G$. Note also that $-\log \Phi(x)$ is continuous on G. We denote by $\Psi(x)$ the function $-\log \Phi(x)$. Note also that Ψ is spherical.

THEOREM 3.9. — A function Φ on G is in the class $\mathfrak{D}$ for the pair (G, K) if and only if

$$\Phi(x) = \exp - \Psi(x) \tag{3.17}$$

where $\Psi(x)$ is a spherical function on G of the following form;

$$\Psi(x) = \lim_{j\to\infty} d_j(1 - \Phi_j(x)), \qquad x \in G, \tag{3.18}$$

where $\Phi_j(x)$ is a spherical normalized positive definite continuous function on G, d_j are real numbers ≥ 0, and the limit is uniform for x in some compact neighborhood of e in G.

Proof (7). — We have already seen above that if $\Phi \in \mathfrak{D}$ for the pair (G, K) then $\Phi(x) = \exp - \Psi(x)$ with $\Psi(x)$ satisfying (3.18) (with $d_j = j$). Conversely let $\Psi(x)$ satisfy (3.18). Writing $\Psi_j(x) = d_j(1 - \Phi_j(x))$, consider the function $\exp - \Psi_j(x) = \exp d_j(\Phi_j(x) - 1)$. Since Φ_j is a normalized continuous spherical positive definite function, it follows from our remarks that $\exp - \Psi_j(x)$ again a normalized continuous spherical positive definite function. Now, if $\Phi(x) = \exp - \Psi(x)$, then $\Phi(x) = \lim_{j\to\infty} \exp - \Psi_j(x)$. Thus $\Phi(x)$ is spherical, normalized and positive definite. Further since (3.18) holds uniformly in a neighborhood of $e \in G$, it follows that Ψ and hence also Φ is continuous in a neighborhood of $e \in G$; hence Φ is continuous on G. Finally Φ is infinitely divisible. Indeed $(\exp - (\Psi(x)/n))^n = \Phi(x)$ for every positive integer n, and the fact that $\exp - (\Psi(x)/n)$ is positive definite may be deduced by noting that if $\Psi(x)$ satisfies (3.18) then so does $\Psi(x)/n$, and then applying the above argument to $\Psi(x)/n$ in place of $\Psi(x)$.

Q. E. D.

COROLLARY 3.10. — If $\Phi \in \mathfrak{D}$ then $\Phi \in I$. Therefore, $\mathfrak{D} = I$ for the pair (G, K).

Indeed, if Ψ is a function of the type described in (3.18) then so is the function $t\Psi$ where t is any real number ≥ 0. Hence $\exp - t\Psi$ is also

(7) Note that this proof implies, by virtue of Gelfand's lemma and by the remark immediately preceding the theorem, that Ψ is continuous on G and the convergence of Ψ_j to Ψ is uniform on compacts in G.

positive definite. This implies that $\Phi = \exp - \Psi$ is in the class I for the pair (G, K).

It remains to obtain a more explicit description of functions such as Ψ envisaged in (3.18). This is our next task.

Let φ be a continuous normalized positive definite function on G, so that $\varphi(x) = (T(x)u, u)$ for some representation T and *unit* vector $u \in H(T)$. Then we may decompose T into its irreducible components and H(T) may be written as a direct sum $\bigoplus_{i \in I} H_i(T)$ (I being some index set) such that T acts irreducibly on each subspace $H_i(T)$. Let u_i be the component of u in $H_i(T)$ and T_i the restriction of T to H_i. Then

$$\varphi(x) = \sum_{i \in I} (T_i(x)u_i, u_i) = \sum_{i \in I} \| u_i \|^2 \cdot \left(T_i(x) \frac{u_i}{\| u_i \|}, \frac{u_i}{\| u_i \|} \right).$$

Thus we see that

$$\varphi(x) = \sum_{i \in I} \lambda_i \varphi_i(x), \tag{3.19}$$

where φ_i is a normalized continuous positive definite function associated with T_i, and $\lambda_i \geq 0$, such that $\sum_{i \in I} \lambda_i = \varphi(e) = 1$. Clearly only a countable number of λ_i are nonzero. Each T_i is, of course, irreducible. But the different T_i may not all be mutually inequivalent. We may amalgamate the equivalent ones (a procedure which offers no difficulty because $\Sigma\lambda_i < \infty$), and redefine the coefficients λ_i to obtain the following proposition:

PROPOSITION 3.11. — Given a continuous normalized positive definite function φ on G, there is a countable subset $\mathfrak{R}^\varphi(G)$ of $\mathfrak{R}(G)$ such that

$$\varphi(x) = \sum_{\alpha \in \mathfrak{R}^\varphi(G)} c_\alpha \varphi_\alpha(x) \qquad (x \in G) \tag{3.20}$$

where φ_α is a normalized continuous positive definite function associated with a representation of class α. Further $c_\alpha \geq 0$, and $\sum_{\alpha \in \mathfrak{R}^\varphi(G)} c_\alpha = \varphi(e) = 1$.

This theorem is, of course, well-known. See e. g. Godement [*1*, proposition 9, p. 52] for a different proof. Note that if, furthermore, φ is spherical, then the above argument shows that each φ_α is likewise so, and therefore in this case the sum in (3.20) need only be taken over a countable subset (say $\mathcal{R}_K^{\varphi}(G)$) of $\mathcal{R}_K(G)$. Note also that since φ_α is elementary, $\varphi_\alpha(x) = (T_\alpha(x)v_\alpha, v_\alpha)$ where T_α is of class α and v_α is a unit vector in $H(T_\alpha)$.

We would like to remark here for future use that the term $a_\alpha \varphi_\alpha$ is determined uniquely by φ. Indeed, if h is any square integrable function, we may regard it as a vector in $L_2(G)$, on which space one can realize the left regular representation S of G. For any $\alpha \in \mathcal{R}(G)$, let $L_2^\alpha(G)$ be the subspace of $L_2(G)$ consisting of those vectors which transform under S according to α. i. e. $L_2^\alpha(G)$ consists of those vectors $v \in L_2(G)$ such that the restriction of S to the closed cyclic subspace generated by v under S decomposes into irreducible subrepresentations all of class α. For each α, $L_2^\alpha(G)$ is finite dimensional, and consists precisely of the *complex conjugates* of all the elementary functions associated with α [8]. Further $L_2^\alpha(G)$ and $L_2^\beta(G)$ are orthogonal if $\alpha \neq \beta$, and $L_2(G) = \bigoplus_\alpha L_2^\alpha(G)$, the sum being direct in the Hilbert space sense. Therefore, the function $h \in L_2(G)$ determines completely its component h^α in $L_2^\alpha(G)$. If now one takes a look at the representation $\varphi = \Sigma a_\alpha \varphi_\alpha$ in proposition 3.11, it is seen that $a_\alpha \varphi_\alpha$ is exactly the component of φ in $L_2^{\bar{\alpha}}(G)$, where $\bar{\alpha}$ is the class contragredient to α. (Of course, we have used here the fact that the complex conjugate of a function associated with α is a function associated with $\bar{\alpha}$). It follows that $a_\alpha \varphi_\alpha$ is determined uniquely by φ. Further, the condition that φ_α is normalized means that for each α both a_α and φ_α are determined uniquely by φ. This will be useful to us later.

LEMMA 3.12. — Let $h \in L_1(G)$. Then given $\varepsilon > 0$, there exists a finite subset $\mathcal{N}$ of $\mathcal{R}(G)$ such that if $\alpha \notin \mathcal{N}$ and if φ_α is an elementary normalized positive definite function associated with a representation of class α, then $\left| \int_G h(x)\varphi_\alpha(x)dx \right| < \varepsilon$.

Proof:

Let $\varphi_\alpha(x) = (T_\alpha(x)v_\alpha, v_\alpha)$, $\| v_\alpha \| = 1$. Then

$$\int_G h(x)\varphi_\alpha(x)dx = (T_\alpha(h)v_\alpha, v_\alpha) \tag{3.21}$$

where

$$T_\alpha(h) = \int_G T_\alpha(x)h(x)dx. \tag{3.22}$$

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[8] This inversion occurs because we are working with the *left* regular representation rather than the right regular representation.

Therefore

$$\left| \int_G h(x)\varphi_\alpha(x)dx \right| \leq \| T_\alpha(h) \| \tag{3.23}$$

where $\| T_\alpha(h) \|$ is the norm of the operator $T_\alpha(h)$ on the Hilbert space $H(T_\alpha)$.

If h is a finite linear combination of elementary functions, then the Schur orthogonality relations show that $T_\alpha(h) = 0$ except for finitely many α. Thus the lemma is proved when h is a finite combination of elementary functions. But because of the Peter-Weyl theorem, such functions are dense in $L_1(G)$. The lemma now follows by an easy approximation argument which is omitted.

LEMMA 3.13. — Let A be a neighborhood of e in G and let $\alpha \in \mathcal{R}(G)$ such that $\alpha \neq 1$. There exists a real number $\delta_\alpha > 0$ such that if φ_α is any normalized positive definite continuous function associated with a representation T_α of class α, then we have

$$1 - \frac{\int_A \text{Re } \varphi_\alpha(x)dx}{\text{vol. A}} \geq \delta_\alpha > 0 \tag{3.24}$$

where $\text{Re}\,\varphi_\alpha$ is the real part of φ_α and vol. A is the Haar measure $\int_A dx$ of A.

Proof :

We have $\varphi_\alpha(x) = (T_\alpha(x)u_\alpha, u_\alpha)$, with $u_\alpha \in H(T_\alpha)$ and $\| u_\alpha \| = 1$. Now

$$\begin{aligned} 1 - \frac{\int_A \text{Re } \varphi_\alpha(x)dx}{\text{vol. A}} & \\ &= \frac{1}{\text{vol. A}} \int_A (1 - \text{Re } \varphi_\alpha(x))\, dx \\ &\geq 0 \qquad \text{because} \qquad \text{Re}\,\varphi_\alpha \leq 1. \end{aligned} \tag{3.25}$$

We claim that actually, strict inequality holds in (3.25). For, if

$$\int_A (1 - \text{Re } \varphi_\alpha(x))dx = 0 \tag{3.26}$$

then we would have $\text{Re}\,\varphi_\alpha(x) = 1$ on A. But φ_α being normalized and positive definite, one has for any $x \in A$ and any $y \in G$, $| \varphi_\alpha(y) - \varphi_\alpha(yx) |^2 \leq 2(1 - \text{Re}\,\varphi_\alpha(x))$. Thus this implies that $\varphi_\alpha(y) = \varphi_\alpha(yx)$ for all $y \in G$ and

$x \in A$. Since G is connected, φ_α turns out to be equal to 1 everywhere. This contradicts the hypothesis that $\alpha \neq \mathfrak{1}$. Thus we have

$$(3.27) \quad 0 < \frac{1}{\text{vol. A}} \int_A (1 - \operatorname{Re} \varphi_\alpha(x))dx$$
$$= \frac{1}{\text{vol. A}} \cdot \int_A (1 - \operatorname{Re}(T_\alpha(x)u_\alpha, u_\alpha))dx\,; \qquad \| u_\alpha \| = 1.$$

But regarded as a function of u_α, the right side is continuous. Since T_α is finite dimensional, the unit sphere in $H(T_\alpha)$ is compact and so, as a function of u_α, the right side is bounded away from 0 by some positive number δ_α. This is the δ_α envisaged by the Lemma.

Q. E. D.

COROLLARY 3.14. — Given any neighborhood A of e in G, there is a $\delta > 0$ such that for any $\alpha \in \mathfrak{R}(G)$, $\alpha \neq \mathfrak{1}$,

$$\frac{1}{\text{vol. A}} \int_A (1 - \operatorname{Re} \varphi_\alpha(x))dx \geq \delta > 0$$

where φ_α is any continuous normalized positive definite function associated with a representation $T_\alpha \in \alpha$.

By lemma 3.12 we have that there is a finite subset $\mathcal{N}$ of $\mathfrak{R}(G)$ such that if $\alpha \notin \mathcal{N}$ then $\left| \frac{1}{\text{vol. A}} \int_A \operatorname{Re} \varphi_\alpha(x)dx \right| < \frac{1}{2}$. Hence for such α,

$$(3.28) \quad \frac{1}{\text{vol. A}} \int_A 1 - \operatorname{Re} \varphi_\alpha(x)dx$$
$$= 1 - \frac{1}{\text{vol. A}} \int_A \operatorname{Re} \varphi_\alpha(x)dx$$
$$> \frac{1}{2}.$$

This, together with lemma 3.13, implies the corollary.

Q. E. D.

We now prove:

THEOREM 3.15. — A function Φ on G is in the class $\mathfrak{D}$ for (G, K) if and only if

$$(3.29) \quad \Phi(x) = \exp - \Psi(x)$$

where $\Psi(x)$ is a function on G having the following representation:

$$(3.30) \quad \Psi(x) = \sum_{\alpha \in \mathfrak{R}_K(G)} a_\alpha(1 - \varphi_\alpha(x))\,; \quad \text{with} \quad a_\alpha \geq 0 \sum_{\alpha \in \mathfrak{R}_K(G)} a_\alpha < \infty\,;$$

where $\varphi_\alpha(x)$ is a normalized continuous spherical positive definite function associated with a representation T_α of class α.

Further, the coefficients a_α in (3.30) are uniquely determined by Φ.

Proof. — Suppose first that Ψ is of the form (3.30). Then Ψ is continuous, because the right side converges uniformly on G; also Ψ is spherical. Next, there are only countably many classes, say $\alpha_1, \alpha_2, \ldots$, such that the corresponding coefficients a_{α_i} are non-zero on the right. Now set

$$d_j = \sum_{i=1}^{j} a_{\alpha_i} \qquad \text{and} \qquad \Phi_j(x) = \sum_{i=1}^{j} a_{\alpha_i}\varphi_{\alpha_i}(x) \Big/ \sum_{i=1}^{j} a_{\alpha_i}.$$

Then $d_j \geq 0$ and we have

$$\sum_{i=1}^{j} a_{\alpha_i}(1 - \varphi_{\alpha_i}(x)) = d_j(1 - \Phi_j(x)).$$

$\Phi_j(x)$ is trivially a continuous normalized spherical positive definite function, and surely,

$$\Psi(x) = \lim_{j\to\infty} d_j(1 - \Phi_j(x)).$$

Thus by Theorem 3.9, $\Phi \in \mathfrak{D}$.

Conversely suppose that $\Phi \in \mathfrak{D}$. Then by Theorem 3.9, $\Phi(x) = \exp -\Psi(x)$,

(3.31) $$\Psi(x) = \lim_{j\to\infty} d_j(1 - \Phi_j(x)), \qquad d_j \geq 0,$$

and Φ are continuous spherical normalized positive definite functions on G.

We want to show that under these conditions, $\Psi(x)$ must be of the form (3.30). Let us write $\Psi_j(x) = d_j(1 - \Phi_j(x))$. So that $\lim_{j\to\infty} \Psi_j(x) = \Psi(x)$, uniformly in some neighborhood B of e.

By proposition 3.11 and the remarks which follow it we see that

(3.32) $$\Phi_j(x) = \sum_{\alpha \in \mathfrak{K}_K^{\Phi_j}(G)} b_\alpha^j \varphi_\alpha^j(x), \qquad j = 1, 2, \ldots \text{ (9)}$$

(9) The j which appears here and in the following proof is a superscript and not an exponent.

where $b^j_\alpha \geq 0$, $\sum_{\alpha \in \mathfrak{K}_K^{\Phi_j}(G)} b^j_\alpha = 1$ for each j, and where for each j, φ^j_α is an elementary positive definite function associated with α.

To avoid summing over a different set each time let us write $\mathcal{S}$ for the set $\bigcup_{j=1}^{\infty} \mathfrak{K}_K^{\Phi_j}(G)$, and let $\mathcal{S}'$ denote the set $\mathcal{S}$ minus the class $\mathit{1}$. We then see that if we define b^j_α to be zero on $\mathcal{S} - \mathfrak{K}_K^{\Phi_j}(G)$, then

$$\Phi_j(x) = \sum_{\alpha \in \mathcal{S}'} b^j_\alpha \varphi^j_\alpha(x), \qquad j = 1, \ldots \tag{3.33}$$

and

$$\Psi_j(x) = \sum_{\alpha \in \mathcal{S}'} c^j_\alpha (1 - \varphi^j_\alpha(x)), \tag{3.34}$$

where $c^j_\alpha = d_j b^j_\alpha \geq 0$, $\sum_\alpha c^j_\alpha = d_j$.

Note that in (3.34) the sum may be taken over $\mathcal{S}'$ because when $\alpha = \mathit{1}$, the function $\varphi^j_\alpha \equiv 1$ so the corresponding term drops out.

We first claim that there exists a constant M independent of j such that

$$\sum_{\alpha \in \mathcal{S}'} c^j_\alpha \leq M \text{ for all } j. \tag{3.35}$$

To see this, note that $\Psi_j(x) \xrightarrow[j \to \infty]{} \Psi(x)$ uniformly in a neighborhood B of e. Hence it follows that for any neighborhood A of e such that $A \subset B$, we have $\frac{1}{\text{vol. } A} \int_A \operatorname{Re} \Psi_j(x) dx \to \frac{1}{\text{vol. } A} \int_A \operatorname{Re} \Psi(x) dx$. In particular there exists a constant M_A such that

$$\frac{1}{\text{vol. } A} \int_A \operatorname{Re} \Psi_j(x) \leq M_A \text{ for all } j. \tag{3.36}$$

But, using $\sum_{\alpha \in \mathcal{S}'} c^j_\alpha < \infty$ for each j, we have

$$\begin{aligned} \sum_{\alpha \in \mathcal{S}'} c^j_\alpha \cdot \frac{1}{\text{vol. } A} \int_A (1 - \operatorname{Re} \varphi^j_\alpha(x)) dx &= \frac{1}{\text{vol. } A} \int_A \operatorname{Re} \Psi_j(x) dx \\ &\leq M_A. \end{aligned} \tag{3.37}$$

But according to Corollary 3.14, there is a $\delta > 0$ such that for all j,

$$\begin{aligned}(3.38)\qquad &\delta \sum_{\alpha \in \mathcal{S}'} c_\alpha^j \\ &\leq \sum_{\alpha \in \mathcal{S}'} c_\alpha^j \cdot \frac{1}{\text{vol. A}} \int_{\text{A}} (1 - \operatorname{Re} \varphi_\alpha^j(x))dx \\ &\leq \text{M}_{\text{A}}\end{aligned}$$

or

$$(3.39)\qquad \sum_{\alpha \in \mathcal{S}'} c_\alpha^j \leq \frac{\text{M}_{\text{A}}}{\delta} = \text{M for all } j.$$

This proves (3.35).

We next claim that given $\varepsilon > 0$, there is a finite subset $\mathcal{N} \subset \mathcal{S}'$ and an integer j_0 such that if $j \geq j_0$, then

$$(3.40)\qquad \sum_{\alpha \in \mathcal{S}' - \mathcal{N}} c_\alpha^j < 4\varepsilon.$$

To see this note that Re $\Psi(x)$ is continuous at e and Re $\Psi(e) = 0$. It follows that for all sufficiently small neighborhoods A of e we have

$$(3.41)\qquad \left| \frac{1}{\text{vol. A}} \int_{\text{A}} \operatorname{Re} \Psi(x)dx \right| \leq \sup_{x \in \text{A}} | \operatorname{Re} \Psi(x) | \leq \varepsilon.$$

Since $\Psi_j(x) \to \Psi(x)$ uniformly in B, we see that for all sufficiently small neighborhoods A $\subset$ B, and $j \geq$ some integer j_0,

$$\begin{aligned}(3.42)\qquad \frac{1}{\text{vol. A}} \int_{\text{A}} \operatorname{Re} \Psi_j(x)dx &\leq \frac{1}{\text{vol. A}} \int_{\text{A}} \operatorname{Re} \Psi(x)dx + \varepsilon \\ &\leq 2\varepsilon \quad \text{by (3.41).}\end{aligned}$$

Next, fix some such neighborhood A, and apply lemma 3.12 to the function $h(x) = \dfrac{\chi_{\text{A}}(x)}{\text{vol. A}}$ where χ_{A} is the indicator function of A. Then there is a finite subset $\mathcal{N}$ of $\mathcal{S}'$ such that if $\alpha \notin \mathcal{N}$, then for all j,

$$(3.43)\qquad \frac{1}{\text{vol. A}} \int_{\text{A}} \operatorname{Re} \varphi_\alpha^j(x)dx \leq \frac{1}{2}$$

or

$$1-\frac{1}{\text{vol. A}}\int_A \operatorname{Re}\varphi_\alpha^j(x)dx \geq \frac{1}{2} \text{ for all } j. \tag{3.44}$$

Therefore

$$\begin{aligned}\frac{1}{2}\sum_{\alpha\in\mathcal{S}'-\mathcal{N}} c_\alpha^j &\leq \sum_{\alpha\in\mathcal{S}'} c_\alpha^j\cdot\frac{1}{\text{vol. A}}\int_A(1-\operatorname{Re}\varphi_\alpha^j(x))dx \\ &= \frac{1}{\text{vol. A}}\int_A \operatorname{Re}\Psi_j(x)dx \\ &\leq 2\varepsilon \quad \text{by } (3.42).\end{aligned} \tag{3.45}$$

This proves (3.40).

On the basis of (3.35), exactly as in the classical case, we may now follow the diagonal procedure (remembering that $\mathcal{S}'$ is countable) and conclude that there is a subsequence $(j_1, j_2, \ldots$ of the integers such that $c_\alpha^j \to a_\alpha$ for each $\alpha\in\mathcal{S}'$ and $a_\alpha \geq 0$ and $\sum_{\alpha\in\mathcal{S}'} a_\alpha < \infty$. Since we are interested only in the limit of $\Psi_j(x)$, we may name this subsequence as c_α^j again.

Next, recall that $\varphi_\alpha^j(x)$ was a normalized positive definite function associated with T_α of class α, i. e. $\varphi_\alpha^j(x) = (T_\alpha(x)u_\alpha^j, u_\alpha^j)$ with $u_\alpha^j \in H(T_\alpha)$ and $\| u_\alpha^j \| = 1$. Now, since $H(T_\alpha)$ is finite dimensional, its unit sphere is compact. Hence a subsequence of $\{ u_\alpha^j \}_{j=1}^\infty$ converges to some $u_\alpha \in H(T_\alpha)$ such that $\| u_\alpha \| = 1$. Again following the diagonal procedure we see that along an appropriate subsequence we have, as $j\to\infty$, $u_\alpha^j \to u_\alpha$ for each α; and so $\varphi_\alpha^j(x) \to (T_\alpha(x)u_\alpha, u_\alpha) = \varphi_\alpha(x)$ say, along this subsequence. Note that $T(k)u_\alpha = u_\alpha$, $k\in K$. The net result is that along an appropriate sequence we have for each α,

$$c_\alpha^j \to a_\alpha, \qquad a_\alpha \geq 0, \quad \sum_{\alpha\in\mathcal{S}'} a_\alpha < \infty, \qquad j\to\infty. \tag{3.46}$$

$$\varphi_\alpha^j(x) \to \varphi_\alpha(x), \quad \text{as} \quad j\to\infty. \tag{3.47}$$

Note that the convergence in (3.47) is taking place *boundedly*, and that $\varphi_\alpha(x)$ is also spherical.

We now have the estimate

$$(3.48)\qquad \Big|\Psi_j(x) - \sum_{\alpha\in\mathcal{S}'} a_\alpha(1-\varphi_\alpha(x))\Big|$$

$$= \Big|\sum_{\alpha\in\mathcal{S}'} c_\alpha^j(1-\varphi_\alpha^j(x)) - a_\alpha(1-\varphi_\alpha(x))\Big|$$

$$\leq \Big|\sum_{\alpha\in\mathcal{N}} c_\alpha^j(1-\varphi_\alpha^j(x)) - a_\alpha(1-\varphi_\alpha(x))\Big|$$

$$+ \sum_{\alpha\in\mathcal{S}'-\mathcal{N}} c_\alpha^j\,|(1-\varphi_\alpha^j(x))| + \sum_{\alpha\in\mathcal{S}'-\mathcal{N}} a_\alpha\,|(1-\varphi_\alpha(x))|$$

$$\leq \Big|\sum_{\alpha\in\mathcal{N}} c_\alpha^j(1-\varphi_\alpha^j(x)) - a_\alpha(1-\varphi_\alpha(x))\Big|$$

$$+2\sum_{\alpha\in\mathcal{S}'-\mathcal{N}} c_\alpha^j + 2\sum_{\alpha\in\mathcal{S}'-\mathcal{N}} a_\alpha$$

where $\mathcal{N}$ is any finite subset of $\mathcal{S}'$ and we used $|\varphi_\alpha^j(x)| \equiv 1$, $|\varphi_\alpha(x)| \leq 1$.

This estimate together with (3.46), (3.47), and (3.40) implies that along the subsequence mentioned, we have $\Psi_j(x) \to \sum_{\alpha\in\mathcal{S}'} a_\alpha(1-\varphi_\alpha(x))$.

Since $\Psi(x) = \lim_{j\to\infty} \Psi_j(x)$, we see that

$$(3.49)\qquad \Psi(x) = \sum_{\alpha\in\mathcal{S}'} a_\alpha(1-\varphi_\alpha(x)), \qquad a_\alpha \geq 0, \qquad \sum_{\alpha\in\mathcal{S}'} a_\alpha < \infty$$

To finish the proof of the theorem we have to check the coefficients a_α are uniquely determined by Φ, or what is the same thing, by Ψ. Now $\Psi(x) = \sum_{\alpha\in\mathcal{S}'} a_\alpha - \sum_{\alpha\in\mathcal{S}'} a_\alpha\varphi_\alpha(x)$. This representation shows that $-a_\alpha\varphi_\alpha$ is the component of Ψ which lies in $L_2^{\overline{\alpha}}(G)$ when $\alpha \neq \mathfrak{1}$. Cf. our remarks after Proposition 3.11, Since $L_2(G)$ is the *direct* sum of L_2^α, it follows that $a_\alpha\varphi_\alpha$ is determined by Ψ, and hence so is a_α, since φ_α is normalized. The theorem is thus proved in full.

Q. E. D.

COROLLARY 3.16. — With G, K as above, a function Φ on G is in the class $\mathfrak{D}$ is and only if

$$(3.50)\qquad \Phi(x) = \exp - (\varphi(e) - \varphi(x))$$

where $\varphi(x)$ is a continuous spherical positive definite function on G. Indeed

$$\Psi(x) = \varphi(e) - \varphi(x) \quad \text{where} \quad \varphi(x) = \sum_{\alpha \in \mathfrak{R}_K(G)} a_\alpha \varphi_\alpha(x).$$

In Gangolli [*3*], we had this stated this theorem only for pairs (G, K) such that G is a connected compact semisimple Lie group and K a closed subgroup such that G/K is symmetric. As is now clear, no such restriction is necessary. Of course, the compact symmetric spaces are the most important special case of the above. For them the above theorem takes even a simpler form, which it may be worthwhile to sketch. So let us now assume that G is a compact connected semisimple Lie group, K is a closed subgroup such that G/K is symmetric. It is a well-known theorem of É. Cartan that in this case, is T is a spherical representation of G, then the subspace $\{ v \mid v \in H(T),\ T(k)v = v \text{ for all } k \in K \}$ is one-dimensional. See e. g. Helgason [*1*, Chapter X].

It follows that there is *just one* normalized elementary spherical function φ_α associated with each $\alpha \in \mathfrak{R}_K(G)$ and this function is automatically positive definite, being just $(T_\alpha(x)v, v)$ where v is a unit vector such that $T_\alpha(k)v = v$ for all $k \in K$. Further $\psi_K(G)$ is countable, so we may enumerate the elementary normalized spherical positive definite functions in the list $\varphi_0, \varphi_1, \varphi_2, \ldots$ with $\varphi_0 \equiv 1$. We then have the following theorem.

THEOREM 3.16. — Let G be a compact connected semisimple Lie group, K a closed subgroup such that G/K is a symmetric space. A function Φ on G is in the class $\mathfrak{D}$ for the pair (G, K) is and only if

$$\Phi(x) = \exp - \Psi(x) \qquad x \in G, \tag{3.51}$$

and $\Psi(x)$ has the representation

$$\Psi(x) = \sum_{n \geq 1} a_n(1 - \varphi_n(x)), \tag{3.52}$$

with $a_n \geq 0$, $\Sigma a_n < \infty$, and where $\{ \varphi_n \}_{n=1}^{\infty}$ is the set of non-constant elementary positive definite spherical functions on G.

This is Theorem 3.5 of Gangolli [*3*].

We saw in section § 2 that Lévy-Schoenberg kernels were in biunique correspondence with real-valued functions Φ in $\mathfrak{D}$. It is therefore important to describe those members of $\mathfrak{D}$ which are real valued. We shall do this briefly now.

For any $\alpha \in \mathfrak{K}_K(G)$ let $\bar{\alpha}$ be the class contragredient to α. If T_α is a representation of class α, on the Hilbert space $H(T_\alpha)$, then we may realize a representative $T_{\bar{\alpha}}$ of class $\bar{\alpha}$ on the same Hilbert space, by passing to the conjugate imaginary representation of T_α, i. e., $T_{\bar{\alpha}}(x) = {}^tT_\alpha(x^{-1})$ where ${}^tT$ means the transpose of T. Now, Ψ is real if and only if φ is real, where φ is the function $\sum_{\alpha \in \mathfrak{K}_K(G)} a_\alpha \varphi_\alpha(x)$. Therefore Ψ is real if and only if $\sum_{\alpha \in \mathfrak{K}_K(G)} a_\alpha \varphi_\alpha(x) = \sum_{\alpha \in \mathfrak{K}_K(G)} a_\alpha \overline{\varphi_\alpha(x)}$. Now, for any α, $\overline{\varphi_\alpha(x)}$ is an elementary normalized positive definite function associated with a representation of class $\bar{\alpha}$. It follow that $a_\alpha \overline{\varphi_\alpha(x)}$ is the component of φ in $L_2^{\bar{\alpha}}(G)$; cf. our remarks immediately following proposition 3.11 above. But this component is also equal to $a_{\bar{\alpha}} \varphi_{\bar{\alpha}}(x)$. By the uniqueness mentioned in the remarks after Proposition 3.11, we must have $a_\alpha = a_{\bar{\alpha}}$ and $\varphi_{\bar{\alpha}}(x) = \overline{\varphi_\alpha(x)}$. Another way of stating this is that if $\varphi_\alpha(x) = (T_\alpha(x)u_\alpha, u_\alpha)$ and $\varphi_{\bar{\alpha}}(x) = (T_{\bar{\alpha}}(x)u_{\bar{\alpha}}, u_{\bar{\alpha}})$, then we must have $u_{\bar{\alpha}} = u_\alpha$, where of course we are assuming T_α and $T_{\bar{\alpha}}$ are realized in the same representation space, as described above. We have proved

THEOREM 3.17. — A function Φ on G is real valued and in the class $\mathfrak{D}$ for (G, K) if and only if

$$\Phi(x) = \exp - \Psi(x) \tag{3.53}$$

where

$$\Psi(x) = \sum_{\alpha \in \mathfrak{K}_K(G)} a_\alpha (1 - \varphi_\alpha(x)) \tag{3.54}$$

where $a_\alpha \geq 0$, $\sum_{\alpha \in \mathfrak{K}_K(G)} a_\alpha < \infty$, $a_\alpha = a_{\bar{\alpha}}$ for each α, and $\varphi_\alpha(x)$ is for each α a normalized spherical positive definite function associated with α, such that $\varphi_{\bar{\alpha}}(x) = \overline{\varphi_\alpha(x)}$.

In the special case when G/K is a symmetric space, as we have remarked above, there is only one normalized elementary positive definite spherical function which is associated with a given member of $\mathfrak{K}_K(G)$. If, as above, we enumerate these spherical functions as $\varphi_0, \varphi_1, \ldots$ with $\varphi_0 \equiv 1$, and if φ_n is the function associated with a given class $\alpha \in \mathfrak{K}_K(G)$, then we may let n^* be the integer such that φ_{n^*} is the unique function in this list which is associated with the class $\bar{\alpha}$. The map $n \to n^*$ is an involution on $(0, 1, 2, \ldots)$ and $0^* = 0$. In this case we get the following theorem, which is the way we had stated it in Gangolli [3].

THEOREM 3.18. — Let G be a connected compact semisimple Lie group, and K a closed subgroup such that G/K is symmetric. A function Φ on G is a real valued function in the class $\mathfrak{D}$ for (G, K) if and only if

$$\Phi(x) = \exp - \Psi(x), \qquad x \in G$$

with

$$\Psi(x) = \sum_{n \geq 1} a_n(1 - \varphi_n(x)) \tag{3.55}$$

where $a_n \geq 0$, $a_n = a_{n^*}$, $\sum_{n \geq 1} a_n < \infty$; $\varphi_1, \varphi_2, \ldots$ are the elementary non-constant positive definite normalized spherical functions, and n^* is defined by $\varphi_{n^*}(x) = \overline{\varphi_n(x)}$. These conditions are equivalent to demanding that the function $\varphi(x)$ defined by $\varphi(x) = \Sigma a_n \varphi_n(x)$ is a real valued spherical continuous positive definite function with $\Sigma a_n < \infty$.

We state without comment the following theorem.

THEOREM 3.19. — A kernel f on G/K is a Lévy-Schoenberg kernel if and only if

$$f(a, b) = \frac{1}{2}(r(a, o) + r(b, o) - r(a, b)), \qquad a, b \in G/K \tag{3.56}$$

where $r(x\mathrm{K}, y\mathrm{K}) = \Psi(y^{-1}x)$, and Ψ is a real valued continuous spherical function on G of the form

$$\Psi(x) = \varphi(e) - \varphi(x), \tag{3.57}$$

with φ positive definite.

The condition that φ is real valued positive definite is equivalent to saying shat $\varphi(x) = \sum_{\alpha \in \mathfrak{K}_{\mathrm{K}}(\mathrm{G})} a_\alpha \varphi_\alpha(x)$ where $a_\alpha \geq o$, $a_{\bar{\alpha}} = a_\alpha$, $\sum_\alpha a_\alpha < \infty$, and $\overline{\varphi_\alpha(x)} = \varphi_{\bar{\alpha}}(x)$. When G/K is a symmetric speace, these conditions can be written $\varphi(x) = \sum_{n \geq 1} a_n \varphi_n(x)$, $a_n \geq 0$, $a_{n^*} = a_n$, $\sum a_n < \infty$, in the notation introduced above.

In many of the important cases of the classical compact symmetric spaces, the involution $n \to n^*$ is trivial, i. e. $n = n^*$ (This will always happen for example when the Weyl group of the symmetric space contains — I). The condition $n = n^*$ is equivalent to saying that φ_n is real valued for each n, or, alternatively, that each $\alpha \in \mathfrak{K}_K(G)$ is self-contragredient, i. e. $\bar{\alpha} = \alpha$. In this case every member of $\mathfrak{D}$ is real valued.

For example, the above property will hold for any compact symmetric space whose simply connected Riemannian covering space decomposes into irreductible factors, none of which is any of the following symmetric spaces: (*i*) A circle; (*ii*) A group manifold $\mathbf{SU}(n)$ $n > 2$; (*iii*) A group manifold $\mathbf{spin}$ $(4n + 2)$ $n > 0$; (*iv*) A group manifold $\mathbf{E}_6$; (*v*) $\mathbf{SU}(n)/\mathbf{SO}(n)$ $n > 2$; (*vi*) $\mathbf{SU}(2n)/\mathbf{sp}(n)$; (*vii*) $\mathbf{SO}(4n + 2)/\mathbf{SO}(2n + 1) \times \mathbf{SO}(2n + 1)$, $n > 0$; (*viii*) $\mathbf{E}_6/\mathbf{F}_4$; (*ix*) $\mathbf{E}_6/(\mathbf{sp}4/(\pm 1))$. For more information on this see Wolf [*1*].

We shall illustrate the results of this case by examples in § 4. For practical purposes, it is important to be able to determine when a given continuous function Ψ on G may be represented as $\varphi(e) - \varphi(x)$ where φ is a continuous positive definite function on G. In this connection we have the following theorem, and its corollaries which will be useful in § 4.

THEOREM 3.20. — Let ζ be a continuous function on G. For the left regular representation S of G on $L_2(G)$, let $L_2^\alpha(G)$ be the subspace of vectors in $L_2(G)$ which transform under S according to α. Let ζ^α be the component of ζ in $L_2^\alpha(G)$, so that $\zeta = \sum_{\alpha \in \mathfrak{K}(G)} \zeta^\alpha$, the sum converging in $L_2(G)$. If for each α, ζ^α is positive definite, then the convergence of $\Sigma\zeta^\alpha$ to ζ is uniform on G [10].

Proof. — Write $\zeta^\alpha = a_\alpha \varphi_\alpha$, where $a_\alpha \geq 0$ and φ_α is a normalized positive definite function associated with a representation of class α. It is clearly enough to show that $\sum_{\alpha \in \mathfrak{K}(G)} a_\alpha < \infty$.

If T_α is a representation of G of class α and μ a finite complex measure on G, write $T_\alpha(\mu) = \int_G T_\alpha(x)\, d\mu(x)$. If h is a complex valued function in $L_1(G)$ we write $T_\alpha(h) = \int_G T_\alpha(x) h(x) dx$.

---

(10) Note that because $L_2^\alpha(G)$ is finite dimensional for each $\alpha \in \mathfrak{K}(G)$, the function ζ^α is automatically continuous.

$L_2(G)$ is an algebra under convolution, and it is well known that by the compactness of G, L_2 is the direct sum of its minimal two-sided ideals. Indeed if $L_2^\alpha(G)$ is the subspace of $L_2(G)$ consisting of vectors which transform under the left regular representation according to α, $L_2^\alpha(G)$ is a minimal two-sided ideal in $L_2(G)$ and dim α. $\bar{\chi}_\alpha$ is the unique idempotent element in this ideal. Here χ_α is the character of α, i. e. $\chi_\alpha(x) = \text{Trace } T_\alpha(x)$. For these facts see Loomis [*1*]. It follows that convolution by (dim α) $\bar{\chi}_\alpha$ effects the projection of $L_2(G)$ onto $L_2^\alpha(G)$. Thus for any $h \in L_2(G)$, if h^α is its component in $L_2^\alpha(G)$, then

$$(3.58) \qquad h^\alpha = (\dim \alpha)\bar{\chi}_\alpha * h$$

$$\begin{aligned} \text{i. e.} \quad h^\alpha(x) &= \dim \alpha \int_G \bar{\chi}_\alpha(xy^{-1})h(y)dy \\ &= \dim \alpha \int_G \chi_\alpha(yx^{-1})h(y)dy \\ &= \dim \alpha . \text{Trace}\left(\int_G T_\alpha(y)h(y)dy . T_\alpha(x^{-1})\right) \\ &= \dim \alpha . \text{ Trace } (T_\alpha(h) . T_\alpha(x^{-1})). \end{aligned}$$

In particular

$$(3.59) \qquad \begin{aligned} \zeta^\alpha(x) &= \dim \alpha . \text{ Trace } (T_\alpha(\zeta)T_\alpha(x^{-1})) \\ &= a_\alpha \varphi_\alpha(x). \end{aligned}$$

Since φ_α is normalized this means that

$$(3.60) \qquad a_\alpha = \zeta^\alpha(e) = \dim \alpha . \text{ Trace } T_\alpha(\zeta).$$

Now let g be any central function in $L_1(G)$, i. e., $g(xy) = g(yx)$ and $g \in L_1(G)$. Then it follows that g is actually in the centre of $L_1(G)$. By the irreducibility of T_α, we have $T_\alpha(g)$ is a scalar operator equal to $\lambda_\alpha(g)I_\alpha$ say, where I_α is the identity on $H(T_\alpha)$. Then

$$(3.61) \qquad \begin{aligned} (g * \zeta)^\alpha(x) &= \dim \alpha . \text{ Trace } (T_\alpha(g)T_\alpha(\zeta) . T_\alpha(x^{-1})) \\ &= \dim \alpha . \lambda_\alpha(g) . \text{Trace } (T_\alpha(\zeta)T_\alpha(x^{-1})) \end{aligned}$$

so

$$(3.62) \qquad \begin{aligned} (g * \zeta)^\alpha(e) &= \dim \alpha . \lambda_\alpha(g) . \text{ Trace } T_\alpha(\zeta) \\ &= a_\alpha \lambda_\alpha(g). \end{aligned}$$

Now let $\{g_n\}_{n=1}^{\infty}$ be an approximate identity in $L_1(G)$ such that ([11])

i) g_n is central for each n.

ii) $g_n \geq 0$, $\int_G g_n(x)dx = 1$.

iii) For each n, $\lambda_\alpha(g_n) \geq 0$ for each α and $\lambda_\alpha(g_n) = 0$ for all but a finite number of α.

Such approximate identities exist, as is well known. For a proof see e. g. Edwards and Hewitt [*1*].

Then by (3.61),

$$(g_n * \zeta)^\alpha(x) = \dim \alpha . \lambda_\alpha(g_n) . \text{Trace } T_\alpha(\zeta)T_\alpha(x^{-1}). \tag{3.63}$$

Therefore, since for fixed n, $\lambda_\alpha(g_n)$ is zero except for a finite number of α, it follows that $(g_n * \zeta)^\alpha \equiv 0$ for all but a finite number of α, so $\sum_\alpha (g_n * \zeta)^\alpha$ is actually a finite sum. Hence

$$(g_n * \zeta)(x) = \sum_{\alpha \in \mathfrak{R}(G)} (g_n * \zeta)^\alpha(x) \qquad \textit{for each } x \tag{3.64}$$

and

$$\begin{aligned} (g_n * \zeta)(e) &= \sum_{\alpha \in \mathfrak{R}(G)} (g_n * \zeta)^\alpha(e) \\ &= \sum_{\alpha \in \mathfrak{R}(G)} a_\alpha \lambda_\alpha(g_n). \end{aligned} \tag{3.65}$$

Now

$$\begin{aligned} |(g_n * \zeta)(e)| &= \left| \int g_n(x)\zeta(x^{-1})dx \right| \\ &\leq \| \zeta \| \int_G g_n(x)dx \\ &\leq \| \zeta \| \quad \text{where} \quad \| \zeta \| = \sup_{x \in G} |\zeta(x)|. \end{aligned} \tag{3.66}$$

Therefore we have for all n

$$(g_n * \zeta)(e) = \sum_{\alpha \in \mathfrak{R}(G)} a_\alpha \lambda_\alpha(g_n) \leq \| \zeta \|, \tag{3.67}$$

([11]) Since we are concerned with a single function $\zeta \in L_1(G)$, we may assume $L_1(G)$ to be separable, so that it is enough to work with an approximate identity which is a sequence rather than a net.

Now, as $n \to \infty$, since g_n is an approximate identity, $\lambda_\alpha(g_n)$ (which is just Trace $T_\alpha(g_n)/\dim \alpha$), approaches 1 (= Trace $T_\alpha(\mu_e)/\dim \alpha$, where μ_e is the point mass at e).

Thus in (3.67), if we let $n \to \infty$, and remember that $a_\alpha \geq 0$, $\lambda_\alpha(g_n) \geq 0$, we easily conclude from (3.67) that we have

$$\sum_{\alpha \in \mathfrak{R}(G)} a_\alpha \leq \| \zeta \| < \infty. \tag{3.68}$$

Q. E. D.

This theorem and its proof are to be compared to a classical theorem on Fourier series, which says that if a continuous function on the unit circle has all its Fourier coefficients nonnegative then its Fourier series converges uniformly on the unit circle. Indeed the above theorem reduces to this in case G is the unit circle. For similar questions see Krein [*1*].

COROLLARY 3.21. — Let Ψ be a continuous function on G. In order that $\Psi(x) = \varphi(e) - \varphi(x)$ where φ is a continuous positive definite function on G, the following conditions are necessary and sufficient: (*i*) $\Psi^{\mathfrak{1}} \geq 0$, (*ii*) $-\Psi^\alpha$ is positive definite for each $\alpha \neq \mathfrak{1} \in \mathfrak{R}(G)$, (*iii*) $\Psi(e) = 0$ [12].

Proof. — If $\Psi(x) = \varphi(e) - \varphi(x)$ with $\varphi(x)$ continuous and positive definite, then $\varphi(x) = \sum_{\alpha \in \mathfrak{R}(G)} a_\alpha \varphi_\alpha(x)$ and so $\varphi(e) = \sum_{\alpha \in \mathfrak{R}(G)} a_\alpha$ [13]. Remembering that $\varphi_{\mathfrak{1}} \equiv 1$ we see that

$$\Psi(x) = \sum_{\substack{\alpha \in \mathfrak{R}(G) \\ \alpha \neq \mathfrak{1}}} a_\alpha - \sum_{\substack{\alpha \in \mathfrak{R}(G) \\ \alpha \neq \mathfrak{1}}} a_\alpha \varphi_\alpha(x). \tag{3.69}$$

By the uniqueness of the expansion $\Psi(x) \stackrel{L_2}{=} \Sigma \Psi^\alpha(x)$ (cf. remarks after Proposition 3.11) we have $\Psi^{\mathfrak{1}} = \sum_{\substack{\alpha \in \mathfrak{R}(G) \\ \alpha \neq \mathfrak{1}}} a_\alpha \geq 0$ and $-\Psi^\alpha = a_\alpha \varphi_\alpha$ which is positive definite. Of course, $\Psi(e) = 0$.

Conversely, if $\Psi^{\mathfrak{1}} \geq 0$ and $-\Psi^\alpha$ is positive definite, let $-\Psi^\alpha = a_\alpha \varphi_\alpha$

([12]) $\Psi^{\mathfrak{1}}$ denotes, as has been mentioned, the component of Ψ in $L_2^{\mathfrak{1}}(G)$; since the only elementary normalized function associated with $\mathfrak{1}$ is the function identically equal to 1, $L_2^{\mathfrak{1}}(G)$ consists of just the constants. It follows that $\Psi^{\mathfrak{1}}$ is a constant.

([13]) Cf. Proposition 3.11.

with $a_\alpha \geq 0$ and φ_α a normalized positive definite function associated with α. Then we have

$$\Psi(x) \overset{L_2}{=} \Psi^{\mathfrak{1}} - \sum_{\substack{\alpha \in \mathfrak{R}(G) \\ \alpha \neq \mathfrak{1}}} a_\alpha \varphi_\alpha(x) \tag{3.70}$$

where the sum converges in $L_2(G)$. Now let $\varphi(x)$ be the function $\sum_{\alpha \neq \mathfrak{1}} a_\alpha \varphi_\alpha(x)$ in $L_2(G)$. Then $\varphi(x) = \Psi^{\mathfrak{1}} - \Psi(x)$, so φ is continuous, and $a_\alpha \varphi_\alpha$ is its component in $L_2^{\alpha}(G)$. But $a_\alpha \varphi_\alpha = - \Psi^{\alpha}$ which is positive definite. By the above theorem, it follows that $\Sigma a_\alpha \varphi_\alpha$ converges uniformly to φ, so that $\Sigma a_\alpha < \infty$. Thus

$$\Psi(x) = \Psi^{\mathfrak{1}} - \varphi(x), \qquad \text{all } x \in G \tag{3.71}$$

and since $\Psi(e) = 0$ we have $\Psi^{\mathfrak{1}} = \varphi(e)$ so

$$\Psi(x) = \varphi(e) - \varphi(x). \tag{3.72}$$

Q. E. D.

Note that in both the above theorems the restriction that the functions involved be spherical can be introduced trivially.

These results will be quite useful to us in the special case when G/K is symmetric. In that case, as we have noted, there is just one elementary normalized spherical function φ_α associated with each $\alpha \in \mathfrak{R}_K(G)$, and this function is positive definite.

Now given continuous spherical function Ψ on G, its component Ψ^α is a constant multiple of the unique elementary spherical function associated with α, i. e. $\Psi^\alpha = b_\alpha \varphi_\alpha$. Since φ_α is positive definite, $-\Psi^\alpha$ is positive definite if and only if $b_\alpha \leq 0$. Further, b_α can be expressed succinctly in terms of Ψ and φ_α; indeed, $b_\alpha = \dim \alpha \int_G \Psi(x)\overline{\varphi_\alpha(x)}dx$. For, $\Psi \overset{L_2}{=} \sum_{\alpha \in \mathfrak{R}(G)} b_\alpha \varphi_\alpha$, and by the Schur orthogonality relations, we have $\int_G \varphi_\alpha(x)\overline{\varphi_\beta(x)}dx = \frac{1}{\dim \alpha} \cdot \delta_{\alpha\beta}$. Thus in this special case we get a more explicit form of the above corollary which is perhaps worth noting, since it is of great usefulness.

COROLLARY 3.22. — Let G be a connected compact semisimple Lie group and let K be a closed subgroup such that G/K is a symmetric space. Let $\varphi_0, \varphi_1, \ldots$ be the elementary spherical normalized positive definite functions on G, $\varphi_0 \equiv 1$. A continuous spherical function Ψ on G is of the form $\Psi(x) = \varphi(e) - \varphi(x)$ with φ a continuous positive definite function on G, if and only if the following three conditions are satisfied.

$$\int_G \Psi(x)dx \geq 0. \tag{3.73}$$

$$\int_G \Psi(x)\overline{\varphi_n(x)}dx \leq 0 \quad \text{for each} \quad n \geq 1 \tag{3.74}$$

$$\Psi(e) = 0. \tag{3.75}$$

Of course, conditions under which Ψ (or φ) is real may be described exactly as before.

The main point of all the work in the last few pages is that they make it unnecessary, in determining whether Ψ will give rise to a Lévy-Schoenberg kernel, to check the crucial condition $\sum_\alpha a_\alpha < \infty$ which occurs in (3.54), and which is often messy to check. Indeed the content of the above theorem is precisely that when a non-positivity condition like (3.74) can be checked, then the condition $\Sigma a_n < \infty$ follows automatically. In practice, as we shall see, this turns out to be very convenient.

CASE IV Begin with a connected non-compact semisimple Lie group G whose center is finite and let K be a maximal compact subgroup of G. Then G/K is a Riemannian symmetric space of non-compact type and every Riemannian symmetric space of non-compact type can be obtained in this way ([14]). By a representation T of G we shall always understand a continuous unitary representation on a Hilbert space H(T). Notions like spherical representations and functions are introduced exactly as in Case III. Thus $\mathfrak{K}(G)$, $\mathfrak{K}_K(G)$ will have the same meanings. Of course, members of any class in $\mathfrak{K}(G)$ are now infinite dimensional in general.

Let $\alpha \in \mathfrak{K}_K(G)$ and T_α be a representation of class α. Then if $H_K(T_\alpha)$ is the subspace of $H(T_\alpha)$ consisting of vectors u such that $T_\alpha(k)u = u$ for all $k \in K$, then it is a result of Gelfand and Naimark that $H_K(T_\alpha)$ is one-dimensional. See e. g. Helgason [*1*, p. 416] (This constitutes a generalization of

([14]) The unfortunate usage of the words *non-compact type* is technical, i. e. the term « symmetric space of *non-compact type* » is reserved for those Riemannian symmetric spaces whose group of isometries is semisimple and non-compact. Thus Euclidean spaces are non-compact Riemannian symmetric but are not Riemannian symmetric spaces of non-compact type. See Helgason [*1*, Chapter V].

the result of É. Cartan mentioned in Case III). It follows that there is exactly one normalized elementary spherical function associated with a class $\alpha \in \mathfrak{K}_K(G)$, namely the function $\varphi_\alpha(x) = (T_\alpha(x)u, u)$ where u is a unit vector in $H_K(T_\alpha)$. Note that φ_α is automatically positive definite. Thus $\mathfrak{K}^K(G)$ is in one-one correspondence with the set of all normalized elementary positive definite spherical functions on G. This set will be denoted by $\mathcal{M}$. Given a $\varphi \in \mathcal{M}$, the construction of Gelfand and Raikov and Godement cited above determines the class of representations with which it is associated.

We shall need several known facts about $\mathcal{M}$. For these reference may be made to Godement [2] and Helgason [1, Chapter X]. These facts are set out here.

The spherical functions which are in $L_1(G)$ form a closed *-subalgebra of $L_1(G)$, which we denote by $L_1(K\backslash G/K)$. The involution is, of course, $h \to h^*$ where $h^*(x) = \overline{h(x^{-1})}$. The most important fact about $L_1(K\backslash G/K)$ is that it is a *commutative* Banach *-algebra. A continuous homomorphism of $L_1(K\backslash G/K)$ onto the complex numbers is of the form

$$h \to \int_G h(x)\varphi(x)dx \tag{3.78}$$

where φ is continuous bounded and satisfies

$$\int_K \varphi(xky)dk = \varphi(x)\varphi(y) \qquad x, y \in G \tag{3.79}$$

$$\varphi(e) = 1. \tag{3.80}$$

Thus the maximal ideal space $\mathcal{M}'$ of $L_1(K\backslash G/K)$ is just the set of all bounded continuous functions φ which satisfy (3.79), (3.80). Actually, a solution of (3.79) is analytic, a fact which we will use below. $\mathcal{M}'$ may be topologized with the topology of uniform convergence on compact subsets of G, and with this topology it is a locally compact space. We denote by ∞ the point at infinity on $\mathcal{M}'$. By the Gelfand structure theory, for a given $h \in L_1(K\backslash G/K)$, the function $\widehat{h}(\varphi) = \varphi(h) = \int_G h(x)\varphi(x)dx$ vanishes at ∞ on $\mathcal{M}'$. $\mathcal{M}$ is a closed subset of $\mathcal{M}'$, and consists of exactly those functions in which are positive definite. It follows that the restriction of $\widehat{h}$ to $\mathcal{M}$ also vanishes at infinity. Finally, in Gelfand [1] we find the following analogue of Bochner's theorem for locally compact abelian groups.

THEOREM 3.23. —Given a continuous positive definite spherical function ζ on G, there exists a unique nonnegative measure μ on $\mathcal{M}$ such that

$$\zeta(x) = \int_{\mathcal{M}} \varphi(x)d\mu(\varphi). \qquad x \in G. \tag{3.81}$$

Here, for fixed x, we are regarding $\varphi(x)$ as a continuous function on $\mathcal{M}$.

We are now in a position to begin our characterization of real valued functions in the class $\mathfrak{D}$ for (G, K). The proof parallels that in Case III modulo some technical complications.

PROPOSITION 3.24. — If $\Phi \in \mathfrak{D}$ then $\Phi(x) \neq 0$ for any $x \in G$.

The proof is identical to that of the corresponding proposition in Case III, and so will not be repeated.

Now since G is a connected Lie group, it is arcwise connected and it follows, exactly as in Case III, that if $\Phi \in \mathfrak{D}$ then the principal value of $\log \Phi(x)$ is continuous on G. Calling it $-\Psi(x)$, we get

THEOREM 3.25. — A continuous spherical function Φ on G is in the class $\mathfrak{D}$ for the pair (G, K) if and only if it admits the representation

$$\Phi(x) = \exp - \Psi(x) \tag{3.82}$$

where $\Psi(x)$ is a continuous spherical function on G such that

$$\Psi(x) = \lim_{j\to\infty} d_j(1 - \Phi_j(x)), \qquad x \in G \tag{3.83}$$

where $d \geq 0$, Φ is, for each j, a normalized continuous positive definite spherical function on G, and the limit in (3.83) is uniform in some neighborhood of e in G.

For the proof once again we refer the reader to the corresponding proposition in Case III. Note that the properties of positive definite functions stated at the outset in Case III also hold when G is noncompact, so the same proof works.

LEMMA 3.26. — Let A be any compact neighborhood of $e \in G$ such that $kAk' \subset A$, $k, k' \in K$. For $\varphi \in \mathcal{M}$ let $Q_A(\varphi)$ be the function on $\mathcal{M}$ defined by

$$Q_A(\varphi) = \frac{1}{\text{vol. A}} \int_A (1 - \operatorname{Re} \varphi(x))dx. \tag{3.84}$$

Then $0 \leq Q_A(\varphi) \leq 2$, and $Q_A(\varphi) = 0$ if and only if $\varphi = \mathfrak{1}$ [15]. Further $Q_A(\varphi) \to 1$ as $\varphi \to \infty$ on $\mathcal{M}$. Here vol. A is the Haar measure of A.

Proof. — Since φ is positive definite, it is clear that $0 \leq Q_A(\varphi) \leq 2$. Next if $Q_A(\varphi) = 0$ then $\operatorname{Re} \varphi(x) = 1$ for x in A because φ is continuous and positive definite. Since G is connected, it follows exactly as in Lemma 3.13

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[15] $\mathfrak{1}$ is the function identically equal to 1. Clearly $\mathfrak{1} \in \mathcal{M}$.

that φ must be identically 1, so $Q_A(\varphi) = 0$ if and only if $\varphi = 1$. Finally

$$Q_A(\varphi) = 1 - \int_G h(x)\varphi(x)dx \qquad \text{where} \qquad h(x) = \frac{1}{\text{vol. A}} \cdot \chi_A(x)$$

and χ_A is the indicator function of A. Since A is compact, $h(x) \in L_1$. Further because $kAk' \subset A$, $h(x) \in L_1(K\backslash G/K)$. Hence $\widehat{h}(\varphi) = \int_G h(x)\varphi(x)dx$ vanishes at infinity on $\mathcal{M}$. This implies that $Q_A(\varphi) \to 1$ as $\varphi \to \infty$ on $\mathcal{M}$.

Q. E. D.

As before, we must characterize functions $\Psi(x)$ which satisfy (3.83). For a given Φ_j which is continuous, normalized, spherical and positive definite, apply Gelfand's theorem 3.23 to obtain a unique measure μ_j on $\mathcal{M}$ such that

$$\Phi_j(x) = \int_{\mathcal{M}} \varphi(x)d\mu_j(\varphi).$$

Then :

$$\mu_j(\mathcal{M}) = 1, \qquad \text{so,} \qquad 1 - \Phi_j(x) = \int_{\mathcal{M}} (1 - \varphi(x))d\mu_j(\varphi).$$

Therefore

$$d_j(1 - \Phi_j(x)) = \int_{\mathcal{M}} (1 - \varphi(x))dL_j(\varphi)$$

where L_j is the measure $d_j\mu_j$ on $\mathcal{M}$. L_j is a nonnegative finite measure on $\mathcal{M}$ of total mass d_j. Thus if we write

$$\Psi(x) = d_j(1 - \Phi_j(x)) = \int_{\mathcal{M}} (1 - \varphi(x))dL_j(\varphi)$$

we see that (3.83) is equivalent to

$$\Psi(x) = \lim_{j \to \infty} \Psi_j(x) \tag{3.85}$$

$$\Psi_j(x) = \int_{\mathcal{M}} (1 - \varphi(x))dL_j(\varphi) \tag{3.86}$$

where $\{L_j\}$ is a sequence of nonnegative finite measures on $\mathcal{M}$, and the convergence in (3.85) is uniform for x in some neighborhood of e in G.

$\mathcal{M}$ has a natural involution defined on it, namely the involution $\varphi \to \overline{\varphi}$, defined by complex conjugation. Thus functions, measures etc. on $\mathcal{M}$ also inherit it. For example given a measure μ on $\mathcal{M}$ we define $\overline{\mu}(A) = \mu(\overline{A})$ where $\overline{A}$ is the image of A under $\varphi \to \overline{\varphi}$. We may say that μ is self-adjoint if $\mu = \bar{\mu}$. Note that if μ is a self-adjoint finite nonnegative measure then $\int_{\mathcal{M}} \varphi(x) d\mu(\varphi)$ is a *real valued* continuous positive definite spherical function on G and conversely every function of the latter description can be written, because of Theorem 3.23, in the form $\int_{\mathcal{M}} \varphi(x) d\mu(\varphi)$ with μ a self-adjoint nonnegative finite measure on $\mathcal{M}$; of course μ is unique.

THEOREM 3.27 ([16]). — A continuous real valued spherical function $\Psi(x)$ on G satisfies (3.85), (3.86) if and only if it admits a representation as follows,

$$\Psi(x) = g(x) + \int_{\mathcal{M}-\{1\}} (1 - \varphi(x)) dL(\varphi), \qquad x \in G \tag{3.87}$$

where g is a function on G and L a measure on $\mathcal{M} - \{1\}$, and g, L meet the requirements (*a*), (*b*), (*c*) below.

(*a*) L is a self-adjoint nonnegative σ-finite measure on $\mathcal{M} - \{1\}$, and gives finite mass to the complement of any neighborhood of $\{1\}$ in $\mathcal{M}$.

(*b*) If A is any compact neighborhood of e in G such that $KAK \subset A$, and $Q_A(\varphi)$ is the function on $\mathcal{M}$ defined by

$$Q_A(\varphi) = \int_A (1 - \operatorname{Re} \varphi(x)) dx / \operatorname{vol.} A,$$

then

$$\int_{\mathcal{M}} Q_A(\varphi) dL(\varphi) < \infty. \tag{3.88}$$

(*c*) g is a nonnegative continuous function on G of the form

$$g(x) = \lim_{r \to \infty} \int_{U_r} (1 - \varphi(x)) d\nu_r(\varphi); \qquad x \in G \tag{3.89}$$

where $\{U_r\}_{r=1}^{\infty}$ is a sequence of compact neighborhoods of 1 in $\mathcal{M}$ such

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([16]) Consideration of all the complex valued functions in $\mathfrak{D}$ gets extremely technical, and involves a point in the representation theory of semisimple Lie groups whose present status must be regarded as dubious in view of some recent work of Kunze and Stein (unpublished). We therefore consider only real valued members of $\mathfrak{D}$. As will be pointed out below, for a fairly large class of spaces, this is actually no restriction, for it turns out in those cases that all members of $\mathfrak{D}$ are real valued.

that $U_{r+1} \subset U_r$ and $\bigcap_r U_r = \{ e \}$, and $\{ \nu_r \}$ is a sequence of finite nonnegative self-adjoint measures on $\mathcal{M}$ such that the support of ν_r is contained in U_r.

Proof. — First suppose that g, **L** fulfill (*a*), (*b*), (*c*) and that (3.87) defines the *continuous* real valued function Ψ. We want to show that Ψ satisfies (3.85), (3.86). Let U_r, ν_r be as guaranteed by (*c*) and define the measure L_r on $\mathcal{M}$ to be the sum of ν_r and the restriction of **L** to the complement of U_r. Then L_r is a self-adjoint finite nonnegative measure. Let $\Psi_r(x) = \int_{\mathcal{M}} (1 - \varphi(x)) dL_r(\varphi)$. It is obvious that $\Psi_r(x) \to \Psi(x)$ pointwise for $x \in G$. So we merely have to show that this convergence is uniform in a neighborhood of e. Now $\exp - \Psi_r(x)$ is a positive definite continuous function on G for each r (because $\exp(\varphi(x) - 1)$ is so for each $\varphi \in \mathcal{M}$ and $\exp - \Psi_r(x)$ is a « continuous » product of such functions), and since $\exp - \Psi_r(x) \to \exp - \Psi(x)$ it follows that $\exp - \Psi(x)$ is also positive definite. Further $\Psi_j(x) \geq 0$, $\Psi(x) \geq 0$, so the convergence of $\exp - \Psi_j(x) \to \exp - \Psi(x)$ is taking place *boundedly* on G. Now Gelfand's Lemma (see our introductory remarks in Case III) may be applied to the the sequence $\exp - \Psi_r(x)$, to conclude that this sequence of continuous positive definite functions on G must converge uniformly on compact subsets of G to the continuous positive definite function $\exp - \Psi(x)$. This clearly implies that Ψ is approached uniformly in a neighborhood of e by Ψ_r.

Conversely, let us start with a real valued $\Psi(x)$ which satisfies (3.85), (3.86). Now,

$$\operatorname{Re} \Psi_j(x) = \int_{\mathcal{M}} 1 - \varphi(x) d\left(\frac{L_j + \overline{L_j}}{2}\right)(\varphi),$$

hence, since Ψ is real valued, we may assume that each Ψ^j is likewise real valued and therefore that each L_j is self-adjoint. Now if B is any compact neighborhood of e in G on which $\Psi_j(x) \to \Psi(x)$ uniformly, then we have

$$\int_A \Psi_j(x) dx \to \int_A \Psi(x) dx$$

for any $A \subset B$. In particular, for any such A, we have a constant M_A such that

$$\frac{1}{\text{vol. A}} \int_A \Psi_j(x) dx \leq M_A \qquad \text{for all } j. \tag{3.88}$$

Now $\Psi_j(x) = \int_{\mathcal{M}} (1 - \varphi(x)) dL_j(\varphi) = \int_{\mathcal{M}} (1 - \text{Re } \varphi(x)) dL_j(\varphi)$ because $L_j = \bar{L}_j$. Hence

$$\begin{aligned}(3.89) \qquad & \frac{1}{\text{vol. A}} \cdot \int_A \Psi_j(x) dx \\ &= \int_{\mathcal{M}} Q_A(\varphi) dL_j(\varphi) \\ &\leq M_A \qquad \text{for all } j.\end{aligned}$$

Let U be any compact neighborhood of $\mathbf{1}$ in $\mathcal{M}$, and U^c its complement in $\mathcal{M}$. Then, by Lemma 3.26, $Q_A(\varphi)$ is bounded and bounded away from zero on U^c. Hence if $\delta_A > 0$ is a lower bound for $Q_A(\varphi)$ on U^c, we have

$$\begin{aligned}(3.90) \qquad & \int_{U^c} dL_j(\varphi) \\ &\leq \frac{1}{\delta_A} \int_{U^c} Q_A(\varphi) dL_j(\varphi) \\ &\leq M_A/\delta_A.\end{aligned}$$

Next, given $\varepsilon > 0$, choose $A \subset B$ so small that $\sup_{x \in A} |\Psi(x)| < \varepsilon$, and the integer j_0 so large that for $j \geq j_0$ we have

$$\frac{1}{\text{vol. A}} \int_A \Psi_j(x) dx \leq \frac{1}{\text{vol. A}} \int_A \Psi(x) dx + \varepsilon.$$

Then we have for any neighborhood N of $\mathbf{1}$ in $\mathcal{M}$,

$$\begin{aligned}(3.91) \qquad & \int_{N^c} Q_A(\varphi) dL_j(\varphi) \\ &= \frac{1}{\text{vol. A}} \int_A \Psi_j(x) dx \\ &\leq 2\varepsilon.\end{aligned}$$

But $Q_A(\varphi) \to 1$ as $\varphi \to \infty$ on $\mathcal{M}$ so N may be chosen so that $Q_A(\varphi) \geq \frac{1}{2}$ for $\varphi \in N^c$. For this choice of N we have

$$\begin{aligned}(3.92) \qquad & \int_{N^c} dL_j(\varphi) \\ &\leq 2 \int_{N^c} Q_A(\varphi) dL_j(\varphi) \\ &\leq 4\varepsilon.\end{aligned}$$

It is immediate from (3.90) and (3.92) that there exists a subsequence of L (again called L_j) such that the sequence of measures L converges weak * on the complement of any neighborhood U of $\mathfrak{1}$ in $\mathcal{M}$ to a measure $\mathbf{L}$. Further, because of (3.90), $\mathbf{L}$ gives finite mass to the complement of every such neighborhood, so $\mathbf{L}$ is σ-finite. Clearly $\mathbf{L}$ is self-adjoint, because each L_j is, and by (3.89) we have $\int_{U^c} Q_A(\varphi)d\mathbf{L}(\varphi) < M_A$ for any compact neighborhood A of e in G, such that $KAK \subset A$. It follows that

$$\int_{\mathcal{M}-\{\mathfrak{1}\}} Q_A(\varphi)d\mathbf{L}(\varphi) < \infty.$$

Now let $U_1, U_2, \ldots$ be a sequence of compact neighborhoods of χ in such that $U_{r+1} \subset U_r$ and $\bigcap_r U_r = \{\mathfrak{1}\}$. Then

$$\text{(3.93)} \qquad \Psi_j(x) = \int_{U_r} (1 - \varphi(x))dL_j(\varphi) + \int_{U_r^c} (1 - \varphi(x))dL_j(\varphi).$$

Letting $j \to \infty$, and remembering that for fixed x, $1 - \varphi(x)$ is a bounded continuous function of φ, we have

$$\text{(3.94)} \qquad \Psi(x) = \lim_{j\to\infty} \int_{U_r} (1 - \varphi(x))dL_j(\varphi) + \int_{U_r^c} (1 - \varphi(x))d\mathbf{L}(\varphi).$$

The limit on the right side exists because the other two terms have a limit. Letting $r \to \infty$, and remembering that $\mathbf{L}$ is self-adjoint, and therefore $\int_{U_r^c} (1 - \varphi(x))\mathbf{L}d(\varphi) = \int_{U_r^c} (1 - \operatorname{Re} \varphi(x))d\mathbf{L}(\varphi)$ is non-decreasing in r, we get

$$\text{(3.95)} \qquad \Phi(x) = g(x) + \int_{\mathcal{M}-\{\mathfrak{1}\}} (1 - \varphi(x))d\mathbf{L}(\varphi)$$

where $g(x) = \lim_{r\to\infty} \lim_{j\to\infty} \int_{U_r} (1 - \varphi(x))dL_j(\varphi)$, and this limit also exists for obvious reasons. This limit is monotone, therefore uniform on compacts in G ; it follows that g is continuous, spherical and nonnegative. Finally we may choose a subsequence $\{j_r\}_{r=1}^{\infty}$ of 1, 2, ... such that

$$\lim_{r\to\infty} \lim_{j\to\infty} \int_{U_r} (1 - \varphi(x))dL_j(\varphi) = \lim_{r\to\infty} \int_{U_r} (1 - (\varphi))dL_{j_r}(\varphi).$$

If ν_r is the restriction of L_{j_r} to U_r we get $g(x) = \lim_{r\to\infty} \int_{U_r} (1 - \varphi(x)) d\nu_r(\varphi)$ and the theorem is proved.

We should now like to get a uniqueness assertion, stating that in the representation (3.87), the function g and the measure $\mathbf{L}$ are uniquely determined by Ψ. To do this we need a lemma.

LEMMA 3.28. — The function g described above satisfies the following functional equation:

$$\int_K g(xky)dk = g(x) + g(y). \tag{3.96}$$

Proof. — Using the functional equation $\int_K \varphi(xky)dk = \varphi(x)\varphi(y)$ for $\varphi \in \mathcal{M}$, we have

$$\begin{aligned} \int_K (1 - \varphi(xky))dk &\\ &= 1 - \varphi(x)\varphi(y) \\ &= (1 - \varphi(x)) + (1 - \varphi(y)) - (1 - \varphi(x))(1 - \varphi(y)). \end{aligned} \tag{3.97}$$

Using the facts proved above and that g is nonnegative, we now have ([17])

$$\begin{aligned} \int_K g(xky)dk &\\ &= \int_K \left(\lim_{r\to\infty} \int_{U_r} (1 - \varphi(xky)) d\nu_r(\varphi) \right) dk \\ &= \lim_{r\to\infty} \int_K \int_{U_r} (1 - \varphi(xky)) d\nu_r(\varphi) dk \end{aligned} \tag{3.98}$$

while

$$\begin{aligned} &\int_K \int_{U_r} (1-\varphi(xky)) d\nu_r(\varphi) dk \\ &= \int_{U_r} \int_K 1-\varphi(xky) dk d\nu_r \\ &= \int_{U_r} (1-\varphi(x)) d\nu_r(x) + \int_{U_r} (1-\varphi(x)) d\nu_r(\varphi) + \int_{U_r} (1-\varphi(x))(1-\varphi(y)) d\nu_r(\varphi). \end{aligned} \tag{3.99}$$

([17]) Because the measures involved are self adjoint, the integrals $\int_{\mathcal{M}} (1 - \varphi(x)) d\nu_r(\varphi)$ etc. may be written $\int_{\mathcal{M}} (1 - \mathrm{Re}\, \varphi(x)) d\nu_r(\varphi)$. Thus the integrands can be assumed nonnegative, and the interchanges of the orders of integration below can be justified.

Next

$$\left|\int_{U_r}(1-\varphi(x)(1-\varphi(y))d\nu_r(\varphi)\right| \leq \sup_{\varphi\in U_r}|1-\varphi(x)|.\int_{U_r}|1-\varphi(y)|\,d\nu_r(\varphi). \tag{3.100}$$

But φ is positive definite, hence $|1-\varphi(y)| \leq 2(1-\mathrm{Re}\,\varphi(y))$. So,

$$\begin{aligned}\left|\int_{U_r}(1-\varphi(x))(1-\varphi(y))d\nu_r(\varphi)\right| &\leq 2\sup_{\varphi\in U_r}|1-\varphi(x)|.\int_{U_r}(1-\mathrm{Re}\,\varphi(y))d\nu_r(\varphi)\\ &= 2\sup_{\varphi\in U_r}|1-\varphi(x)|.\mathrm{Re}\left(\int_{U_r}(1-\varphi(y))d\nu_r(\varphi)\right).\end{aligned} \tag{3.101}$$

As $r\to\infty$, we have $U_r \downarrow \{1\}$ so $\sup_{\varphi\in U_r}|1-\varphi(x)|\to 0$, while the 2nd factor on the right side in (3.101) converges to $\mathrm{Re}\,g(y)=g(y)$. It follows that

$$\lim_{r\to\infty}\int_{U_r}(1-\varphi(x))(1-\varphi(y))d\nu_r(\varphi)=0. \tag{3.102}$$

By (3.98), (3.99) and (3.102), we get

$$\begin{aligned}\int_K g(xky)dk &= \lim_{r\to\infty}\int_{U_r}(1-\varphi(x))d\nu_r(\varphi)+\lim_{r\to\infty}\int_{U_r}(1-\varphi(y))d\nu_r(\varphi)\\ &= g(x)+g(y).\end{aligned} \tag{3.103}$$

Q. E. D.

LEMMA 3.29. — Let μ, ν be totally finite signed real measures on $\mathcal{M}$ such that for each $x\in G$, $\int_{\mathcal{M}}\varphi(x)d\mu(\varphi)=\int_{\mathcal{M}}\varphi(x)d\nu(\varphi)$. Then $\mu=\nu$.

Proof. — Let $\mu=\mu_+-\mu_-$ and $\nu=\nu_+-\nu_-$ be the Hahn-Jordan decompositions of μ and ν. Then we have

$$\int_{\mathcal{M}}\varphi(x)d(\mu_++\nu_-)(\varphi)=\int_{\mathcal{M}}\varphi(x)d(\nu_++\mu_-)(\varphi).$$

Now it follows by theorem 3.23 that $\mu^++\nu^-=\nu^++\mu^-$ so that $\mu=\nu$.

Q. E. D.

THEOREM 3.30. — In the representation

$$(3.104) \qquad \Psi(x) = g(x) + \int_{\mathcal{M}-\{\mathbf{1}\}} (1 - \varphi(x)) d\mathbf{L}(\varphi)$$

where g, $\mathbf{L}$ satisfy the conditions of theorem 3.27, the function g and the measure $\mathbf{L}$ are uniquely determined by Ψ.

Proof. — Let g', $\mathbf{L}'$ be another pair satisfying the conditions of theorem 3.27 for which $\Psi(x) = g'(x) + \int_{\mathcal{M}-\{\mathbf{1}\}} (1 - \varphi(x)) d\mathbf{L}'(\varphi)$. We want to show that $g = g'$, $\mathbf{L} = \mathbf{L}'$. Let $\tilde{g} = g - g'$ and $\tilde{\mathbf{L}} = \mathbf{L}' - \mathbf{L}$. We extend $\mathbf{L}, \mathbf{L}', \tilde{\mathbf{L}}$ to all of $\mathcal{M}$ by setting $\mathbf{L}(\{\mathbf{1}\}) = \mathbf{L}'(\{\mathbf{1}\}) = \tilde{\mathbf{L}}(\{\mathbf{1}\}) = 0$. Then we have.

$$(3.105) \qquad \tilde{g}(x) = \int_{\mathcal{M}} (1 - \varphi(x)) d\tilde{\mathbf{L}}(\varphi).$$

Now since g, g' satisfy (3.96), so does $\tilde{g}$. Hence

$$\begin{aligned}
(3.106) \quad 0 &= \int_{\mathcal{M}} \tilde{g}(xky)dk - \tilde{g}(x) - \tilde{g}(y) \\
&= \int_{\mathcal{M}} \left(1 - \int_{\mathrm{K}} \varphi(xky)dk - (1 - \varphi(x)) - (1 - \varphi(y))\right) d\tilde{\mathbf{L}}(\varphi) \\
&= \int_{\mathcal{M}} (1 - \varphi(x)\varphi(y) - (1 - \varphi(x)) - (1 - \varphi(y))) d\tilde{\mathbf{L}}(\varphi) \\
&= - \int_{\mathcal{M}} (1 - \varphi(x))(1 - \varphi(y)) d\tilde{\mathbf{L}}(\varphi).
\end{aligned}$$

This being true for each y, we may change y to y^{-1}, add the resulting equation to the above, and use $\varphi(y^{-1}) = \overline{\varphi}(y)$, to get

$$(3.107) \qquad \int_{\mathcal{M}} (1 - \varphi(x))(1 - \operatorname{Re} \varphi(y)) d\tilde{\mathbf{L}}(\varphi) = 0 \qquad x, y \in \mathrm{G}.$$

Now let $\tilde{\mathbf{L}}_y$ be the signed measure on G defined by

$$\int_{\mathrm{D}} d\tilde{\mathbf{L}}_y(\varphi) = \int_{\mathrm{D}} (1 - \operatorname{Re} \varphi(y)) d\tilde{\mathbf{L}}(\varphi)$$

where D is a Borel set of $\mathcal{M}$. Then, because $\int_{\mathcal{M}} 1 - \operatorname{Re} \varphi(y) d\mathbf{L}$ and

$\int_{\mathcal{M}} 1 - \operatorname{Re} \varphi(y) d\mathbf{L}'$ are finite, we see that $\tilde{\mathbf{L}}_y$ is a totally finite signed real measure. Also, (3.107) says

$$\int_{\mathcal{M}} \varphi(x) d\tilde{\mathbf{L}}_y(\varphi) = \tilde{\mathbf{L}}_y(\mathcal{M}). \tag{3.108}$$

The right side of this is also equal to $\int_{\mathcal{M}} \varphi(x) d\nu(\varphi)$, where ν is a measure degenerate at $\mathit{1} \in \mathcal{M}$ and has total mass equal to $\tilde{\mathbf{L}}_y(\mathcal{M})$. It follows by lemma 3.29 that $\tilde{\mathbf{L}}_y = \nu$. But $\tilde{\mathbf{L}}_y$ gives zero mass to $\{\mathit{1}\}$ and ν is degenerate at $\mathit{1}$, so $\tilde{\mathbf{L}}_y = \nu = 0$. Thus $\tilde{\mathbf{L}}_y(\mathcal{M}) = 0$, i. e. $\int_{\mathcal{M}} 1 - \operatorname{Re} \varphi(y) d\tilde{\mathbf{L}}(\varphi) = 0$, which means that $\int_{\mathcal{M}} (1 - \operatorname{Re} \varphi(y)) d\mathbf{L}(\varphi) = \int_{\mathcal{M}} (1 - \operatorname{Re} \varphi(y)) d\mathbf{L}'(\varphi)$. Remembering that $\mathbf{L}, \mathbf{L}'$ are self-adjoint we have $g(y) = g'(y)$ for sach $y \in \mathrm{G}$. So $g = g'$. Further, if D is any Borel set in $\mathcal{M}$ we have $\int_{\mathrm{D}} d\tilde{\mathbf{L}}_y(\varphi) = 0$ so

$$\int_{\mathrm{D}} (1 - \operatorname{Re} \varphi(y)) d\mathbf{L}(\varphi) = \int_{\mathrm{D}} (1 - \operatorname{Re} \varphi(y)) d\mathbf{L}'(\varphi). \tag{3.109}$$

Integrating this over a compact neighborhood of e in G we get

$$\int_{\mathrm{D}} \mathrm{Q}_{\mathrm{A}}(\varphi) d\mathbf{L}(\varphi) = \int_{\mathrm{D}} \mathrm{Q}_{\mathrm{A}}(\varphi) d\mathbf{L}'(\varphi). \tag{3.110}$$

This, together with the fact $\mathrm{Q}_{\mathrm{A}}(\varphi) \neq 0$ when $\varphi \neq \mathit{1}$ implies that $\mathbf{L} = \mathbf{L}'$.

Q. E. D.

THEOREM 3.31. — A function Φ is a real valued function in the class $\mathfrak{D}$ for (G, K) if and only if it admits a representation

$$\Phi(x) = \exp - \Psi(x) \tag{3.111}$$

where

$$\Phi(x) = g(x) + \int_{\mathcal{M} - \{x\}} (1 - \varphi(x)) d\mathbf{L}(\varphi) \tag{3.112}$$

and g, $\mathbf{L}$ satisfy the requirements (*a*), (*b*), (*c*) of theorem 3.27. Further g, $\mathbf{L}$ are uniquely determined by Φ.

COROLLARY 3.32. — Φ is a real-valued function in the class I for (G, K) if and only if it is a real-valued function in the class $\mathfrak{D}$ for (G, K).

This follows because if g, $\mathbf{L}$ satisfy the requirements (*a*), (*b*), (*c*), then so do tg, $t\mathbf{L}$ where t is a nonnegative real number.

THEOREM 3.33. — A kernel f on G/K is a Lévy-Schoenberg kernel if and only if

$$(3.113)\quad f(a, b) = \frac{1}{2}(r(a, o) + r(b, o) - r(a, b))\,; \qquad a, b \in G/K$$

where $r(xK, yK) = \Psi(y^{-1}x)$, and Ψ is a continuous real-valued spherical function on G such that

$$(3.114)\qquad \Psi(x) = g(x) + \int_{\mathcal{M}-\{1\}} (1 - \varphi(x))d\mathrm{L}(\varphi)$$

and the function g and the measure **L** satisfy the conditions (a), (b), (c) of theorem 3.27.

I would now like to make several comments on the results of this case.

The functional equation (3.96), which should be compared to (3.4), in this study arose somewhat unexpectedly for me. It arises in other contexts as well, see e. g. Furstenberg [*1*] who calls solutions to it the A-spherical functions. It is the additive analogue of (3.79), and because of its frequent occurrence may be worthy of further study.

The analogue of (3.96) in case II is entertaining. There G is the group of all proper rigid motions of R^d and K is $\mathbf{SO}(d)$. It is easy to check that if h is a function on G which satisfies

$$(3.115)\qquad \int_{\mathrm{K}} h(xky)dk = h(x) + h(y)$$

then h is spherical, i. e. can be identified with a radial function on $\mathrm{R}^d = \mathrm{G/K}$. Now if xK, yK are points of R^d whose distances from $0 \in \mathrm{R}^d$ are s, t respectively, and k is a rotation which sends the north pole of S^{d-1} into a point whose colatitude is θ, then the distance of xkyK from 0 can be computed to be $\sqrt{s^2 + t^2 - 2ts\cos\theta}$. Considering h as a radial function on R^d, and thereby identifying it with a function h^* on $(0, \infty)$, we see that h^* satisfies.

$$(3.116)\quad \frac{1}{\mathrm{C}}\int_0^\pi h^*\left(\sqrt{s^2 + t^2 - 2ts\cos\theta}\right)\sin^{d-2}\theta d\theta = h^*(s) + h^*(t)$$

where C is the area of the unit sphere in R^d. It is easily checked that the function $h^*(r) = r^2$ is a solution of this equation, for

$$(3.117)\qquad \frac{1}{\mathrm{C}}\int_0^\pi (s^2 + t^2 - 2ts\cos\theta)\sin^{d-2}\theta d\theta = s^2 + t^2$$

and by a more complicated analysis, it can be shown that $h^*(r) = r^2$ is the *only* solution, up to constant multiples, of (3.116); thus, in case II, an alternative description of the radial function $g^*(a)$ could be as a continuous nonnegative solution of (3.116) (or, what is the same thing (3.115), with appropriate reinterpretation).

Of course, the point is that the result of theorem 3.31 is now seen to be somewhat less satisfactory than one could have wished. Namely, while we have been able to show that the Gaussian part $g(x)$ in (3.87) must be a nonnegative continuous solution of (3.96), we have not been able to show that *every* nonnegative continuous solution of (3.96) can serve as $g(x)$ for some Ψ. Such a result would be as nice a result as one could expect in this situation, and it is a lacuna to be further studied. It seems to be related to the present state of the art in the representation theory of semisimple Lie groups. It is fairly easy to describe all continuous complex valued solutions to (3.96). Indeed, let $G = KAN$ be the Iwasawa decomposition of G, and for $x \in G$ let $H(x)$ be the unique element in the Lie algebra $\mathfrak{h}_{\mathfrak{p}_0}$ of A such that $x = k(x) \exp H(x).n(x)$, $k(x) \in K$, $n(x) \in N$ (See Helgason [*1*, Chapter X] for details). If λ is a complex-valued linear functional on $\mathfrak{h}_{\mathfrak{p}_0}$, then the function $h(x) = \int_K \lambda(H(xk))dk$ is a solution of $\int_K h(xky)dk = h(x) + h(y)$, as may be checked, and every solution of this functional equation is of the form $\int_K \lambda(H(xk))dk$ for some λ (18). Finally it can be shown that λ, λ' give rise to the same solution if and only if they are in the same orbit of the action of the Weyl group of G/K on $\mathfrak{h}_{\mathfrak{p}_0}$. Now the difficulty is to identify those λ which give rise to real nonnegative solutions $h(x)$, and then to relate them to positive definite functions in $\mathcal{M}$. This difficulty is related to the following well-known lacuna in representation theory. Namely, each complex-valued linear functional λ on $\mathfrak{h}_{\mathfrak{p}_0}$ is known to give rise to a solution of (3.79) (See again Helgason [*1*, chapter X]). But it is not known *precisely* which linear functionals λ lead to *positive-definite* functions. Indeed, it is known that if λ is real-valued, then it gives rise to a positive definite function, but there are also complex λ which do the same thing and little is known about which they are. This question is tantamount to asking about the supplementary series of spherical unitary representations of G, and the fine structure of $\mathcal{M}$ near the point $\mathfrak{1} \in \mathcal{M}$.

In view of this, it is tempting to conjecture that there is some connection between linear functionals λ which give rise to nonnegative solutions of $\int_K h(xky)dk = h(x) + h(y)$ and the linear functional λ which give rise to

(18) This is proved in Furstenberg [*1*, p. 406].

positive definite solutions of $\int_K \varphi(xky)dk = \varphi(x)\varphi(y)$. We shall leave this subject here.

For the groups $G = SL(2, R)$ and $G = SL(2, C)$, this information is available and it is possible to fill the gap mentioned above. We shall indicate how one may do this in § 4, where some examples of the above theory are discussed in detail.

We have already remarked above as to why we have restricted ourselves to the description only of the real-valued functions in $\mathfrak{D}$. However, in a number of cases, this is not a restriction at all. Namely, in these cases a function in the class $\mathfrak{D}$ is necessarily real-valued. This happens whenever every $\alpha \in \mathfrak{R}_K(G)$ is self-contragredient, or what is the same thing, $\varphi = \overline{\varphi}$ for each $\varphi \in \mathcal{M}$. For example, G/K has this property if the Weyl group of (G/K) contains $-$ I.

Further, it can be shown that if the compact symmetric space U/K which is dual (in the sense of É. Cartan) to G/K has the property that every $\alpha \in \mathfrak{R}_K(U)$ is self-contragredient, then each $\alpha \in \mathfrak{R}_K(G)$ is also self-contragredient. In case III, we had mentioned a list of compact symmetric spaces U/K for which the property « $\alpha = \overline{\alpha}$ for each $\alpha \in \mathfrak{R}_K(U)$ » holds. It follows that the non-compact symmetric spaces which are dual to spaces in that list have the above property, and in these cases $\mathfrak{D}$ consists only of real valued functions. In particular, one could show that if G/K is two-point homogeneous, the $\mathfrak{D}$ has only real valued functions in it. All the classical hyperbolic spaces are of this sort.

The above cases have all been concerned with locally compact groups. It is pertinent to ask if the class $\mathfrak{D}$ cannot be studied on some groups which are not locally compact. This can be done in some situations, if, instead of local compactness, one has a linear structure on G. For example, suppose G is a separable Hilbert space and K a closed subspace. By an argument similar to that of case I, we can confine attention to the case when $K = \{0\}$.

Now it turns out that the class $\mathfrak{D}$ for (G, $\{0\}$) may be defined relative to a variety of topologies on G. Namely, recall that each member of $\mathfrak{D}$ is to be a normalized, infinitely divisible continuous positive definite function on G. The requirement of continuity on a member of $\mathfrak{D}$ can be interpreted in various topologies, and different kinds of Lévy-Khinchine formulas then result. For example, if the class $\mathfrak{D}$ is to consist of functions which are normalized, positive definite, infinitely divisible and continuous in the usual strong topology on the Hilbert space G, then one can show that a function Φ on G is in the class $\mathfrak{D}$ for (G, $\{0\}$) if and only if $\Phi = \exp - \Psi$ and Ψ has the following representation

$$(3.118)\quad \Psi(x) = i(y_0, x) + (\mathrm{A}x, x) = \int_{\widehat{G}-\{0\}} (1 - \exp i(y, x) - i\lambda_x(y))d\mathbf{L}(y)$$

where $y_0 \in \widehat{G}$, the dual of $\widehat{G}$ (of course, $\widehat{G} \cong G$ here), A is a continuous positive operator on G, and $\mathbf{L}$ is a (not necessarily normalized) weak distribution on $\widehat{G}$, whose restrictions to finite dimensional subspaces of $\widehat{G}$ satisfy conditions analogous to those satisfied by the Lévy measure in the Lévy-Khinchine formula for R^d. Further, $\lambda_x(y)$ here is for each x, a certain weakly continuous bounded function of y which behaves like (x, y) near $y = 0$ in G. For an account of the notion of weak distribution and related concepts see Prohorov [*1*] [*2*], and Gelfand-Vilenkin [*1*]. In the latter book, weak distributions are termed cylinder set measures essentially.

The fact that weak distributions occur in (3.118) rather than measures is due to the circumstance that Bochner's theorem does not hold for a positive-definite function on G which is continuous in the strong topology on G. That is, a normalized positive-definite function on G, continuous in this topology, is not necessarily the Fourier transform (characteristic functional) of a countably additive probability *measure* on $\widehat{G}$. It is, however, the Fourier transform of a weak distribution.

On the other hand, if one alters the requirement of continuity and studies it in a suitably different topology, then it is well-known that Bochner's theorem can be recovered. For example, if the topology $\mathfrak{J}$ is the weakest topology on G which makes all the Hilbert-Schmidt operators on G continuous, then a $\mathfrak{J}$-continuous normalized positive definite function on G turns out to be the Fourier transform of a unique countably additive probability measure on $\widehat{G}$.

Now if one studies the class $\mathfrak{D}$ for (G, $\{0\}$), with the proviso that $\mathfrak{D}$ is to consist of $\mathfrak{J}$-continuous, normalized, infinitely divisible positive definite functions, then the members of $\mathfrak{D}$ can be characterized with the help of results of Varadhan [*1*]. A formula similar to (3.118) results, but now the operator A is Hilbert-Schmidt, and $\mathbf{L}$ is actually a countably additive measure on $\widehat{G}$ such that $\int_{\|y\|>0} \| y \|^2/1 + \| y \|^2 d\mathbf{L}(y) < \infty$. See e. g. Varadhan [*1*].

Finally, we would like to mention that we have also obtained a description of the class $\mathfrak{D}$ for (G, K) when G is a nuclear countably Hilbertian space and K a closed subspace such that G/K is complete, and therefore nuclear

(For information about nuclear spaces the reader is referred to Gelfand and Vilenkin [1]). Of course, if G/K is not complete, one can complete it and then it is again a nuclear countably Hilbertian space. As above, one can again restrict oneself to the case $K = \{0\}$. The class $\mathfrak{D}$ for $(G, \{0\})$ is then defined as the class of all normalized positive definite infinitely divisible functions on G, which are continuous in the topology of G. A version of Bochner's theorem is valid in this setting; namely, a continuous positive definite function on G is the Fourier transform of a unique countably additive nonnegative finite measure on the dual $\widehat{G}$ of G. It turns out that $\Phi \in \mathfrak{D}$ if and only if $\Phi = \exp - \Psi$ and Ψ enjoys a representation

$$(3.119) \quad \Psi(x) = i(y_0, x) + B(x, x) + \int_{\widehat{G}-\{0\}} (1 - \exp i(y, x) - i\lambda_x(y))\, dL(y)$$

where $y_0 \in \widehat{G}$, B is a continuous positive definite bilinear form on G, and L is a nonnegative, σ-finite measure on $\widehat{G}$ which assigns finite mass to the complement of every neighborhood of 0 in $\widehat{G}$ and such that $\int_{\widehat{G}-(0)} (1 - \operatorname{Re} \exp i(y, x))\, dL(y)$ is a finite, continuous function of x on G. Here, $\lambda_x(y)$ is a certain continuous function of y, which is, for fixed x, bounded on G and which behaves like (x, y) near $y = 0$.

It is to be noted that the Schwarz kernel theorem can be applied to the form $B(x, x)$ and one can then show that $B(x, x) = (Ax, x)$ where A is a linear operator of a specific kind. See Gelfand and Vilenkin [1, p. 74].

Naturally, in each of these cases, one can characterize the corresponding Lévy-Schoenberg kernels on G/K.

The methods which are useful in arriving at (3.118) and (3.119) differ from those used in cases I-IV insofar as the absence of local compactness makes compactness properties of families of measures a more delicate matter than in cases I-IV. The relevant methods have been surveyed by Prohorov in [1], and Gelfand and Vilenkin [1]. We have not embarked upon the full proof of (3.118) and (3.119) because the latter part of this paper uses the finite dimensionality of the space G/K in an essential way, and therefore does not apply to Hilbert or nuclear spaces.

§ 4. — Some examples.

The theory above will now be illustrated by some examples. I shall confine attention to the cases III and IV.

Let S^d be the unit sphere in R^{d+1}, and let $G = SO(d+1)$, the special orthogonal group. Let K be the subgroup of G leaving $o = (0, 0, \ldots, 1)$ fixed. Then $K \cong SO(d)$, and $S^d = G/K$. The action of K on G/K is transitive on each set of points on S^d having the same colatitude. Thus a function h on G is K-spherical if and only if its value at $x \in G$ depends only on the colatitude of xK. If θ designates the colatitude of a point $a = xK$ on S^d, then the range of variation of θ as a runs over S^d is $[0, \pi]$. Thus a continuous K-spherical function h on G can be identified with a continuous function h^* on $[0, \pi]$, and conversely, given a continuous function h^* on $[0, \pi]$, it gives rise to a K-spherical function (which we shall call h) on G.

The elementary positive definite spherical functions on G are well known and go back to Cartan [*1*]. They are the functions φ_n on G for which $\varphi_n^*(\theta) = c_n P_n^{(d-1)/2}(\cos\theta)$; $n = 0, 1, \ldots$ Here $P_n^{(d-1)/2}$ is the ultraspherical or Gegenbauer polynomial, and c_n is a normalizing constant chosen so that $\varphi_n(e)(= \varphi_n^*(0)) = 1$. Cf. Erdelyi *et al.* [*1*] for the definition of these polynomials. Of course, $\varphi_0 \equiv 1$.

Given a function K-spherical Ψ on G, theorem 3.19 tells us that $\Phi(x) = \exp - \Psi(x)$ will be in the class $\mathfrak{D}$ for (G, K) if and only if $\Psi(x) = \varphi(e) - \varphi(x)$, where $\varphi(x)$ is a positive definite continuous spherical function on G. Such a function φ must be of the form $\varphi(x) = \sum_{n \geq 0} a_n \varphi_n(x)$ with $a_n \geq 0$ and $\sum_{n \geq 0} a_n < \infty$. Interpreting this on $[0, \pi]$, we see that we must have $\Psi^*(\theta) = \varphi^*(0) - \varphi^*(\theta)$ with $\varphi^*(\theta) = \sum_{n \geq 0} a_n \varphi_n^*(\theta)$, with $a_n \geq 0$, $\sum_{n \geq 0} a_n < \infty$. Since $\varphi_n^*(\theta) = c_n P_n^{(d-1)/2}(\cos\theta)$ is real valued, it follows that the class $\mathfrak{D}$ for (G, K) consists in this case of real valued functions only. Finally, by virtue of theorem 3.21 and its corollary, $\Psi^*(\theta)$ is of the form $\varphi^*(0) - \varphi^*(\theta)$ if and only if

$$\Psi^*(0) = 0 \tag{4.1}$$

$$\int^{\pi} \Psi^*(\theta) \sin^{d-1}\theta \, d\theta \geq 0 \tag{4.2}$$

$$\int_0^{\pi} \Psi^*(\theta) P_n^{(d-1)/2}(\cos\theta) \sin^{d-1}\theta \, d\theta \leq 0\,; \quad n \geq 1. \tag{4.3}$$

Note that up to a positive constant multiple, $\sin^{d-1}\theta\, d\theta$ is the element of measure induced on $[0, \pi]$ by the normalized Haar measure of G.

Let Ψ^* be a continuous function on $[0, \pi]$ satisfying the above three conditions, and let Ψ be the corresponding spherical function on G. If $a = x\mathrm{K}$, $b = y\mathrm{K}$ are two points of $\mathrm{S}^d = \mathrm{G}/\mathrm{K}$, let $d(a, b)$ be the geodesic distance between a, b. Clearly $d(a, b) =$ colatitude of $y^{-1}x\mathrm{K}$. Now set $r(a, b) = \Psi^*(d(a, b)) = \Psi(y^{-1}x)$. Then our work of the last section assures us that the kernel $f(a, b) = \frac{1}{2}(r(a, o) + r(b, o) - r(a, b))$ is a Lévy-Schoenberg kernel on S^d. Conversely, if f is a Lévy-Schoenberg kernel, and $r(x\mathrm{K}, y\mathrm{K}) = \Psi(y^{-1}x)$, then Ψ^* satisfies (4.1)-(4.3).

Lévy's kernel (cf. (1.4) above) on S^d viz:

$$(4.4) \qquad f(a, b) = \frac{1}{2}(d(a, o) + d(b, o) - d(a, b)); \qquad a, b \in \mathrm{S}^d,$$

arises by choosing $\Psi^*(\theta) = \theta$. Therefore, to get an analytical proof, without the use of white noise integrals, of the fact that (4.4) is a positive definite kernel, it suffices to check that the function $\Psi^*(\theta) = \theta$ satisfies (4.1)-(4.3). Of these (4.1), (4.2) are trivially satisfied by this function. We shall check (4.3). Setting $\cos\theta = t$, $(d-1)/2 = \alpha$, we will have to use the following facts about $\mathrm{P}_n^\alpha(t)$. For these we refer to Erdelyi *et al.* [*1*].

$$(4.5) \qquad \frac{d}{dt}(1-t^2)^{\alpha+\frac{1}{2}}\frac{d}{dt}.\mathrm{P}_n^\alpha(t) = -n(n+2\alpha).(1-t^2)^{\alpha-\frac{1}{2}}.\mathrm{P}_n^\alpha(t).$$

$$(4.6) \qquad \mathrm{P}_n^\alpha(-t) = (-1)^n\mathrm{P}_n^\alpha(t)$$

$$(4.7) \qquad \frac{d}{dt}.\mathrm{P}_n^\alpha(t) = 2\alpha\mathrm{P}_{n-1}^{\alpha+1}(t), \qquad n \geq 1$$

$$(4.8) \qquad 2(n+\alpha)t\mathrm{P}_n^\alpha(t) = (n+2\alpha-1)\mathrm{P}_{n-1}^\alpha(t) + (n+1)\mathrm{P}_{n+1}^\alpha(t); \qquad n \geq 1$$

Now write $a_n^\alpha = \int_0^\pi \theta\mathrm{P}_n^\alpha(\cos\theta)\sin^{2\alpha}\theta d\theta$; $n \geq 1$. Using (4.5) and integrating by parts twice, we get

$$\begin{aligned}(4.9) \qquad a_n^\alpha &= \int_{-1}^{+1} \operatorname{arc\,cos} t\mathrm{P}_n^\alpha(t)(1-t^2)^{\alpha-\frac{1}{2}}dt \\ &= \frac{-1}{n(n+2\alpha)}\cdot\int_{-1}^1 \operatorname{arc\,cos} t.\frac{d}{dt}.(1-t^2)^{\alpha+\frac{1}{2}}.\frac{d}{dt}.\mathrm{P}_n^\alpha(t).dt \\ &= \frac{-2\alpha}{n(n+2\alpha)}\int_{-1}^1 t\mathrm{P}_n^\alpha(t).(1-t^2)^{\alpha-1}dt.\end{aligned}$$

Now use (4.8) to get

$$(4.10)\quad a_n^\alpha = -\frac{2\alpha}{n(n+2\alpha)}\left[\frac{n+2\alpha-1}{2(n+\alpha)}\int_{-1}^{1} P_{n-1}^\alpha(t).(1-t^2)^{\alpha-1}dt + \frac{n+1}{2(n+\alpha)} . \int_{-1}^{1} P_{n+1}^\alpha(t)(1-t^2)^{\alpha-1}dt\right]$$

$$= -\frac{\alpha(n+2\alpha-1)}{n(n+\alpha)(n+2\alpha)} . b_{n-1}^{\alpha-1} - \frac{\alpha(n+1)}{n(n+\alpha)(n+2\alpha)} . b_{n+1}^{\alpha-1}$$

where we have written $b_n^{\alpha-1} = \int_{-1}^{1} P_n^\alpha(t).(1-t^2)^{\alpha-1}dt.$

Note that since $n \geq 1$ and $\alpha = (d-1)/2 \geq 1/2$, the coefficients of $b_{n-1}^{\alpha-1}$ and $b_{n+1}^{\alpha-1}$ in (4.10) are negative. Next we examine $b_n^{\alpha-1}$. By (4.6) it is clear that $b_n^{\alpha-1} = 0$ if n is odd. Let therefore n be even, so that $n \geq 2$. Then, using (4.7), and integrating by parts, we get, as long as $\alpha - 1 > 0$

$$(4.11)\quad \begin{aligned} b_n^{\alpha-1} &= \int_{-1}^{1} P_n^\alpha(t).(1-t^2)^{\alpha-1}dt \\ &= \frac{1}{2(\alpha-1)} . \int_{-1}^{1} \frac{d}{dt} P_{n+1}^{\alpha-1}(t).(1-t^2)^{\alpha-1}dt \\ &= \int_{-1}^{1} t.P_{n+1}^{\alpha-1}(t).(1-t^2)^{\alpha-2}dt \\ &= \frac{n+2\alpha-2}{2(n+\alpha)} . b_n^{\alpha-2} + \frac{n+2}{2(n+\alpha)} . b_{n+2}^{\alpha-2}. \end{aligned}$$

Note that the coefficients are positive, and that the subscripts on the right are greater than or equal to n, and have the same parity as n. Therefore, if the superscript $\alpha - 1$ of a given $b_n^{\alpha-1}$ is positive, then that $b_n^{\alpha-1}$ can be written as a sum, with positive coefficients, of $b_m^{\alpha-2}$, $m \geq n$, i. e. with a smaller superscript. It follows that $b_n^{\alpha-1}$ is a sum, with positive coefficients, of various terms of the type b_m^0 or $b_m^{-1/2}$, according as α is an integer of a half-integer (i. e. according as d is odd or even), and m ranges over the integers $n, n+2, n+4, \ldots, n+2[\alpha-1]$. Now

$$(4.12)\quad \begin{aligned} b_m^0 &= \int_{-1}^{1} P_m^1(t)dt \\ &= \int_0^\pi P_m^1(\cos\theta)\sin\theta d\theta \\ &= \int_0^\pi \sin(m+1)\theta d\theta \\ &= \begin{cases} 0 \text{ if } m \text{ is odd} \\ 2/(m+1) \text{ if } m \text{ is even} \end{cases} \\ &\geq 0 \end{aligned}$$

where, for the last step but one, we used the fact that

$$\mathrm{P}_m^1(\cos\theta) = \sin(m+1)\theta/\sin\theta.$$

Again

$$b_m^{-1/2} = \int_{-1}^{1} \mathrm{P}_m^{\frac{1}{2}}(t)(1-t^2)^{-\frac{1}{2}}dt \tag{4.13}$$

$$= \int_0^{\pi} \mathrm{P}_m^{\frac{1}{2}}(\cos\theta)\mathrm{I}d\theta.$$

We now use the fact that $\mathrm{P}_m^{\frac{1}{2}}(\cos\theta)$ is just the Legendre polynomial of degree m, and recall the expansion of this polynomial in terms of $\cos l\theta$, $l \geq 0$, to get

$$b_m^{-1/2} = \begin{cases} 0 \text{ if } m \text{ is odd} \\ \pi \cdot \left(\dfrac{1\cdot 3\cdot 5\cdot \ldots (m-1)}{2\cdot 4\cdot 6\cdot \ldots m}\right)^2 \text{ if } m \text{ is even} \end{cases} \tag{4.14}$$

$$\geq 0.$$

For the expansion referred to, again see Erdelyi *et al.* [*1*, p. 174 ff.].

It follows from (4.12), (4.14) that $b_n^{\alpha-1} = 0$ for odd n and $b_n^{\alpha} > 0$ for even n, and therefore from (4.10) we get

$$a_n^{\alpha} = 0 \text{ if } n \text{ is even} \qquad \text{and} \qquad n \geq 1 \tag{4.15}$$
$$< 0 \text{ if } n \text{ is odd.}$$

This proves that $\Psi^*(\theta) = \theta$ satisfies (4.3), so that Lévy's kernel (4.4) is positive definite.

Similar computations can be carried out for other choices of Ψ^*. One choice is $\Psi^*(\theta) = \theta(\pi - \theta)$. It follows, after computations similar to the above, and which we will not reproduce, that this $\Psi^*(\theta)$ also satisfies (4.1)-(4.3), and therefore gives rise to a Lévy-Schoenberg kernel. We shall refer to this fact again later.

We shall now examine the situation when G/K is a compact symmetric space symmetric space of rank one; S^d is a special case of spaces of this type. These spaces, also known as two-point homogenous spaces, were classified by Wang [*1*]. They have the characteristic property that the linear isotropy group (induced by K on the tangent space to G/K at the coset eK) acts transitively on the unit sphere in this tangent space.

Here is a complete list of the compact symmetric spaces of rank one.

i) The spheres S^d $\quad d = 1, 2, 3 \ldots$

ii) The real projective spaces $P^d(R)$ $d = 2, 3 \ldots$

iii) The complex projective spaces $P^d(C)$ $d = 4, 6, 8, \ldots$

iv) The quaternionic projective spaces $P^d(H)$ $d = 8, 12, \ldots$

v) The Cayley elliptic plane P^{16} (Cay).

The superscripts here denote the dimension over the reals, of the underlying manifols. The G, K which yield these are as follows

i) $G = SO(d+1)$ $K = SO(d)$

ii) $G = SO(d+1)$ $K = O(d)$

iii) $G = SU(l+1)$ $K = S(U(l) \times U(1))$ $l = d/2$

iv) $G = Sp(l+1)/Sp(l) \times Sp(1)$ $l = d/4$

v) $G = F_{4(-52)}/SO(9)$. See Helgason [*1*, p. 354].

For information on this point see e. g. Helgason [*2*]. The geometry of these spaces is, in many respects, similar. For example, all geodesics in a given one of these spaces are closed, and have the same length, say 2L. Then L is the diameter of G/K, i. e. the maximum distance between any two points. If G/K is one such space, and a, b are any two points in G/K at a distance θ from the point $o = e$K, then there exists $k \in K$ such that $ka = b$. Thus a function on G/K is invariant under the left action of K on G/K if and only if it depends only on the distance of its argument from the point eK. Since the distance of any point of G/K from eK is at most L, it follows as before that a K-spherical function Ψ on G can be identified with a function Ψ^* on [0, L]. Conversely, given a continuous function Ψ^* on [0, L], one identifies it with a K-spherical function Ψ on G.

The geometry of these spaces is discussed in some detail in Helgason [*2*]. We shall also need some facts, proved in that paper, which we now set forth.

Letting θ be the distance of a point from eK, we may choose a geodesic polar coordinate system (θ, u) (where u is an « angular » (multi)-parameter) with pole at the point eK. This coordinate system is valid for $0 < \theta < L$. In this coordinate system, the radial part of the Laplace Beltrami operator of G/K has the expression

$$\Delta_\theta = \frac{1}{A(\theta)} \cdot \frac{d}{d\theta}\left(A(\theta) . \frac{d}{d\theta}\right) \tag{4.16}$$

where $A(\theta)$ is the area of the sphere of radius θ in G/K. $A(\theta)$ can be computed in terms of the structure of the Lie algebras of G and K. It turns out, cf. Helgason [*2*], that

$$A(\theta) = \Omega_{p+q+1} \lambda^{-p} (2\lambda)^{-q} \sin^p \lambda\theta \sin^q 2\lambda\theta \tag{4.17}$$

where p, q are certain nonnegative integers depending on the structure of G, K, Ω_d is the area of the unit sphere in R^d, and λ is a number depending on the metric in G/K. We quote the following list from Helgason [2, p. 171].

i)	$G/K = S^d$:	$p = 0$	$q = d - 1$,	$\lambda = \pi/2L$	$d = 1, 2, \ldots$
ii)	$G/K = P^d(R)$:	$p = 0$	$q = d - 1$	$\lambda = \pi/4L$	$d = 2, 3, \ldots$
iii)	$G/K = P^d(C)$:	$p = d - 2$	$q = 1$	$\lambda = \pi/2L$	$d = 4, 6, \ldots$
iv)	$G/K = P^d(H)$:	$p = d - 4$	$q = 3$	$\lambda = \pi/2L$	$d = 4, 8, \ldots$
v)	$G/K = P^{16}(Cay)$:	$p = 8$	$q = 7$	$\lambda = \pi/2L$.	

It is to be noted especially that p is *always even* in all these cases.

By virtue of (4.17), we may write Δ_θ as

$$\Delta_\theta = \frac{1}{\sin^p \lambda\theta \sin^q 2\lambda\theta} \cdot \frac{d}{d\theta} \cdot \sin^p \lambda\theta \sin^q 2\lambda\theta \cdot \frac{d}{d\theta}. \tag{4.18}$$

Using $x = \cos 2\lambda\theta$, this operator takes the form, up to a positive multiple,

$$\Delta_x = \frac{1}{(1-x)^\alpha(1+x)^\beta} \cdot \frac{d}{dx} \cdot (1-x)^{\alpha+1} \cdot (1+x)^{\beta+1} \cdot \frac{d}{dx} \tag{4.19}$$

where $\alpha = \dfrac{p+q-1}{2}$, $\beta = \dfrac{q-1}{2}$; so that $\alpha - \beta = p/2 = l$, say, and note that l is an integer.

The operator (4.19) is the familiar Jacobi operator. Its eigenfunctions are just the Jacobi polynomials $P_n^{\alpha,\beta}(x)$ (For the notation we follow Erdelyi *et al.* [*1*]). Since the elementary spherical functions on G are just the eigenfunctions of the Laplace-Beltrami operator, we can now identify the elementary normalized positive definite spherical functions on G in each of the five cases above. In each case we call them $\varphi_0, \varphi_1, \ldots$, with $\varphi_0 \equiv 1$, and let $\varphi_n^*(\theta)$ be the corresponding function induced on $[0, L]$ by φ_n; we have, remembering the special nature of $P^\alpha(R)$, the following list.

i)	$G/K = S^d$,	$\varphi_n^*(\theta) = c_n P_n^{\alpha,\beta}(\cos 2\lambda\theta)$,	$n = 0, 1, \ldots$
ii)	$G/K = P^d(R)$,	$\varphi_n^*(\theta) = c_n P_n^{\alpha\beta}(\cos 2\lambda\theta)$,	$n = 0, 2, 4, \ldots$
iii)	$G/K = P^d(C)$,	$\varphi_n^*(\theta) = c_n P_n^{\alpha\beta}(\cos 2\lambda\theta)$,	$n = 0, 1, 2, \ldots$
iv)	$G/K = P^d(H)$,	$\varphi_n^*(\theta) = c_n P_n^{\alpha\beta}(\cos 2\lambda\theta)$,	$n = 0, 1, 2, \ldots$
v)	$G/K = P^{16}(Cay)$,	$\varphi_n^*(\theta) = c_n P^{\alpha\beta}(\cos 2\lambda\theta)$,	$n = 0, 1, 2, \ldots$

In each of these cases $\alpha = (p + q - 1)/2$, $\beta = (q - 1)/2$, and p, q, λ have the meanings already described. In *i)* and *ii)*, we have $p = 0$, $q = d - 1$ so $\alpha = \beta = (d - 2)/2$ and the polynomials $P_n^{\alpha,\beta}$ reduce to $P_n^{(d-2)/2,(d-2)/2}$ which just the Gegenbauer polynomial $P_n^{(d-1)/2}$; (this is the standard notation, i. e. the Gegenbauer polynomial P_n^{μ} is defined as the Jacobi polynomial $P_n^{\left(\mu-\frac{1}{2},\mu-\frac{1}{2}\right)}$. Note that in *ii*) only the polynomials of even degree appear because, due to the identification of antipodal points on S^d, only the even order polynomials P_n^{μ} can be lifted to be functions on $P^d(R)$. That one gets all the Jacobi polynomials in *iii*) was pointed out by Cartan [*1*]. *iv*) and *v*) can be arrived at by similar methods.

It is now natural to ask if the function $\Psi^*(\theta) = \theta, 0 \leq \theta \leq L$, gives in these cases a Lévy-Schoenberg kernel. This amounts to asking if $\Psi^*(\theta)$ satisfies

$$(4.20) \qquad \Psi^*(0) = 0$$

$$(4.21) \qquad \int_0^L \Psi^*(\theta)\, A(\theta) d\theta \geq 0$$

$$(4.22) \qquad \int_0^L \Psi^*(\theta)\, \varphi_n^*(\theta)\, A(\theta) d\theta \leq 0 \qquad n \geq 1$$

where we have used the fact that to a positive constant multiple, $A(\theta)d\theta$ is the measure induced on $[0, L]$ by the Haar measure of G.

Again it is trivial to verify that $\Psi^*(\theta) = \theta$ satisfies (4.20), (4.21). We have already seen that for the spheres S^d (4.22) holds. It remains to check if (4.22) holds for spaces in *ii*), *iii*), *iv*), *v*).

For the real projective spaces, i. e., those in *ii*), then, we are asking whether or not

$$(4.23) \qquad \int_0^L \theta P_{2n}^{(d-1)/2} (\cos 2\lambda\theta) \sin^{d-1} 2\lambda\theta d\theta \leq 0 \qquad n \geq 1 \quad \text{where} \quad \lambda = \pi/4L$$

or, what is the same thing, whether or not we have:

$$(4.24) \qquad \int_0^{\pi/2} \theta P_{2n}^{(d-1)/2} (\cos\theta) \sin^{d-1}\theta d\theta \leq 0\ ? \qquad n = 1, 2, \ldots$$

Unfortunately, I am not able to answer this question at the present time, for the spaces of *ii*). Some experimentation seems to indicate that (4.24) does *not* hold. *Added in proof : See the end of this paper.*

Finally in the case *iii*), *iv*), *v*) we have to check whether or not

$$(4.25) \qquad \int_0^L \theta P_n^{\alpha,\beta} (\cos 2\lambda\theta)\, A(\theta) d\theta \leq 0.$$

Setting $\cos 2\lambda\theta = t$, and remembering $\lambda = \pi/2L$, we get the following integral

$$(4.26) \qquad \int_{-1}^{+1} \operatorname{arc\,cos} t \, P_n^{\alpha,\beta}(t)\cdot(1-t)^{\alpha}(1+t)^{\beta}\cdot dt.$$

Now there is the formula

$$(4.27) \quad (1-t)\, P_n^{\alpha,\beta}(t) = \frac{2(n+\alpha)}{2n+\alpha+\beta+1}\cdot P_n^{\alpha-1,\beta}(t) - \frac{n+1}{2n+\alpha+\beta+1}\cdot P_{n+1}^{\alpha-1,\beta}(t)$$

which connects the Jacobi polynomials. Now recall that in *iii*), *iv*), *v*) we have $\alpha - \beta = l$ an integer. It follows by inductive use of (4.27) that

$$(4.28) \quad (1-t)^l\, P_n^{\alpha,\beta}(t) = \sum_r a(n, \alpha, \beta, r)\cdot P_{n+2r}^{\beta,\beta}(t) - \sum_s b(n, \alpha, \beta, s)\cdot P_{n+1+2s}^{\beta,\beta}$$

where $a(n, \alpha, \beta, r)$, $b(n, \alpha, \beta, s)$ are certain positive coefficients; of course, both sums are over finitely many nonnegative values of r and s. From (4.28) we have

$$\begin{aligned}(4.29) \quad & \int_{-1}^{+1} \operatorname{arc\,cos} t\cdot P_n^{\alpha,\beta}(t)\cdot(1-t)^{\alpha}(1+t)^{\beta} dt \\ &= \sum_r a(n, \alpha, \beta, r) \int_{-1}^{+1} \operatorname{arc\,cos} t . P_{n+2r}^{\beta,\beta}(t)\cdot\left(1-t^2\right)^{\beta} dt \\ &- \sum_s b(n, \alpha, \beta, s) \int_{-1}^{+1} \operatorname{arc\,cos} t\cdot P_{n+1+2s}^{\beta,\beta}(t)\left(1-t^2\right)^{\beta} dt.\end{aligned}$$

The integrals that appear here are of the type

$$\int_{-1}^{+1} \operatorname{arc\,cos} t\, P_j^{\mu}(t)\left(1-t^2\right)^{\mu-\frac{1}{2}} dt \qquad j \geq 1,$$

with $\mu = \beta + \dfrac{1}{2} = q/2$ and P_j^{μ} is the Gegenbauer polynomial. Note that μ is a half integer. In the first part of this section, during the discussion of S^d, we have evaluated these integrals, and have found that if j is even then such an integral is zero, and if j is odd it is strictly negative (Cf. (4.15)).

Therefore in (4.29), it is clear that if n is even then every term in the first sum is zero while every term in the second sum is *negative*. It follows that for even n, (4.29) is *positive*. For odd n, every term in the first sum in (4.29) is negative and every term in the second sum is zero. Hence (4.29) is *negative* for odd n. That is, for $n \geq 1$, we have

$$(4.30)\quad \int_{-1}^{+1} \text{arc cos}\, t\, P_n^{\alpha,\beta}(t)\cdot(1-t)^{\alpha}(1+t)^{\beta}dt \begin{cases} <0 & n \text{ odd} \\ >0 & n \text{ even} \end{cases}$$

It follows that $\Psi^*(\theta) = \theta$ does *not* give us a Lévy-Schoenberg kernel on the spaces $P^d(C)$, $P^d(H)$ and $P^{16}(\text{Cay})$. Indeed this argument did not use the special nature of this function Ψ^*. Generalizing it, one has the result that if Ψ^* is a nonzero function on $[0, \pi]$ determining a Lévy-Schoenberg kernel on S^d for each d, then Ψ^* cannot determine a Lévy-Schoenberg kernel on any of the spaces $P^d(C)$, $P^d(H)$, $P^{16}(\text{Cay})$. Even more is clearly true but we shall not insist on it.

Note that in the above, we only had to check the nonpositivity conditions on the « Fourier » coefficients of Ψ^*, and did not have to check the finiteness of their sum; checking the finiteness would have been very troublesome to carry out. This is the main virtue of theorem 3.21 and its corollary.

The above examples will be of interest to us later when we discuss the stochastic processes to which Lévy-Schoenberg kernels give rise. For the present we may leave this question aside, and turn now to some other examples in all of which G is a noncompact semisimple classical group.

Let $G = \mathbf{SL}(2, R)$, the group of all 2×2 matrices with real entries and determinant 1. G acts transitively via fractional linear maps on the complex upper half plane: $\{ z, \text{Im}\, z > 0 \}$. A matrix leaves the point i fixed if and only if it is of the form $\begin{pmatrix} \cos u & \sin u \\ -\sin u & v \cos u \end{pmatrix}$, $0 \leq u \leq 2\pi$. Thus the isotropy subgroup of i is just $K \cong \mathbf{SO}(2)$. Then the upper half plane $= G/K$, and $eK = i$. As a Riemannian symmetric space (with Riemannian structure induced by the Killing form of G), this space is exactly the Poincaré plane with its hyperbolic metric.

K acts transitively on each Riemannian sphere in G/K with centre at eK. Thus a function h on G is spherical if and only if its value at $x \in G$ depends only on the hyperbolic distance of xK from eK. We shall write $\zeta(xK, yK)$ for the hyperbolic distance of xK from yK. As x ranges over G, $\zeta(xK, eK)$ ranges over $[0, \infty)$. Thus the set of continuous spherical functions on G can be identified with the set of continuous functions on $[0, \infty)$; a continuous spherical function h on G determines and is determined by a continuous function h^* on $[0, \infty)$. Further, since $[0, \infty)$ is mapped onto $[1, \infty)$ in one-to-one fashion by the map $\zeta \to \text{ch}\, \zeta$ [19], we may think of h^* on

[19] ch, sh stand respectively for the hyperbolic cosine and the hyperbolic sine functions.

$[0, \infty)$ as being given by a continuous function $\tilde{h}$ on $[1, \infty)$; thus we have, $h(x) = h^*(\zeta) = \tilde{h}(\operatorname{ch}\zeta)$, where $\zeta = \zeta(x\mathrm{K}, e\mathrm{K})$ and h is a continuous spherical function on G, and h^*, $\tilde{h}$ are the corresponding functions on $[0, \infty)$, $[1, \infty)$ respectively. This notation will be employed consistently throughout the rest of this section.

If $x\mathrm{K}$ is at a distance ξ from $e\mathrm{K}$ and $y\mathrm{K}$ at a distance η from $e\mathrm{K}$, and $k \in \mathrm{K}$ is given by $u \in [0, 2\pi]$, then the point $xky\mathrm{K}$ is at a distance ζ from $e\mathrm{K}$, where

$$\operatorname{ch}\zeta = \operatorname{ch}\xi \operatorname{ch}\eta + \operatorname{sh}\xi \operatorname{sh}\eta \cos u. \tag{4.31}$$

The functional equation

$$\int_{\mathrm{K}} \varphi(xky)dk = \varphi(x)\varphi(y) \tag{4.32}$$

can now be written in the radial form:

$$\frac{1}{2\pi}\int_0^{2\pi} \tilde{\varphi}(\operatorname{ch}\xi \operatorname{ch}\eta + \operatorname{sh}\xi \operatorname{sh}\eta \cos u)du = \tilde{\varphi}(\operatorname{ch}\xi)\tilde{\varphi}(\operatorname{ch}\eta). \tag{4.33}$$

This is the functional equation of the Legendre functions, thus the solutions to (4.32) are given by $\tilde{\varphi}(\operatorname{ch}\zeta) = \mathrm{P}_t(\operatorname{ch}\zeta)$, where t is a complex number. The radial part Δ_ζ of the Laplace-Beltrami operator is given by

$$\Delta_\zeta = \frac{1}{\operatorname{sh}\zeta} \cdot \frac{d}{d\zeta} \cdot \operatorname{sh}\zeta \frac{d}{d\zeta} \tag{4.34}$$

and $\mathrm{P}_t(\operatorname{ch}\zeta)$ is an eigenfunction of this with eigenvalue $t(t+1)$.

The positive definite solutions of (4.32) were determined essentially by Bargmann [1]. They are those P_t for which $t(t+1)$ is nonpositive. This means that either $\operatorname{Re} t = -1/2$; or $\operatorname{Im} t = 0$, and $-1/2 \leq \operatorname{Re} t \leq 0$. Further, since $\mathrm{P}_t = \mathrm{P}_{-t-1}$, $-1/2 + i\tau$ and $-1/2 - i\tau$ determine the same positive definite function. Thus the set $\mathcal{M}$ of elementary positive definite spherical functions on G consists exactly of those φ for which $\tilde{\varphi}(\operatorname{ch}\zeta) = \mathrm{P}_t(\operatorname{ch}\zeta)$ with $t \in \{-1/2 + i\tau; \tau \geq 0\} \cup \{\sigma; -1/2 < \sigma \leq 0\}$. A convenient way of parametrizing $\mathcal{M}$ is by letting $t(t+1) = -\alpha^2$, $\alpha \geq 0$. Then $\mathcal{M}$ is in one-one correspondence with $\tilde{\mathcal{M}} = [0, \infty)$ under the correspondence $\alpha \in [0, \infty) \longleftrightarrow \mathrm{P}_{-1/2+i\sqrt{\alpha^2-1/4}}$. For typographical ease, we will write $t(\alpha) = -1/2 + i\sqrt{\alpha^2 - 1/4}$. We then have $t(0) = 0$, and $\mathrm{P}_{t(0)} = \mathrm{P}_0 \equiv 1$.

The work of § 3 tells us that a function Φ on G is a real valued function in the class $\mathfrak{D}$ for (G, K) is and only if $\Phi = \exp - \Psi$, where Ψ is a continuous spherical function on G such that the corresponding function $\Psi^*(\zeta)$ is given by

$$\Psi^*(\zeta) = g^*(\zeta) + \int_{\mathcal{M}} (1 - \Phi^*(\zeta)) d\mathbf{L}(\varphi) \tag{4.35}$$

with obvious notation. Transferring this to $[1, \infty)$ under $\zeta \to \operatorname{ch} \zeta$, and remembering our identification of $\mathcal{M}$ with $[0, \infty)$, we get

$$\tilde{\Psi}(\operatorname{ch} \zeta) = \tilde{g}(\operatorname{ch}\zeta) + \int_0^\infty (1 - \mathrm{P}_{t(\alpha)}(\operatorname{ch} \zeta)) d\tilde{\mathbf{L}}(\alpha) \tag{4.36}$$

where $\tilde{g}$ is continuous, and $\tilde{g}$ and $\tilde{\mathbf{L}}$ satisfy the conditions given below.

$$\begin{aligned} \tilde{g}(\operatorname{ch} \zeta) &= \lim_{j\to\infty} \int_0^{\alpha_j} (1 - \operatorname{Re} \mathrm{P}_{t(\alpha)}(\operatorname{ch} \zeta)) d\nu_j(\alpha) \\ &= \lim_{j\to\infty} \int_0^{\alpha_j} (1 - \mathrm{P}_{t(\alpha)}(\operatorname{ch} \zeta)) d\nu_j(\alpha) \end{aligned} \tag{4.37}$$

where $\alpha_j \to 0$ and ν_j is a measure on $[0, \alpha_j]$, $j = 1, 2, \ldots$ Further $\tilde{\mathbf{L}}$ is a measure on $\mathcal{M}$ ($= [0, \infty)$ under the above identification) such that $\tilde{\mathbf{L}}$ gives finite mass to the complement of any half-interval containing 0, and such that for each $a > 0$,

$$\int_0^\infty \mathrm{Q}_a(\alpha) d\mathbf{L}(\alpha) < \infty \tag{4.38}$$

where

$$\mathrm{Q}_a(\alpha) = \int_0^a (1 - \mathrm{P}_{t(\alpha)}(\operatorname{ch} \zeta)) \operatorname{sh} \zeta d\zeta \Big/ \int_0^a \operatorname{sh} \zeta d\zeta. \tag{4.39}$$

Of course, to gain some concreteness, it is now necessary to try and determine the form of $\tilde{g}$ and Q_a.

As to $\tilde{g}$, it satisfies the functional equation

$$\frac{1}{2\pi} \int_0^{2\pi} \tilde{h}(\operatorname{ch} \xi \operatorname{ch} \eta + \operatorname{sh} \xi \operatorname{sh} \eta \cos u) du = \tilde{h}(\operatorname{ch} \xi) + \tilde{h}(\operatorname{ch} \eta) \tag{4.40}$$

which is the radial form of $\int_{\mathrm{K}} h(xky) dk = h(x) + h(y)$.

Recall that we remarked at the end of § 3, case IV, that the solutions of this equation were of the form $h(x) = \int_K \lambda(H(xk))dk$. In the present instance, the Iwasawa decomposition of G is $G = KAN$ where

$$A = \left\{ \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} ; d > 0 \right\} \quad \text{and} \quad N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} z \text{ real} \right\}.$$

For details of this as well as a part of the material above, see Helgason [*1*, chapter X]. Computing $H(xk)$, we get

$$\tilde{h}(\operatorname{ch} \zeta) = \frac{c}{2\pi} \int_0^{2\pi} \log(\operatorname{ch} \zeta + \operatorname{sh} \zeta \cos 2u) du \tag{4.41}$$

where c is a constant. To compute $\frac{1}{2\pi} \int_0^{2\pi} \log(\operatorname{ch} \zeta + \operatorname{sh} \zeta \cos 2u) du$, we recall that

$$P_t(\operatorname{ch} \zeta) = \frac{1}{2\pi} \int_0^{2\pi} (\operatorname{ch} \zeta + \operatorname{sh} \zeta \cos 2u)^t du \tag{4.42}$$

and that for fixed $\zeta \geq 0$, P_t is an entire function of t. It follows that

$$\frac{d}{dt} P_t(\operatorname{ch} \zeta) \Big|_{t=0} = \frac{1}{2\pi} \int_0^{2\pi} \log(\operatorname{ch} \zeta + \operatorname{sh} \zeta \cos 2u) du. \tag{4.43}$$

The left side may be computed by using known expansions of $P_t(\operatorname{ch} \zeta)$ in terms of t. See e. g. Robin [*1*, p. 170, formula (329)]. The result is

$$\frac{1}{2\pi} \int_0^{2\pi} \log(\operatorname{ch} \zeta + \operatorname{sh} \zeta \cos 2u) du = 2 \log\left(\operatorname{ch} \frac{\zeta}{2}\right). \tag{4.44}$$

Therefore

$$\tilde{h}(\operatorname{ch} \zeta) = c \log(\operatorname{ch}(\zeta/2)) \qquad c \geq 0 \tag{4.45}$$

are the only nonnegative solutions of (4.40). Thus the function $\tilde{g}$ in (4.36) is of this form. In the present case we can also prove the converse. Namely that if $c \geq 0$ then $\exp - c \log(\operatorname{ch}(\zeta/2))$ is in the class $\mathfrak{D}$ (when thought of as a spherical function on G). To show this let t be real such that $-\frac{1}{2} \leq t < 0$. Then $P_t(\operatorname{ch} \zeta)$ is positive definite. It follows that $\exp - \frac{c}{2}(P_t(\operatorname{ch} \zeta) - 1)/t$ is in the class $\mathfrak{D}$; as $t \to 0$ we see that this has the limit $\exp - c \log(\operatorname{ch}(\zeta/2))$, because

$$\frac{d}{dt} P_t(\operatorname{ch} \zeta) \Big|_{t=0} = \lim_{t \to 0} \frac{(P_t \operatorname{ch} \zeta) - 1)}{t} = 2 \log \operatorname{ch}(\zeta/2).$$

Hence for each $c \geq 0$, the function $\exp - c \log(\text{ch}(\zeta/2))$ is in $\mathfrak{D}$. Finally, the behaviour of $Q_a(\alpha)$ near $\alpha = 0$ can be shown to be like $t(\alpha)\left(= -\frac{1}{2} + i\sqrt{\alpha^2 - \frac{1}{4}}\right)$. Putting together all these facts we get the following theorem, which is as explicit as could be wished.

THEOREM 4.1. — Let $G = \mathbf{SL}(2, R)$, $K = \mathbf{SO}(2)$. A function Φ on G is in the class for (G, K) if and only if $\Phi(x) = \exp - \Psi(x)$, where Ψ is a continuous spherical function on G such that

$$(4.46) \qquad \tilde{\Psi}(\text{ch}\,\zeta) = c \log(\text{ch}(\zeta/2)) + \int_0^\infty (1 - P_{t(\alpha)}(\text{ch}\,\zeta)) d\tilde{L}(\alpha)$$

where $c \geq 0$, and $\tilde{L}$ is a nonnegative measure on $[0, \infty)$ such that

$$(4.47) \qquad \int_0^\infty \frac{|t(\alpha)|}{1 + |t(\alpha)|} d\tilde{L}(\alpha) < \infty \quad \text{where} \quad t(\alpha) = -1/2 + i\sqrt{\alpha^2 - 1/4}.$$

Note the analogy with the classical Lévy-Khinchine formula.

Using this theorem, we can write down many Lévy-Schoenberg kernels on G/K quite explicitly. For example, if μ is any finite measure on $[0, \infty)$ then the function $\exp - \left\{ \int_0^\infty (1 - P_{-1/2+i\tau}(\text{ch}\,\zeta)) d\mu(\tau) \right\}$ is in the class $\mathfrak{D}$. This integral may be evaluated explicitly for various choices of μ, since the functions $P_{-1/2+i\tau}(\text{ch}\,\zeta)$ are the so-called conical functions and have been extensively studied. See e. g. Robin [*2*] and Erdelyi *et al.* [*1*].

Thus the function $c \log(\text{ch}(\zeta/2))$ here plays a role similar to the function $c\,|a|^2$ in the Euclidean case. It is interesting to note that $\log(\text{ch}(\zeta/2))$ is essentially the only solution of $\Delta_\zeta \varphi = \text{const.}$ which is bounded at $\zeta = 0$, where Δ_ζ is the radial Laplacian. The same is true of the function $|a|^2$ in Euclidean space.

The above work can be extended to all the higher dimensional hyperbolic spaces. All these spaces are of rank one and there is one for each dimension $d = 2, 3, \ldots$ The above example is the one in which $d = 2$. They arise as follows: $G = \mathbf{SO}_0(d, 1)$, $K = \mathbf{SO}(d)$. Here $\mathbf{SO}_0(d, 1)$ stands for the connected component of the identity in the group of all $d + 1 \times d + 1$ real matrices which preserve the quadratic form $x_0^2 - x_1^2 - x_2^2 \ldots - x_d^2$, and have determinant 1.

Here the analogues of (4.33), (4.34) are

$$(4.48) \qquad \frac{1}{\Omega_d} \int_0^\pi \tilde{\varphi}(\text{ch}\,\xi\,\text{ch}\,\eta + \text{sh}\,\xi\,\text{sh}\,\eta \cos u) \sin^{d-2} u\,du = \tilde{\varphi}(\text{ch}\,\xi)\tilde{\varphi}(\text{ch}\,\eta)$$

where Ω_d is the area of the unit sphere in R^d.

$$(4.49)\qquad \Delta_\zeta = \frac{1}{\mathrm{sh}^{d-1}\zeta}\cdot\frac{d}{d\zeta}\cdot \mathrm{sh}^{d-1}\zeta\,\frac{d}{d\zeta}.$$

The solutions to (4.47) are, with $\lambda = (d-2)/2$,

$$(4.50)\qquad \begin{aligned} \mathrm{R}_t^\lambda(\mathrm{ch}\,\zeta) &= 2^\lambda\Gamma(\lambda+1)(\mathrm{sh}\,\zeta)^{-\lambda}\mathrm{P}_t^\lambda(\mathrm{ch}\,\zeta)\\ &= \left(\frac{1+\mathrm{ch}\,\zeta}{2}\right)^{-\lambda}.\mathrm{F}(-t, t+1\,;1+\lambda, (1-\mathrm{ch}\,\zeta)/2)\end{aligned}$$

where P_t^λ is the usual associated Legendre function, F is the hypergeometric function and t is a complex number (The notation of Erdelyi *et al.* [*1*]).

These functions are eigenfunctions of (4.49), with eigenvalue $(t-\lambda)(t+\lambda+1)$. The positive definite functions among these arise from values of t for which $(t-\lambda)(t+\lambda+1)$ is nonpositive. Further $\mathrm{R}_t^\lambda = \mathrm{R}_{-t-1}^\lambda$, so the set $\mathcal{M}$ of positive definite functions arises from t in $\{-\lambda-1/2+i\tau;\ \geq 0\}\cup\{\sigma;\ -\lambda-1/2<\sigma\leq\lambda\}$. If we set $(\lambda-t)(\lambda+t+1)=\alpha^2$, $\alpha\geq 0$, then the map $\alpha\rightarrow \mathrm{R}_{t(\alpha)}^\lambda$ with $t(\alpha) = -1/2+i\sqrt{\alpha^2-(\lambda+1/2)^2}$ gives a parametrization of $\mathcal{M}$, so can be identified with $\tilde{\mathcal{M}} = [0,\infty)$. The functional equation $\int_{\mathrm{K}} h(xky)dk = h(x) + h(y)$ takes the form

$$(4.51)\qquad \frac{1}{\Omega_d}\int_0^\pi \tilde{h}(\mathrm{ch}\,\xi\,\mathrm{ch}\,\eta + \mathrm{sh}\,\xi\,\mathrm{sh}\,\eta\cos u)\sin^{d-2}u\,du = \tilde{h}(\mathrm{ch}\,\xi) + \tilde{h}(\mathrm{ch}\,\eta).$$

Its solutions which are of the form $h(x) = \int_{\mathrm{K}} \lambda(\mathrm{H}(xk))dk$, here look like

$$(4.52)\qquad \tilde{h}(\mathrm{ch}\,\zeta) = \frac{c}{\Omega_d}\int_0^\pi \log(\mathrm{ch}\,\zeta + \mathrm{sh}\,\zeta\cos u)\sin^{d-2}u\,du$$

with c a constant. The integral $\frac{1}{\Omega_d}\int_0^\pi \log(\mathrm{ch}\,\zeta+\mathrm{sh}\,\zeta\cos u)\sin^{d-2}u\,du$ is precisely $\frac{d}{dt}\cdot\mathrm{R}_t^\lambda(\mathrm{ch}\,\zeta)\Big|_{t=\lambda}$. Thus every Gaussian part g appearing in the formula for a function in the class $\mathfrak{D}$ is of the form

$$(4.53)\qquad \tilde{g}(\mathrm{ch}\,\zeta) = \frac{c}{\Omega_d}\int_0^\pi \log(\mathrm{ch}\,\zeta+\mathrm{sh}\,\zeta\cos u)\sin^{d-2}u\,du.$$

Further, the same method followed in the case $d=2$ shows that for $c\geq 0$, the function $\exp -c\,\{(\mathrm{R}_t^\lambda(\mathrm{ch}\,\zeta)-1)/(t-\lambda)\}$ is in $\mathfrak{D}$ for $-\lambda-\frac{1}{2}\geq t<\lambda$, and this converges to $\exp -c\left\{\frac{d}{dt}\mathrm{R}_t^\lambda(\mathrm{ch}\,\zeta)\Big|_{t=\lambda}\right\}$ as $t\rightarrow\lambda$.

Thus for *any* $c \geq 0$, the function $\frac{c}{\Omega_d}\int_0^\pi \log(\operatorname{ch}\zeta + \operatorname{sh}\zeta\cos u)\sin^{d-2} u\,du$ can arise as the Gaussian part of a function Φ in $\mathfrak{D}$. One can write down the following expression for $\frac{d}{dt}\cdot R_t^\lambda(\operatorname{ch}\zeta)\Big|_{t=\lambda}$.

$$(4.54)\quad \frac{d}{dt}\cdot R_t^\lambda(\operatorname{ch}\zeta)\Big|_{t=\lambda}$$
$$= 2^\lambda\Gamma(\lambda+1)(\operatorname{sh}\zeta)^{-\lambda}\frac{d}{dt}F\left(-t,\, t+1\,;\,\lambda+1,\frac{1-\operatorname{ch}\zeta}{2}\right)\Big|_{t=\lambda}$$
$$= \Gamma(\lambda+1)(\tanh\zeta/2)^\lambda\sum_{p=0}^{\infty}(-1)^p\frac{\Gamma(p+1)\Gamma(\lambda+1)}{\Gamma(p+\lambda+2)}(\operatorname{sh}^2(\zeta/2))^{p+1}.$$

The last series converges for small ζ, but for other ζ one must define this by analytic continuation of the integral for $\frac{d}{dt}R_t^\lambda(\operatorname{ch}\zeta)\Big|_{t=\lambda}$.

The condition (4.47) remains exactly the same, with the new meaning of $t(\alpha)$; i. e. $d\tilde{L}(\alpha)$ must now satisfy

$$(4.55)\quad \int_0^\infty \frac{|t(\alpha)|}{1+|t(\alpha)|}d\tilde{L}(\alpha) < \infty\,; \quad t(\alpha) = -1/2 + i\sqrt{\alpha^2-(\lambda+1/2)^2}.$$

With these meanings for $\tilde{g}$ and $\tilde{L}$, theorem 4.1 holds true verbatim for the present case, and is again as explicit as one could have wished.

The case $d = 3$ is of special interest. Then the space $G/K = SO_0(3,1)/SO(3)$ is the Minkowski space, obtained from the proper homogeneous Lorentz group $SO_0(3,1)$. It may also be realized (by passing to the covering group of $SO_0(3,1)$) as $SL(2,C)/SU(2)$. In this case the functions R_t^λ can be expressed in terms of elementary functions; indeed

$$(4.56)\quad R_t^\lambda(\operatorname{ch}\zeta) = \operatorname{sh}(t+1/2)\zeta/(t+1/2)\operatorname{sh}\zeta.$$

The integral in (4.52) can be explicitly evaluated, and the functions $\tilde{g}$ are seen to be

$$(4.57)\quad \tilde{g}(\operatorname{ch}\zeta) = c(\zeta\coth\zeta - 1); \qquad c \geq 0.$$

It is again interesting to note that $\tilde{g}$ is a solution of $\Delta_\zeta\tilde{g} = \text{const}$, bounded near $\zeta = 0$. These results could perhaps have some relevance for the study of relativistic turbulence.

The spaces considered here are the noncompact symmetric spaces which are dual in Cartan's sense to the compact symmetric spaces S^d. One could similarly do computations like the above for the noncompact spaces dual to $P^d(C), P^d(H)$. The functions which arise would be the Jacobi functions $P_t^{\alpha\beta}$ of a complex parameter t. We have not, however, carried this out. For computations leading to (4.50), see Krein [*1*], Vilenkin [*1*]; see also Gelfand and Naimark [*1*].

§ 5. — Subordinate kernels; Brownian motions.

Among the Lévy-Schoenberg kernels on R^d (1.2), occupies a somewha. distinguished place. It is worth inquiring what its analogue is in generalt Fix a pair (G, K) as envisaged in the preceding sections and suppose $\Phi \in \mathfrak{D}$ for the pair (G, K) and that Φ is real valued; then Φ is nonnegative, and $\Phi = \exp - \Psi$, for an appropriate Ψ, which is again nonnegative, since $|\Phi| \leq 1$. Now, if $\Phi \in \mathfrak{D}$, then $\Phi^t \in \mathfrak{D}$ since $\mathfrak{D} = I$, and it follows that $\exp(\Phi^t - 1) \in \mathfrak{D}$ also. Hence that $\exp \alpha(\Phi^t - 1) \in \mathfrak{D}$ for each $\alpha \geq 0$, and if $\alpha_1, \ldots, \alpha_n \geq 0$, and $t_1, \ldots, t_n \geq 0$ then $\exp \sum_{i=1}^{n} \alpha_i(\Phi^{t_i} - 1)$ is again in $\mathfrak{D}$. It is only a step from this to see that if ν is a nonnegative measure on $[0, \infty)$ for which $\int_0^\infty (\Phi^t(x) - 1) d\nu(t)$ makes sense as a continuous functions of x, then $\exp \int_0^\infty (\Phi^t(x) - 1) d\nu(t)$ is also a function in $\mathfrak{D}$. This function is nothing but $\exp - \left\{ \int_0^\infty (1 - \exp - t\Psi(x)) d\nu(t) \right\}$. Now the point is that for various choices of ν, one may evaluate the integral here explicitly. E. g. we have the well known fact that for each α such that $0 \leq \alpha < 2$, there is a measure ν_α on $[0, \infty)$ such that

$$\int_0^\infty (1 - \exp - tr^2)\, d\nu_\alpha(t) = r^\alpha. \tag{5.1}$$

Using this, and remembering that Ψ is nonnegative, we see that along with $\Phi = \exp - \Psi(x)$, the function on G given by

$$\exp - \int_0^\infty (1 - \exp - t\Psi(x)) d\nu_\alpha(t) = \exp - (\Psi(x))^{\alpha/2} \quad 0 \leq \alpha < 2 \tag{5.2}$$

is in the class $\mathfrak{D}$ also. It follows that along with Ψ, the function $\Psi^{\alpha/2}$ also gives rise to Lévy-Schoenberg kernels; i. e., if

$$f(a, b) = \frac{1}{2}(r(a, o) + r(b, o) - r(a, b)) \tag{5.3}$$

is a Lévy-Schoenberg kernel then so is

$$f_\alpha(a, b) = \frac{1}{2}(r(a, o)^{\alpha/2} + r(b, o)^{\alpha/2} - r(a, b)^{\alpha/2}) \tag{5.4}$$

for any α such that $0 \leq \alpha \leq 2$.

The kernels (5.4) may be called subordinate to (5.3). In R^d, if we start with $r(a, b) = | a - b |^2$ which arises from the Gaussian part $\Psi(x) = | xK |^2$, (cf. § 3), we see that

$$f_\alpha(a, b) = \frac{1}{2}(| a |^\alpha + | b |^\alpha - | a - b |^\alpha) \qquad 0 \leq \alpha \leq 2 \tag{5.5}$$

are all Lévy-Schoenberg kernels, and Lévy's Brownian motion arises from the kernel with $\alpha = 1$ here. That all the kernels (5.5) are positive definite is probably known to many people.

Now the point is that in any situation where one may reasonably single out the so-called Gaussian parts of Ψ, one may arrive at an analogue of (5.5) which is intrinsic. For example, when we are in case IV of § 3, we may let $\Psi(x) = g(x)$, and conclude that $\exp - (g(x))^{\alpha/2}$ is in $\mathfrak{D}$ for each $0 \leq \alpha \leq 2$. Thus $g(x)^{\alpha/2}$ also gives rise to a Lévy-Schoenberg kernel. Thus if $a = xK$, $b = yK$ are points of G/K then

$$f_\alpha(a, b) = \frac{1}{2}(g(x)^{\alpha/2} + g(y)^{\alpha/2} - g(y^{-1}x)^{\alpha/2}) \qquad 0 \leq \alpha \leq 2 \tag{5.6}$$

are all Lévy-Schoenberg kernels. The analogue to (1.2) could now be taken to be the above kernel with $\alpha = 1$. Thus for $G = \mathbf{SL}(2, R)$, $K = \mathbf{SO}(2)$, we have that if $a = xK$, $b = yK \in G/K$, then

$$\begin{aligned} f_1(a, b) = \frac{1}{2}\Big(&\sqrt{\log \operatorname{ch}(\zeta(a, o))/2} + \sqrt{\log \operatorname{ch}(\zeta(b, o)/2)} \\ &- \sqrt{\log \operatorname{ch}(\zeta(a, b)/2)}\Big) \end{aligned} \tag{5.7}$$

(where $\zeta(a, b)$ is the hyperbolic distance between a, b and $o = eK$) is a

Lévy-Schoenberg kernel on G/K analogous to (1.2), and when G = SL(2, C), K = SU(2), the analogue is

$$(5.8)\quad f_1(a,b)=\frac{1}{2}\{(\zeta(a,o)\coth\zeta(a,o)-1)^{1/2}+(\zeta(b,o)\coth\zeta(b,o)-1)^{1/2}-(\zeta(a,b)\coth\zeta(a,b)-1)^{1/2}\}$$

with similar conventions.

When G is compact, there is no natural way of singling out the « Gaussian parts », due to the « disconnectedness » of the Fourier analytic dual of G/K. It is therefore pertinent to ask if (1.4) is quite the natural thing to look at from our point of view. I should think that Lévy's consideration of the kernel

$$(5.9)\qquad f(a,b)=\frac{1}{2}(d(a,o)+d(b,o)-d(a,b))\qquad a,b\in S^d$$

was prompted by the fact that if one constructs the corresponding stochastic process ξ, then as t runs along a geodesic in S^d, $\xi(t)$ is essentially just the classical Brownian motion of one parameter on $[0,\pi]$. Now from our point of view, we have seen in § 4 that this is quite special to S^d. Namely, that for the spaces $P^d(C)$, $P^d(H)$ there is no possibility of having a stochastic process ξ parametrized by points on these spaces, such that the restriction of ξ to geodesics is the classical Brownian motion of one parameter (This is essentially the content of the result that $\exp-\theta$ is not positive definite on these spaces, cf. § 4). It appears then that if one wants to study these processes in a general framework, one has to pay the price of admitting that the kernel (1.4) is somewhat accidental.

As has been mentioned above, it would be interesting to see if in case IV, the kernel

$$(5.10)\qquad f(a,b)=\frac{1}{2}(d(a,o)+d(b,o)-d(a,b))$$

is a Lévy-Schoenberg kernel. This amounts to asking, for example, when G = SL(2, R), K = SO(2), whether there exists a function Φ in the class such that $\Phi^*(\zeta)=\exp-\zeta$, or equivalently, whether ζ admits the representation

$$(5.11)\qquad \zeta=c\tilde{g}(\mathrm{ch}\,\zeta)+\int_0^\infty(1-P_{t(\alpha)}(\mathrm{ch}\,\zeta))d\tilde{L}(\alpha)$$

for a suitable choice of c and $\tilde{L}$ (cf. theorem 4.1). I do not know the answer. More generally one ought to ask if for arbitrary G, K as in case IV, one can find a function g and a measure $\mathbf{L}$ satisfying the requirements of theorem 3.27 such that

$$\zeta(xK, eK) = g(x) + \int_{\mathcal{M}-\{1\}} (1 - \varphi(x))d\mathbf{L}(\varphi) \tag{5.12}$$

where $\zeta(xK, eK)$ is the distance on G/K of xK from eK. This is, of course, asking a great deal.

The existence of a representation (5.11) for ζ would imply the existence of a Gaussian process ξ with parameter running over G/K such that the restriction of ξ to a geodesic emanating from eK is just the Brownian motion on $[0, \infty)$.

§ 6. — Markov semigroups determined by members of $\mathfrak{D}$.

This section is something of a digression, and matters will be dealt with perfunctorily here.

The classical Lévy-Khinchine formula arose from the consideration of infinitely divisible measures on the line, and characterizes the Fourier transforms of such measures. These Fourier transforms are, in the sense of the present paper, in the class $\mathfrak{D}$ for (R, $\{ 0 \}$). It is natural to inquire if there is any possibility of dualizing the class $\mathfrak{D}$ in general to yield probability measures which are infinitely divisible in some sense on the appropriate dual object for G/K.

The answer is yes whenever there is available some variant of Bochner's theorem, so that positive definite spherical functions on G can be identified with measures on the appropriate dual object. The dual objects which we can call $(G/K)^\wedge$ which arise are as follows:

Case I: $(G/K)^\wedge$ = the dual group of the locally compact abelian group G/K.

Case II: $(G/K)^\wedge = [0, \infty)$.

Case III: $(G/K)^\wedge$ = the set of all normalized elementary positive definite spherical functions on G.
= $\{ 0, 1, 2, \dots \}$ when G/K is a compact symmetric space.

Case IV: $(G/K)^\wedge = \mathcal{M}$.
= the set of all normalized elementary positive definite spherical functions.

It is to be noted that in each case, the dual object in question can actually be identified with the set of continuous positive definite elementary normalized spherical functions on G.

In each case, a continuous positive definite spherical function Φ on G can be written $\Phi(x) = \int \varphi(x)d\mu(\varphi)$ (note that each element of $(G/K)^\frown$ is a spherical function on G), where the integral is over $(G/K)^\frown$ in an appropriate sense, μ is a nonnegative finite measure on $(G/K)^\frown$, and the correspondence $\Phi \longleftrightarrow \mu$ is one-one. Now if Φ_1, Φ_2 are two continuous positive-definite spherical functions on G, then so is their product $\Phi_1\Phi_2$. If μ_i is the measure on $(G/K)^\frown$ determined by Φ_i $i = 1, 2$, then the measure determined by $\Phi_1\Phi_2$ can be denoted by $\mu_1 * \mu_2$. Then the operation $(\mu_1, \mu_2) \to \mu_1 * \mu_2$ is a commutative convolution on $(G/K)^\frown$. Of course whenever $(G/K)^\frown$ is naturally a group (i. e. in all the abelian situations), this reduces to ordinary convolution. This convolution operation has a very nice representation theoretic meaning. Namely let $\psi \in (G/K)^\frown$ so that ψ is actually an elementary continuous normalized positive definite functions, and let μ_ψ be the point mass on $(G/K)^\frown$ giving unit mass to the point ψ. Then $\psi(x) = \int \varphi(x)d\mu_\psi(\varphi)$. Then if $\mu_{\psi_1}\mu_{\psi_2}$ are two such point masses, their convolution $\mu_{\psi_1} * \mu_{\psi_2}$ is the de composition of $\psi_1\psi_2$ as an integral over $(G/K)^\frown$. But it ψ_i is a positive definite function associated with the unitary representation T_i of G then $\psi_1\psi_2$ is associated with the representation $T_1 \otimes T_2$ of G. Thus the convolution $\mu_{\psi_1} * \mu_{\psi_2}$ corresponds to tensoring representations! When there is a one-one correspondence between irreducible spherical representations of G and the points of $(G/K)^\frown$, as for example is the case when G/K is a symmetric space, then the measure $\mu_{\psi_1} * \mu_{\psi_2}$ describes how $T_1 \otimes T_2$ breaks up into irreducible spherical subrepresentations. We have here an example of the generalized translation operation of Delsarte [*1*] and Levitan [*1*]; this time the operation lives on the space $(G/K)^\frown$.

The elements Φ of class $\mathfrak{D}$ for (GK) correspond exactly to those measures on $(G/K)^\frown$ which are infinitely divisible under the convolution operation. From this point of view, such a function Φ in $\mathfrak{D}$, therefore gives rise to a semigroup $\{\mu_t, t \geq 0\}$ of probability measures on $(G/K)^\frown$ such that $\mu_t * \mu_s = \mu_{t+s}$, $\mu_0 =$ point mass at $1 \in (G/K)^\frown$. This may be interpreted as a stochastic process on $(G/K)^\frown$.

These semigroups and the processes to which they give rise are dual in the Fourier analytic sense to the processes discussed in an earlier paper; Gangolli [*2*].

When G/K is a compact symmetric space, the dual $(G/K)^\frown$ is essentially the integers, since it consists of the countable set of elementary positive definite normalized spherical functions $\varphi_0, \varphi_1, \ldots$ Multiplying any two of them and decomposing the product, we have $\varphi_n\varphi_m = \sum_j a(n, m, j)\varphi_j$ where $a(n, m, j)$ are nonnegative rationals with sum equal to one, and $a(n, m, j)$ are nonzero for only finitely many j (for fixed n, m). This gives on the integers a generalized translation operation, dual to the translation on G. For $G/K = S^d$, this sort of generalized translation has been implicitly studied classically by Gegenbauer, Sonine, and others (For recent illustrations of this operation see g. e. Kennedy [*1*], Hirschman [*1*]; but these authors do not make any reference to the underlying group structure). We shall discuss this subject elsewhere.

CHAPTER II

GAUSSIAN PROCESSES WHOSE COVARIANCES ARE LÉVY-SCHOENBERG KERNELS

§ 7. — Continuity of sample functions.

Let f be a Lévy-Schoenberg kernel on G/K. Because f is real valued and positive definite, it is possible to construct a centered Gaussian real stochastic process ξ with parameter running over G/K, such that the covariance of ξ is f. In other words, we may construct a probability space $(\Omega, \mathcal{S}, \mathbf{P})$ and a map ξ from G/K into real valued random variables defined on Ω such that

(7.1) For each $a \in G/K$, $\xi(a)$ is a real valued function in $L_2(\Omega, P)$.

(7.2) $\quad \mathbf{E}(\xi(a)) = 0 \quad$ for all $\quad a \in G/K$

(7.3) $\quad \mathbf{E}(\xi(a)\xi(b)) = f(a, b) \quad\quad a, b \in G/K$

(7.4) Given any points $a_1, a_2, \ldots a_n$ in G/K, the joint distribution of $\xi(a_1); \ldots \xi(a_n)$ is Gaussian.

If ξ is such a process, we may, for a fixed $\omega \in \Omega$, consider the function (on G/K) $a \to \xi(a, \omega)$ as a sample function of the stochastic process ξ. It is our purpose now to discuss certain almost sure properties of the sample functions of ξ.

For this purpose it seems useful to assume that G/K is locally Euclidean.

We are therefore henceforth assuming that we are in one of the cases II, III, IV. Further, in case III, we assume that G/K is Riemannian ([20]).

The problem to be considered is one of getting simple conditions on a Lévy-Schoenberg kernel f which will ensure that the process ξ will have almost surely continuous sample functions. Now, we have seen that f is determined by a real valued function Φ in the class $\mathfrak{D}$ for (G, K) and this function Φ is in turn determined by the function Ψ admitting a representation of the type (3.6) or (3.55) or (3.87) depending on which situation one is considering. It is therefore possible to seek conditions on Ψ which will ensure that the process ξ will have continuous sample functions. In what follows we fix a Lévy-Schoenberg kernel f, and the corresponding centered Gaussian process ξ whose parameter runs over G/K. r will denote the polarized kernel of f i. e. $r(a, b) = f(a, a) + f(b, b) - 2f(a, b)$, and Φ, Ψ, will have the meaning above. i. e. $\Psi(x) = r(xk, k)$, $\Phi = \exp - \Psi$.

We shall begin with a Lemma.

LEMMA 7.1. — Let $\tilde{U}$ be the cube in d-dimensional Euclidean space R^d, defined by $\tilde{U} = \{ a = (a_1, \dots, a_d) \mid |a_i| \leq 1\ i = 1, \dots d\}$. Let $\tilde{\xi}$ be a centred real Gaussian stochastic process defined on $\tilde{U}$ (i. e. with $\tilde{U}$ as parameter set).

Suppose that there exist constants C, $\beta > 0$ such that

$$\tilde{r}(a, b) = \mathbf{E}((\tilde{\xi}(a) - \tilde{\xi}(b))^2) \leq C\,|a - b|^{\beta},\ a, b \in U.$$

Then the process $\tilde{\xi}$ has almost surely continuous sample functions.

This Lemma, and even a stronger result, is known when $d=1$ (See e. g. Belayev [*1*] or Loéve [*1*, p. 520]). Proofs of most such results are modelled after Lévy's proof of the continuity of the sample functions of his Brownian motion with parameter in R^d, see Lévy [*1*, p. 279]. The generalization to the case $d > 1$, given below, offers no essential difficulty.

Proof. — For any integer n, let L_n be the lattice of points $a \in \tilde{U}$ such that each coordinate a_i of a is of the form $a_i = m_i 2^{-n}$, m_i an integer, $-2^n \leq m_i \leq 2^n$. We shall write (m) for $(m_1, m_2, \dots m_d)$ and $(m)2^{-n}$ for $(m_1 2^{-n}, m_2 2^{-n} \dots, m_d 2^{-n})$. L_n divides $\tilde{U}$ into $2^{(n+1)d}$ cubes all of the same size, and the diameter of each cube is $\sqrt{d}.2^{-n}$.

([20]) This means that G is to be a Lie group.

We now have the estimate.

$$(7.5)\quad \mathbf{P}\Big(\max_{\substack{a,b\in L_n\\ |a-b|=2^{-n}}} |\tilde{\xi}(a)-\tilde{\xi}(b)|\geq \varepsilon\Big)$$

$$\leq \sum_{\substack{a,b\in L_n\\ |a-b|=2^{-n}}} \mathbf{P}(|\tilde{\xi}(a)-\tilde{\xi}(b)|\geq \varepsilon)$$

$$\leq \sum_{\substack{a,b\in L_n\\ |a-b|=2^{-n}}} \frac{2}{\sqrt{2\pi\tilde{r}(a,b)}}\int_{\varepsilon}^{\infty} \exp-(t^2/2\tilde{r}(a,b))dt.$$

$$\leq \sum_{\substack{a,b\in L_n\\ |a-b|=2^{-n}}} \frac{2}{\sqrt{2\pi C\,|a-b|^{\beta}}}\int_{\varepsilon}^{\infty} \exp-(t^2/2C\,|a-b|^{\beta})dt.$$

where we used the fact that $\tilde{\xi}(a)-\tilde{\xi}(b)$ is Gaussian with mean zero and variance $\tilde{r}(a, b)$, and also used the fact that the function

$$(2\pi x)^{-1/2}\int_{\varepsilon}^{\infty} \exp-t^2/2x\,dt$$

is an increasing function of x for $x > 0$.

It follows from (7.5) that

$$(7.6)\quad \mathbf{P}\Big(\max_{\substack{a,b\in L_n\\ |a-b|=2^{-n}}} |\tilde{\xi}(a)-\tilde{\xi}(b)|\geq \varepsilon\Big)$$

$$\leq \sum_{\substack{a,b\in L_n\\ |a-b|=2^{-n}}} 2(2\pi C2^{-n\beta})^{-1/2}\int_{\varepsilon}^{\infty} \exp-(t^2/2C2^{-n\beta})dt.$$

$$\leq \sum_{\substack{a,b\in L_n\\ |a-b|=2^{-n}}} 2(2\pi C2^{-n\beta})^{-1/2}.\ \frac{C2^{-n\beta}}{\varepsilon}.\exp-(\varepsilon^2/2C2^{-n\beta}).$$

$$\leq M_1 2^{n(d-\beta/2)}\varepsilon^{-1}\exp-c_2.\varepsilon^2 2^{n\beta}$$

where for the last step but one we used the estimate

$$\int_a^{\infty} e^{-t^2/2x}\,dt \leq \frac{x}{a}\exp-(a^2/2x),$$

and for the last step, estimated, very crudely, the number of pairs $a, b \in L_n$

such that $| a - b | = 2^{-n}$. Here M_1, c_2 are certain constants, depending only on d perhaps, but *not* on n. Now let $\varepsilon = 2^{-n\beta/3}$ to get

$$
\begin{aligned}
(7.7) \qquad & \mathbf{P}\Big(\operatorname*{Max}_{\substack{a,b \in L_n \\ |a-b| = 2^{-n}}} | \tilde{\xi}(a) - \tilde{\xi}(b) | \geq 2^{-n\beta/3} \Big) \\
& \leq M_1 2^{n(d-\beta/6)} \exp - c^2 . 2^{n\beta/3}. \\
& = J_n \quad \text{say.}
\end{aligned}
$$

It is clear that $\sum_{n \geq 1} J_n < \infty$, since $\beta > 0$. So it follows by the Borel-Cantelli Lemma, that we have almost surely, for $n \geq n_0$ say

$$
(7.8) \qquad \operatorname*{Max}_{\substack{a,b \in L_n^{\sim} \\ |a-b| = 2^{-n}}} | \tilde{\xi}(a) - \tilde{\xi}(b) | < 2^{-n\beta/3}.
$$

Now let $t \in L_{n+m}$ where $m \geq 0$. Then we can write $t = a^0 + s$ where $a^0 \in L_n$ and $s = (s_1, s_2, \ldots, s_d)$ is such that each s_i is a dyadic rational and $0 \leq s_i < 2^{-n}$, $i = 1, \ldots, d$. Then $s_i = \sum_{j=1}^{m} \varepsilon_{ij}\, 2^{-n-j}$ where $\varepsilon_{ij} = 0$ or ± 1.

It follows that we can get points $a^0, a^1, \ldots, a^m$ in U, such that $a^m = t$, $a^q \in L_{n+q}$, and $| a^q - a^{q-1} | \leq d 2^{-(n+q)}$, $q = 1, 2, \ldots m$. For example we can define $a^q = (a_1^q, \ldots, a_d^q)$ by $a_i^q = a_i^0 + \sum_{j=1}^{q} \varepsilon_{ij} 2^{-n-j}$. Now use (7.8) and the triangle inequality repeatedly to get for $n \geq n_0$,

$$
\begin{aligned}
(7.9) \qquad & | \tilde{\xi}(t) - \tilde{\xi}(a^0) | \\
& \leq | \tilde{\xi}(a^m) - \tilde{\xi}(a^0) | \\
& \leq \sum_{q=1}^{m} | \tilde{\xi}(a^q) - \tilde{\xi}(a^{q-1}) | \\
& \leq \sum_{q=1}^{m} d . 2^{-\beta(n+q)/3} \\
& \leq d . 2^{-n\beta/3} \sum_{q=1}^{\infty} 2^{-\beta q/3} \\
& \leq M_2 2^{-n\beta/3}.
\end{aligned}
$$

Let t' be another point of L_{n+m} $m \geq 0$. If $|t' - t|$ is sufficiently small, then considering the collection of cubes its which L_n divides $\tilde{U}$, it becomes clear that t and t' belong to adjacent cubes in the collection, i. e. to cubes with a common vertex, a^0 say. We then have $t = a^0 + s$, $t' = a^0 + s'$,

$$s_i = \sum_{j=1}^{m} \varepsilon_{ij} 2^{-n-j}, \quad s'_i = \sum_{j=1}^{m} \varepsilon'_{ij} 2^{-n-j} \text{ and } \varepsilon_{ij} = 0 \text{ or } \pm 1, \quad \varepsilon'_{ij} = 0 \text{ or } \pm 1.$$

Then

$$(7.10) \qquad |\tilde{\xi}(t') - \tilde{\xi}(a^0)| \leq M_2 . 2^{-n\beta/3}, \qquad n \geq n_0$$

and hence

$$(7.11) \qquad |\tilde{\xi}(t) - \tilde{\xi}(t')| \leq 2M_2 . 2^{-n\beta/3}, \qquad n \geq n_0.$$

It follows that $\tilde{\xi}$ is almost surely uniformly continuous on $\bigcup_{n \geq 1} L_n$ and since this is dense in $\tilde{U}$, that $\tilde{\xi}$ is continuous in $\tilde{U}$ almost surely.

Q. E. D.

With the help of the above Lemma it is fairly easy now to get a useful condition on Ψ which will guarantee the continuity of the sample functions of the corresponding process ξ.

We fix a G-invariant Riemannian metric d on G/K, and for $x \in G$, set $|x| = d(xK, K)$. Then clearly $d(xK, yK) = |y^{-1}x|$.

THEOREM 7.2. — Let f be a Lévy-Schoenberg kernel on G/K and ξ the corresponding centred Gaussian process. Suppose that the function Ψ satisfies a Hölder condition of the form

$$(7.12) \qquad |\Psi(x)| \leq C |x|^{\beta} \qquad C, \beta > 0$$

for x in some neighbourhood B of e in G.

Then ξ has continuous sample functions almost surely.

Proof. — Since $r(xK, yK) = \Psi(y^{-1}x)$, the condition (7.12) implies that if x, y are close together (i. e. for $y^{-1}x \in B$), then $r(xK, yK) \leq C\, d(xK, yK)^{\beta}$ $c, \beta > 0$.

Now if $a = xK$, $b = yK$, and if $y^{-1}x \in B$,

$$(7.13) \qquad \mathbf{E}((\xi(a) - \xi(b))^2) = f(a, a) + f(b, b) - 2f(a, b) = r(a, b) \leq C\, d(a, b)^{\beta}.$$

It follows that there is a small ball U around eK in G/K such that if $a, b \in$ U then.

$$\mathbf{E}((\xi(a) - \xi(b))^2) \leq C\, d(a, b)^{\beta}. \tag{7.14}$$

Let M_{eK} be the tangent space to G/K at eK and let Exp be the exponential map from M_{eK} to G/K. We assume that U is small enough so that $\mathrm{Exp}^{-1}(U)$ is diffeomorphic to U. Let $\tilde{U}$ be a cube in Exp^{-1} (U) defined by

$$\tilde{U} = \{ x = (x_1, \ldots, x_d) \mid x_i \leq t \}$$

for some $t > 0$, and x_i are coordinates in M_{eK}. Define the process $\tilde{\xi}$ on $\tilde{U}$ by $\tilde{\xi}(a) = \xi$ (Exp.a), $a \in v$. Then clearly $\tilde{\xi}$ is centred and Gaussian, Further ξ has almost surely continuous sample functions on Exp $\tilde{U}$ if and only if $\tilde{\xi}$ has almost surely continuous sample functions on $\tilde{U}$. Now

$$\begin{aligned} \mathbf{E}((\tilde{\xi}(a) - \tilde{\xi}(b))^2) &= \mathbf{E}((\xi(\mathrm{Exp}\, a) - \xi(\mathrm{Exp}\, b))^2) \\ &\leq C\, d(\mathrm{Exp}\, a, \ \mathrm{Exp}\, b)_{\beta}. \end{aligned} \tag{7.15}$$

Now, it can be shown fairly easily that $d(\mathrm{Exp}\ a, \mathrm{Exp}\ b)/ \mid a - b \mid$ is bounded above and away from 0 on U (This is clear except at points of the diagonal of U $\times$ U; at such points one uses the fact that $d(\mathrm{Exp}\ a, \mathrm{Exp}\ b)/ \mid a - b \mid$ approaches a positive constant as $b \to a$ along geodesics, and the fact that d is G-invariant metric). It follows that (7.15) is equivalent to

$$\mathrm{E}((\xi(\tilde{a}) - \xi(\tilde{b}))^2) \leq C' \| a - b \|^{\beta} \text{ [21]}. \tag{7.16}$$

One may now apply lemma 7.1 to $\tilde{\xi}$ to conclude that it has almost surely continuous sample functions on $\tilde{U}$ (The cube $\tilde{U}$ of lemma 7.1 was the unit cube, but it should be clear that this is merely a matter of typographical convenience). Hence ξ has almost surely continuous sample functions on Exp $\tilde{U}$.

Now G/K can be covered by countably many left-translates (by elements of G) of Exp $\tilde{U}$. For each such translate one can repeat the above argument, and thus get the continuity of sample functions of ξ on G/K. The argument leading to this is quite similar, except for notation, to the above and is therefore omitted.

Q. E. D.

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(21) $\|.\|$ is a Euclidean norm on the tangent space. There is a canonical choice for this norm which is described later in the paper.

The condition $\mathbf{E}((\tilde{\xi}(a) - \tilde{\xi}(b))^2) \leq C \| a - b \|^\beta$ is certainly not the weakest possible condition which will ensure the conclusion of Lemma 7.1. Considerably weaker conditions of the form

$$\mathbf{E}((\tilde{\xi}(a) - \tilde{\xi}(b))^2) \leq C / | \log \| a - b \| |^\gamma$$

for appropriate $\gamma > 0$ can be shown to be sufficient to guarantee the conclusion of lemma 7.1. It follows that the condition (7.12) of the above theorem can be also weakened correspondingly. Further, one can even get estimates for the moduli of continuity of ξ in terms of the metric d. For example, ξ will almost surely satisfy Hölder conditions of the form $| \xi(a) - \xi(b) | \leq Cd(a, b)^\gamma$ for appropriate γ, uniformly on compacts in G/K. The technique of proving such results is to reduce the problem to one in a Euclidean space R^d by devices similar to the one used in theorem 7.2, and then get estimates for processes $\tilde{\xi}$ on R^d. Since we shall not use the extra information to be gained by doing this, we do not insist on these details here.

When does the function Ψ satisfy (7.12)? We shall study this question now.

Given Ψ, let Ψ^* be the function on G/K defined by $\Psi^*(x\mathrm{K}) = \Psi(x)$. This makes sense since Ψ is spherical. Clearly the condition $| \Psi(x) | \leq C | x |^\beta$ is equivalent to

$$(7.17) \qquad | \Psi^*(x\mathrm{K}) | \leq C(d(x\mathrm{K}, \mathrm{K}))^\beta.$$

Suppose now that we are considering case II, where $\mathrm{G/K} = \mathrm{R}^d$. Then if $a \in \mathrm{R}^d$, we know by (3.55) that

$$(7.18) \qquad \Psi^*(a) = g^*(a) + \int_{\lambda > 0} (1 - \mathrm{Y}_d(\lambda | a |)) d\mathrm{L}^*(\lambda)$$

with $g^*(a) = c | a |^2$, $a \leq 0$ and $\mathbf{L}^*$ satisfies $\int_{\lambda > 0} \lambda^2/1 + \lambda^2 \, d\mathbf{L}^*(\lambda) < 0$.

Clearly $g^*(a)$ satisfies a Hölder condition like (7.17), with $\beta = 2$. Further if Ψ^* satisfies a Hölder condition (7.17) for some $\beta > 0$, then $(\Psi^*)^\alpha$ when $0 < \alpha \leq 1$ also satisfies (7.17) for some other $\beta > 0$.

It follows that in case II, all the processes which arise from the kernels

$$(7.19) \qquad f(a, b) = \frac{1}{2}(| a |^\alpha + | b |^\alpha - | a - b |^\alpha) \; 0 \leq \alpha \leq 2$$

have continuous sample functions almost surely.

Examining the expression (7.18), one sees that a condition like (7.17) on Ψ^* is related to the rate at which the Lévy measure $\mathbf{L}$ decreases at ∞

For example if Ψ^* is of class C^2 near $0 \in R^d$, then $|\Psi^*(a)| \leq C|a|^\beta$ will surely hold for some $\beta > 0$. Now such a smoothness condition on Ψ^* can be procured by putting conditions on the measure L^* near ∞. For example, if $\int_{\lambda>0} \lambda^2 dL^*(\lambda) < \infty$, then, using the fact that $Y_d(\lambda |a|)$ is an eigenfunction of the Laplace operator of R^d with eigenvalue equal to a negative multiple of λ^2, and that $Y_d(\lambda |a|)$ is bounded for $a \in R^d$, one can conclude that $\int_{\lambda>0} (1 - Y_d(\lambda |a|))dL^*(\lambda)$ is then a function of class C^2 as a function of $a \in R^d$. It follows that if the Lévy measure L^* determined by Ψ (or Ψ^*) has this property $\int_{\lambda>0} \lambda^2 dL^*(\lambda) < \infty$, then Ψ^* satisfies (7.17) and therefore the process determined by the corresponding Lévy-Schoenberg kernel has almost surely continuous sample functions.

Next suppose we are in case III. Then

$$(7.20) \qquad \Psi(x) = \sum_{n\geq 1} a_n(1 - \varphi_n(x)); \qquad a_n \geq 0 \sum_{n\geq 1} a_n < \infty.$$

It can be again seen that the smoothness of Ψ near the origin is related to the rate at which the coefficients a_n tend to zero as $n \to \infty$. For example, if Δ is the Laplace-Beltrami operator of G/K, and $-\lambda_n$ is the eigenvalue of the eigenfunction φ_n of Δ i. e. $\Delta\varphi_n = -\lambda_n\varphi_n$, then the condition $\sum_{n\geq 1} \lambda_n a_n < \infty$ is enough to guarantee that $\Psi^*(xK)$ is of class C^2 near $eK \in G/K$ (Here again we are writing Ψ^* for the function on G/K determined by Ψ). Using Taylor's theorem, it is then easy to show that $\Psi^*(xK)$ satisfies (7.13) for xK near eK.

Finally let us turn to case IV. In this case

$$(7.21) \qquad \Psi(x) = g(x) + \int_{\mathcal{M}-\{1\}} (1 - \varphi(x))dL(\varphi).$$

Here we first of all claim that the function g is of class C^∞ on G (Actually, we shall see in § 9 that it is even analytic). To see this recall that g is satisfies $\int_K g(xky)dk = g(x) + g(y)$ and that therefore $g(x) = \int_K \lambda(H(xk))dk$ (Recall the remarks made at the end of our treatment of case IV in § 3). Now the function $x \to H(x)$ is of class C^∞ (See e. g. Helgason [*1*, chapter X]), and it therefore follows that $x \to \lambda(H(xk))$ is, for fixed $k \in K$, a function of class C^∞. Since K is compact, it is seen that $g(x)$ is of class C^∞

on G. Let g^* be the function on G/K determined by g, i. e. $g^*(x\mathrm{K}) = g(x)$. We now propose to estimate g^* near eK. For this we use a technique used before by us in Gangolli [*1*, p. 225, equation (6.5)], to which the reader is referred to details.

Let $\mathfrak{p}_0$ be the tangent space to G/K at eK. $\mathfrak{p}_0$ can and will be identified with the orthogonal complement of the Lie algebra $\mathfrak{k}_0$ of K in the Lie algebra $\mathfrak{g}_0$ of G (The orthogonal complement being with respect to the Killing form of $\mathfrak{g}_0$). The Killing form of $\mathfrak{g}_0$ is positive definite on $\mathfrak{p}_0 \times \mathfrak{p}_0$ so it defines a norm $\| \cdot \|$ on $\mathfrak{p}_0$. Now given $x \in \mathrm{G}$, let $\mathrm{X} \in \mathfrak{p}_0$ be such that $x = \exp \mathrm{X}.k$, $k \in \mathrm{K}$ where exp is the exponential map of $\mathfrak{g}_0$ to G. Also let $\| \mathrm{X} \| = r$, and $\tilde{\mathrm{X}} = \mathrm{X}/\| \mathrm{X} \|$. Then since g is spherical, we have $g(x) = g(\exp \mathrm{X}) = g(\exp r\tilde{\mathrm{X}}) = g(\exp \mathrm{Ad}(k).r\tilde{\mathrm{X}})$ for all $k \in \mathrm{K}$. Here A$d(k)$ stands for the adjoint action of $k \in \mathrm{K}$ on $\mathfrak{p}_0$. Hence,

$$g(x) = \int_{\mathrm{K}} g\,(\exp r\, \mathrm{A}d(k)\mathrm{X})dk \tag{7.22}$$

Now, since g is of class C^∞, the function $g(\exp r\, \mathrm{A}d\,(k)\tilde{\mathrm{X}})$ is of class C^∞ as a function of r, so using Taylors theorem on it, we get

$$\begin{aligned} g\,(\exp r\, \mathrm{A}d(k)\tilde{\mathrm{X}}) &= g(e) + r((\mathrm{A}d(k)\tilde{\mathrm{X}})g)\,(e) \\ &\quad + \frac{r^2}{2}((\mathrm{A}d(k)\tilde{\mathrm{X}})^2 g)\,(e) \\ &\quad + r^2 \cdot \theta_x(k) \end{aligned} \tag{7.23}$$

with $\theta_x(k) = o(1)$ as $r \to 0$ uniformly for $k \in \mathrm{K}$, and for x in a sufficiently small neighbourhood of e.

Of course, here we are interpreting the elements $\mathrm{A}d(k)\tilde{\mathrm{X}}$ of $\mathfrak{p}_0$ as differential operators as usual.

Integrating the above over K, we get

$$\begin{aligned} g(x) &= g(e) + r\int_{\mathrm{K}} ((\mathrm{A}d(k)\tilde{\mathrm{X}})g)(e)dk \\ &\quad + r^2 \int_{\mathrm{K}} ((\mathrm{A}d(k)\tilde{\mathrm{X}})^2 g)(e)dk \\ &\quad + o_x(r^2) \\ &= g(e) + r\left(\left(\int_{\mathrm{K}} \mathrm{A}d(k)\tilde{\mathrm{X}}dk\right)g\right)(e) \\ &\quad + \frac{r^2}{2}\left(\left(\int_{\mathrm{K}} (\mathrm{A}d(k)\tilde{\mathrm{X}})^2 dk\right)g\right)(e) \\ &\quad + o_x(r^2). \end{aligned} \tag{8.24}$$

Now, since $\int_K g(xky)dk = g(x) + g(y)$, and $g(k) = g(e)$, we see setting $x = y = e$ that $2g(e) = \int_K g(k)dk = g(e)\int_K dk = g(e)$ so $g(e) = 0$. Also, $\int_K \mathrm{A}d(k)\tilde{\mathrm{X}}dk = 0$, as is well-known. Hence it follows that

$$g(x) = \text{const. } r^2 + o(r^2) \text{ as } r \downarrow 0. \tag{7.25}$$

The canonical G-invariant Riemannian metric d on G/K is induced by the restriction of the Killing form of $\mathfrak{g}_0$ to $\mathfrak{p}_0 \times \mathfrak{p}_0$, and one may check that $r^2 \sim d(x\mathrm{K}, e\mathrm{K})^2$ as $r \to 0$. It follows that $g(x) \sim (d(x\mathrm{K}, e\mathrm{K}))^2$ as $x\mathrm{K} \to e\mathrm{K}$, so that we get

$$| g^*(x\mathrm{K}) | \leq \mathrm{C}(d(x\mathrm{K}, e\mathrm{K}))^2 \text{ for } x\mathrm{K} \text{ near } e\mathrm{K}. \tag{7.26}$$

It follows from this and our work above that if f is the Lévy-Schoenberg kernel on G/K defined by

$$f(x\mathrm{K}, y\mathrm{K}) = \frac{1}{2}(g(x)^\alpha + g(y)^\alpha - g(y^{-1}x)^\alpha) \tag{7.27}$$

with $0 \leq \alpha \leq 1$, then the corresponding process ξ has continuous sample functions almost surely.

Finally, just as in cases II, III above, we can give conditions on the rate of decrease of the measure $\mathbf{L}$ on which will ensure that the function Ψ defined by (7.21) satisfies a Hölder condition $| \Psi(x) | \leq \mathrm{C} | x |^\beta$ near $e \in \mathrm{G}$. Namely, Let Δ be the Laplace operator of G/K, and let $-\lambda(\varphi)$ be the eigenvalue of the eigenfunction φ of Δ i. e. $\Delta\varphi = -\lambda(\varphi)\varphi$. If the measure $\mathbf{L}$ on $\mathcal{M}$ satisfies the condition $\int_{\mathcal{M}} | \lambda(\varphi) | d\mathrm{L}(\varphi) < \infty$, it can be shown that $\int_{\mathcal{M}-\{1\}} (1 - \varphi(x))d\mathbf{L}(\varphi)$ is, as a function of x, of class C^2 near $e \in \mathrm{G}$. It follows by methods similar to the above that in that case, the function Ψ defined by (7.21) satisfies a Hölder condition like (7.12), and therefore determines a process ξ whose sample functions are almost surely continuous.

To summarize, we have shown in case II, IV, that if Ψ^*, Ψ are as above and $(g^*, \mathbf{L}^*)$ $(g, \mathbf{L})$ correspond to Ψ^*, Ψ, then the conditions

$$\int_{\lambda > 0} \lambda^2 d\mathbf{L}^*(\lambda) < \infty, \qquad \int_{\mathcal{M}} | \lambda(\varphi) | d\mathbf{L}(\varphi) < \infty$$

will ensure that the corresponding process ξ will have continuous sample functions almost surely, while in case III, the conditions $\sum_{n \geq 1} \lambda_n a_n < \infty$, will ensure that Ψ satisfies the Hölder condition (7.12), and hence that the process ξ corresponding to Ψ will have continuous sample paths. Further, in cases II and IV, if $\Psi(x) = g(x)$ (i. e. if $L = 0$) then the Hölder condition $| g^*(xK) | \leq C(d(xK, eK))^2$ is satisfied for xK near eK; Finally, if Ψ satisfies a Hölder condition such as (7.12), then so does Ψ^α for any $0 < \alpha \leq 1$ so that when α is thus, the process ξ corresponding to Ψ^α also has almost surely continuous paths.

It must of course be possible to give less stringent conditions on the Lévy-measure **L** which will ensure that satisfies (7.12). We have not investigated what these might be precisely, first because it would lead us too far afield and second because we have no use for such refinements in the later parts of this paper. Anyhow, the proof of theorem 7.2 makes plain what sort of conditions one would like to seek for Ψ.

§ 8. — An orthogonal decomposition.

We shall now work with a fixed centred Gaussian process with parameter running over G/K (case II, III, or IV), whose covariance os a Lévy-Schoenberg kernel f. We shall assume that the structure function Ψ of f satisfies a Hölder condition $| \Psi(x) | \leq C | x |^\beta$, $C, \beta > 0$, near $e \in G$, so that ξ has almost surely continuous sample functions. Moreover, while some of the results below can be proved without any further assumption on G/K, it seems that a natural domain for them is when G/K is a two-point homogeneous space. We shall therefore make this assumption, and indicate after each result whether and how the result might be salvaged in general.

As has been mentioned above, two-point homogeneous of the compact type were classified by Wang [*1*]. In the non-compact case they were classified by Tits [*1*]. Apart from the Euclidean spaces (case I) the two-point homogeneous spaces of non-compact type are the real, complex and quaternionic hyperbolic spaces and the hyperbolic analogue of the Cayley elliptic plane (case IV). Here is a list of groups G, K which give rise to these.

i) Real hyperbolic spaces $L^d(R)$; $d = 2, 3, \ldots$

$$G = \mathbf{SO}_0(d, 1) \qquad K = \mathbf{SO}(d).$$

ii) Complex hyperbolic spaces $L^d(C)$; $d = 2, 4, 6, \ldots$

$$G = \mathbf{SU}(p, 1) \qquad K = \mathbf{S}(\mathbf{U}_p \times \mathbf{U}_1) \qquad p = d/2.$$

iii) Quaternionic hyperbolic spaces $L^d(H)$; $d = 4, 8, 12, 16, \ldots$

$$G = \mathbf{Sp}(p, 1) \qquad K = \mathbf{Sp}(p) \times \mathbf{Sp}(1) \qquad p = d/4.$$

iv) Cayley hyperbolic space $L^{16}(\mathrm{Cay})$.

$$G = \mathbf{F}_{4,(-20)} \qquad K = \mathbf{SO}(9).$$

In each case d is the real dimension of the space. For the notation see Helgason [*1*, Chapter IX]. Except for the real projective space $P^d(R)$, the rest of the compact two-point homogeneous spaces in the list in § 4 are dual in Cartan's sense to the spaces in the present list.

From now on we let G/K be a compact or non-compact two point homogeneous space *except* a real projective space. For that space, the disconnectedness of K raises problems with which we do not wish to deal. We shall also exclude G/K = a circle, this case being trivial.

Fix once and for all the canonical G-invariant metric d on G/K. If G/K is Euclidean, this is just the usual metric. In the other cases, this metric arises in the following way. Let $\mathfrak{g}_0$ be the Lie algebra of G and $\mathbf{B}$ its killing form. Let $\mathfrak{k}_0$ be the Lie algebra of K, and $\mathfrak{p}_0$ the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$ with respect to $\mathbf{B}$. If G/K is compact then $-\mathbf{B}$ is positive definite on $\mathfrak{p}_0 \times \mathfrak{p}_0$ while if G/K is non-compact, then $\mathbf{B}$ is positive definite on $\mathfrak{p}_0 \times \mathfrak{p}_0$. In each case we endow $\mathfrak{p}_0$ with the inner product and norm induced by this positive definite form. $\mathfrak{p}_0$ can be identified with the tangent space to G/K at eK, and now, since G acts transitively on G/K, this inner product on $\mathfrak{p}_0$ can be used to introduce a G-invariant Riemannian structure on G/K.

Let L be the diameter of G/K computed via the above G-invariant metric; i. e. $L = \sup_{x \in G} d(xK, eK)$. If G/K is non-compact then $L = \infty$. Otherwise L is finite. It is well-known that in the present set-up, the exponential map Exp from $\mathfrak{p}_0$ to G/K maps the open ball $\{ X \mid \| X \| < L \}$ in $\mathfrak{p}_0$ homeomorphically onto the open ball $B(L) = \{ xK \mid d(xK, eK) < L \}$ in G/K (Of course, when G/K is non-compact B(L) = G/K, and this mapping Exp gives a homeomorphism of $\mathfrak{p}_0$ onto G/K). We may therefore introduce geodesic polar coordinates (t, θ) with pole at eK, valid on B(L). Here a point $xK \in B(L)$ has coordinates (t, θ) where $t = d(xK, eK)$ and θ is an « angular » parameter running over the unit sphere in $\mathfrak{p}_0$.

Choose a unit vector $Z \in \mathfrak{p}_0$ once and for all, and let γ be the geodesic in G/K issuing from eK with tangent vector Z. For $t \in [0, L)$, let $c_t = \operatorname{Exp} tZ$. Then $c_t \in \gamma$, and c_t is at distance t from $o(= eK)$. Let us fix $z_t \in G$ such that $z_t K = c_t$. Since G/K is two point homogeneous, K acts [22] transitively and effectively on the unit sphere in $\mathfrak{p}_0$. Let U be the subgroup of K which leaves the point Z fixed. It is clear that U leaves every point of γ fixed, for since $U \subset K$, U fixes eK, and also fixes Z.

Thus U leaves fixed the point $c_t \in S(t)$ [23]. Also, since K acts transitively on $S(t)$, we can set $S(t) = K/U$ for $0 \leq t < L$. For example if $a = xK \in S(t)$ and $k \in K$ such that $k.c_t = a$, then the correspondence $a \longleftrightarrow kU$ identifies $S(t)$ with K/U.

Finally we need some conventions about function spaces on K/U. As usual, functions on K/U may be thought of as functions on K which are invariant under right translations by elements of U. As in § 3 (case III), $\mathfrak{K}(K)$ will denote the set of equivalence classes of irreductible unitary representations of K and $\mathfrak{K}_U(K)$ the subset consisting of those elements of $\mathfrak{K}(K)$ which are U-spherical i. e. $\alpha \in \mathfrak{K}_U(K)$ if and only if whenever T_α is a representation of class α on a Hilbert space $H(T_\alpha)$, there exists a unit vector $v \in H(T_\alpha)$ such that $T_\alpha(u)v = v$ for all $u \in U$. This is the same thing as demanding that the reduction of the restriction of T_α to U contains the trivial representation $u \rightarrow 1$ of U. For a given $\alpha \in \mathfrak{K}_U(K)$, fix a representation T_α of class α on $H(T_\alpha)$, and let $H_U(T_\alpha)$ be the subspace of $H(T_\alpha)$ defined by $H_U(T_\alpha) = \{ v \mid v \in H(T_\alpha); T_\alpha(u)v = v \text{ for all } u \in U \}$. Clearly $H_U(T_\alpha) \neq \{ 0 \}$. Define $n_\alpha = \dim (H_U(T_\alpha))$, $d_\alpha = \dim \alpha = \dim H(T_\alpha)$. Further let us choose an orthonormal basis $v_1, \dots v_{d_\alpha}$ of $H(T_\alpha)$ such that $v_1, \dots v_{n_\alpha}$ is a basis of $H_U(T_\alpha)$. Define the functions $\chi_{\alpha pq}$ on K by $\chi_{\alpha pq}(k) = (T_\alpha(k)v_p, v_q)$ $1 \leq p, q \leq d_\alpha$. Clearly $\chi_{\alpha pq}$ are the elementary representative functions associated with α. Further, if $1 \leq p \leq d_\alpha$, then $\chi_{\alpha pq}(ku) = \chi_{\alpha pq}(k)$ forall $k \in K$, $u \in U$, and if $1 \leq p, q \leq n_\alpha$ then $\chi_{\alpha pq}$ is U-spherical. If we denote by $L_2(K/U)$ (resp. $C(K/U)$) the space of functions in $L_2(K)$ (resp. $C(K)$) which are invariant under right translations by elements of U; i. e. $f(ku) = f(k)$, then the functions $\{ \chi_{\alpha pq}; 1 \leq p \leq n_\alpha, 1 \leq q \leq d_\alpha, \alpha \in \mathfrak{K}_U(K) \}$ form an orthogonal basis of $L_2(K/U)$, finite linear combinations of these functions are uniformly dense in $C(K/U)$ and one has the Schur orthogonality relations

$$(8.1) \qquad \int_K \chi_{\alpha pq}(k)\overline{\chi_{\beta rs}(k)}dk = \delta_{\alpha\beta}\delta_{pr}\delta_{qs}/d_\alpha$$

which will be useful below.

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(22) We are here identifying K with the linear isotropy group induced by K on the tangent space $\mathfrak{p}_0$. This may be done because K acts effectively on each sphere of radius $t < L$ in G/K.

(23) $S(t)$ is the sphere $\{ xK \mid d(xK, eK) = t \}$ in G/K.

For a T_α of class α, consider the operator $P_U = \int_U T_\alpha(u)du$. It is easy to show that P_U is a self-adjoint projection in $H(T_\alpha)$ whose range is $H_U(T_\alpha)$. Thus if $1 \leq p \leq n_\alpha$ then $P_U v_p = v_p$, and $P_U v_p = 0$ if $n_\alpha < p \leq d_\alpha$. Therefore, if $1 \leq p \leq n_\alpha$,

$$\begin{aligned}(8.2)\qquad &\int_U \chi_{\alpha pq}(k'uk)du \\ &= \left(\int_U T_\alpha(k')T_\alpha(u)T_\alpha(k)\cdot v_p,\ v_q\right) \\ &= (T_\alpha(k')P_U T_\alpha(k)v_p,\ v_q) \\ &= (T_\alpha(k')P_U T_\alpha(k)P_U v_p,\ v_q)\end{aligned}$$

and if we now use the fact that

$$\begin{aligned}(8.3)\qquad T_\alpha(k)P_U v_p &= \sum_{r=1}^{d_\alpha} (T_\alpha(k)P_U v_p,\ v_r)v_r \\ &= \sum_{r=1}^{d_\alpha} \chi_{\alpha pr}(k)\cdot v_r,\end{aligned}$$

we get

$$\begin{aligned}(8.4)\qquad &\int_U \chi_{\alpha pq}(k'uk)du \\ &= \sum_{r=1}^{d_\alpha} (T_\alpha(k')P_U v_r,\ v_q)\chi_{\alpha pr}(k) \\ &= \sum_{r=1}^{n_\alpha} (T_\alpha(k')P_U v_r,\ v_q)\chi_{\alpha pr}(k) \qquad P_U v_r = 0 \text{ if } r > n_\alpha \\ &= \sum_{r=1}^{n_\alpha} \chi_{\alpha rq}(k')\cdot\chi_{\alpha pr}(k)\end{aligned}$$

which we shall use below.

We are now ready to describe an orthogonal decomposition of our process ξ. The reader should compare our method to that of McKean [*1*] which is our model. Let xK be a point of G/K, and suppose $xK \in S(t)$. Then as we have seen, xK can be written $xK = kz_tK$ for some $k \in K$, and k is unique modulo U. Thus $\xi(xK) = \xi(kz_tK)$. Now, as xK ranges over

$S(t)$, k ranges over a cross section of K/U; further, since $\xi(xK)$ is a continuous function of xK, it follows that $\xi(kz_tK)$ is a continuous, hence square integrable, function of k, which is invariant under right translations by elements of U (for clearly $\xi(kuz_tK) = \xi(kz_tK)$, since $uz_tK = z_tK$ for all $u \in U$). The idea is now to perform, for fixed t, a Fourier expansion of this function $\xi(kz_tK)$ as a function in $L_2(K/U)$. As we shall see, this will give us an orthogonal decomposition for ξ in terms of which the Markov property for ξ can be discussed.

Recalling that $\chi_{\alpha pq}$; $1 \leq p \leq n_\alpha$, $1 \leq q \leq d_\alpha$; $\alpha \in \mathfrak{R}_U(K)$ gives us an orthogonal basis of $L_2(K/U)$, we define now for $t \in [0, L)$, the process $\xi_{\alpha pq}(t)$ by

$$(8.5) \quad \xi_{\alpha pq}(t) = \int_K \xi(kz_tK)\chi_{\alpha pq}(k)dk \qquad 1 \leq p \leq n_\alpha,\ 1 \leq q \leq d_\alpha,\ \alpha \in \mathfrak{R}_U(K).$$

so that we have

$$(8.6) \qquad \xi(kz_tK) = \sum_{\alpha \in \mathfrak{R}_U(K)} \sum_{p=1}^{n_\alpha} \sum_{q=1}^{d_\alpha} d_\alpha \xi_{\alpha pq}(t)\bar{\chi}_{\alpha pq}(k) \quad \text{in} \quad L_2(K/U).$$

Since ξ was centred and Gaussian, it is clear that $\xi_{\alpha pq}(t)$ is also a centred (complex-valued) [24] Gaussian process. We now want to compute the covariance of $\xi_{\alpha pq}(t)$, and also to determine the relation between the various processes $\xi_{\alpha pq}(t)$.

To this end let $x'K \in S(t')$, $x'K = k'z_{t'}K$. Then, remembering that $\mathbf{E}(\xi(a)\xi(b)) = f(a, b)$ $a, b \in G/K$, we get [25]

$$(8.7) \qquad \mathbf{E}(\xi_{\alpha pq}(t)\xi(x'K))$$

$$= \int_K \mathbf{E}(\xi(kz_tK)\xi(k'z_{t'}K))\chi_{\alpha pq}(k)dk$$

$$= \int_K f(kz_tK, k'z_{t'}K)\chi_{\alpha pq}(k)dk$$

$$= \int_K f(kc_t, k'c_{t'})\chi_{\alpha pq}(k)dk$$

$$= 1/2\int_K (r(kc_t, o) + r(k'c_{t'}, o) - r(kc_t, k'c_{t'}))\chi_{\alpha pq}(k)dk$$

$$= 1/2\int_K (r(c_t, o) + r(c_{t'}, o) - r(kc_t, k'c_{t'}))\chi_{\alpha pq}(k)dk$$

---

[24] A complex-valued random variable is said to be Gaussian if its real and imaginary parts are Gaussian.

[25] The interchange of the order of integration offers no difficulty, since K is compact and the random variables involved are Gaussian.

where we used the invariance of r (cf. (2.18)) and the fact that $k.o = o$, for all $k \in \mathrm{K}$. Now, $r(c_t, o) = r(z_t\mathrm{K}, e\mathrm{K}) = \Psi(z_t)$, and $r(c_{t'}, o) = \Psi(z_{t'})$. This enables us to evaluate the first two terms above.

Also,

$$
\begin{aligned}
(8.8) \qquad & r(kc_t, k'c_{t'}) \\
&= r(kc_t, k'uc_{t'}) \\
&= r(kz_t\mathrm{K}, k'uz_{t'}\mathrm{K}) \\
&= \Psi\big(z_{t'}^{-1}u^{-1}k'^{-1}kz_t\big) \qquad \text{for all} \quad u \in \mathrm{U}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(8.9) \qquad & \int_{\mathrm{K}} r(kc_t, k'c_{t'})\chi_{\alpha pq}(k)dk \\
&= \int_{\mathrm{K}} \Psi\big(z_{t'}^{-1}u^{-1}k'^{-1}kz_t\big)\chi_{\alpha pq}(k)dk \\
&= \int_{\mathrm{K}} \Psi\big(z_{t'}^{-1}kz_t\big)\chi_{\alpha pq}(k'uk)dk \quad \text{for all} \quad u \in \mathrm{U}.
\end{aligned}
$$

Since the left side is independent of $u \in \mathrm{U}$, we can integrate w. r. t. u to get

$$
\begin{aligned}
(8.10) \qquad & \int_{\mathrm{K}} r(kc_t, k'c_{t'})\chi_{\alpha pq}(k)dk \\
&= \int_{\mathrm{K}} \Psi\big(z_{t'}^{-1}kz_t\big)\left(\int_{\mathrm{U}} \chi_{\alpha pq}(k'uk)du\right)dk.
\end{aligned}
$$

So we finally have, using this in (8.7),

$$
\begin{aligned}
(8.11) \qquad & \mathbf{E}(\xi_{\alpha pq}(t)\xi(x'\mathrm{K})) \\
&= \mathbf{E}(\xi_{\alpha pq}(t)\xi(k'z_{t'}\mathrm{K})) \\
&= 1/2\,\Psi(z_t)\int_{\mathrm{K}} \chi_{\alpha pq}(k)dk + 1/2\,\Psi(z_{t'})\int_{\mathrm{K}} \chi_{\alpha pq}(k)dk \\
&- 1/2 \int_{\mathrm{K}} \Psi\big(z_{t'}^{-1}kz_t\big)\int_{\mathrm{U}} \chi_{\alpha pq}(k'uk)du.dk \\
&= \begin{cases} 1/2\,(\Psi(z_t) + \Psi(z_{t'}) - \int_{\mathrm{K}} \Psi(z_{t'}^{-1}kz_t)dk) & \text{if} \quad \alpha = \mathfrak{1} \\ -1/2 \int_{\mathrm{K}} \Psi\big(z_{t'}^{-1}kz_t\big)\int_{\mathrm{U}} \chi_{\alpha pq}(k'uk)du.dk & \text{if} \quad \alpha \neq \mathfrak{1} \end{cases}
\end{aligned}
$$

where we used (8.1), and the fact that the constant function 1 is associated with the class $\mathbb{1}$. It follows that

$$(8.12)\quad \mathbf{E}(\xi_{\alpha pq}(t)\bar{\xi}_{\alpha' p'q'}(t'))$$

$$= \int_{\mathrm{K}} \mathbf{E}(\xi_{\alpha pq}(t)\bar{\xi}(k'z_{t'}\mathrm{K}))\bar{\chi}_{\alpha' p'q'}(k')dk'$$

$$= \begin{cases} 1/2\left(\Psi(z_t) + \Psi(z_{t'}) - \int_{\mathrm{K}} \Psi(z_{t'}^{-1}kz_t)dk\right)\int_{\mathrm{K}} \bar{\chi}_{\alpha'p'q'}(k')\,dk'\Big) & \text{if} \quad \alpha \neq \mathbb{1} \\ -\,1/2 \int_{\mathrm{K}} \Psi(z_{t'}^{-1}kz_t)\left(\int_{\mathrm{K}}\left(\int_{\mathrm{U}} \chi_{\alpha pq}(k'uk)du\right)\bar{\chi}_{\alpha'p'q'}(k')dk'\right) & \text{if} \quad \alpha \neq 1. \end{cases}$$

Now recall (8.4), use it on the integral in the parentheses to get, with the help of (8.1),

$$(8.13)\qquad \int_{\mathrm{K}}\int_{\mathrm{U}} \chi_{\alpha pq}(k'uk)du\bar{\chi}_{\alpha'p'q'}(k')dk'$$

$$= \sum_{r=1}^{n_\alpha} \chi_{\alpha pr}(k)\int_{\mathrm{K}} \chi_{\alpha rq}(k')\bar{\chi}_{\alpha'p'q'}(k')dk'.$$

$$= \sum_{r=1}^{n_\alpha} \chi_{\alpha pr}(k)\delta_{\alpha\alpha'}\delta_{rp'}\delta_{qq'}/\delta_\alpha$$

$$= \chi_{\alpha pp'}(k)\delta_{\alpha\alpha'}\delta_{qq'}/d_\alpha$$

It follows that

$$(8.14)\quad \mathbf{E}(\xi_{\alpha pq}(t)\bar{\xi}_{\alpha'p'q'}(t'))$$

$$= \begin{cases} 1/2\left(\Psi(z_t) + \Psi(z_{t'}) - \int_{\mathrm{K}} \Psi(z_{t'}^{-1}kz_t)dk\right) & \text{if} \quad \alpha = \alpha' = \mathbb{1} \\ -\,1/2\,\dfrac{\delta_{\alpha\alpha'}\cdot\delta_{qq'}}{d_\alpha}\displaystyle\int_{\mathrm{K}} \Psi(z_{t'}^{-1}kz_t)\chi_{\alpha pp'}(k)dk & \text{otherwise.} \end{cases}$$

The main point of this is that when $\alpha \neq \alpha'$ or $q \neq q'$, $\xi_{\alpha pq}$ and $\xi_{\alpha'p'q'}$ are orthogonal, while if $\alpha = \alpha'$ $q = q'$ then the covariance between $\xi_{\alpha pq}(t)$ $\xi_{\alpha p'q}(t')$ is expressed in terms of Ψ.

For fixed α, and $1 \leq q \leq d_\alpha$, let $\eta_{\alpha q}(t)$ be the n_α-dimensional complex vector whose components are $\xi_{\alpha pq}(t)$ $1 \leq p \leq n_\alpha$. Then $t \rightarrow \eta_{\alpha q}(t)$ is a centred Gaussian vector valued process defined over $[0, \mathrm{L})$. Further if $\lambda_{\alpha q}(k)$ is the n_α-vector whose coordinates are $d_\alpha\chi_{\alpha pq}(k)$, $1 \leq p \leq n_\alpha$, then using the usual definition of scalar product on n_α-dimensional vectors with complex coordinates, we see that

$$(8.15) \qquad (\eta_{\alpha q}(t), \lambda_{\alpha q}(k)) = \sum_{p=1}^{n_\alpha} d_\alpha \xi_{\alpha pq}(t)\chi_{\alpha pq}(k)$$

so that (8.6) reads

$$(8.16) \qquad \xi(kz_t\mathrm{K}) = \sum_{\alpha \in \mathfrak{R}_{\mathrm{U}}(\mathrm{K})} \sum_{q=1}^{d_\alpha} (\eta_{\alpha q}(t), \lambda_{\alpha q}(k)).$$

Note that $t \to \eta_{\alpha q}(t)$ is a vector valued stochastic process with parameter $t \in [0, \mathrm{L})$ such that each coordinate of $\eta_{\alpha q}(t)$ is a complex valued random variable whose real and imaginary parts have Gaussian distributions. The vector $\eta_{\alpha q}(t)$ is n_α-dimensional. Further when $(\alpha, q) \neq (\alpha', q')$, $\eta_{\alpha q}$ and $\eta_{\alpha' q'}$ are orthogonal processes in the sense that the components of $\eta_{\alpha q}$ are orthogonal to the components of $\eta_{\alpha' q'}$ as complex-valued random variables in $\mathrm{L}_2(\Omega, \mathcal{S}, \mathbf{P})$ [26].

A substantial simplification takes place if K/U is a symmetric space. This happens for example when G/K is a Euclidean space R^d or a sphere S^d or a real hyperbolic space $\mathrm{L}^d(\mathrm{R})$. Indeed, in each of these three cases, $\mathrm{K} = \mathrm{SO}(d)$ and $\mathrm{U} = \mathrm{SO}(d-1)$ so that K/U is just S^{d-1} (Unfortunately K/U is not symmetric in general; for example when $\mathrm{G/K} = \mathrm{P}^d(\mathrm{C})$ or $\mathrm{L}^d(\mathrm{C})$, K/U is not symmetric). When K/U has this extra feature, $n_\alpha = 1$ for each $\alpha \in \mathfrak{R}_{\mathrm{U}}(\mathrm{K})$ (cf. Cartan's theorem quoted in § 3). This means that the process $\eta_{\alpha q}(t)$ are now all one-dimensional. Further, $\chi_{\alpha 11}$ is then the unique U-spherical

---

[26] It is possible to give a more intrinsic description of the process $\eta_{\alpha q}$. Indeed, in the terminology of § 3, it is possible to show that $\sum_{q=l}^{d_\alpha} (\eta_{\alpha q}(t), \lambda_{\alpha q}(k))$ is exactly the component of the function $\xi(kz_t\mathrm{K})$ ($\in \mathrm{L}_2(\mathrm{K/U})$) which lies in $\mathrm{L}_2^\alpha(\mathrm{K/U})$. Here $\mathrm{L}_2^\alpha(\mathrm{K/U})$ is the subspace of $\mathrm{L}_2(\mathrm{K/U})$ consisting of vectors which transform according to α under the left regular representation of K on $\mathrm{L}_2(\mathrm{K/U})$. Actually, $\mathrm{L}_2^\alpha(\mathrm{K/U})$ is a direct sum of n_α subspaces each of dimension d_α on each of which the left regular representation acts like a member of α. The functions $\{ d_\alpha \chi_{\alpha pq},\ 1 \leq p \leq n_\alpha,\ 1 \leq q \leq d_\alpha \}$ form a basis of $\mathrm{L}_2^\alpha(\mathrm{K/U})$, and $\xi_{\alpha pq}(t)$ are the components of $\sum_{q=1}^{d_\alpha} (\eta_{\alpha q}(t), \lambda_{\alpha q}(k))$ with respect to this basis.

function associated with the class α and this function is automatically a positive definite normalized U-spherical continuous function on K. $\{ \chi_{\alpha 1 q} 1 \leq q \leq d_\alpha, \alpha \in \mathfrak{R}_U(K) \}$ form an orthogonal basis of $L^2(K/U)$. They are just the « spherical harmonics » of K/U. We now have

$$(8.17) \qquad \xi(kz_t K) = \sum_{\alpha \in \mathfrak{R}_U(K)} d_\alpha \sum_{q=1}^{d_\alpha} \eta_{\alpha q}(t) \bar{\chi}_{\alpha 1 q}(k)$$

where $\eta_{\alpha q}(t)$ are certain one dimensional complex valued centred Gaussian process with parameter $t \in [0, L)$. Further, if $(\alpha, q) \neq (\alpha', q')$ then $\eta_{\alpha q}$, $\eta_{\alpha' q'}$ are orthogonal. Finally (8.14) says here that

$$(8.18) \quad E(\eta_{\alpha q}(t)\eta_{\alpha' q'}(t'))$$
$$= \begin{cases} 0 & \text{if} \quad (\alpha, q) \neq (\alpha', q') \\ 1/2\left(\Psi(z_t) + \Psi(z_{t'}) - \int_K \Psi(z_{t'}^{-1} k z_t) dk\right) & \text{when} \quad \alpha = 1, q = 1 \\ -\dfrac{1}{2d_\alpha} \int_K \Psi(z_{t'}^{-1} k z_t) \chi_{\alpha 11}(k) dk & \text{otherwise} \end{cases}$$

where of course $\chi_{\alpha 11}$ is the unique elementary normalized U-spherical function associated with α.

In case $G/K = R^d$ or S^d or $L^d(R)$, the space K/U is more than just symmetric. It is itself a two-point homogeneous space. This has the consequence that the function $\chi_{\alpha 11}$ is real valued. Now $\chi_{\alpha 1 q}$ are a basis for the space spanned by the left translates, by elements of K, of the function $\chi_{\alpha 11}$. In this case the basis $v_1, \ldots, v_{d_\alpha}$ may be chosen in such a way that the functions $\chi_{\alpha 1 q}$ are all real valued. It follows that the process $\eta_{\alpha q}(t)$ are in this case all real valued centred Gaussian processes with parameter $t \in [0, L)$. Moreover, $\eta_{\alpha q}$, $\eta_{\alpha' q'}$ being Gaussian, real and orthogonal, are independent. Thus (8.17) describes a decomposition of into independent processes depending on a one dimensional parameter $t \in [0, L)$.

In this case, as we have remarked above, $K/U = S^{d-1}$. As in § 4, we now use the explicit description of the U-spherical functions on K, which is known for this case. Each U-spherical elementary function $\chi_{\alpha 11}$ is a Gegenbauer polynomial $P_n^{(d-2)/2}$ (cos θ), upto a constant multiple, and the function $\chi_{\alpha 1 q}$ are then the ultraspherical harmonics of weight n. We may thus identify $\mathfrak{R}_U(K)$ with $\{0, 1, 2, \ldots\}$ by the requirement $\alpha \longleftrightarrow n$ if $\chi_{\alpha 11} = P_n^{(d-2)/2}$ and rename $\eta_{\alpha q}$ as η_{nq}. $n = 0, 1, 2, \ldots$ Further if Ψ is the structure function of our process ξ, it is possible now to write down explicitly the formulas (8.18) in these special cases. As in § 4, if h is K-spherical function on G, we let h^* be the function it determines on $[0,L)$. Further, when $G/K = R^d$, we define $\tilde{h}$ on $[0, \infty)$ by $\tilde{h}(t) = h^*(\sqrt{t})$; when $G/K = S^d$, then $L = \pi$, and we define $\tilde{h}$ on $[-1, +1]$ by $\tilde{h}(\cos t) = h^*(t)$. Finally, when $G/K = L^d(R)$, we define $\tilde{h}$ on $[1, \infty)$ by $\tilde{h}(\cosh t) = h^*(t)$. With these conventions the formulas (8.18) take the following shape in these cases.

For $G/K = R^d$,

(8.19) $E(\eta_{nq}(t)_{n'q'}(t'))$

$$= \begin{cases} 0 \quad \text{if} \quad (n,q) \neq (n',q') \\ 1/2(\tilde{\Psi}(t^2) + \tilde{\Psi}(t'^2) - \dfrac{1}{A_d}\displaystyle\int_0^\pi \tilde{\Psi}(t^2 + t'^2 - 2tt'\cos\theta)\sin^{d-2}\theta d\theta) \\ \qquad\qquad \text{when} \quad n = 0, \\ -1/2C_n \displaystyle\int_0^\pi \tilde{\Psi}(t^2 + t'^2 - 2tt'\cos\theta)P_n^{d(-2)/2}(\cos\theta)\sin^{d-2}\theta d\theta \\ \qquad\qquad \text{otherwise} \end{cases}$$

where $C_n = (d_\alpha A_d P_n^{(d-2)/2}(1))^{-1}$, d_α is the dimension of the representation corresponding to n, and A_d is the area of the unit sphere in R^d. If $G/K = S^d$, we get

(8.20) $E(\eta_{nq}(t)\eta_{n'q'}(t'))$

$$= \begin{cases} 0 \quad \text{if} \quad (n, q) \neq (n', q') \\ 1/2(\tilde{\Psi}(\cos t) + \tilde{\Psi}(\cos t') \\ -\dfrac{1}{A_d}\displaystyle\int_0^\pi \tilde{\Psi}(\cos t \cos t' + \sin t \sin t' \sin^{d-2}\theta d\theta) \quad \text{if} \quad n = 0, \\ -1/2C_n \displaystyle\int_0^\pi \tilde{\Psi}(\cos t \cos t' + \sin t \sin t' \cos\theta) \\ \qquad P_n^{(d-2)/2}(\cos\theta)\sin^{d-2}\theta d\theta \quad \text{for} \quad n > 0. \end{cases}$$

and when $G/K = L^d(R)$, we have

(8.21) $E(\eta_{nq}(t)\eta_{n'q'}(t'))$

$$= \begin{cases} 0 \quad \text{if} \quad (n, q) \neq (n', q') \\ 1/2(\tilde{\Psi}(\mathrm{ch}\, t) + \tilde{\Psi}(\mathrm{ch}\, t') - \dfrac{1}{A_d}\displaystyle\int_0^\pi \tilde{\Psi}(\mathrm{ch}\, t\, \mathrm{ch}\, t' + \mathrm{sh}\, t\, \mathrm{sh}\, t' \cos dt)) \\ \qquad\qquad \text{if} \quad n = 0 \\ -1/2C_n \displaystyle\int_0^\pi \tilde{\Psi}(\mathrm{ch}\, t\, \mathrm{ch}\, t' + \mathrm{sh}\, t\, \mathrm{sh}\, t' \cos\theta) \\ \qquad P_n^{(d-2)/2}(\cos\theta)\sin^{d-2}\theta d\theta \quad \text{if} \quad n > 0. \end{cases}$$

Where C_n, A_d have the same meanings.

Note that in these spaces η_{nq} and $\eta_{nq'}$ have the same distribution; i. e. the coefficient processes of the same « weight » n are all identical in distribution, their covariance being as above.

The reader should compare these formulas with those given by McKean [*1*, p. 361]. McKean is there concerned with Lévy's Brownian motion with parameter in R^d. For this process, $\Psi^*(t) = t$. Thus $\tilde{\Psi}(t) = \sqrt{t}$, and (8.19) specializes to McKean's formulas except for a constant factor which arises due to a difference in normalization of the functions $P_n^{(d-2)/2}$.

In conclusion, we should mention that one can generalize the orthogonal decomposition (8.17) to a quite general symmetric space, i. e. to the case where G/K is not necessarily two-point homogeneous. Each point x on the sphere $S(t)$ of radius t in G/K will now lie on an orbit of K; one may use the notion of « complex distance » due to Cartan, and parametrize the set of orbits of K on $S(t)$ by l nonnegative parameters $t_1, \dots t_l$ where l is the rank of G/K. Two points in G/K are at the same complex distance $(t_1, \dots t_l)$ if and only if they lie on an orbit of K. The process $\eta_{\alpha q}$ now turn out to be processes whose parameter set is the set of all possible complex distances $(t_1, \dots t_l)$. In the noncompact case, this set is just the positive octant in l dimensional Euclidan space, while in the compact case this set is a certain Polyhedron in a Euclidean space of l dimensions, of the form $\{ (t_1, \dots t_l); 0 \leq t_i < L_i \}$. This Polyhedron is what Cartan calls the fundamental Polyhedron of G/K. Thus in the general case when G/K is not two-point homogeneous, the component process $\xi_{\alpha pq}$ are Gaussian processes with an l-dimensional parameter $(t_1, \dots t_l)$; the orthogonal decomposition can be recovered in this situation. We have not done so, because it involves quite a few details of technique which would lead us too far afield.

§ 9. — The Markov property.

This section is not intended to be systematic. It rather serves to outline some ideas whose full developments may be undertaken in the future. We continue with the setting of the preceding section; further, we shall only discuss the cases $G/K = R^d$ or $G/K = S^d$ or $G/K = L^d(R)$.

Let $(\Omega, \mathcal{S}, \mathbf{P})$ be the probability space on which the process ξ is constructed. For each $a \in G/K$, $\xi(a) \in L_2(\Omega)$, and we may assume, without loss of generality that $L_2(\Omega)$ is generated by $\{ \xi(a), a \in G/K \}$. For any Borel set $B \subset G/K$, let $\mathcal{S}(B)$ be the smallest sub-σ-field of $\mathcal{S}$ with respect to which the random variables $\{ \xi(a), a \in B \}$ are all measurable; further, let $\mathbf{H}(B)$ be the closed subspace generated by these random variables, in $L_2(\Omega)$.

Let C be a closed smooth hypersurface in G/K which disconnects G/K into an interior I (such that $eK \in I$ and the closure $I \cup C$ of I is compact), and an exterior E. Define $\mathcal{S}_+(C)$ to be the smallest sub-σ-field of $\mathcal{S}$ with respect to which the random variables $\{ \xi(a), a \in C \cup E \}$ are all measurable i. e. $\mathcal{S}_+(C) = \mathcal{S}(C \cup E)$. Also let $\mathcal{S}_-(C) = \bigcap_{n \geq 1} \mathcal{S}(I_n)$ where I_n is a sequence of open neighbourhoods of I such that $I_{n+1} \subset I_n$ and $\bigcap_{n \geq 1} I_n = I \cup C$. Along with McKean [*1*, section 6], we call $\mathcal{S}_+(C), \mathcal{S}_-(C)$ respectively the future and the past of ξ with respect to C. The process ξ is said to have the Markov property of order $p+1$ with respect to C if, roughly speaking, the future $\mathcal{S}_+(C)$ and the past $\mathcal{S}_-(C)$ are independent conditionally on the knowledge of ξ on C and p derivatives of ξ normal to C (Other definitions of markov properties are also possible). Of course, the chief difficulty with the above definition is to give a suitable interpretation to the normal derivatives of ξ on C, which do not exist in the usual sense. However, in many instances, a suitable interpretation can be given to the above formulation, for, even though the normal derivatives of ξ on C may not exist in the usual sense, they may exist if one carries out an appropriate smoothing operation first.

For example, let C be $S(t)$, the Riemannian sphere of radius t to around eK; $t \in [0, L)$. $S(t)$ is a smooth submanifold of G/K and hence inherits a differentiable as well as Riemannian structure from G/K. We have seen that $S(t) \cong K/U$. Now let $g \in C^\infty(K/U)$, $t \in [0, L)$, and define the random variable $\xi(g, t)$ by

$$\xi(g, t) = \int_K \xi(kz_tK)g(k)dk. \tag{9.1}$$

Now, consider the geodesic γ joining eK and z_tK. It is a consequence of a well-known Lemma of Gauss, that this geodesic meets $S(t)$ perpendicularly. For example see Ambrose [*1*, p. 34]. Thus, differentiation along this geodesic is exactly the same as differentiation in the direction normal to $S(t)$. Since t is a parameter along γ, it follows that, formally at any rate, differentiating $\xi(kz_tK)$ at the point kz_tK, in the direction normal to $S(t)$ is precisely the same as differentiating $\xi(kz_tK)$ with respect to the variable t. Now as we have noted, $\xi(kz_tK)$ may not itself be differentiable with respect to t (for fixed k). But it may happen that $\xi(g, t)$ *is* differentiable with respect to t. Indeed, we can get, by following the method of § 8,

$$
\begin{aligned}
(9.2)\quad & \mathbf{E}(\xi(g, t)\xi(h, t')) \\
&= \int_K \int_K \mathbf{E}(\xi(kz_t K)\xi(k'z_{t'}K))g(k)g(k')dkdk' \\
&= \int_K \int_K f(kz_t K, k'z_{t'})g(k)g(k')dkdk' \\
&= 1/2 \int_K \int_K (\Psi(z_t) + \Psi(z_{t'}) - \Psi(z_{t'}^{-1}k'^{-1}kz_t))g(k)g(k')dkdk'
\end{aligned}
$$

It follows that for $n \geq 1$, we have

$$
\begin{aligned}
(9.3)\quad & \frac{\partial^n}{\partial t^n}\frac{\partial^n}{\partial t'^n}\mathbf{E}(\xi(g, t)\xi(g, t')) \\
&= -1/2 \frac{\partial^n}{\partial t^n}\frac{\partial^n}{\partial t'^n} \int_K \int_K \Psi(z_{t'}^{-1}k'^{-1}kz_t)g(k)g(k')dkdk' \\
&= -1/2 \frac{\partial^n}{\partial t^n}\frac{\partial^n}{\partial t'^n} \int_K \int_K \Psi(z_{t'}^{-1}kz_t)g(k'k)g(k')dkdk'
\end{aligned}
$$

when the right side exists.

Now it can come about that if the function Ψ is nice enough, then the right side of (9.3) may exist for $(t, t') \in [0, L) \times [0, L)$, including points on the diagonal $t = t'$. In that case, as is well known, the process $\xi(g, t)$, as a process with parameter in $[0, L)$, has sample functions which are n times differentiable.

Nor is it very hard to give conditions on Ψ which will accomplish this. For example, if $\frac{\partial^n}{\partial t^n}\frac{\partial^n}{\partial t'^n}\Psi(z_{t'}^{-1}kz_t)$ (which is a function on K/U) belongs to $L_1(K/U)$ for fixed $(t, t') \in (0, L) \times (0, L)$, then clearly the right side in (9.3) exists because of Lebesgue's theorem.

In view of this we may define

$$
(9.4)\quad m = \sup \left\{ n \,\middle|\, \frac{\partial^n}{\partial t^n}\frac{\partial^n}{\partial t'^n}\Psi(z_{t'}^{-1}kz_t) \in L_1(K/U) \quad \text{for each} \quad (t, t') \in (0, L) \times (0, L) \right\}
$$

Of course m may be infinite, in which case the sample functions of $\xi(g, t)$ are of class C^∞. For any integer $p \leq m$ we may now define the Dirichlet field $\mathcal{S}(t, p)$ as the smallest sub-σ-field of $\mathcal{S}$ with respect to which the random variables

$$
\left\{ \frac{d^n}{ds^n}(\xi(g, s))\bigg|_{s=t} ; \qquad 0 \leq n \leq p,\ g \in C^\infty(K/U) \right\}
$$

are all measurable. This field $\mathcal{S}(t, p)$ is the field which contains the information about ξ along $S(t)$ and its p derivatives normal to $S(t)$. With this definition, we may say that ξ is markovian of order $p+1$ with respect to $S(t)$ if the σ-fields $\mathcal{S}_+(S(t))$ and $\mathcal{S}_-(S(t))$ are independent conditional on $\mathcal{S}(t, p)$, and p is the smallest integer which will do this. In McKean's terminology we are saying that $\mathcal{S}(t, p)$ is to be a splitting field for $\mathcal{S}^+(S(t))$, $\mathcal{S}_-(S(t))$.

We may then say that ξ has the markov property of order $p+1$ with respect to the family of concentric spheres about eK if ξ has the markov property of order $p+1$ with respect to each sphere $S(t)$ about eK.

Though we have restricted ourselves above to the family of spheres about eK, the reader should note that it is possible to define similarly the Dirichlet field $\mathcal{S}(C, p)$ where C is any smooth hypersurface, and p is a suitable integer. This is done for $G/K = R^d$ by McKean [*1*] in a somewhat different language. All one has to do to carry over his definition, is to use differentiation in directions normal to the hypersurface C, and impose conditions on Ψ which will guarantee that the analogue of (9.3) makes sense (of course, instead of integration with respect to $dkdk'$, one has to use integration with respect to the surface area on C induced by the Riemannian metric on G/K, in order to define the analogue of (9.1)). There is however no point in pursuing this at the present level of detail in our exposition. Anyhow it should be clear that the markov property can be given a formulation in the above fashion.

It turns out that the orthogonal decomposition studied in § 8, is rather well suited for the discussion of the markov property of a process ξ as formulated above, with respect to the family of concentric spheres centred around eK in G/K. We shall not give detailed proofs but the situation may be outlined as follows; we use the terminology introduced by McKean [*1*, sections 6, ff.]. Namely, in the Gaussian case, if $B \subset G/K$, the σ-field $\mathcal{S}(B)$ generated by $\{ \xi(a), a \in B \}$ coincides with the σ-field generated by all the random variables in $\mathbf{H}(B)$ (recall that $\mathbf{H}(B)$ is the closed linear subspace of $L_2(\Omega)$ generated by $\{ \xi(a), a \in B \}$). This has the consequence that if C is any smooth surface, then the minimal splitting field $\dot{\mathcal{S}}(C)$ of $\mathcal{S}_+(C)$ and $\mathcal{S}_-(C)$ coincides with the smallest σ-field generated by the random variables in the orthogonal projection $\dot{\mathbf{H}}(C)$ of $\mathbf{H}_+(C) = \mathbf{H}(C \cup E)$ on $\mathbf{H}_-(C) = \bigcap_{n \geq 1} \mathbf{H}(I_n)$ (the notation as above).

The proof of this is given in McKean [*1*, p. 367-368]. Now, because of the orthogonal decompositions obtained in § 8, it can be shown that

(9.5)
$$\begin{aligned} \mathbf{H}_{+}(\mathrm{S}(t)) &= \bigoplus_{n,q} \mathbf{H}^{+}_{nq}(t) \\ \mathbf{H}_{-}(\mathrm{S}(t)) &= \bigoplus_{n,q} \mathbf{H}^{-}_{nq}(t) \\ \dot{\mathbf{H}}(\mathrm{S}(t)) &= \bigoplus_{n,q} \dot{\mathbf{H}}_{nq}(t) \end{aligned}$$

where $\mathbf{H}^{+}_{nq}(t)$ is the Hilbert space generated by all the random variables $\{ \eta_{nq}(s), s \geq t \}$, $\mathbf{H}^{-}_{nq}(t)$ is the intersection of the subspaces $\{ \mathbf{H}_{nq}(t+\varepsilon), \varepsilon > 0 \}$ where $\mathbf{H}_{nq}(t+\varepsilon)$ is generated by the random variables $\{ \eta_{nq}(s), s \leq t + \varepsilon \}$, and $\dot{\mathbf{H}}_{nq}(t)$ is the minimal splitting field of $\mathbf{H}^{+}_{qn}(t)$ and $\mathbf{H}^{-}_{nq}(t)$ (Recall here that we are working with $\mathrm{G/K} = \mathrm{R}^d$ or $\mathrm{G/K} = \mathrm{S}^d$ or $\mathrm{G/K} = \mathrm{L}^d(\mathrm{R})$ and so all the processes involved are real valued).

The functions χ_{n1q} of § 8 belong to $\mathrm{C}^{\infty}(\mathrm{K/U})$ and if as in (8.1) we form $\xi(\chi_{n1q}, t)$ we see that

(9.6)
$$\begin{aligned} \xi(\chi_{n1q}, t) &= \int_{\mathrm{K}} \xi(kz_t\mathrm{K})\chi_{n1q}(k)dk \\ &= \eta_{nq}(t). \end{aligned}$$

Thus if $p \leq m$ where m is defined by (9.4), then $\eta_{nq}(t)$ is p times differentiable.

Now let $\mathbf{H}(t, p)$ be the Hilbert space spanned by the random variables

$$\left\{ \frac{d^r}{ds^r} \eta_{nq}(s) \bigg|_{s=t} ; \qquad 0 \leq r \leq p, n \geq 0, 1 \leq q \leq d_\alpha \right\}.$$

Because the processes η_{nq} are mutually orthogonal, and because the functions χ_{n1q} are an orthogonal basis of $\mathrm{L}_2(\mathrm{K/U})$, and because ξ is Gaussian it can be shown that the Dirichlet field $\mathcal{S}(t, p)$ is in this case identical with the smallest σ-field which measures all the random variables in $\mathbf{H}(t, p)$, and further that

(9.7)
$$\mathbf{H}(t, p) = \bigoplus_{n,q} \mathbf{H}_{nq}(t, p)$$

where $\mathbf{H}_{nq}(t, p)$ is the Hilbert space generated by the random variables

$$\left\{ \frac{d^r}{ds^r} \eta_{nq}(s) \bigg|_{s=t}, 0 \leq r \leq p \right\}.$$

Now it is easy to see that

(9.8)
$$\begin{aligned} \mathbf{H}_{nq}(t, p) &\subset \dot{\mathbf{H}}_{nq}(t) \\ \mathbf{H}(t, p) &\subset \dot{\mathbf{H}}(\mathrm{S}(t)) \end{aligned}$$

and since $\mathcal{S}_+(\mathrm{S}(t))$ and $\mathcal{S}_-(\mathrm{S}(t))$ are split over $\dot{\mathcal{S}}(\mathrm{S}(t))$ (which is generated by the random variables in $\dot{\mathrm{H}}(\mathrm{S}(t))$), one can conclude that the process ξ has the markov property of order $p+1$ in the sense formulated above, if and only if each component process $\eta_{nq}(t)$ is Markovian of order $p+1$ i. e. its future $\mathcal{S}^+_{nq}(t) = \mathcal{S}(\eta_{nq}(s);\, s \geq t)$ and the past $\mathcal{S}^-_{np}(t) = \bigcap_{\varepsilon>0} \mathcal{S}(\eta_{nq}(s);\, s < t+\varepsilon)$ are split by the σ-field $\mathcal{S}_{nq}(t, p)$ generated by the derivatives of $\eta_{nq}(t)$ upto order p. Of course this amounts to saying that in (9.8), the inclusion is an equality.

The remarks above serve to reduce the problem of studying the Markov property of ξ with respect to the family of concentric spheres around $e\mathrm{K}$, to the problem of studying the markov property of a countable collection of Gaussian processes of a single parameter, namely the processes $\eta_{nq}(t)$. Now this latter problem is one which is a fairly long-standing unsolved problem. See e. g. Hida [*1*] for information about the results available on this point.

Of course, if ξ does not have the markov property of order p with respect to the family of concentric spheres about $e\mathrm{K}$, then it cannot have the markov property with respect to the family of *all* smooth hypersurfaces as envisaged above. The point is that in the cases at hand, fairly explicit expressions are available, as written down in (8.19), (8.20), (8.21) for the covariances of the processes η_{nq}. Thus, in special cases, it may be possible to arrive at some conclusion regarding the markov property for η_{nq} by studying this covariance. For example, if $\mathrm{G/K} = \mathrm{R}^d$ and ξ is the process whose covariance is

$$(9.9) \qquad f^\alpha(a, b) = 1/2(|a|^\alpha + |b|^\alpha - |a-b|^\alpha) \qquad 0 \leq \alpha \leq 2,$$

then

$$(9.10) \quad \mathrm{E}(\eta_{nq}(t)\eta_{nq}(t'))$$

$$= \begin{cases} 1/2\left(t^\alpha + t'^\alpha - \dfrac{1}{\mathrm{A}_d}\displaystyle\int_0^\pi (t^2 + t'^2 - 2tt'\cos\theta)^{\alpha/2} \sin^{d-2}\theta\, d\theta\right) & \text{if} \quad n = 0, \\ -1/2\mathrm{C}_n \displaystyle\int_0^\pi (t^2 + t'^2 - 2tt'\cos\theta)^{\alpha/2} \mathrm{P}_n^{(d-2)/2}(\cos\theta) \sin^{d-2}\theta\, d\theta & \text{when} \quad n > 0. \end{cases}$$

When $\mathrm{G/K} = \mathrm{S}^d$, and ξ is the process whose covariance is, say

$$(9.11) \qquad f(a, b) = 1/2(d(a, o) + d(b, o) - d(a, b))$$

then

(9.12) $\mathrm{E}(\eta_{nq}(t)\eta_{nq}(t'))$

$$= \begin{cases} 1/2\left(t + t' - \dfrac{1}{\mathrm{A}_d}\displaystyle\int_0^\pi \cos^{-1}(\cos t \cos t' + \sin t \sin t' \cos\theta)\right) \\ \qquad\qquad \sin^{d-2}\theta d\theta \quad \text{if} \quad n = 0 \\ -\,1/2\mathrm{C}_n \displaystyle\int_0^\pi \cos^{-1}(\cos t \cos t' + \sin t \sin t' \cos\theta) \mathrm{P}_n^{(d-2)/2}(\cos\theta) \\ \qquad\qquad \sin^{d-2}\theta d\theta \quad \text{if} \quad n > 0 \end{cases}$$

Finally, when $\mathrm{G/K} = \mathrm{L}^d(\mathrm{R})$ and ξ is given by the covariance

(9.13) $$f_\alpha(x\mathrm{K}, y\mathrm{K}) = 1/2(g(x)^{\alpha/2} + g(y)^{\alpha/2} - g(y^{-1}x)^{\alpha/2}) \quad \text{for some} \quad 0 \leq \alpha \leq 2,$$

where g is the Gaussian part arrived at in § 4, and if we define $\tilde{g}$ on $[1, \infty)$ by $\tilde{g}(\mathrm{ch}\,\zeta) = g^*(\zeta) = g(x)$ where ζ is the distance of $x\mathrm{K}$ from $e\mathrm{K}$. Then we have

(9.14) $\mathrm{E}(\eta_{nq}(t)\eta_{nq}(t'))$

$$= \begin{cases} 1/2(\tilde{g}(\mathrm{ch}\, t)^{\alpha/2} + \tilde{g}(\mathrm{ch}\, t')^{\alpha/2} - \dfrac{1}{\mathrm{A}_d}\displaystyle\int_0^\pi \tilde{g}(\mathrm{ch}\, t\, \mathrm{ch}\, t' + \mathrm{sh}\, t\, \mathrm{sh}\, t' \cos\theta)^{\alpha/2} \\ \qquad\qquad \sin^{d-2}\theta d\theta) \quad \text{if} \quad n = 0 \\ -\,1/2\mathrm{C}_n \displaystyle\int_0^\pi \tilde{g}(\mathrm{ch}\, t\, \mathrm{ch}\, t' + \mathrm{sh}\, t\, \mathrm{sh}\, t' \cos\theta)^{\alpha/2} \mathrm{P}_n^{(d-2)/2}(\cos\theta) \\ \qquad\qquad \sin^{d-2}\theta d\theta \quad \text{when} \quad n > 0. \end{cases}$$

We would like to end this with a few comments. Since one knows what g is explicitly when $d = 2$ or 3, one can try to compute (9.14) in those cases. When $d = 2$ one gets rather complicated elliptic integrals; but for $d = 3$ the situation is simpler. As for (9.12), again it is fairly easy to see that in general one has elliptic integrals there when d is even; but for odd d the situation simplifies. Finally note that the expressions in (9.10) are homogeneous in t, t', This enables one to make a time change in the process η_{nq} after which the process becomes stationary. In principle, the markov property for a stationary Gaussian process can be decided by looking at its spectral density function. This should, after some computations not here undertaken, enable us to decide the markov property for the processes given by (9.9). We have not been able to get any such stationarity for (9.12) or (9.14), by means of a time change.

By now the reader is aware of the fragmentary nature of our results in this direction. We have stated them here with the hope of perhaps stimulating further work in these directions.

As a final remark we may state that if in (9.13) we let $\alpha = 2$, then we are able to show that for the spaces $L^d(R)$ and also some others, the processes η_{nq} are actually analytic, and so, the corresponding process ξ has no markov property whatever (This accords well with the situation, in Euclidean space R^d, for the process whose covariance is given as in (9.9) but with $\alpha = 2$). The proof of this assertion rests on showing that the function $g(x)$ is in all these cases, a solution of $\Delta g =$ constant, where Δ is the Laplace-Beltrami operator of G/K. It follows that $\Delta^2 g = 0$ so that g is analytic by virtue of a theorem of S. Bernstein. It can now be shown that the covariance of $\dot{\eta}_{nq}$ is zero if $n = 0$, and, when $n \geq 1$, it is annihilated by the radial part of Δ, applied to either variable t or t'. One can conclude from this that these covariances are analytic in $(0, L) \times (0, L)$, leading to the desired conclusion. However, the full description of this proof is technical enough, and the result fragmentary enough, so that it does not seem worth reproducing here.

§ 10. — Conclusion.

A problem which remains to be tackled is that of obtaining representations for our processes by means of white-noise integrals. Given a measure space $(X, \mathcal{A}, \mu)$, a Gaussian white noise on X based on μ is a map W which assigns to each μ-finite subset $A \in \mathcal{A}$, a random variable in $L_2(\Omega)$, such that *i*) The distribution of W(A) is Gaussian with $\mathbf{E}(W(A)) = 0$, $\mathbf{E}(W(A)^2) = \mu(A)$. *ii*) W(A), W(B) are independent if $A \cap B = \emptyset$. *iii*) If A_i, $i = 1, 2, \ldots$ is an increasing sequence of members of such that $A = \bigcup_i A_i$ is μ-finite then $W(A) = \lim_{i \to \infty} W(A_i)$ in $L_2(\Omega)$.

Given such a white noise, one can define stochastic integrals

$$W(h) = \int_X h(u) dW(u) \qquad \text{for} \qquad h \in L_2(X, \mu)$$

in a standard and well-known manner.

A centred Gaussian process ξ with parameter running over a set S is said to have a representation in terms of an integral of the white noise W, if there exists a function $h(a, x)$ on $S \times X$ such that $\xi(a) = \int_X h(a, x)\, dW(x)$ for each $a \in S$.

Of course, there is a good deal of arbitrariness in the choice of X, and different representations result for different choices of X, when at all possible.

Naturally, to be useful for specific purposes, one must seek representations whose kernels $h(a, x)$ have further properties.

For example, in the theory of Gaussian stationary processes on R^1, one choice for X is to make X the Fourier-analytic dual of R^1 namely R^1 itself. This results in the Cramér-Khinchine representation

$$\xi(t) = \int_{R^1} \exp itx dZ(x)$$

for the process. See e. g. Bochner [*1*]. On the other hand, this choice for X and the representations it yields are essentially spectral in character, and are virtually totally useless for discussing temporal properties of the process such as the Markov property.

For studying these properties, Lévy has initiated the study of canonical representations of a Gaussian process with parameter in R^1. He poses the problem of getting white noise integral representation for ξ in terms of white noise built over some appropriate measure on R^1 i. e. with the choice $X = R^1$, but with several further restrictions on the kernels $h(a, x)$, one of these being the requirement that for each fixed $a \in R^1$, $h(a, x)$ as a function of x has support in $(-\infty, a]$. Thus the kernel should be supported on the « past » of the process ξ. For an account of these ideas, the reader is referred to Lévy [*2*] or Hida [*1*].

In the setting of the present paper, let us consider the various representations of a centred Gaussian process ξ whose parameter runs over G/K and whose covariance is a Lévy-Schoenberg kernel f.

Since G/K has a natural Fourier-analytic dual object $(G/K)^\frown$, it is pertinent to ask if some spectral representation is possible for ξ. Supposing, for instance, that G/K has a differentiable structure, let $C_0^\infty(G/K)$ be the space of infinitely differentiable functions with compact support, carrying the usual topology of Schwarz. G acts on $C^\infty(G/K)$ in the usual way and we write xg for the image of $g \in C_0^\infty(G/K)$ under the action of $x \in G$. Now extend the map ξ to $C_0^\infty(G/K)$ by setting $\xi(g) = \int_{G/K} \xi(a)g(a)da$, $g \in C_0^\infty(G/K)$, where da is the (essentially) unique measure on G/K which is G invariant. Then $g \to \xi(g)$ is a so-called generallized random field in the terminology of Gelfand and Itô. Now let $B(g, h)$ be the covariance functional of this field, so that $B(g, h) = \mathbf{E}(\xi(g)\xi(h))$. B is naturally symmetric bilinear and positive definite. Moreover, let us consider the restriction of B to $D_0 \times D_0$ where D_0 is the subspace

$$(10.1) \qquad D_0 = \left\{ g \mid g \in C_0^\infty(G/K), \int g(a)da = 0 \right\}$$

Then one has,

$$(10.2) \qquad B(g, h) = - \int_{G/K}\int_{G/K} r(a, b)g(a)h(b)dadb \,; \qquad g, h \in D_0$$

$$(10.3) \quad B(xg, xh) = B(g, h) \,; \qquad g, h \in D_0,\ x \in G$$

$$(10.4) \qquad B(g, g) \geq 0.$$

(10.3) is a consequence of the invariance $r(xa, xb) = r(a, b)$ of r. Thus on D_0, B is a symmetric positive definite invariant bilinear form. In Kolmogorov's terminology, the process ξ is locally homogeneous and locally isotropic. Adopting this point of view, and using known aspects of harmonic analysis on G/K, it is possible to perform a spectral decomposition of the form B, which constitutes a correlation theory for the process ξ. For analogous questions for processes whose parameters are in a Euclidean space, see for example, Gelfand and Vilenkin [*1*] and Yaglom [*1*], [*2*]. However, it is one thing to get spectral decompositions for B and quite another to get from them a white noise integral for ξ. The latter problem seems to be a difficult one, and involves deeper factorization problems (In the theory of stationary processes with parameter in R^1, the analogous problem is essentially that of factoring a non-negative function in $L_2(R^1)$ as a product $g\bar{g}$ with $g \in$ the Hardy class H_1).

Turning to other candidates for X, it is worth recalling the paper of Chentsov [*1*] where he gets a white noise integral for Lévy's Brownian motion ξ with parameter in R^d. Chentsov chooses X to be the projective dual of R^d i. e. he takes X to be the set of all hyperplanes in R^d. Then X is in a natural way a homogeneous space of the Euclidean group G, and Chentsov's construction amounts to expressing $\xi(a)$, $a \in R^d$, as an integral over X with respect to white noise based on the G-invariant measure on X. Further the kernel of the white noise integral for $\xi(a)$ is just the indicator function of the set of hyperplance which intersect the line segment oa, where o is the origin of R^d. This formulation of Chentsov's result has the advantage that this formulation makes sense in any situation where there is a projective duality between points of G/K and elements of some other homogeneous space of G (For example, consider Grassmann manifolds of

p-planes and q-planes in R^d, where $p + q + 1 = d$). Generalizations of the projective duality between points and hyperplanes in R^d, which are applicable to the setting of § 8 have been considered by several people. See e. g. Helgason [2] and the literature cited there. It seems to be a fruitful context within which one could examine Chentsov's construction, and could possibly result in white noise integral representations for a Lévy-Schoenberg process ξ with parameter in G/K, resembling Chentsov's representation.

Finally, there is the result of Itô [*1*], pointing out that the gradient of Lévy's Brownian motion is an isotropic solenoidal random current, whose structure was determined by Itô. The question arises of examining the relation of this circle of ideas to the ideas of the present paper, and trying to express some of the Lévy-Schoenberg processes by means of solutions to invariant partial differential equations on G/K, forced by white noise.

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Manuscrit reçu le 30 juin 1966.

ADDED IN PROOF

After this paper was written, Professor R. Askey communicated to me a proof that (4.24) does not hold. That is to say, for each $d \geq 2$, there is some n for which (4.24) is violated. Professor Askey's proof is reproduced below, with his generous permission.

We see by an integration by parts, that

$$(*) \qquad 8 \int_0^{\pi/2} \theta P_4 (\cos \theta) \sin \theta d\theta = 1/15 > 0$$

where P_4 (cos θ) is the Legendre polynomial of degree 4. This means that when $d = 2$, (4.24) does not hold. Now there is the result of Askey in

Proc. Amer. Math. Soc., 16 (1965), p. 1191-1194 that if $(\lambda - 1)/2 < \mu < \lambda$ then

$$(\sin \theta)^{2\mu} P_n^{\mu} (\cos \theta) = \sum_{k=0}^{\infty} a_k P_{n+2k}^{\lambda} (\cos \theta) \cdot (\sin \theta)^{2\lambda}$$

with $a_k > 0$ for each k. Therefore, if

$$(**) \qquad \int_0^{\pi/2} \theta P_{2n}^{\lambda} (\cos \theta) \sin^{2\lambda} \theta d\theta \leq 0 \qquad n = 1, 2, \ldots, \lambda = (d-1)/2,$$

then also

$$\int_0^{\pi/2} \theta P_{2n}^{\mu} (\cos \theta) \sin^{2\mu} \theta d\theta \leq 0 \qquad n = 1, 2, \ldots$$

for all μ such that $(\lambda - 1)/2 < \mu < \lambda$.

Using this a number of times we see that the truth of (**) would contradict (*). Thus (4.24) does not hold, and we get the conclusion that $\Psi^* (\theta) = \theta$ does not give us a Lévy-Schoenberg kernel in any of the spaces of *ii*.