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Stark resonances for random potentials of Anderson type

by

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ABSTRACT. – For the Schrödinger operator on $L^2(\mathbf{R})$ corresponding to a particle subject to a constant electric field, moving in a disordered potential of Anderson type, we prove the existence of resonances with a width exponentially small with respect to the intensity of the field. © Elsevier, Paris

Key words: Schrödinger operators; Disordered systems; Stark–Wannier resonances

RÉSUMÉ. – On considère l'opérateur de Schrödinger sur $L^2(\mathbf{R})$ correspondant à une particule soumise à un champ électrique constant et se déplacant dans un potentiel désordonné du type Anderson. On montre l'existence de résonances dont la largeur est exponentiellement petite par rapport à l'intensité du champ. © Elsevier, Paris

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1. INTRODUCTION

This paper is devoted to the study of the spectral properties of the Hamiltonian of an electron moving in a random potential and subject to an exterior constant electric field,

$$H_\omega(F) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\omega + Fx, \qquad F > 0,$$
$$V_\omega = \sum_{i\in\mathbf{Z}} \omega_i u_i(x-i)$$

on $\mathcal{H} = L^2(\mathbf{R})$. The functions u_i, called the atomic potentials, are supposed to be negative and vanishing at ∞. The coupling constants ω_i, $i \in \mathbf{Z}$, are independent, identically distributed random variables. The distribution of probability of ω_i has a density which is continuous and supported on $[\omega_m, \omega_M]$, $0 < \omega_m < \omega_M$ (see remark below). In the following, we denote by $(\Omega, \tau, \mathbf{P})$ the probability space generated by $\{\omega_i,\ i \in \mathbf{Z}\}$.

It has been known for many years that adding a linear potential to a regular bounded potential gives rise to absolutely continuous spectrum [3] (see also [7] for a more general setting), and that for some models with singular potentials the spectrum is pure point [15,2,16]. Several papers have been devoted to the analytic periodic potential case and to the existence of resonances (called Bloch oscillators), in this sense we mention the works [19,4,9,11].

It is now natural to address the question of the resonances in the regular random situation and we mention an earlier work [17], where the case of large interatomic distance is considered.

Without external field, contrary to the periodic case, the spectrum is pure point and dense, for almost all potentials, see, e.g., [29,26] and [10] for a complete discussion of this point. So, in some sense we are close to the atomic situation (see [20,5] and references therein) where the resonances come from the eigenvalues. However, there is an important difference, as the eigenvalues form a dense set, it is not clear that we can obtain well separated resonances, except if, when adding the exterior field, some of the eigenvalues remain close to the real axis while some others move far apart. Intuitively this can occur because at zero field the eigenfunctions are localized uniformly on the real line [14]. If we consider the situation near the zero energy, when adding a linear potential with a small positive slope, the eigenfunctions located close to the origin will be less affected by the exterior field than those located far apart. They give rise to sharp resonances instead the eigenfunctions located far to the left give rise to resonances far apart from the real axis because for them the perturbation is in some sense larger.

In this paper we define a resonance as a pole of some matrix elements of the resolvent operator as in [27] and we will use the analytic distortion method introduced by Hunziker in [21], the parameter index will be called θ. The complex eigenvalues of the distorted operator $H_\omega(F,\theta)$ are the resonances of the original one.

Let us make precise here our assumptions on the atomic potential:

[H1]. – *There exists $b \leqslant 0$ and $a > 0$ such that the atomic potentials u_i are analytic in the half strip* $S_{a,b} = \{z \in \mathbf{C},\ |\mathrm{Im}\, z| < a,\ \mathrm{Re}\, z < b - i\}$

denoting by $C_i = [i - 1/2, i + 1/2]$ the ith cell and by $\mathbf{1}_{C_i}$ the characteristic function of C_i;

[H2]. – *There exists some strictly positive constants c_1, c_2 and $\alpha \geqslant 6$, such that,*

$$c_1 \mathbf{1}_{C_0}(x) \leqslant |u_i(x)| \leqslant \frac{c_2}{1+|x|^\alpha}, \quad x \in \mathbf{R},$$

and the upper estimate has to be satisfied in all of $S_{a,b}$.

Remark. – (i) In general the support of the density probability is chosen as a compact subset of $\mathbf{R}$ or as the total real line with a sufficiently decaying density. This could be done here, but we prefer a support strictly positive to avoid inessential technicalities.

(ii) Analyticity of the u_i's for $x \leqslant b - i$ is needed because the distorted part of the operator $H_\omega(F,\theta)$ is supported in a neighbourhood of $-\infty$ which is included in $(-\infty, b)$.

In the atomic case, i.e., when the potential goes to zero at infinity, there exists, for negative energies, a classically forbidden region sufficiently large, separating the interior well from the infinite exterior well. In our case the existence of a similar forbidden region is also essential. As it will be explained in Section 2, this geometric assumption can be interpreted as one on the exponential behavior of the local Green function which takes here the following form. Let H_Λ be the restriction of $H_\omega(F)$ on an interval Λ, which we define as the self-adjoint operator on $L^2(\Lambda)$ with Dirichlet boundary conditions on $\partial\Lambda$ and $\forall \varphi \in \mathbf{C}^\infty(\Lambda)$, $\varphi(x) = 0$, $x \in \partial\Lambda$, $H_\Lambda \varphi = H_\omega(F)\varphi$, we denote by $R_\Lambda(z)$ the corresponding resolvent. Let χ_l, χ_r be open intervals having an F independent size, located respectively near the left boundary and the right boundary of Λ

and $\Delta = [-E_0, 0]$, $E_0 > 0$, an energy interval, suppose F small, we will say that

the operator $H_\omega(F)$ satisfies the condition C on Δ, if for all intervals $\Lambda = [0, E^+/F]$, $E^+ > 0$, there exits a complex neighbourhood of Δ, Δ', independent of F and some uniform constants $c, \gamma, \nu > 0$ and $p \geqslant 1$, such that,

$$\mathbf{P}\big(\forall z \in \Delta',\ \|\mathbf{1}_{\chi_l} R_\Lambda(z) \mathbf{1}_{\chi_r}\| \leqslant c\mathrm{e}^{-\gamma|\Lambda|} \|R_\Lambda(z)\|^p\big) \geqslant 1 - \mathrm{O}(F^\nu). \quad (1.1)$$

In Section 2, we exhibit an Anderson model for which the condition C is satisfied. This model consists in choosing for $i \geqslant 0$, the functions $u_i = u$ for some function u with compact support. Work is in progress to overcome some of these conditions on the u_i's to cover situations in which the random potential presents long range correlations.

Denoting by

$$v_t = \sup_{x \in \mathbf{R}} \Big| \sum_i (1 - \mathbf{1}_{C_i})(x) u_i(x - i) \Big|$$

and

$$U_m = \omega_m \inf_i \Big\{ \sup_{x \in \mathbf{R}} |u_i(x)| \Big\},$$

we also suppose:

[H3]. – *The potential V_ω and the energy interval $\Delta = [-E_0, 0]$ have to satisfy*:

$$4\omega_M v_t < |\Delta| < U_m - \omega_M v_t.$$

If we admit that $H_\omega(F)$ satisfies the condition C above, we only need in the rest of the article that the u_i's satisfy [H1], [H2] and [H3] in particular we get:

THEOREM 1.1. – *Let $\Delta = [-E_0, 0]$ be an energy interval and suppose that* [H1], [H2], [H3] *and condition C are satisfied. Then there exists a set of full measure, $\Gamma \subset \Omega$, such that for each $\omega \in \Gamma$, there exists a sequence F_n, $n \in \mathbf{N}$, $F_n \to 0$, as $n \to \infty$ such that the operator $H_\omega(F_n)$ has at least one resonance Z_n whose real part is in Δ and the imaginary part satisfies*

$$|\mathrm{Im}\, Z_n| \leqslant \mathbf{c} \exp(-\tau/F_n)$$

for some uniform strictly positive constants $\mathbf{c}$ *and* τ.

This result, together with the lower bound obtained in [1] for completely analytic Stark–Wannier systems gives the following estimate on the width of resonance (or the inverse resonance life time),

$$\widetilde{\mathbf{c}}\exp(-\widetilde{\tau}/F_n) \leqslant |\mathrm{Im}\, Z_n| \leqslant \mathbf{c}\exp(-\tau/F_n)$$

for some constants $0 < \widetilde{\mathbf{c}} < \mathbf{c}$ and $0 < \tau < \widetilde{\tau}$.

To analyze the spectrum of the distorted operator, we will use the so-called "decoupling method" which in our case goes as follows. We choose on the real line $x^{\text{ext}} > x^{\text{int}} = 0$, x^{ext} is smaller than the last turning point to the right (defined for energy 0) and $x^{\text{ext}} - x^{\text{int}} = \mathrm{O}(1/F)$. Notice that with our conditions on V_ω, the region of turning points has a size $\mathrm{O}(1/F)$. We introduce two operators $H_\omega^{\text{ext}}(F,\theta)$ on $\mathcal{H}^{\text{ext}} = L^2((-\infty, x^{\text{ext}}))$ and $H_\omega^{\text{int}}(F)$ on $\mathcal{H}^{\text{int}} = L^2((x^{\text{int}}, +\infty))$ defined as the restriction of the original operator $H_\omega(F,\theta)$, respectively on $(-\infty, x^{\text{ext}})$ and $(x^{\text{int}}, +\infty)$ with Dirichlet boundary conditions. Because of the potential increase at $+\infty$, $H_\omega^{\text{int}}(F)$ has pure point spectrum and we will show in Section 3 that to some of its eigenvalues correspond eigenvectors which are localized to the right of x^{ext}. We will see in Section 4 that in a neighbourhood of size $\mathrm{O}(F^4)$ of one of these eigenvalues, the operator $H_\omega^{\text{ext}}(F,\theta)$ has no spectrum. When we focus on one of these eigenvalues, going from the decoupled operator $H_\omega^d(F,\theta) = H_\omega^{\text{ext}}(F,\theta) \oplus H_\omega^{\text{int}}(F)$ to $H_\omega(F,\theta)$ is a small perturbation, because what occurs in the interval $(x^{\text{int}}, x^{\text{ext}})$ is similar to the presence of a barrier. In fact we will show in Section 2 that in this interval, due to the disorder, the Green function for energy close to 0 is exponentially decreasing with probability close to 1.

So from the formula linking the resolvents for $H_\omega(F,\theta)$, $H_\omega^{\text{ext}}(F,\theta)$ and $H_\omega^{\text{int}}(F)$ (see Appendix) we deduce that there exists an eigenvalue of $H_\omega(F,\theta)$ and then a resonance for $H_\omega(F)$, exponentially close to the real eigenvalue of $H_\omega^{\text{int}}(F)$, this is done in Section 5.

All that has been written, up to now, is sketchy, and will be made precise later, since special attention has to be paid to the probabilistic aspects.

Remark. – We look at the spectrum near 0 energy for notational convenience, let us notice that because of the stationarity property of V_ω and the linearity of Fx, looking at energies near some E is the same as looking at energies near 0, performing a translation in space by E/F.

In the previous papers on Stark–Wannier resonances one needs additional assumptions on the smallness of the Planck constant or the distance separating the atomic centers, here, only smallness on F is required.

2. EXPONENTIAL DECAY OF THE GREEN FUNCTION AND DECOUPLING

The existence of sharp Stark resonances in atoms or molecules is due to the presence of a potential barrier which becomes larger as the electric field goes to zero. One shows that the Green function decreases exponentially in the barrier, this implies that the coupling between the interior part, where the bounded states live, in absence of the field, and the exterior is exponentially small. This is part of the so–called "decoupling" method used to prove exponentially small resonance widths. In our case, the condition C means that in an interval to the right of 0, the Green function decreases exponentially with a rate larger than γ. Consider a eigenstate localized, at zero field, to the right of 0, at some distance from it. When the field is turned on, it will be trapped to the right of 0, due to the increasing potential, and partially trapped to the left by some effective barrier associated with an interval in the vicinity of 0, in which the Green function decreases exponentially. So in our decoupling method we will distinguish an exterior part $\mathbf{R}^{\text{ext}} = (-\infty, x^{\text{ext}})$ and an interior part $\mathbf{R}^{\text{int}} = (x^{\text{int}} = 0, +\infty)$, as above we associate to them the operators $H_{\omega}^{\text{ext}}(F)$, $H_{\omega}^{\text{int}}(F)$. The x^{ext} has to be chosen in such a way that, some eigenvectors of $H_{\omega}^{\text{int}}(F)$ corresponding to eigenvalues close to the energy 0, live in $(x^{\text{ext}}, +\infty)$. By Theorems 3.1 and 3.2 below, then $x^{\text{ext}} = c^{\text{ext}}/F$ where the energy c^{ext} satisfies, $0 < c^{\text{ext}} < c_{\max} = U_m - |\Delta| - \omega_M v_t$. Notice that for F small enough, the point $c_{\max}/F$ is smaller than the last turning point to the right (defined for energy 0). Such a state will have at its left a barrier whose width is at least $[0, x^{\text{ext}}]$ and whose length increases as F decreases. In the sequel we will first give a model for which the condition C is satisfied and secondly prove the existence of eigenstates for $H_{\omega}^{\text{int}}(F)$ localized to the right of x^{ext}.

Let us sketch here, rapidly why the Green function decays exponentially on the interval $[0, x^{\text{ext}}] = [0, c^{\text{ext}}/F]$. If we choose some point x_0, in this interval and replace the potential Fx by Fx_0 in some interval of size $2l_0$ included in $[0, x^{\text{ext}}]$, the solutions for the corresponding Schrödinger equation for negatives energies E close to 0 will behave exponentially due to the disorder. The exponential rate will be close to the Lyapunov exponent corresponding to the "effective" energy $E - Fx_0$. Notice that in general the Lyapunov decreases when the energy increases, so, if x_0 goes to the left, the effective energy increases and the Green function will have a slower decrease. Then, reintroducing the perturbation $Fx - Fx_0$ in the interval of size $2l_0$ does not change drastically the exponential behavior of the Green function if F is not too large. This will be done using the stability of solutions of differential equations under small perturbations. After that, we will enlarge the size of the box using multiscale analysis and reach sizes of the order of $1/F$ with a control on the probability measure of the set of potentials for which we can show the exponential behavior. So we prove now,

THEOREM 2.1. – *Suppose* [H1], [H2], $u_i = u$ *for* $i \geqslant 0$ *for some* $u \in \mathbf{C}_0^\infty$ *and the tail* v_t *small enough. Let* $\Delta = [-E_0, 0]$, *there exists* F_0 *such that for* $0 < F < F_0$, $H_\omega(F)$ *satisfies the condition* C *on* Δ.

Remark. – The smallness of the tail evoked in Theorem 2.1 is explicit through the condition (2.19) of the proof of Lemma 2.1 below.

Sketch of the proof. – Firstly, we note that for all interval Λ around the origin and $|\Lambda| = 0(1/F)$, we have the following Wegner estimate [23]: let $E \in \mathbf{R}$, $\eta > 0$, then

$$\mathbf{P}\big[\operatorname{dist}\big(E, \sigma(H_\Lambda)\big) < \eta\big] \leqslant c_W \eta |\Lambda| \tag{2.2}$$

for some $c_W > 0$.

The proof of the theorem is based on Lemma 2.1, which is the step 0 in the multiscale analysis. This corresponds to verify that (1.1) of C is valid in all boxes $\Lambda_{x_0,l_0} = [x_0 - l_0, x_0 + l_0]$ included in $[0, +\infty)$ for l_0 large enough and F independent. After that, we follow the same steps as in [12]. Let $E \in \Delta$, by choosing an adapted scale, this procedure shows the existence of $\xi > 2$ and $\gamma > 0$ such that,

$$\begin{aligned}\mathbf{P}\big[\forall z \in \widetilde{\Delta},\ \big\|\mathbf{1}_{\chi_l} R_\Lambda(z) \mathbf{1}_{\chi_r}\big\| \leqslant \mathrm{e}^{-\gamma|\Lambda|}\big\|R_\Lambda(z)\big\|^2\big] \\ \geqslant 1 - \mathrm{O}\big(F^{4\xi/3 - 2/3}\big)\end{aligned} \tag{2.3}$$

for some complex neighborhood, $\widetilde{\Delta}$ of E and $|\widetilde{\Delta} \cap \mathbf{R}| = \mathrm{O}(F^{2(\xi+1)/3})$. From (2.3) the theorem easily follows.

LEMMA 2.1. – *Let* $E \in \Delta$, *there exists* $l_0^* > 0$, *such that for all* $l_0 > l_0^*$, *there exists* $\gamma_0 > 0$, $\xi > 2$, *field* $F_0 > 0$ *and a* F *independent complex neighbourhood of* E, $\widetilde{\Delta}_0$ *such that for all* $0 < F < F_0$, *and for all* *intervals* $\Lambda_{x_0,l_0} \subset [0,+\infty)$, *if* $\chi_0 \subset \Lambda_{x_0,l_0}$, *is an interval centered around* x_0 *and* $\chi_b \subset \Lambda_{x_0,l_0}$, *with* $\mathrm{dist}(\chi_0,\chi_b) \geqslant l_0/3$ *then*

$$\mathbf{P}\big[\forall z \in \widetilde{\Delta}_0,\ \|\mathbf{1}_{\chi_0} R_{x_0,l_0}(z)\mathbf{1}_{\chi_b}\| < \mathrm{e}^{-\gamma_0|\Lambda|}\big] \geqslant 1 - l_0^{-\xi}. \tag{2.4}$$

Proof. – Denote $x^* = 2V_M/F$, $V_M = \omega_M \sup_{x\in\mathbf{R}} \sum_i |u_i(x-i)|$, then x^* is surely in the classically forbidden region for non-positive energies. All finite intervals included in $[x^*, +\infty)$ are such that the event in (2.4) is realized, as a consequence of a result by [6,18] and the Wegner estimate. Hence in the following we will only consider intervals having a nonempty intersection with $[0, x^*)$. To study the Green function on a given interval Λ_{x_0,l_0} to the right of 0 we will proceed by the following steps.

First in $H_\omega(0)$, we replace $V_\omega(x)$ by $V_t(x) = \sum_i \omega_i \mathbf{1}_{C_i}(x)u_i(x-i)$ then we restrict it to Λ_{x_0,l_0} and call $H_{t,\omega}(0)$ the operator obtained with DBC at the borders. For $H_{t,\omega}(0)$, the random monodromy matrices from $i-1/2$ to $i+1/2$, are independent, and have the same distribution. So we can apply the large deviation theorem as presented in [8]. It says that we can construct locally a solution of

$$H_{t,\omega}(0)\phi = (E - Fx_0)\phi, \quad E \in \Delta, \tag{2.5}$$

with given initial condition at $x_0 - l_0$: $\phi(x_0 - l_0) = 0$, $\phi'(x_0 - l_0) = 1$, and $\forall\varepsilon > 0$, there exists $0 < N_0(\varepsilon, E - Fx_0) < \infty$ and $a(\varepsilon, E - Fx_0)$ such that for all integer, $n > N_0(\varepsilon, E - Fx_0)$

$$\mathbf{P}\Big[\mathrm{e}^{(\gamma'-\varepsilon)n} \leqslant \sqrt{|\phi(x_0 - l_0 + n)|^2 + |\phi'(x_0 - l_0 + n)|^2} \leqslant \mathrm{e}^{(\gamma'+\varepsilon)n}\Big] \geqslant 1 - \mathrm{e}^{-an}, \tag{2.6}$$

where $a = a(\varepsilon, E - Fx_0)$ and γ' is the Lyapunov exponent. At the second step we want to study the behavior of the solutions

$$H_\omega(F)\psi = \widetilde{E}\psi, \tag{2.7}$$

on the interval Λ_{x_0,l_0} for all $\widetilde{E}$ in a small neighbourhood of E. Considering $H_\omega(F) - H_{t,\omega}(0) - Fx_0 + E - \widetilde{E}$, as a small perturbation, we will use the theory of stability of differential equations. In their book Daleckii and Krein [13] addressed the question of the stability of Bohl exponents (which describe the exponential behavior of the solutions at infinity) for the differential system

$$\frac{\mathrm{d}y_0}{\mathrm{d}x} = A(x)y_0 \tag{2.8}$$

and the perturbed system

$$\frac{dy}{dx} = A(x)y + B(x)y \tag{2.9}$$

when the unperturbed differential system admits an exponential splitting of order n, the dimension of the system. In our case we do not use directly this theorem since it concerns a property at $+\infty$, but looking at its proof we see that it contains the exponential bounds on the solutions of the differential equation on finite intervals.

So we are faced with checking the exponential splitting for the differential equation $H_{t,\omega}(0)\phi = (E - Fx_0)\phi$. On Λ_{x_0,l_0} we consider a solution ϕ_+ which increases exponentially and a solution ϕ_- which decreases exponentially and we denote $\theta_+ = (\phi_+(x_0 - l_0), \phi'_+(x_0 - l_0))$ and $\theta_- = (\phi_-(x_0 - l_0), \phi'_-(x_0 - l_0))$. If we call in $\mathbf{R}^2$, p_+ and p_- resp. the projections on the directions θ_+ and θ_- and denote by $U(.)$ the monodromy matrix, by definition, exponential splitting means that $\|U(x)p_+U^{-1}(x)\| \leqslant M$ and $\|U(x)p_-U^{-1}(x)\| \leqslant M$, $0 < M < \infty$.

Now, let us show that we can find ϕ_+ and ϕ_- such that these inequalities are true with probability close to 1. Let $\Omega_\Lambda, \tau_\Lambda, \mathbf{P}_\Lambda$ be the probability space corresponding to the sites $x_0 - l_0, \ldots, x_0 + l_0$ of $\Lambda = \Lambda_{x_0,l_0}$, we denote also $\Omega_i, \tau_i, \mathbf{P}_i$ the probability space corresponding to the single site i. Consider Ω_Λ^+ the subset of Ω_Λ such that the solution of (2.5), which verifies the initial conditions $\phi_+(x_0 - l_0) = 0$, $\phi'_+(x_0 - l_0) = 1$ is an exponential solution in the following sense, $\forall n > \widetilde{N}_0 > N_0$

$$\begin{aligned} c_+ e^{(\gamma'-\varepsilon)n} &< \sqrt{|\phi_+(x_0 - l_0 + n)|^2 + |\phi'_+(x_0 - l_0 + n)|^2} \\ &< c'_+ e^{(\gamma'+\varepsilon)n}. \end{aligned} \tag{2.10}$$

By the large deviation theorem $\mathbf{P}_\Lambda(\Omega_\Lambda^+) \geqslant 1 - (1/a)\exp(-a(\widetilde{N}_0 - 1))$. Write now $\Omega_\Lambda = \Omega_{x_0 - l_0} \times \Omega_{\Lambda'}$ and consider the subset $\Omega_{\Lambda'}^-$ of $\Omega_{\Lambda'}$ such that for given final condition $\phi_-(x_0 + l_0) = 0$ and $\phi'_-(x_0 + l_0) = 1$, ϕ_- is a the solution of (2.5) with exponential backward behavior, i.e, $\forall n > \widetilde{N}_0 > N_0$,

$$\begin{aligned} c_- e^{(\gamma'-\varepsilon)(l_0-n)} &< \sqrt{|\phi_-(x_0 + l_0 - n)|^2 + |\phi'_-(x_0 + l_0 - n)|^2} \\ &< c'_- e^{(\gamma'+\varepsilon)(l_0-n)}, \end{aligned} \tag{2.11}$$

then as a consequence of the large deviation theorem $\mathbf{P}_{\Lambda'}(\Omega_{\Lambda'}^-) \geqslant 1 - (1/a)\exp(-a(\widetilde{N}_0 - 1))$.

We will say that the *"splitting property"*, S holds if the angle between θ^- and $\theta^+ = (0,1)$ is not smaller than $e^{-\varepsilon l_0}$. For a given $\omega_{\Lambda'} = \omega_{x_0-l_0+1}, \omega_{x_0-l_0+2}, \ldots, \omega_{x_0+l_0}$ moving $\omega_{x_0-l_0}$ causes an analytic change of θ^-. Some standard arguments of [26] imply $d\theta^-/d\omega_{x_0-l_0} > \text{const}$ uniformly. Then the probability, once $\omega_{\Lambda'}$ is fixed that S is satisfied, is $\mathbf{P}_{x_0-l_0} \geqslant 1 - O(e^{-\varepsilon l_0})$. By the Fubini's theorem, Ω_Λ^- being the subset of Ω_Λ for which (2.11) and S are satisfied,

$$\begin{aligned}\mathbf{P}_\Lambda(\Omega_\Lambda^-) &= \mathbf{P}_{\Lambda'}(\Omega_{\Lambda'})\mathbf{P}_{x_0-l_0} \geqslant \left(1 - O\left(\exp(-a\widetilde{N}_0)\right)\right)\left(1 - O\left(e^{-\varepsilon l_0}\right)\right) \\ &= 1 - 1/4 l_0^{-\xi} \end{aligned} \tag{2.12}$$

for some $\xi > 2$ and suitable $\widetilde{N}_0$. Here l_0 has to be sufficiently large in order the calculated $\widetilde{N}_0$ becomes larger than N_0 as necessary and in the other hand $l_0 > 2\widetilde{N}_0$. Then if we denote by $\Omega_\Lambda^{+,-} = \Omega_\Lambda^+ \cap \Omega_\Lambda^-$, we get

$$\mathbf{P}(\Omega_\Lambda^{+,-}) = \mathbf{P}_\Lambda(\Omega_\Lambda^{+,-}) \geqslant 1 - 1/2 l_0^{-\xi}. \tag{2.13}$$

In the proof of the Daleckii–Krein theorem, it is necessary to bound ϕ_+ and ϕ_- by exponentials all over Λ_{x_0,l_0}, this can be done using the fact that in $(x_0 - l_0, x_0 - l_0 + \widetilde{N}_0)$ if $\|T\|$ is the sup of the norms of the monodromy matrices from site to site, the solutions are bounded above by $\|T\|^{\widetilde{N}_0}$. So if $\omega_\Lambda \in \Omega_\Lambda^+$ we have, $\forall n > \widetilde{N}_0$,

$$\begin{aligned} c_+\|T\|^{-\widetilde{N}_0} e^{(\gamma'-\varepsilon)n} &< \sqrt{|\phi_+(x_0 - l_0 + n)|^2 + |\phi'_+(x_0 - l_0 + n)|^2} \\ &< c'_+\|T\|^{\widetilde{N}_0} e^{(\gamma'+\varepsilon)n}. \end{aligned} \tag{2.14}$$

By using (2.14), the analog inequality for ϕ_- and property S, for l_0 large enough, a straightforward analysis leads to,

$$\begin{aligned} \left\|U(x)p_+U^{-1}(y)\right\| &\leqslant \|T\|^{\widetilde{N}_0} e^{2\varepsilon l_0} e^{(\gamma'+\varepsilon)(x-y)}, \\ \left\|U(x)p_+U^{-1}(x)\right\| &\leqslant e^{2\varepsilon l_0} \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \left\|U(x)p_-U^{-1}(y)\right\| &\geqslant \|T\|^{-\widetilde{N}_0} e^{-2\varepsilon l_0} e^{-(\gamma'+\varepsilon)(x-y)}, \\ \left\|U(x)p_-U^{-1}(x)\right\| &\leqslant e^{2\varepsilon l_0}. \end{aligned} \tag{2.16}$$

From these estimates, we can find $F_0 > 0$, such that if $0 < F < F_0$, it exists ψ_+ and ψ_- solutions of (2.7) such that $\forall n > \widetilde{N}_0$,

$$C_+ e^{(\gamma'-\varepsilon_-)n} < \sqrt{|\psi_+(x_0-l_0+n)|^2 + |\psi'_+(x_0-l_0+n)|^2}$$
$$< C'_+ e^{(\gamma'+\varepsilon_+)n} \tag{2.17}$$

$$C_- e^{-(\gamma'+\varepsilon_-)n} < \sqrt{|\psi_-(x_0-l_0+n)|^2 + |\psi'_-(x_0-l_0+n)|^2}$$
$$< C'_- e^{-(\gamma'-\varepsilon_-)n} \tag{2.18}$$

with probability bounded below by $1-1/2l_0^{-\xi}$. The constants that appear in these formula are under the condition (2.19) below, given by: $\varepsilon_+ = \varepsilon + \|T\|^{\widetilde{N}_0} e^{2\varepsilon l_0} w$; $\varepsilon_- = \varepsilon + 2\|T\|^{\widetilde{N}_0} e^{4\varepsilon l_0} w$, $C_-, C_+ = \text{const}\,\|T\|^{-\widetilde{N}_0} e^{-4\varepsilon l_0}$ and $C'_-, C'_+ = \text{const}\,\|T\|^{\widetilde{N}_0} e^{4\varepsilon l_0}$, here const denotes a strictly positive and uniform constant, w is the bound of the perturbation, $w = Fl_0 + \omega_M v_t + |E-\widetilde{E}|$. Here appears the fact that the atomic tails, the complex energy $\widetilde{E}$ and the field F have to be chosen such that,

$$v_t, Fl_0, |\widetilde{E}-E| \leqslant \frac{1}{2}(\gamma'-\varepsilon)\|T\|^{-2\widetilde{N}_0} \exp(-6\varepsilon l_0) \tag{2.19}$$

in order $(\gamma'-\varepsilon_-)$ to be positive. Notice that this condition fixes the value F_0 that the field cannot exceed.

For a given energy $\widetilde{E}$, satisfying (2.19), there exists a linear combination, $\widetilde{\psi}_+$ of ψ_+ and ψ_- which increases exponentially in the positive direction and satisfy $\widetilde{\psi}_+(x_0-l_0)=0$ and $\widetilde{\psi}'_+(x_0-l_0)=1$, there also exists a solution $\widetilde{\psi}_-$ which increases exponentially in the negative direction and satisfies $\widetilde{\psi}_-(x_0+l_0)=0$ and $\widetilde{\psi}'_-(x_0+l_0)=1$, then the Green function associated to the operator $H_{x_0,l_0} = H_{\Lambda_{l_0}}$ is given by

$$G_{x_0,l_0}(x,y,\widetilde{E}) = \frac{\widetilde{\psi}_+(x)\widetilde{\psi}_-(y)}{W(\widetilde{\psi}_+,\widetilde{\psi}_-)} = \frac{\widetilde{\psi}_+(x)\widetilde{\psi}_-(y)}{\widetilde{\psi}_-(x_0-l_0)} \quad \text{for } x<y, \tag{2.20}$$

where we use the fact that the wronskian $W(\widetilde{\psi}_+,\widetilde{\psi}_-) = \widetilde{\psi}_-(x_0-l_0) = \widetilde{\psi}_+(x_0+l_0)$. Notice that by using the Prüfer variables, we have,

$$\widetilde{\psi}_-(x,\widetilde{E}) = \widetilde{r}_-(x,\widetilde{E})\sin\widetilde{\theta}_-(x,\widetilde{E}),$$
$$\widetilde{\psi}'_-(x,\widetilde{E}) = \widetilde{r}_-(x,\widetilde{E})\cos\widetilde{\theta}_-(x,\widetilde{E}). \tag{2.21}$$

Now suppose that there exists a gap around E of size $\delta = 2c_W{}^{-1}(l_0)^{-\xi-1}$, for some uniform constant, this event has a probability given by the Wegner estimate (2.2),

$$\mathbf{P}\big[\text{dist}(E,\sigma(H_{x_0,l_0})) \geqslant 2c_W{}^{-1}(l_0)^{-\xi-1}\big] \geqslant 1-1/2(l_0)^{-\xi}. \tag{2.22}$$

Let $\widetilde{E}$ satisfying (2.19), then $|E - \widetilde{E}| < 1/2c_W{}^{-1}(l_0)^{-\xi-1}$, for l_0 large enough, by standard arguments of the theory of differential equations (see, e.g., [26]),

$$\left|\sin\widetilde{\theta}_-(x_0 - l_0, E)\right| \geqslant \text{const}\,(l_0)^{-\xi-2} \tag{2.23}$$

if $|\delta|$ is small enough and then l_0 big enough. Suppose that if $x \in \chi_0$ and $y \in \chi_b$ then $x < y$, the other case is established in a same way, by using $W(\widetilde{\psi}_+, \widetilde{\psi}_-) = \widetilde{\psi}_+(x_0 + l_0)$. In this case, a straightforward computation using (2.19) together with (2.22) and the estimates on the solutions obtained above give (2.3) for some $\gamma_0(E) > 0$ if l_0 big enough and F small enough. Taking now $N_0 = \sup_{E \in S} N_0(E)$, $a = \inf_{E \in S} a(E)$ and $\gamma_0 = \inf_{E \in S} \gamma_0(E)$, where S is the following energy compact set $S = \{E - Fx_0,\ E \in \Delta,\ 0 < x_0 < x^*\}$, the lemma is proven. □

Remark. – By standard arguments see, e.g., Agmon type estimates [6], the lemma is valid for all energy $\widetilde{E} \in \mathbf{C}$ with any imaginary part and $|\mathrm{Re}\,\widetilde{E} - E|$ satisfying (2.19).

The arguments evoked in the proof show as a by-product that $H_\omega(0)$ restricted to $[0, \infty)$ has only pure point spectrum with exponentially decaying eigenfunctions. The theorem of large deviation which gives the local exponential behavior of the solutions plays an essential role. A similar fact has already been noticed in [24] (see also [28]) for discrete random operators.

3. EIGENVALUES ESTIMATES FOR THE INTERIOR PART AND EIGENFUNCTION LOCALIZATION

To use perturbation theory to recover the spectrum of the distorted operator, $H_\omega(F, \theta)$ (see Section 4), we need to know, in the vicinity of zero energy, the distance between the eigenvalues of operators $H_\omega(F)$ restricted to some intervals $\mathbf{I}$ of the form $\mathbf{I} = (x_a, +\infty)$. x_a is chosen such that $x_a = a/F$, $0 \leqslant a < c_{\max}$, $c_{\max} = U_m - |\Delta| - \omega_M v_t$, the last turning point is then always larger than x_a for F small. We denote by $H_{\mathbf{I}}(F)$ such an operator. It is intuitively clear that the number of eigenvalues of $H_{\mathbf{I}}(F)$ in the interval $\Delta = [-E_0, 0]$ increases as F goes to 0 since the positive region for which the potential is smaller than 0, has a size $\mathrm{O}(1/F)$. We want to find some constants $C_1 < C_2$ such that this number, denoted in the following by $N(\Delta)$, is bounded by C_1/F and C_2/F. Calling $\mathcal{N}_0(E)$, the integrated density of states for $H_\omega(0)$, we get:

THEOREM 3.1. – *Let $\Delta = [-E_0, 0]$ be a real energy interval and $1/2 < \xi < 1$. Suppose* [H2], [H3] *and choose an interval $\mathbf{I} = (x_a, \infty)$ for which $x_a = a/F$, $0 < a < c_{\max}$. Then there exists F_0 and two strictly positive constants C_1, C_2 uniform with respect to $\omega \in \Omega$, such that for $0 < F < F_0$,*

$$\mathbf{P}\left[|\Delta|\frac{\mathcal{N}_0(-a_-)}{4F} - \frac{C_1}{F^\xi} < N(\Delta) < 2|\Delta|\frac{\mathcal{N}_0(-a'_-)}{F} - \frac{C_2}{F^\xi}\right] > 1 - \mathrm{O}(F^{2\xi-1}), \tag{3.1}$$

where $a_- = a + |\Delta| + \omega_M v_t$ and $a'_- = a + \omega_M v_t$.

Remark. – The condition $a + |\Delta| + \omega_M v_t < U_m$ insures the positivity of $\mathcal{N}_0(-a_-)$.

Proof. – Choose a point b/F, $b > V_M$, $V_M = \omega_M \sup_{x\in\mathbf{R}} \sum_i |u_i(x - i)|$, by the classical results on the spectral stability [6,18], the operators $H_\mathbf{I}$ and $H_{\widetilde{\mathbf{I}}}$, $\widetilde{\mathbf{I}} = (a/F, b/F)$ have the same number of eigenvalues in the interval Δ. Let us first replace the potential V_ω by $V_t(x) = \sum_i \omega_i \mathbf{1}_{C_i} u_i(x - i)$ and denote by $H_{t,\widetilde{\mathbf{I}}}$ the corresponding operator. We cut the interval $\widetilde{\mathbf{I}}$ in $2(b-a)/|\Delta|$ intervals of length $|\Delta|/2F$ and consider the restriction of $H_{t,\widetilde{\mathbf{I}}}(F)$ on these intervals with Neumann boundary condition (NBC)and call them $h_i(F)$, $i = 1\ldots 2(b-a)/|\Delta|$. For each, we will study the number of eigenvalues inside the energy interval Δ.

Let us call $N_F^i(E)$ and $N_0^i(E)$ the number of eigenvalues smaller than E of $h^i(F)$ and $h^i(0)$ respectively. Since the variation of Fx in the first interval is $|\Delta|/2$ and in the form sense, $h^1(0) + a < h^1(F) < h^1(0) + a + |\Delta|/2$, one obtains,

$$N_0^1\left(E - a - \frac{|\Delta|}{2}\right) < N_F^1(E) < N_0^1(E - a), \tag{3.2}$$

and then

$$N_0^1\left(-a - \frac{|\Delta|}{2}\right) - N_0^1(-a - |\Delta|) < N_F^1(0) - N_F^1(-|\Delta|). \tag{3.3}$$

Similarly, in the second interval, since

$$h^2(0) + a + \frac{|\Delta|}{2} < h^2(F) < h^2(0) + a + 2\frac{|\Delta|}{2},$$

one obtains,

$$N_0^2(-a-|\Delta|) - N_0^2\left(-a-3\frac{|\Delta|}{2}\right) < N_F^2(0) - N_F^2(-|\Delta|) \qquad (3.4)$$

and so on, for all the intervals. Taking the sum for all the intervals of the right hand side terms, this quantity is larger than

$$\begin{aligned} &N_0^1(-a-|\Delta|/2) - N_0^1(-a-|\Delta|) + \cdots - N_0^{i-1}(-a-i|\Delta|/2) \\ &\quad + N_0^i(-a-i|\Delta|/2) - \cdots \end{aligned} \qquad (3.5)$$

Due to the independence, $\mathbf{E}N_0^i(E) = \mathbf{E}N_0^j(E)$, ($\mathbf{E}$ denoting the expectation value with respect to the probability space), so expression (3.5) can be written as,

$$\begin{aligned} &\mathbf{E}N_0^1(-a-|\Delta|/2) + (N_0^1(-a-|\Delta|/2) - \mathbf{E}N_0^1(-a-|\Delta|/2)) \\ &\quad - (N_0^1(-a-|\Delta|) - \mathbf{E}N_0^1(-a-|\Delta|)) \\ &\quad + (N_0^2(-a-|\Delta|) - \mathbf{E}N_0^2(-a-|\Delta|)) + \cdots \\ &\quad - (N_0^{i-1}(-a-i|\Delta|/2) - \mathbf{E}N_0^{i-1}(-a-i|\Delta|/2)) \\ &\quad + (N_0^i(-a-i|\Delta|/2) - \mathbf{E}N_0^i(-a-i|\Delta|/2)) - \cdots . \end{aligned} \qquad (3.6)$$

To evaluate a lower bound for the sum (3.6) at F small, we compute the probability that the differences are smaller than const $F^{-\xi}$ for some $0 < \xi < 1$. In fact, let $1/2 < \xi < 1$, we divide the ith interval in c/F^{ξ} intervals of length $\Delta/(2cF^{1-\xi})$, called $I_{i,j}$ and consider the number of eigenvalues $N_0^{i,j}$ smaller than E for the operators $h^{i,j}$, defined as the restrictions to these intervals of the operator $h_i(0)$. For fixed i, the $N_0^{i,j}$ are independent. So their sum has a variance which is proportional to the number of intervals c/F^{ξ} times the variance of one of them, i.e., const$/F^{1-\xi}$. Then by Chebyshev,

$$\mathbf{P}\left(\left|\sum_j (N_0^{i,j} - \mathbf{E}N_0^{i,j})\right| > \nu F^{-\xi}\right) < \frac{c'}{\nu^2} F^{2\xi-1} \qquad (3.7)$$

for all $\nu > 0$ and some constant $c' > 0$. Let $\widetilde{N}^i$ be the quantity $\sum_j N_0^{i,j}$, to go from the $h^{i,j}$ to $h^i(0)$ we have to take off c/F^{ξ} boundary conditions. It is easy to see that the condition $|N_0^i - \mathbf{E}N_0^i| > \beta F^{-\xi}$, $\beta > 0$, implies $|\widetilde{N}^i - \mathbf{E}\widetilde{N}^i| > (\beta - 2c)F^{-\xi}$ if β is large enough, then

$$\mathbf{P}(|N_0^i - EN_0^i| > \beta F^{-\xi}) < c' F^{2\xi-1}/(\beta - 2c)^2. \qquad (3.8)$$

Using formula (3.8) in the bound (3.6), the sum of eigenvalues of the operator $\bigoplus_i h_i(F)$ in the interval Δ is greater than (here we have neglected the first positive term in (3.6)),

$$\mathbf{E}N_0^2(-a-|\Delta|/2)-\frac{2(b-a)}{|\Delta|}\cdot\frac{\beta}{F^{\xi}} \tag{3.9}$$

with a probability bounded below by $1-\mathrm{O}(F^{2\xi-1})$, or introducing the integrated density of states $\mathcal{N}_0$, by standard arguments see, e.g., [10] it is greater than

$$\frac{|\Delta|}{2F}\mathcal{N}_0(-a-|\Delta|/2)-\frac{2(b-a)}{|\Delta|}\cdot\frac{\beta}{F^{\xi}}. \tag{3.10}$$

To get the number of eigenvalues for $H_{t,\tilde{\mathbf{I}}}(F)$ we have to take off the NBC. This can modify the total number number of eigenvalues by the number of NBC, i.e., by $2(b-a)/\Delta$. So the number of eigenvalues of $H_{t,\tilde{\mathbf{I}}}(F)$ is greater than

$$\frac{|\Delta|}{2F}\mathcal{N}_0(-a-|\Delta|/2)-\frac{2(b-a)}{|\Delta|}\beta\frac{c}{F^{\xi}}-\frac{2(b-a)}{\Delta} \tag{3.11}$$

with a probability bounded below by $1-\mathrm{O}(F^{2\xi-1})$. To evaluate the number of eigenvalues for $H_{\tilde{\mathbf{I}}}(F)$ we have to take into account the tails $V_\omega-V_t<0$. They introduce a correction which is small in norm. From the inequalities

$$H_{t,\tilde{\mathbf{I}}}(F)-\omega_M v_t<H_{\tilde{\mathbf{I}}}(F)<H_{t,\tilde{\mathbf{I}}}(F), \tag{3.12}$$

we can deduce that the number of eigenvalues for $H_{\tilde{\mathbf{I}}}(F)$ in the interval Δ is greater than the number of eigenvalues for $H_{t,\tilde{\mathbf{I}}}(F)$ in the interval $[-E_0+\omega_M v_t,0]$. So finally, we get by (3.11) that this number is bounded below by,

$$\frac{|\Delta|}{2F}\mathcal{N}_0(-a-|\Delta|-\omega_M v_t)-\frac{2(b-a)}{|\Delta|-\omega_M v_t}\cdot\frac{\beta}{F^{\xi}}-\frac{2(b-a)}{|\Delta|-\omega_M v_t} \tag{3.13}$$

with the probability above, this proves the lower bound part of theorem. For the upper bound, we follow from the same scheme as the previous one, but here we divide the interval $(a/F,b/F)$ in $(b-a)/\Delta$ intervals of length Δ/F and we use the DBC instead NBC to define the operators on these intervals. □

Remark. – We have not a precise control on the distribution of eigenvalues, even if we would guess that they obey a law close to Poisson, like for $F=0$. In particular we cannot assert that from place to place they do not accumulate on intervals exponentially small with respect to F. We denote by Cl a set of eigenvalues with exponentially small distance between them. Using some standard arguments of [23], we get that the number of eigenvalues of $H_{\widetilde{\mathbf{I}}}(F)$ in the energy interval Δ, which cannot exceed $n_{\max}/F$ for some $n_{\max}>0$.

We now want to prove that some eigenvectors live at the right side of some point $\widetilde{x}_l^{\text{int}} > x^{\text{ext}}$, of course $\widetilde{x}_l^{\text{int}}$ has to be smaller than the last turning point. It has been proven recently in [14] that for the discrete Anderson model the distribution of the maxima of the eigenvectors is uniform on the line and we guess that the proof could be adapted in our case to some large intervals even if we do not have the translational invariance for the potential. As in fact, we do not need in Section 5 such a strong result, here we only prove the following theorem, let $H_I(F)$ is the operator restricted to $I=(x_a,\infty)$ and P_Δ its spectral projector on the energy interval Δ, then:

THEOREM 3.2. – *Under the same assumptions as in Theorem* 3.1, *there exists some constants,* $0<F_0<\infty$, $0<\widetilde{C}_a<\infty$ *and a point* $\widetilde{x}_a$, $x_a<\widetilde{x}_a<\infty$ *with* $\operatorname{dist}(\widetilde{x}_a,x_a)=c_a/F$ *for some* $0<c_a<\infty$, *uniformly with respect to* $\omega\in\Omega$, *such that for all* $0<F<F_0$,

$$\mathbf{P}\big[\operatorname{Tr}(\mathbf{1}_{\widetilde{\chi}}P_\Delta\mathbf{1}_{\widetilde{\chi}})\geqslant\widetilde{C}_aF^{-1}\big]>1-\mathrm{O}\big(F^{2\xi-1}\big) \tag{3.14}$$

here $\mathbf{1}_{\widetilde{\chi}}$ *is the characteristic function of interval* $\widetilde{\chi}=[\widetilde{x},+\infty)$ *with* $\widetilde{x}\leqslant\widetilde{x}_a$. *Moreover there exists* $\widetilde{c}>0$ *such that the event:* $H_I(F)$ *has at least one eigenvector* ϕ *associated to an eigenvalue in* Δ *satisfying,*

$$\big|(\phi,\mathbf{1}_{\widetilde{\chi}}\phi)\big|\geqslant\widetilde{c} \tag{3.15}$$

has a probability bounded below by $1-\mathrm{O}(F^{2\xi-1})$.

Proof. – By the standard arguments of [23], we can see that

$$\operatorname{Tr}\big((1-\mathbf{1}_{\widetilde{\chi}})P_\Delta(1-\mathbf{1}_{\widetilde{\chi}})\big)\leqslant\operatorname{const}c_a/F.$$

Then

$$\begin{aligned}\operatorname{Tr}(\mathbf{1}_{\widetilde{\chi}}P_\Delta\mathbf{1}_{\widetilde{\chi}})&=\operatorname{Tr}P_\Delta-\operatorname{Tr}\big((1-\mathbf{1}_{\widetilde{\chi}})P_\Delta(1-\mathbf{1}_{\widetilde{\chi}})\big)\\&\geqslant\operatorname{Tr}P_\Delta-\operatorname{const}c_aF^{-1}.\end{aligned} \tag{3.16}$$

So by using Theorem 3.1, we get (3.14), choosing c_a small enough. On the other hand, if one writes $\mathrm{Tr}(\mathbf{1}_{\tilde{\chi}} P_\Delta \mathbf{1}_{\tilde{\chi}}) = \sum_i (\phi_i, \mathbf{1}_{\tilde{\chi}} P_\Delta \phi_i)$, where the ϕ_i are the eigenvectors of $H_{\tilde{\mathbf{I}}}(F)$, then by (3.1) and (3.14), there exists a constant $\tilde{c}$ and at least an eigenvector $\phi = \phi_i$ for which $|(\phi_i, \mathbf{1}_{\tilde{\chi}} P_\Delta \phi_i)| \geqslant \tilde{c}$.

In Section 5, we will also use the family of operators $h_{\omega>}(F)$ defined on $L^2(\mathbf{R})$ as,

$$
\begin{aligned}
h_{\omega>}(F) &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{i>x_M} \omega_i u_i(x-i) + Fx \\
&= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v_{\omega>}(x) + Fx
\end{aligned}
\tag{3.17}
$$

for some point $0 < x_M < \infty$ which will be given in Section 5. Let $(\Omega_>, \tau_>, \mathbf{P}_>)$ the probability space corresponding to sites $i \geqslant x_M$. Then clearly Theorems 3.1 and 3.2 hold with respect to the probability space $\Omega_>$, for $h_{\omega>}(F)$.

For a given potential configuration, $v_{\omega>}(x)$ and a point $x_a = E_a/F$, we will use the notation Cl^+ for a cluster of eigenvalues which contains at least one eigenvalue, corresponding to an eigenvector localized at the right side of x_a in the sense of Theorem 3.2. □

4. SPECTRAL DEFORMATION

In this section, we consider an energy interval, Δ as in Theorem 1.1. To define resonances, we use the complex transformation method of [21], so we have to define an appropriate family of complexified operators $H(\theta) = H_\omega(F, \theta)$, $\theta \in \mathbf{C}$, with $H = H_\omega(F)$. In particular we construct for $\theta \in \mathbf{R}$ a distortion such that, for $\theta \in \mathbf{C}$ the exterior operator $H^{\mathrm{ext}}(\theta)$ has with good probability, a complex neighbourhood of some $E \in \Delta$ in its resolvent set $\rho(H^{\mathrm{ext}}(\theta))$. This problem has been originally solved in [5]. For further applications in the next section we need to consider firstly a family of exterior operators which is slightly different from the operator $H^{\mathrm{ext}} = H_\omega^{\mathrm{ext}}(F)$ defined in Section 2, let

$$
h^{\mathrm{ext}} = h_{\omega<}^{\mathrm{ext}}(F) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v_{\omega<} + Fx, \qquad F > 0, \tag{4.1}
$$

on $\mathcal{H}^{\text{ext}}$ with DBC at $x^{\text{ext}} = c^{\text{ext}}/F$, where

$$v_{\omega_<} = \sum_{i<x_M} \omega_i u_i(x-i) \tag{4.2}$$

for some point x_M, $x^{\text{ext}} < x_M$ and $\operatorname{dist}(x^{\text{ext}}, x_M) = \mathrm{O}(1/F)$. The random operator h^{ext} only depends on the random variables $\{\omega_i,\ i < x_M\}$, we denote by $(\Omega_<, \tau_<, \mathbf{P}_<)$ the corresponding probability space. Notice also that the operators h^{ext} and H^{ext} are essentially selfadjoint on $\mathcal{C}^{\text{ext}} = \{u \in \mathbf{C}_0^\infty((-\infty, x^{\text{ext}}]),\ u(x^{\text{ext}}) = 0\}$. Return now to the definition of the family $H(\theta)$, $\theta \in \mathbf{C}$, suppose F small enough, let $E \in \Delta$, $0 < \varepsilon < 2$ and consider a nonincreasing function $s \in \mathbf{C}^\infty(\mathbf{R})$ satisfying,

$$\begin{aligned} s(x) &= 1 \quad \text{for } x \leqslant x_s,\ x_s = E/F + a_s/F^{2+\varepsilon}, \\ s(x) &= 0 \quad \text{for } x \geqslant x_s',\ x_s' = E/F + a_s'/F^{2+\varepsilon}, \end{aligned} \tag{4.3}$$

where a_s, a_s' are strictly negative constants such that $x_s < x_s' < 0$. Define

$$f = f_{E,\omega} = \frac{1}{2(E-v)^{1/2}} \int_{x_s'+E/F}^{x} \frac{s}{(E-v)^{1/2}}\, \mathrm{d}t, \tag{4.4}$$

where $v = v_{\omega<} + Fx$. The field f is a $\mathbf{C}^\infty$ solution of

$$2f'(v-E) + fv' = -s. \tag{4.5}$$

Let $f_\theta(x) = x + \theta f(x)$, $\theta \in \mathbf{R}$, be the associated distortion. Formula (4.5) is a type of virial equality which implies that the corresponding classical particle moving in the potential v at energy E is non-trapped (see, e.g., [6,18]), we will see that (4.5) is an essential ingredient in our analysis. It holds, for $0 < F < F_0$ and F_0 small enough,

$$\begin{aligned} f &= \mathrm{O}(s/F) \quad \text{near } x = E/F + x_s' \quad \text{and} \\ f &= 1/F\big(1 + \mathrm{O}(x^{-1/2})\big) \quad \text{as } x \to -\infty, \end{aligned} \tag{4.6}$$

which shows that f_θ is a translation in a neighbourhood of $-\infty$. On the other hand a straightforward analysis from (4.5) yields to,

$$\begin{aligned} &f' = \mathrm{O}(sF^\varepsilon), \quad f'' = \mathrm{O}(sF^\varepsilon) + \mathrm{O}(F^{3+2\varepsilon}), \\ &f''' = \mathrm{O}(sF^\varepsilon) + \mathrm{O}(F^{4+3\varepsilon}), \end{aligned} \tag{4.7}$$

near $x = E/F + x_s'$ and

$$f', f'', f''' = \mathrm{O}(x^{-1}) \tag{4.8}$$

in a neighbourhood of $-\infty$, these estimates are valid for all $E \in \Delta$ and $\omega_< \in \Omega_<$. For F and θ small enough f_θ is a $\mathbf{C}^\infty$ diffeomorphism on $\mathbf{R}$ which implements a family of unitary operators on $\mathcal{H}$ by

$$U(\theta)u = |f_\theta'|^{1/2} u \circ f_\theta. \tag{4.9}$$

Let

$$V_\omega(\theta) = U(\theta)V_\omega U(\theta)^{-1} = V_\omega \circ f_\theta. \tag{4.10}$$

By assumption [H1], then for F small enough and some constant $c_\theta > 0$, if $|\theta| < c_\theta F$, the operators $V_\omega(\theta)$ have an analytic extension as bounded operators on $\mathcal{H}$. For $\theta \in \mathbf{R}$ and small,

$$T(\theta) = -U(\theta)\frac{\mathrm{d}^2}{\mathrm{d}x^2}U(\theta)^{-1} = -f_\theta'^{-1/2}\frac{\mathrm{d}}{\mathrm{d}x}f_\theta'^{-1}\frac{\mathrm{d}}{\mathrm{d}x}f_\theta'^{-1/2} \tag{4.11}$$

then for $|\theta| < c_\theta F,\ 0 < F < F_0,\ F_0$ small enough,

$$\begin{aligned} H(\theta) &= U(\theta)HU(\theta)^{-1} \\ &= T(\theta) + V_\omega\big(x + \theta f(x)\big) + F\big(x + \theta f(x)\big) \end{aligned} \tag{4.12}$$

is a type A analytic family of operators. Similarly we define the family $H^{\text{ext}}(\theta)$ and $h^{\text{ext}}(\theta)$. It must be noticed that by some familiar arguments of perturbation theory, we get easily that these operators have a nonempty resolvent set, in particular it contains the half plane $\{z \in \mathbf{C},\ \operatorname{Im} z \geqslant y_0\}$ for some $y_0 > 0$ and big enough. For the sequel, we need more than this last simple spectral estimate, our analysis is based on a Wegner estimate for h^{ext}. Let $\mathbf{I}$ be a real interval and $h_{\mathbf{I}}$ be the restriction of h^{ext} to $\mathbf{I}$, we have:

LEMMA 4.1. – *Let $E \in \Delta$ be a real energy and $\delta > 0$, then there exists an uniform constant $c_W > 0$ with respect to the energy $E \in \Delta$, such that*

$$\mathbf{P}_<\big\{\operatorname{dist}\big(E, \sigma(h_{\mathbf{I}})\big) \leqslant \delta\big\} \leqslant \sup\big\{|E - v|,\ x \in I,\ E \in \Delta\big\} c_W \delta |\mathbf{I}|. \tag{4.13}$$

The proof of this lemma is the same as in [23] where a Wegner estimate for a general Anderson model is given, notice that this fact is also true for the restriction of H^{ext} to the interval $\mathbf{I}$.

Let $\delta(E,t) = \{\lambda \in \mathbf{R},\ |E - \lambda| \leqslant t\},\ \delta^c(E,t)$ its complement in $\mathbf{R}$ and

$$\nu(E,t,t') = \big\{z \in \mathbf{C},\ \operatorname{re} z \in \delta(E,t),\ \operatorname{Im}\big(f_\theta'^2(E - z)\big) \leqslant t'\big\}. \tag{4.14}$$

Then the main result of this section is the following. Let $r^{\text{ext}}(\theta, z) = r^{\text{ext}}(\theta, F, z),\ z \in \rho(h^{\text{ext}}(\theta))$ be the resolvent of $h^{\text{ext}}(\theta)$, we have:

THEOREM 4.1. – *Let $E \in \Delta$ be a real energy, $0 < \varepsilon < 1$, $0 < 4\varepsilon' < \varepsilon$, then there exists some strictly positive constants F_0 and C_{ext} uniform with respect to $\omega_< \in \Omega_<$ and $E \in \Delta$, such that for $\theta = i\beta$, $\beta = F^{3/2+3\varepsilon/8}$, $0 < F < F_0$, the event $\xi(E, F)$:*

$$\text{(i)} \quad \nu = \nu\big(E, 2F^{3+2\varepsilon+\varepsilon'}, -2F^{11/2+3\varepsilon+2\varepsilon'}\big) \subset \rho\big(h^{\text{ext}}(\theta)\big),$$

$$\begin{aligned}\text{(ii)} \quad \forall z \in \nu, \ \|r^{\text{ext}}(\theta, z)\| \leqslant C_{\text{ext}}\big(&\text{dist}^{-1}\big(z, \delta^c\big(E, 2F^{3+2\varepsilon+\varepsilon'}\big)\big)\big) \\ &+ \beta\big(1 + F^p\big)^2 \text{dist}^{-2}\big(z, \delta^c\big(E, 2F^{3+2\varepsilon+\varepsilon'}\big)\big),\end{aligned} \tag{4.15}$$

where $p = -1/2 + 3\varepsilon/8 < 0$, has a probability measure satisfying, $\mathbf{P}_<(\xi(E, F)) \geqslant 1 - \mathrm{O}(F^{\varepsilon'})$.

To prove Theorem 4.1 we introduce for $\theta = i\beta$, $0 \leqslant \beta \leqslant F^{3/2+3\varepsilon/8}$ and $0 < F < F_0$, $z \in \mathbf{C}$ the family of operators,

$$h(z) = h(F, z) = f'_\theta\big(h^{\text{ext}}(\theta) - z\big) f'_\theta \tag{4.16}$$

on $\mathcal{H}^{\text{ext}}$. For θ small enough, the operators $h(z)$ and $(h^{\text{ext}}(\theta) - z)$ are "quasi-similar", in the sense that they differ from each other by the multiplication by the weight function f'_θ which is a bounded one to one mapping on the domain of $h^{\text{ext}}(\theta)$, $\mathcal{D}(h^{\text{ext}}(\theta))$, with a bounded inverse. We have,

$$\begin{aligned}h(z) = h^{\text{ext}} - E + f_\theta'^2(E - z) - \theta s + \{f_\theta, x\} \\ + \theta^2 \mathcal{R}_2 + \theta^3 \mathcal{R}_3 + \theta^4 \mathcal{R}_4,\end{aligned} \tag{4.17}$$

where for some $0 \leqslant |\theta'| \leqslant |\theta|$,

$$\begin{cases} \{f_\theta, x\} = 1/2\big(\theta f''' - \theta^2\big(\frac{f' f'''}{f'_\theta} + \frac{3f''^2}{2f_\theta'^2}\big)\big), \\ \mathcal{R}_2 = f^2 v'' - f'^2(v - E) + 2f' f v', \\ \mathcal{R}_3 = 2f' f^2 v'' - f'^2 f v' + f^3 v'''_{\theta'}, \\ \mathcal{R}_4 = -f'^2 f^2 v'' + f^3 f'(2 + \theta f') v'''_{\theta'}. \end{cases} \tag{4.18}$$

Notice that in (4.17) we have used,

$$f_\theta'^2\big(v(\theta) - E\big) = v - E - \theta s + \theta^2 \mathcal{R}_2 + \theta^3 \mathcal{R}_3 + \theta^4 \mathcal{R}_4 \tag{4.19}$$

which by the conditions of Theorem 4.1 and by (4.7) verifies:

$$f_\theta'^2\big(v(\theta) - E\big) = v - E - \theta s + \theta s \mathrm{O}\big(F^p\big) \tag{4.20}$$

we also have

$$\operatorname{Im}\big(f_\theta'^2(v(\theta)-E)\big) = -\beta s\big(1+O(F^{3\varepsilon/4})\big) < 0. \tag{4.21}$$

Finally, we will denote by $r(z)=r(\theta,z)$, if it exists, the inverse of $h(z)$. The spectral analysis of the operator $h(z)$ has to take into account the singularity of the potential at $-\infty$ which is controlled by the choice of the distortion and the possible eigenstates localized in the region of turning points which are controlled by Wegner estimates. These two types of regimes can be described by the geometric perturbation theory for two wells potentials of the appendix. Hence, let $0<\varepsilon<1$ and consider the following non-disjoint partition of $\mathbf{R}^{\text{ext}}$, $\mathbf{R}^{\text{ext}}=\mathbf{I}_1\cup\mathbf{I}_2$,

$$\mathbf{I}_1 = (-\infty, E/F + x_1), \quad x_1 = a_1/F^{2+\varepsilon},\ a_1 < a_s, \tag{4.22}$$

$$\mathbf{I}_2 = (E/F + x_2, x^{\text{ext}}), \quad x_2 = a_2/F^{2+\varepsilon},\ a_2 < a_1, \tag{4.23}$$

and the operators $h_k(z)$, $k=1,2$, defined on $\mathcal{H}_k = L^2(\mathbf{I}_k)$ as the restriction of the operator $h(z)$ on $\mathbf{I}_k$. We will denote by $r_k(z)$, $k=1,2$, if they exist, the inverse of these operators.

LEMMA 4.2. – *Let $E\in\Delta$ be a real energy, $0<\varepsilon<1$ and $0<4\varepsilon'<\varepsilon$. Then there exists some strictly positive constants F_0 and C_2 uniform with respect $\omega_<\in\Omega_<$ and $E\in\Delta$, such that for $\theta=i\beta$, $0\leqslant\beta\leqslant F^{3/2+3\varepsilon/8}$, $0<F<F_0$, the event, $\xi_2(E,F)$:*

(i) $\nu_2 = \nu\big(E, 4F^{3+2\varepsilon+\varepsilon'}, -4F^{11/2+3\varepsilon+2\varepsilon'}\big)\subset\rho\big(h_2(z)\big)$,

$$\text{(ii)}\quad \forall z\in\nu_2,\ \|r_2(z)\|\leqslant C_2\big(\operatorname{dist}^{-1}\big(z,\delta^c\big(E,4F^{3+2\varepsilon+\varepsilon'}\big)\big)\big) + \beta\big(1+F^p\big)^2\operatorname{dist}^{-2}\big(z,\delta^c\big(E,4F^{3+2\varepsilon+\varepsilon'}\big)\big) \tag{4.24}$$

has a probability, $\mathbf{P}_<(\xi_2(F))\geqslant 1-O(F^{\varepsilon'})$.

Proof. – Consider first the operator

$$h_{2,0}(z) = h_{\mathbf{I}_2} - E + f_\theta'^2(E-z) \tag{4.25}$$

on $\mathcal{H}_2$, here $h_{\mathbf{I}_2}$ is the restriction of h^{ext} on $\mathbf{I}_2=(E/F+x_2,x^{\text{ext}})$. Let $\xi_2'(F)$ be the event,

$$\operatorname{dist}\big(E,\sigma(h_{\mathbf{I}_2})\big)\geqslant 8F^{3+2\varepsilon+\varepsilon'}, \tag{4.26}$$

i.e., $\delta(E, 8F^{3+2\varepsilon+\varepsilon'})$ is a gap for h_{I_2}, then for $z \in \nu_2$, F small enough, the Neumann series imply that $h_{2,0}(z)$ has a bounded inverse, $r_{2,0}(z)$ on $\mathcal{H}_2$ and

$$\|r_{2,0}(z)\| \leqslant 2\,\mathrm{dist}\big(z, \delta^c\big(E, 6F^{3+2\varepsilon+\varepsilon'}\big)\big), \tag{4.27}$$

by Lemma 4.1, the event $\xi_2'(E, F)$ has a probability $\mathbf{P}_<(\xi_2') \geqslant 1 - \mathrm{O}(F^{\varepsilon'})$. In the sequel, we will use $\{f_\theta, x\} = i\beta S_1 + S_2$, where $S_1, S_2 \in \mathbf{C}^\infty(\mathbf{R})$ and verify, $S_1 = \mathrm{O}(sF^\varepsilon)$, $S_2 = \mathrm{O}(F^{6+3\varepsilon+p})$ (see, e.g., (4.7)). Consider now, for $\eta > 0$, the family of operators,

$$\begin{aligned} h_{2,1}(z,\eta) &= h_2(z) - i\eta - S_2 \\ &= h_{2,0}(z) - i\eta - i\beta s + i\beta S_1 - \beta^2\mathcal{R}_2 - i\beta^3\mathcal{R}_3 + \beta^4\mathcal{R}_4 \end{aligned} \tag{4.28}$$

on $\mathcal{H}_2$. For F small enough and z such that $\mathrm{Im}(f_\theta'^2(E-z)) - \eta/4 < 0$, from (4.20), we have,

$$\begin{aligned} \mathrm{Im}\big(h_{2,1}(z,\eta)\big) &= -\beta s\big(1 + \mathrm{o}\big(F^{3\varepsilon/4}\big)\big) + \mathrm{Im}\big(f_\theta'^2(E-z)\big) - \eta \\ &< 0 \end{aligned} \tag{4.29}$$

in the quadratic form sense on $D(h_{2,1}(z,\eta))$. This shows by a straightforward calculus [25] that $\forall u \in D(h_{2,1}(z,\eta))$,

$$\begin{aligned} \|h_{2,1}(z,\eta)u\|^2 &\geqslant -2\eta\,\mathrm{Im}\big(\big(h_{2,1}(z,\eta/4)u, u\big)\big) + \eta^2/2\|u\|^2 \\ &\geqslant \eta^2/2\|u\|^2. \end{aligned} \tag{4.30}$$

Clearly, we get the same estimate for the adjoint operator, $(h_{2,1}(z,\eta))^*$ and by standard arguments [22], this implies that $h_{2,1}(z,\eta)$ has a bounded inverse on $\mathcal{H}_2$, $r_{2,1}(\eta, z)$ and

$$\|r_{2,1}(z,\eta)\| \leqslant 2^{1/2}\eta^{-1}. \tag{4.31}$$

On the other hand by (4.21), for F small enough, we have

$$\begin{aligned} \mathrm{Im}\big(s^{1/2}r_{2,1}(z,\eta)s^{1/2}\big) &= s^{1/2}r_{2,1}(z,\eta)\,\mathrm{Im}\big(h_{2,1}(z,\eta)\big)\big(r_{2,1}(z,\eta)\big)^* s^{1/2} \\ &\leqslant -\beta/2\|s^{1/2}r_{2,1}(z,\eta)s^{1/2}\|^2 \end{aligned} \tag{4.32}$$

which shows by the Cauchy–Schwartz inequality,

$$\|s^{1/2}r_{2,1}(z,\eta)s^{1/2}\| \leqslant 2\beta^{-1}. \tag{4.33}$$

Then let $z \in \nu(E, 6F^{3/2+3\varepsilon/8}, -\eta/4)$, by (4.31), (4.33) and the following resolvent equation,

$$r_{2,1}(\eta, z) = r_{2,0}(z) - r_{2,0}(z) W r_{2,0}(z) + r_{2,0}(z) W r_{2,1}(\eta, z) W r_{2,0}(z) \tag{4.34}$$

where $W = h_{2,1}(z, \eta) - h_{2,0}(z) = i\beta s(-1 + \mathrm{O}(F^p)) - i\eta$, we get,

$$\|r_{2,1}(\eta, z)\| \leqslant \text{const}\left(\|r_{2,0}(z)\| + \left(\beta(1 + F^p)^2 + \eta\right)\|r_{2,0}(z)\|^2\right) \tag{4.35}$$

for a strictly positive constant uniform with respect $\omega_< \in \Omega_<$. Suppose $\eta = 4F^{11/2+3\varepsilon+2\varepsilon'}$ and $z \in \nu_2$, for F small enough, by standard arguments of the regular perturbation theory and (4.27), the operator $h_2(z) = h_{2,1}(z, \eta) + S_2 - i\eta$ has a bounded inverse and

$$\|r_2(z)\| \leqslant \text{const}\left(\|r_{2,0}(z)\| + \left(\beta(1 + F^p)^2 + \eta\right)\|r_{2,0}(z)\|^2\right)$$

together with (4.27) implies (4.24). Finally since the event $\xi_2'(E, F)$ implies $\xi_2(E, F)$, the lemma is proven. □

We now give some spectral estimates for the operator $h_1(z)$. We have:

LEMMA 4.3. – *Let $E \in \Delta$ be a real energy, $0 < \varepsilon < 1$, then there exists $F_0 > 0$, such that for $\theta = i\beta$, $0 \leqslant \beta \leqslant F^{3/2+3\varepsilon/8}$, $0 < F < F_0$, $\nu_1 = \nu(E, F^\varepsilon, \beta/4) \subset \rho(h_1(z))$ and $\forall z \in \nu_1$,*

$$\|r_1(z)\| \leqslant \left(\operatorname{dist}(z, \nu_1^c) + \beta/4\right)^{-1}. \tag{4.36}$$

Notice that this result is deterministic, this is due to the fact that on $\mathbf{I}_1$ the system is weakly affected by the random potential. The proof of this statement follows easily from [5] since on $\mathbf{I}_1$ by (4.21), $\operatorname{Im}(f_\theta'^2(V(\theta) - E)) < -\beta/4 < 0$ for F small enough.

Let χ_a and χ_b be two bounded intervals contained in $\{x < x_s\}$, such that $\operatorname{dist}(\chi_a, \chi_b) > 0$ and ψ a $\mathbf{C}^\infty$ characteristic function of the set $\{x;\ \operatorname{dist}(x, \chi_a) \text{ and } \operatorname{dist}(x, \chi_b) \leqslant \operatorname{dist}(\chi_a, \chi_b)\}$.

LEMMA 4.4. – *With the same conditions as in Lemma 4.3, χ_a and χ_b being as above, then*

$$\|\mathbf{1}_{\chi_a} r_1(z) \mathbf{1}_{\chi_b}\| \leqslant c_1 \exp(-\mu_1 d(\chi_a, \chi_b))\|r_1(z)\| \tag{4.37}$$

for some constant, $c_1 > 0$ and $1 > \mu_1 > 0$, where $d(.,.)$ denotes the semi-distance on $\mathbf{I}_1$ associated with the metric, $\mathrm{d}s^2 = (\beta\psi/16)^2(E - Fx)^{-1}$.

The proof of this lemma is based on the usual boost technique (see, e.g., [6]), by noticing as before that by (4.21), on $\mathbf{I}_1, s = 1$ and then the imaginary part of the operator $h_1(z)$ has a definite sign for F small enough.

We can give now the

Proof of Theorem 4.1. – Suppose F small enough, let z with $\mathrm{Im}\, z > 0$, big enough, $|\mathrm{Re}\, z - E| \leqslant 2F^{3+2\varepsilon+\varepsilon'}$ and

$$h_d(z) = h_1(z) \oplus h_2(z), \qquad r_d(z) = r_1(z) \oplus r_2(z) \tag{4.38}$$

acting on $\mathcal{H}_d = \mathcal{H}_1 \oplus \mathcal{H}_2$. Recall that

$$\mathcal{C}_1 = \{u \in \mathbf{C}^\infty((\infty, E/F + x_1]),\ u(E/F + x_1) = 0\}$$

is a core for $h_1(z)$ and

$$\mathcal{C}_2 = \{u \in \mathbf{C}^\infty([E/F + x_2, x^{\mathrm{ext}}]),\ u(E/F + x_2) = u(x^{\mathrm{ext}}) = 0\}$$

is a core for $h_2(z)$. To compare operators $h(z)$ and $h_d(z)$, we use here, a slightly different perturbation theory than the one described in appendix. Hence let $U_k = \mathbf{R}^{\mathrm{ext}} \setminus \mathbf{I}_{\bar{k}},\ \bar{k} \neq k,\ k = 1, 2$, and define the functions $J_k \in \mathbf{C}^\infty(\mathbf{R})$ such that:

$$\begin{aligned} J_k(x) &= 0 \quad \text{if } \mathrm{dist}(x, U_{\bar{k}}) \leqslant \eta,\ k \neq \bar{k}, \\ J_k(x) &= 1 \quad \text{if } \mathrm{dist}(x, U_k) \geqslant 2/3\, \mathrm{dist}(U_1, U_2) \end{aligned} \tag{4.39}$$

here and in the sequel η denotes a generic, strictly positive F independent constant. Clearly $\|J_k'\|_\infty = \mathrm{O}(F^{2+\varepsilon})$. On the other hand we choose $\widetilde{J}_k,\ k = 1, 2$, as the characteristic functions of $\{x \in \mathbf{R},\ \mathrm{dist}(x, U_k) \leqslant \mathrm{dist}(U_1, U_2)/2\}$, we also denote by the same symbol the corresponding identification operators. For $k = 1, 2$ and (u_k, u) in a dense subset of $\mathcal{H}_k \times \mathcal{H}^{\mathrm{ext}}$,

$$\begin{aligned} &(r(z) J_k - J_k r_k(z) u_k, u) \\ &\quad = ((r_k(z) u_k)', J_k' r(\bar{z}) u) - (J_k' r_k(z) u_k, (r(\bar{z}) u)') \end{aligned} \tag{4.40}$$

this identity is valid in the sense of bounded operators from $\mathcal{H}_k$ to $\mathcal{H}$, if the map on $\mathcal{H}$, $u \to \mathbf{1}_{\mathrm{support}\, J_k'}(r(\bar{z})u)'$ and the map on $\mathcal{H}_k$, $u_k \to \mathbf{1}_{\mathrm{support}\, J_k'}(r_k(z)u_k)'$ are bounded operators respectively on $\mathcal{H}, \mathcal{H}_k$, these facts are considered below. The formula (4.39) leads to define $r(z)$ from the following GPF (see Appendix),

$$\begin{aligned} r(z) = J r_d(z) \widetilde{J}^\star &+ (J_1 r_1(z) \mathcal{F}_2 \sigma, J_2 r_2(z) \mathcal{F}_1 \sigma) \cdot \mathcal{A}_\theta(z) \\ &\times (\mathcal{F}_2^\star r_2(z) \widetilde{J}_2, \mathcal{F}_1^\star r_1(z) \widetilde{J}_1), \end{aligned} \tag{4.41}$$

where we have chosen $\mathcal{F}_k = \mathcal{F}_k(F^{(5+3\varepsilon)/2})$, (4.41) is valid in the sense of bounded operators on $\mathcal{H}$ provided that $\|K_1(z)\|\,\|K_2(z)\| < 1$ with $K_k(z) = \mathcal{F}_k^\star r_k(z)\mathcal{F}_{\bar{k}}\sigma_{\bar{k}}$ (see (A.10) and (A.11)). To justify (4.41) the following energy inequality is useful, obtained from a straightforward calculus, let $k = 1,2,\ l = 1,2,\ u \in \mathcal{H}_l,\ 0 < F < F_0,\ \operatorname{Im} z > 0$ big enough and $|\operatorname{Re} z - E| \leqslant 2F^{3+2\varepsilon+\varepsilon'}$, then

$$\|\mathcal{F}_k^\star r_l(z)u\|^2 \leqslant \text{const}\, F^{3/2+\varepsilon/2}\big(\operatorname{Re}(\zeta_k^2 r_l(z)u, u) + \kappa_\theta(z)\|\zeta_k r_l(z)u\|^2\big) \\ + \|\zeta_k' r_l(z)u\|^2\big) \tag{4.42}$$

valid uniformly on $\Omega_<$ with

$$\kappa_\theta(z) = 1 + \sup\{|F^{1+\varepsilon}\operatorname{Re}(f_\theta'^2(E-z))|,\ x \in \chi_k\},$$

the functions $\zeta_k \in \mathbf{C}^\infty(\mathbf{R}),\ k = 1,2$, are such that

$$\begin{aligned} \zeta_k(x) &= 0 \quad \text{if } \operatorname{dist}(x, \text{support } J_k') \geqslant \eta, \\ \zeta_k(x) &= 1 \quad \text{if } x \in \text{support } J_k', \end{aligned} \tag{4.43}$$

and we denote $\chi_k =$ support of ζ_k. On one hand (4.42) implies, in the norm operator sense,

$$\|\mathcal{F}_k^\star r_l(z)\|^2 \leqslant F^{(3+\varepsilon)/2}\text{const}\,(\|r_l(z)\| + \kappa_\theta(z)\|r_l(z)\|^2) \tag{4.44}$$

for another strictly positive constant, uniform with respect to $\omega_< \in \Omega_<$. In particular the map on $\mathcal{H},\ u \to \mathbf{1}_{\text{support } J_k'}(r(\bar{z})u)'$, and the map on $\mathcal{H}_k,\ u_k \to \mathbf{1}_{\text{support } J_k'}(r_k(z)u_k)'$, are bounded operators respectively on $\mathcal{H}, \mathcal{H}_k$ and these estimates extend to all $z \in \nu$. On the other hand, in the same conditions as for (4.43) we get from (4.42),

$$\|\mathcal{F}_k^\star r_k(z)\mathcal{F}_{\bar{k}}\sigma\| \leqslant F^{(3+\varepsilon)/2}\text{const}\,\kappa_\theta(z)\|\mathbf{1}_{\chi_k} r_k(z)\mathbf{1}_{\chi_{\bar{k}}}\| \tag{4.45}$$

for $\bar{k} \neq k$ and then by Lemma 4.3,

$$\|\mathcal{F}_1^\star r_1(z)\mathcal{F}_2\| \leqslant \text{const}\, F^{(3+\varepsilon)/2}\kappa_\theta(z)\exp\{-\tilde{\mu}_1 F^{-\varepsilon/8}\}\|r_1(z)\| \tag{4.46}$$

for some $0 < \tilde{\mu}_1 < \mu_1$ which is E and $\omega_<$ independent, consequently

$$\|K_1\|\|K_2\| \leqslant \text{const}\, F^{3+\varepsilon}\kappa_\theta(z)^2\exp\{-\tilde{\mu}_1 F^{-\varepsilon/8}\} \\ \times \|r_1(z)\|\|r_2(z)\|. \tag{4.47}$$

Suppose now that the configuration $\omega_<$ are such that $\xi_2(F)$ is realized and let $z \in \nu$. For F small enough, by Lemmas 4.1, 4.2 and (4.47), (4.41) defines $r(z)$ as a bounded operator on $\mathcal{H}^{\mathrm{ext}}$ so $\nu \subset \rho(h(z))$ (notice that for $z \in \nu$, $\kappa_\theta(z)\|r_k(z)\|$ are uniformly bounded). We prove (4.15), from (4.42), in fact, we get

$$\|\mathcal{F}_1^\star r_1(z)\widetilde{J}_1\| \leqslant \mathrm{const}\, F^{(3+\varepsilon)/4} \exp\{-\widetilde{\mu}_1 F^{-\varepsilon/8}\}\|r_1(z)\|, \quad (4.48)$$

$$\|\mathcal{F}_2^\star r_2(z)\widetilde{J}_2\| \leqslant \mathrm{const}\, F^{(3+\varepsilon)/4}\|r_2(z)\| \quad (4.49)$$

for some another $0 < \widetilde{\mu}_1 < \mu_1$ and where the constants are strictly positive and uniform with respect to $E \in \Delta$. On the other hand for $z \in \nu$, $\|r_1(z)\| \leqslant \mathrm{dist}^{-1}(z, \delta^c(E, F^{3+2\varepsilon+\varepsilon'}))$, then by using Lemma 4.1 and formulas (4.41), (4.45)–(4.49) we obtain (4.15). Finally since the event $\xi_2(E, F)$ implies $\xi'(E, F)$, Theorem 4.1 is proven. □

One consequence of Theorem 4.1 concerns the operator $H^{\mathrm{ext}}(\theta)$. Since by construction we have $H^{\mathrm{ext}}(\theta) - h^{\mathrm{ext}}(\theta) = \mathrm{O}(F^\alpha)$ for some $\alpha \geqslant 6$ (see [H2]), if we choose in the sequel $\alpha \geqslant 6 > 11/2 + 3\varepsilon + 2\varepsilon'$ $(3\varepsilon + 2\varepsilon' < 1/2)$, then:

THEOREM 4.2. – *Let $E \in \Delta$ be a real energy,* $0 < 4\varepsilon' < \varepsilon$, $3\varepsilon + 2\varepsilon' < 1/2$ *there exists some strictly positive constants, F_0 and C_{ext}, uniform with respect to $\omega \in \Omega$ and $E \in \Delta$, such that for $0 < F < F_0$, for all $\omega = (\omega_<, \omega_>) \in \Omega$, $\omega_< \in \xi(E, F)$, and $\theta = i\beta$, $\beta = F^{3/2+3\varepsilon/8}$,*

(i) $\nu = \nu(E, F^{3+2\varepsilon+\varepsilon'}, -F^{11/2+3\varepsilon+2\varepsilon'}) \subset \rho(H^{\mathrm{ext}}(\theta))$,

(ii)
$$\forall z \in \nu,\ \|R^{\mathrm{ext}}(\theta, z)\| \leqslant C_{\mathrm{ext}}(\mathrm{dist}^{-1}(z, \delta^c(E, F^{3+2\varepsilon+\varepsilon'}))) \quad (4.50)$$
$$+ \beta(1+F^p)^2 \mathrm{dist}^{-2}(z, \delta^c(E, F^{3+2\varepsilon+\varepsilon'})).$$

The probability of this last event is bounded below by $1 - \mathrm{O}(F^{\varepsilon'})$. The condition on the random potential V_ω, $\alpha \geqslant 6$, can be removed, if we follow the arguments of the proof of Theorem 4.1 with the operator $H^{\mathrm{ext}}(\theta)$ instead, but for further application in Section 5, we need to use this procedure which consists to construct first the event for $h^{\mathrm{ext}}(\theta)$ and then show that this event is stable under the perturbation $H^{\mathrm{ext}}(\theta)$.

We finish this section by giving technical inequalities, [12] needed to use the perturbation theory of Section 5. Suppose F small enough, let $\Lambda = \mathbf{R}^{\mathrm{ext}} \cap \mathbf{R}^{\mathrm{int}}$ and $\chi_{\mathrm{int}}, \chi_{\mathrm{ext}} \subset \Lambda$ be the two disjoint intervals defined in Section 2, recall that $\mathrm{dist}(\chi_{\mathrm{int}}, \chi_{\mathrm{ext}}) = \mathrm{const}/F$. Let $H_{\mathbf{I}_2}$ be the restriction of H on $\mathbf{I}_2$ and $\widetilde{\chi}_{\mathrm{ext}} = \{x \in \mathbf{R};\ \mathrm{dist}(x, \chi_{\mathrm{ext}}) \leqslant \eta\}$, we have, for F small enough, uniformly on Ω

$$\|\mathbf{1}_{\chi_{\mathrm{ext}}} R^{\mathrm{ext}}(\theta, F, z)\mathbf{1}_{\chi_{\mathrm{int}}}\| \leqslant \mathrm{const}\,\|\mathbf{1}_{\widetilde{\chi}_{\mathrm{ext}}} R_{\mathbf{I}_2}(z)\mathbf{1}_{\chi_{\mathrm{int}}}\|\|R^{\mathrm{ext}}(\theta, F, z)\|. \quad (4.51)$$

Both formulas (4.50) and (4.51) will be used together, this leads to suppose that there exists a spectral gap for $H_{\mathbf{I}_2}$ around E and according to the remark above we first assume the existence of a spectral gap around the real energy E for the operator $h_{\mathbf{I}_2}$, in fact this last event is implicit in the proof of Lemma 4.2 and Theorem 4.1, see (4.26). Notice also that for F small enough and all $\omega_< \in \xi(E, F)$ the existence of a spectral gap for $h_{\mathbf{I}_2}$ around E of size $O(F^{3+2\varepsilon+\varepsilon'})$ implies a spectral gap for $H_{\mathbf{I}_2}$ around E with a size $O(F^{3+2\varepsilon+\varepsilon'})$. Hence, we define the exterior event $\xi_{\text{ext}}(E, F)$ as it will be used in Section 5, ε, ε' being as above.

The potential configuration belongs to $\xi_{\text{ext}}(E, F)$ *if*:

(i) *the spectral deformation of* $h^{\text{ext}}_{\omega_<}(F)$, $h^{\text{ext}}_{\omega_<}(F)(\theta)$, *in the sense defined above is such that there exists some constants* F_0 *small enough and* C_1, C_2, C_3 *uniform with respect to* $\omega_< \in \Omega_<$ *and the energy* E *such that for* $0 < F < F_0$, $\theta = i\beta$, $\beta = F^{3/2+3\varepsilon/8}$, $p = -1/2 + 3\varepsilon/8$, $0 < 4\varepsilon < \varepsilon'$, $3\varepsilon + 2\varepsilon' < 1/2$,

$$\nu(E, C_1, C_2) := \nu\big(E, C_1 F^{3+2\varepsilon+\varepsilon'}, -C_2 F^{11/2+3\varepsilon+2\varepsilon'}\big) \subset \rho\big(h^{\text{ext}}(\theta)\big),$$

$$\begin{aligned}\forall z \in \nu,\ \big\|r^{\text{ext}}(\theta, z)\big\| \leqslant{}& C_3\big(\text{dist}^{-1}\big(z, \delta^c\big(E, C_1 F^{3+2\varepsilon+\varepsilon'}\big)\big)\big)\\ &+ \beta\big(1 + F^p\big)^2 \text{dist}^{-2}\big(z, \delta^c\big(E, C_1 F^{3+2\varepsilon+\varepsilon'}\big)\big);\end{aligned}$$

(ii) *the restriction of* h_{ext} *on the interval* $\mathbf{I}_2$, $h_{\mathbf{I}_2}$ *has a spectral gap around* E, *with size* $O(F^{3+2\varepsilon+\varepsilon'})$

and for F small enough, we have the estimate $\mathbf{P}_<(\xi_{\text{ext}}(E, F)) \geqslant 1 - O(F^{\varepsilon'})$.

5. SPECTRAL STABILITY AND EXISTENCE OF RESONANCES

In this section, we suppose that the assumptions of Theorem 1.1 are satisfied. One of the main technical problems here, is the dependance of the random operators $H^{\text{int}} = H^{\text{int}}(F)$ and $H^{\text{ext}} = H^{\text{ext}}(F)$ through the potential. Given a small interval around an energy $E \in \Delta$, if we choose a potential configuration such that H^{int} has an eigenvalue in this interval and the associated eigenstate is well localized, this configuration has to be such that there exists a complex deformation, $H^{\text{ext}}(\theta)$ which contains in its resolvent set a complex neighbourhood of the interval. Such a choice will allow to show that this eigenvalue which is also an eigenvalue of

$H^{\rm int} \oplus H^{\rm ext}(\theta)$ turns into a resonance when going to $H = H_\omega(F)$. In the sequel we will denote by Γ the set of these good configurations.

In the discussion below, we suppose F small. To construct Γ, we divide the probability space in the following way. Let $x_r^{\rm int} = c_r^{\rm int}/F \in \mathbf{R}^{\rm int}$ be a point at the right of $x^{\rm ext}$: $c^{\rm ext} < c_r^{\rm int} < c_{\rm max}$. We denote $\mathbf{R}_r^{\rm int}$ the interior region $(x_r^{\rm int}, \infty)$. Let x_M be the middle point between $x^{\rm ext}$ and $x_r^{\rm int}$ and $(\Omega_<, \tau_<, \mathbf{P}_<)$, $(\Omega_>, \tau_>, \mathbf{P}_>)$, the probability space corresponding respectively to the sites $i < x_M$, $i \geqslant x_M$, so

$$(\Omega, \tau, \mathbf{P}) = (\Omega_<, \tau_<, \mathbf{P}_<) \otimes (\Omega_>, \tau_>, \mathbf{P}_>). \tag{5.1}$$

To the interior region $\mathbf{R}^{\rm int}$, we associate the interior operator,

$$h_{\omega_>}^{\rm int} = h_{\omega_>}^{\rm int}(F) = H^{\rm int} - \sum_{i<x_M} \omega_i u_i(x-i) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v_{\omega_>} + Fx \tag{5.2}$$

acting on $L^2(\mathbf{R}_r^{\rm int})$ with DBC at $x = x_r^{\rm int}$. Due to the large separation between $\mathbf{R}_r^{\rm int}$ and $\mathbf{R}^{\rm ext}$ the random operators $h_{\omega_>}^{\rm int}$ and the one defined in Section 4, $h_{\omega_<}^{\rm ext} = h_{\omega_<}^{\rm ext}(F)$ are weakly dependant. Our strategy consist first to construct the set of good configurations, denoted by Γ^d, for $h^d = h_{\omega_<}^{\rm ext}(\theta) \oplus h_{\omega_>}^{\rm int}$. This operator plays here, a central role, then modulo a set of small probability, we will show that Γ^d is stable under the perturbation $H^{\rm ext}(\theta) \oplus H^{\rm int}$.

Let $\Gamma_>$ be the set of configurations $\omega_> \in \Omega_>$ for which the operator $h_{\omega_>}^{\rm int}$ has const$/F$ eigenvalues in the energy interval Δ, and whose at least one of them is associated to a well localized state with respect to the point $x_r^{\rm int}$ (see Section 3). This event has a probability measure satisfying $\mathbf{P}(\Gamma_>) \geqslant 1 - \mathrm{O}(F^{2\xi-1})$, for some $1/2 < \xi < 1$. On the other hand for technical reasons we also have to consider together the eigenvalues which are not well separated. Hence let $\eta > 0$ and define a cluster $Cl \subset \sigma(h_{\omega_>}^{\rm int})$ as:

$$\lambda \in Cl \quad \text{iff} \quad \operatorname{dist}\big(\lambda, Cl \setminus \{\lambda\}\big) < \exp(-\eta/F) \tag{5.3}$$

and denote Cl^+ the cluster which contains at least one eigenvalue whose associated eigenstate is well localized with respect to $x_r^{\rm int}$. For each $\omega_< \in \Gamma_<$ there exists a cluster Cl^+ of eigenvalues of $h_{\omega_>}^{\rm int}$ such that $Cl^+ \cap \Delta \neq \emptyset$, eventually this cluster contains only one eigenvalue. Notice that for F small, $|Cl^+| < (n_{\rm max})/F \exp(-\eta/F)$, uniformly on $\Gamma_<$. Let $\Gamma_>^n$, $n \in 0 \ldots N(F)$, for some finite integer $N(F)$, a disjoint partition of $\Gamma_>$,

$$\Gamma_> = \bigcup_{n=0}^{N(F)} \Gamma_>^n \tag{5.4}$$

where the sets $\Gamma_>^n$ are inductively defined in the following way. Let $E \in \Delta$, and $x^* = \exp(\eta^*/F)$, where $\eta^* > \eta$. Notice that the interval $[x^*, +\infty)$ is contained in the classically forbidden region,

$$CF = \{x \in \mathbf{R} \text{ s.t. } Fx + v_{\omega_>} > 0\} \tag{5.5}$$

for all $\omega_> \in \Omega_>$. Now, fix $\omega_>^0 \in \Gamma_>$ and let Cl_0^+ be a cluster of eigenvalues for $h_{\omega_>^0}^{\text{int}}$ such that $Cl_0^+ \cap \Delta \neq \emptyset$ and define,

$$\Gamma_>^0 = \{\omega_> \in \Gamma_>;\ |\omega_> - \omega_>^0|_* < F^2 \exp(-\eta/F)\}, \tag{5.6}$$

where we use the notation, $|\omega_> - \omega_>^0|_* = \sup_{x_M < i < x^*} |\omega_i - \omega_i^0|$. Then we have:

LEMMA 5.1. – *There exits F_0 small enough, uniform with respect to $\omega_> \in \Omega_>$, and $\forall F$, $0 < F < F_0$, there exists a real neighbourhood of Cl_0^+, $\mathcal{N}_0$ with $|\mathcal{N}_0| \leqslant 2(n_{\max}/F)\exp(-\eta/F)$ such that for all $\omega_> \in \Gamma_>^0$ the operator $h_{\omega_>}^{\text{int}}$ has a cluster Cl^+ of eigenvalues in $\mathcal{N}_0$.*

Proof. – Let $\omega_> \in \Gamma_>^0$, writing

$$v_{\omega_>} = v_{\omega_>^0} + v_* + v_*' \tag{5.7}$$

where

$$v_*' = \mathbf{1}_{[x_*, +\infty)} \sum_{i > x^*} (\omega_i - \omega_i^0) u_i(x - i)$$

and

$$x_* = 2V_M/F, \quad V_M = \omega_M \sup_{x \in \mathbf{R}} \sum_i |u_i(x - i)|.$$

The perturbation v_*' is uniformly supported in the classically forbidden region CF, consequently by the general results on the spectral stability see, e.g., [6,18], for F small enough, there exists a cluster Cl^+ of eigenvalues of $h_* = h_{\omega_>^0}^{\text{int}} + v_*'$ satisfying the statement of the lemma. On the other hand, by noticing that for $\omega_> \in \Gamma_>^0$, we have by construction of the random potential $v_{\omega_>}$, $|v_*| = O(F^2 \exp(-2\eta/F))$ uniformly on $\Omega_>$.

This due to (5.6) and the fact that the sites $i > x^*$ have an exponentially small contribution to the potential v_*. Then for F small enough, by standard arguments of the regular perturbation theory [27], the statement also follows for the operators $h^{\text{int}}_{\omega_>} = h_* + v_*$ and this proves the lemma. Then for some $\omega^1_> \in \Gamma_> \setminus \Gamma^0_>$, let

$$\Gamma^1_> = \{\omega_> \in (\Gamma_> \setminus \Gamma^0_>);\ |\omega_> - \omega^1_>|_* < F^2 \exp(-\eta/F)\}$$

and successively for some $\omega^n_> \in (\Gamma_> \setminus \bigcup_{k=0}^{n-1} \Gamma^k_>)$, we construct the set

$$\Gamma^n_> = \left\{\omega_> \in \left(\Gamma_> \setminus \bigcup_{k=0}^{n-1} \Gamma^k_>\right);\ |\omega_> - \omega^n_>|_* < F^2 \exp(-\eta/F)\right\}. \tag{5.8}$$

So we obtain a disjoint partition of $\Gamma_>$, $\bigcup_{k=0}^{N} \Gamma^k_> = \Gamma_>$ for some $N = N(F)$. For each $k = 0 \ldots N(F)$, it exists an interval $\mathcal{N}_k$ of size $|\mathcal{N}_k| \leqslant (2n_{\max}/F) \exp(-\eta/F)$ around a cluster Cl^+_k of eigenvalues of $h^{\text{int}}_{\omega^k_>}$, such that $\mathcal{N}_k$ contains a cluster Cl^+ of eigenvalues of the operator $h^{\text{int}}_{\omega_>}$, $\omega_> \in \Gamma^k_>$. From these uniform spectral estimates for the interior part on each $\Gamma^k_>$, we now complete these right configurations by choosing the left configurations $\omega_< \in \Gamma^k_< \subset \Omega_<$ for which the exterior operator $h^{\text{ext}}_{\omega_<}$ has the desired property.

In the cluster Cl^+_k of eigenvalues of $h^{\text{int}}_{\omega^k_>}$ we choose λ_k. For $\omega_< \in \Omega_<$, consider the complexified operator $h^{\text{ext}}_{\omega_<}(\theta)$; $|\theta|$ small enough, obtained through the spectral deformation defined at $E = \lambda_k$ in the sense of Section 4, recall that for $|\theta| < c_\theta F,\ 0 < F < F_0,\ F_0$ small enough, $h^{\text{ext}}_{\omega_<}(\theta)$ is a type A analytic family of operators on $\mathcal{H}^{\text{ext}}$. Let $\Gamma^k_< \subset \Omega_<$ be the event $\xi_{\text{ext}}(\lambda_k, F)$, we have $\mathbf{P}_>(\Gamma^k_<) = 1 - \mathrm{O}(F^{\varepsilon'})$ for some $\varepsilon' > 0$. In particular for $\omega_< \in \Omega_<$, $\operatorname{Im}\theta = F^{3/2+3\varepsilon/8}$, the resolvent set of $h^{\text{ext}}_{\omega_<}(\theta)$ contains the set $\nu(\lambda_k, 1, 1) = \nu(\lambda_k, F^{3+2\varepsilon+\varepsilon'}, -F^{11/2+3\varepsilon+2\varepsilon'})$ which clearly contains $\mathcal{N}_k$ for F small enough. Define now the disjoint sets $\Gamma^k = \Gamma^k_< \times \Gamma^k_>$ and

$$\Gamma^d = \bigcup_{k=0}^{N} \Gamma^k \tag{5.9}$$

which is a measurable set, we have,

$$\begin{aligned}\mathbf{P}(\Gamma^d) &= \sum_{k=0}^{N} \mathbf{P}(\Gamma^k) \geqslant (1 - \mathrm{O}(F^{\varepsilon'}))(1 - \mathrm{O}(F^{2\xi-1})) \\ &= 1 - \mathrm{O}(F^{\varepsilon'}).\end{aligned} \tag{5.10}$$

with a suitable choice of the constant ε'. We summarize this discussion by the following property denoted by $\mathcal{P}(F)$ for the operator $h^{\rm ext}_{\omega_<}(\theta) \oplus h^{\rm int}_{\omega_>}$, $\omega \in \Gamma^d$:

There exists an uniform F_0 small enough, such that for $0 < F < F_0$, there exists some finite constants $C_1, C_2, C_3, C_4 > 0$, uniform with respect to F, the energy $E \in \Delta$ and all $\omega \in \Gamma^d$, such that,

(i) *the event $\xi_{\rm ext}(\lambda, F)$ is realized for some $\lambda \in \Delta$ and for the constants, C_1, C_2, C_3,*

(ii) *there exists real neighbourhood $\mathcal{N}(\lambda)$ of λ, with $|\mathcal{N}(\lambda)| \leqslant C_4$, $\mathcal{N}(\lambda) \subset \nu(\lambda, C_1, C_2)$ and $\mathcal{N}(\lambda)$ contains a cluster Cl^+ of eigenvalues of the operator $h^{\rm int}_{\omega_>}$.*

In this first step $C_1 = C_2 = 2$, $C_3 = C_{\rm ext}$ and $C_4 = 2(n_{\max}/F)\exp(-\frac{\eta}{F})$.

We want to show that the property $\mathcal{P}(F)$ is stable under the perturbation $H^{\rm ext}(\theta) \oplus H^{\rm int}_r$, where $H^{\rm int}_r$ is the restriction of $H^{\rm int}$ on $\mathbf{R}^{\rm int}_r$, i.e.,

$$H^{\rm int}_r = h^{\rm int}_{\omega_>} + \sum_{i<x_M} \omega_i u_i(x-i) \tag{5.11}$$

acting on $L^2(\mathbf{R}^{\rm int}_r)$ with DBC at $x = x^{\rm int}_r$. This is done by using again the regular perturbation theory and the estimates, obtained from [H2],

$$\|H^{\rm int}_r - h^{\rm int}_{\omega_>}\| \leqslant \sup_{x\in\mathbf{R}^{\rm int}_r} \left|\sum_{i<x_M} \omega_i u_i(x-i)\right| = \mathrm{O}(F^\alpha), \tag{5.12}$$

$$\|H^{\rm ext}(\theta) - h^{\rm ext}_{\omega_<}(\theta)\| \leqslant \sup_{x\in\mathbf{R}^{\rm ext}} \left|\sum_{i\geqslant x_M} \omega_i u_i(x-i)\right| = \mathrm{O}(F^\alpha) \tag{5.13}$$

which are uniform with respect to $\omega \in \Omega$. Hence, let $\omega \in \Gamma^k$ for some $k = 1\ldots N(F)$, due to our choice, the operator $h^{\rm int}_{\omega^k_>}$ has only const$/F$ eigenvalues in all bounded real interval. Then there exist some gap $g_{r,-}(g_{r,+})$, respectively in

$$\sigma\left(h^{\rm int}_{\omega_>}\right) \cap \nu\left(\lambda_k, F^{3+2\varepsilon+\varepsilon'}, -F^{11/2+3\varepsilon+2\varepsilon'}\right) \cap \{e < \lambda_k\} \\ \left(\{e \geqslant \lambda_k\}\right), \tag{5.14}$$

satisfying, $|g_{r,-}|, |g_{r,+}| = \mathrm{O}(F^{5+2\varepsilon+2\varepsilon'})$ and

$$\mathrm{dist}\left(g_{r,-}(g_{r,+}), \nu^c\right) \geqslant 1/2F^{4+2\varepsilon+2\varepsilon'}$$

uniformly with respect to $\omega_< \in \Omega_<$. For F small enough, then by (5.12) $g_{r,-}(g_{r,+})$ are stable when going to H_r^{int} in the sense that there exists some gaps for H_r^{int}, $G_{r,-} \subset g_{r,-}$ and $G_{r,+} \subset g_{r,+}$, satisfying $|G_{r,-}|, |G_{r,+}| = O(F^{5+2\varepsilon+2\varepsilon'})$ uniformly on Γ. We then define the spectral projectors associated respectively to the operators h^{int}, H_r^{int}, by

$$p^{\text{int}} = -(2i\pi)^{-1} \oint_{\mathcal{C}} \left(h_{\omega_>}^{\text{int}} - z\right)^{-1} \mathrm{d}z,$$

$$P_r^{\text{int}} = -(2i\pi)^{-1} \oint_{\mathcal{C}} \left(H_r^{\text{int}} - z\right)^{-1} \mathrm{d}z \tag{5.15}$$

for some suitable contour $\mathcal{C}$, we have,

$$\left\| P_r^{\text{int}} - p^{\text{int}} \right\| = O\left(F^{\alpha-(5+2\varepsilon+2\varepsilon')}\right) \tag{5.16}$$

uniformly on Ω. Notice that here, our assumptions and [H2] imply $\alpha - (5 + 2\varepsilon + 2\varepsilon') > 0$. These considerations together with Theorem 4.2 proves

LEMMA 5.2. – *Let* $1 > \varepsilon, \varepsilon' > 0$, $\varepsilon > 4\varepsilon'$ *and* $3\varepsilon + 2\varepsilon' < 1/2$. *There exists* F_0, *such that for* $0 < F < F_0$, $\theta = i\beta$, $\beta = F^{3/2+3\varepsilon/8}$, $\omega \in \Gamma^d$, *the property* $\mathcal{P}(F)$ *is true for* $H^{\text{ext}}(\theta) \oplus H_r^{\text{int}}$. *In this case for each* $\omega \in \Gamma^d$, $|\mathcal{N}(\lambda)| = O(F^{4+2\varepsilon+2\varepsilon'})$ *and satisfy the uniform estimate on* Γ^d,

$$\begin{aligned} &\operatorname{dist}\left(\mathcal{N}(\lambda), \sigma\left(H_r^{\text{int}}\right) \cap \mathcal{N}^c(\lambda) \cap \nu\left(\lambda, F^{3+2\varepsilon+\varepsilon'}, -F^{11/2+3\varepsilon+2\varepsilon'}\right)\right) \\ &\quad \geqslant \text{const}\, F^{5+2\varepsilon+2\varepsilon'} \quad \textit{and} \\ &\operatorname{dist}\left(\mathcal{N}(\lambda), \nu^c\left(\lambda, F^{3+2\varepsilon+\varepsilon'}, -F^{11/2+3\varepsilon+2\varepsilon'}\right)\right) \geqslant 1/2 F^{4+2\varepsilon+2\varepsilon'}, \end{aligned} \tag{5.17}$$

here A^c *denoting the complement of the set* A.

The third step of our method consists in showing that $\mathcal{P}(F)$ is true for $H^{\text{ext}}(\theta) \oplus H^{\text{int}}$ on some set $\Gamma' \subset \Gamma^d$ with a probability $\mathbf{P}(\Gamma') \geqslant 1 - O(F^{\varepsilon'})$, for F small enough. Let $\tilde{x}_r^{\text{int}} = \tilde{c}_r^{\text{int}}/F$, $x_l^{\text{int}} = c_l^{\text{int}}/F \in \mathbf{R}^{\text{int}}$ such that, $c_r^{\text{int}} < \tilde{c}_r^{\text{int}} < c_l^{\text{int}} < c_{\max}$ where $\tilde{x}_r^{\text{int}}$ corresponds in Theorem 3.2 to $\tilde{x}_a$ while x_r^{int} corresponds to x_a and in the other hand $\operatorname{dist}(x_r^{\text{int}}, x_l^{\text{int}}) = 2\operatorname{dist}(x_r^{\text{int}}, \tilde{x}_r^{\text{int}})$.

Let H_l^{int} be the restriction of the operator $H_\omega(F)$ on $\mathbf{R}_l^{\text{int}} = (0 = x^{\text{int}}, x_l^{\text{int}})$. By Theorem 3.2 at least one eigenstate of H_r^{int} is localized at the right of $\tilde{x}_r^{\text{int}}$. By using the geometric perturbation theory, described in Appendix, we compare the operator H^{int} with $H_l^{\text{int}} \oplus H_r^{\text{int}}$ and if $J, \tilde{J}$ denote the identification operators defined according to the decomposition, $\mathbf{R}^{\rm int} = \mathbf{R}_l^{\rm int} \cup \mathbf{R}_r^{\rm int}$, we have for F small enough, $z \in \rho(H_l^{\rm int}) \cup \rho(H_r^{\rm int})$ and $\|K_r^{\rm int}(z)\|\|K_l^{\rm int}(z)\| < 1$,

$$R^{\rm int}(z) = J R^d(z)\widetilde{J}^\star + \left(J_l^{\rm int} R_l^{\rm int}(z)\mathcal{F}_r^{\rm int}\sigma,\ J_r^{\rm int} R_r^{\rm int}(z)\mathcal{F}_l^{\rm int}\sigma\right)\cdot \mathcal{A}(z) \\ \times \left((\mathcal{F}_r^{\rm int})^\star R_r^{\rm int}(z)\widetilde{J}_r^{\rm int},\ (\mathcal{F}_l^{\rm int})^\star R_l^{\rm int}(z)\widetilde{J}_l^{\rm int}\right). \tag{5.18}$$

In (5.18), $J_l^{\rm int}$, $J_r^{\rm int}$, $\widetilde{J}_l^{\rm int}$, $\widetilde{J}_r^{\rm int}$ denote the cut-off functions and $R^d(z) = R_l^{\rm int}(z) \oplus R_r^{\rm int}(z)$. Formula (5.18) defined $R^{\rm int}(z)$ as bounded operator on $\mathcal{H}^{\rm int}$ which implies that $z \in \rho(H^{\rm int})$.

LEMMA 5.3. – *Let $1 > \varepsilon, \varepsilon' > 0$, $\varepsilon > 4\varepsilon'$ and $3\varepsilon + 2\varepsilon' < 1/2$. There exits F_0, such that for $0 < F < F_0$, $\theta = i\beta$, $\beta = F^{3/2+3\varepsilon/8}$, the property $\mathcal{P}(F)$ is true for the operator $H^{\rm ext}(\theta) \oplus H^{\rm int}$ on a set $\Gamma' \subset \Gamma$ and*

$$\mathbf{P}(\Gamma') \geqslant 1 - \mathrm{O}(F^{\varepsilon'}). \tag{5.19}$$

For each $\omega \in \Gamma'$, $|\mathcal{N}(\lambda)| = \mathrm{O}(F^{3+2\varepsilon+\varepsilon'})$ and verifies the following uniform estimates on Γ',

$$\mathrm{dist}\left(\mathcal{N}(\lambda), \sigma\left(H^{\rm int}\right) \cap \mathcal{N}(\lambda)^c \cap \nu\left(\lambda, F^{3+2\varepsilon+\varepsilon'}, -F^{11/2+3\varepsilon+2\varepsilon'}\right)\right) \\ \geqslant \text{const}\ F^{6+2\varepsilon+2\varepsilon'} \quad \textit{and} \\ \mathrm{dist}\left(\mathcal{N}(\lambda), \nu^c\left(\lambda, F^{4+2\varepsilon+\varepsilon'}, -F^{11/2+3\varepsilon+2\varepsilon'}\right)\right) \geqslant 1/2 F^{4+2\varepsilon+2\varepsilon'}. \tag{5.20}$$

Proof. – By using (A.16) we have, for F small enough,

$$\left\|K_l^{\rm int}(z)\right\| \leqslant \text{const}\left\|\mathbf{1}_{\chi_l^{\rm int}} R_l^{\rm int}(z)\mathbf{1}_{\chi_r^{\rm int}}\right\| \quad \text{and} \\ \left\|K_r^{\rm int}(z)\right\| \leqslant \text{const}\left\|R_r^{\rm int}(z)\right\|. \tag{5.21}$$

uniformly on Γ, by our general condition C, the event,

$$\left\|K_l^{\rm int}(z)\right\| \leqslant \text{const}\ \exp(-\gamma/F)\left\|R_l^{\rm int}(z)\right\|^p, \tag{5.22}$$

$\mathrm{Re}\, z \in \Delta$ and for some $\gamma > 0$, is realized with a probability satisfying, $\mathbf{P} \geqslant 1 - \mathrm{O}(F^\nu)$. So, let Γ' be the set of $\omega \in \Gamma^d$ for which (5.22) is satisfied, clearly,

$$\mathbf{P}(\Gamma') \geqslant 1 - \mathrm{O}(F^{\varepsilon'}) - \mathrm{O}(F^\nu) \geqslant 1 - \mathrm{O}(F^{\varepsilon'}) \tag{5.23}$$

for suitable choice of ε'. Then for such configuration, by (5.18), if $z \in \rho(H_l^{\rm int}) \cup \rho(H_r^{\rm int})$ and

$$\mathrm{dist}\left(z, \sigma\left(H_l^{\rm int}\right)\right)\mathrm{dist}\left(z, \sigma\left(H_r^{\rm int}\right)\right) \geqslant \text{const}\ \exp(-\gamma/F), \tag{5.24}$$

$z \in \rho(H^{\text{int}})$. For $\omega \in \Gamma'$, let $\lambda \in \Delta$ and $\mathcal{N}(\lambda)$ as given in Lemma 5.2, recall that on both sides of $\mathcal{N}(\lambda)$, there exist some gaps, $G_{r,-}$, $G_{r,+}$ in the spectrum of H_r^{int} satisfying $|G_{r,-}|, |G_{r,+}| = \mathrm{O}(F^{5+2\varepsilon+2\varepsilon'})$ uniformly on Γ'. On the other hand, since the operator H_l^{int} has only a finite number on eigenvalues in all real bounded interval, there exists some gaps $G_-^d \subset G_{r,-}$, $G_+^d \subset G_{r,+}$ for the operator $H^d = H_l^{\text{int}} \oplus H_r^{\text{int}}$ verifying, $G_-^d, G_+^d = \mathrm{O}(F^{6+2\varepsilon+2\varepsilon'})$ uniformly on Γ' and by (5.24) this also holds for the operator H^{int}. We denote by G_-, G_+ the gaps of H^{int} and again by $\mathcal{N}(\lambda)$, the spectral interval of H^{int} between G_-, G_+. The lemma is proven if we show that $\mathcal{N}(\lambda)$ contains a cluster Cl^+ of eigenvalues for H^{int}. By construction of the function $\widetilde{J}_l^{\text{int}}$ and the spectral interval $\mathcal{N}(\lambda)$, there exists $u \in \mathcal{H}^{\text{int}}$, $\|u\| = 1$, $\widetilde{J}_r^{\text{int}} u = u$ $(\widetilde{J}_l^{\text{int}} u = 0)$ and $\|P_r^{\text{int}} u\| \geqslant$ const > 0 where P_r^{int} denotes the spectral projector on the interval $\mathcal{N}(\lambda)$ associated to the operator H_r^{int}. Then if $\mathcal{C}$ denotes a suitable contour in $\mathbf{C}$, by (5.18) for F small enough, we get,

$$\|P^{\text{int}} u\|^2 \geqslant \|P_r^{\text{int}} u\|^2 - \text{const} \oint_{\mathcal{C}} \|(\mathcal{F}_l^{\text{int}})^* R_r^{\text{int}}(z) u\| \, \|K_l^{\text{int}}(z)\| \, \|\mathcal{F}_r^{\text{int}} R_r^{\text{int}}(z) u\| |\mathrm{d}z|. \tag{5.25}$$

By estimates (A.15), (5.22) and $|\mathcal{C}| = \mathrm{O}(F^{3+2\varepsilon+\varepsilon'})$ we have for F small enough,

$$\oint_{\mathcal{C}} \|(\mathcal{F}_l^{\text{int}})^* R_r^{\text{int}}(z) u\| \, \|K_l^{\text{int}}(z)\| \, \|\mathcal{F}_r^{\text{int}} R_r^{\text{int}}(z) u\| \, \||\mathrm{d}z| \leqslant \text{const}\, F^{-(7+3\varepsilon+\varepsilon')} \mathrm{e}^{-\gamma/F} \tag{5.26}$$

which proves that $\|P^{\text{int}} u\| \geqslant$ const > 0. We prove now one of the main results of this section (see Section 3 for the definition of $n_{\max}$):

THEOREM 5.1 (on the spectral stability). – *Suppose that the assumptions of Theorem* 1.1 *are satisfied. Let* $\Delta = [-E_0, 0]$, $1 > \varepsilon, \varepsilon' > 0$, $3\varepsilon + 2\varepsilon' < 1/2$ *and* $\varepsilon > 4\varepsilon'$. *There exits* F_0, *so that if* $0 < F < F_0$, *there exists* $\Gamma \subset \Omega$ *with a probability measure satisfying*

$$\mathbf{P}(\Gamma) \geqslant 1 - \mathrm{O}(F^{\varepsilon'}) \tag{5.27}$$

and for all $\omega \in \Gamma$ *there exists* $\lambda \in \Delta$ *such that for* $\theta = i\beta$, $\beta = F^{3/2+3\varepsilon/8}$ *the complexified operator* $H(\theta)$, *has at least one and at most* $n_{\max}/F$ *eigenvalues in a complex neighbourhood of* λ. *On the other hand if* z_l, $l \in 1 \ldots n_{\max}/F$, *denotes such an eigenvalue, then,*

$$|\mathrm{Im}\, z_l| \leqslant c_l \exp(-\tau/F). \tag{5.28}$$

for some uniform constants $\tau > 0$ and $c_l > 0$.

Proof. – Let J, $\widetilde{J}$ be the identification operators defined according to the decomposition $\mathbf{R} = \mathbf{R}^{\mathrm{int}} \cup \mathbf{R}^{\mathrm{ext}}$. Let $0 < F < F_0$, $\theta = i\beta$, $\beta = F^{3/2+3\varepsilon/8}$ and choose first $\omega \in \Omega$. For $z \in \rho(H^{\mathrm{ext}}(\theta)) \cap \rho(H^{\mathrm{int}})$ and $\|K^{\mathrm{int}}(z)\|$, $\|K^{\mathrm{ext}}(\theta, z)\| < 1$ we have (see A.11)

$$\begin{aligned} R(\theta, z) = {} & J R^d(\theta, z)\widetilde{J}^\star + \left(J^{\mathrm{int}} R^{\mathrm{int}}(z)\mathcal{F}^{\mathrm{ext}}\sigma,\ J^{\mathrm{ext}} R^{\mathrm{ext}}(\theta, z)\mathcal{F}^{\mathrm{int}}\sigma\right) \cdot \mathcal{A}(\theta, z) \\ & \times \left((\mathcal{F}^{\mathrm{ext}})^\star R^{\mathrm{ext}}(\theta, z)\widetilde{J}^{\mathrm{ext}},\ (\mathcal{F}^{\mathrm{int}})^\star R^{\mathrm{int}}(z)\widetilde{J}^{\mathrm{int}}\right) \end{aligned} \tag{5.29}$$

where $R^d(\theta, z)$ denotes the resolvent of the operator

$$H^d(\theta, z) = H^{\mathrm{ext}}(\theta, z) \oplus H^{\mathrm{int}}(z),$$

formula (5.29) holds in a bounded operator sense on $\mathcal{H}$. By the estimates of the appendix, we have,

$$\begin{aligned} \|K^{\mathrm{int}}(z)\| &\leqslant \kappa(z)\|\mathbf{1}_{\chi_{\mathrm{int}}} R^{\mathrm{int}}(z)\mathbf{1}_{\chi_{\mathrm{ext}}}\|, \\ \|K^{\mathrm{ext}}(\theta, z)\| &\leqslant \kappa_\theta(z)\|\mathbf{1}_{\chi_{\mathrm{ext}}} R^{\mathrm{ext}}(\theta, z)\mathbf{1}_{\chi_{\mathrm{int}}}\|, \end{aligned} \tag{5.30}$$

the second term of the r.h.s. of (5.30) is estimated by (4.51) as

$$\|K^{\mathrm{ext}}(z)\| \leqslant \mathrm{const}\, \|\mathbf{1}_{\widetilde{\chi}_{\mathrm{ext}}} R_{\mathbf{I}_2}(z)\mathbf{1}_{\chi_{\mathrm{int}}}\|\, \|R^{\mathrm{ext}}(z)\| \tag{5.31}$$

and the arguments of the proof of Lemma 4.5 applied to the operator $\mathbf{1}_{\chi_{\mathrm{int}}} R^{\mathrm{int}}(z)\mathbf{1}_{\chi_{\mathrm{ext}}}$ instead give,

$$\|K^{\mathrm{int}}(z)\| \leqslant \mathrm{const}\, \|\mathbf{1}_{\widetilde{\chi}_{\mathrm{int}}} R_l^{\mathrm{int}}(z)\mathbf{1}_{\chi_{\mathrm{ext}}}\|\, \|R^{\mathrm{int}}(z)\| \tag{5.32}$$

uniformly on Ω, where $\widetilde{\chi}_{\mathrm{int}}$ being defined in the same way as $\widetilde{\chi}_{\mathrm{ext}}$. Notice that by construction we have good spectral information on $H_{\mathbf{I}_2}$ and H_l^{int} near the energy $\lambda \in \Delta$ and then some estimate on their resolvent. For $\mathrm{Re}\, z \in \Delta$, we consider the following events,

$$\|\mathbf{1}_{\widetilde{\chi}_{\mathrm{int}}} R_l^{\mathrm{int}}(z)\mathbf{1}_{\chi_{\mathrm{ext}}}\| \leqslant \mathrm{const}\, \exp(-\gamma/F)\|R_l^{\mathrm{int}}(z)\|^p \tag{5.33}$$

and

$$\|\mathbf{1}_{\widetilde{\chi}_{\mathrm{ext}}} R_{\mathbf{I}_2}(z)\mathbf{1}_{\chi_{\mathrm{int}}}\| \leqslant \mathrm{const}\, \exp(-\gamma/F)\|R_{\mathbf{I}_2}(z)\|^p \tag{5.34}$$

for some uniform constants $\gamma > 0$, by the condition C, this event has a probability $\mathbf{P} \geqslant 1 - \mathrm{O}(F^{\nu})$). Define the set $\Gamma'' = \{\omega \in \Gamma'$ s.t. formulas (5.32) and (5.33) hold$\}$, then

$$\mathbf{P}(\Gamma'') \geqslant 1 - \mathrm{O}(F^{\varepsilon'}). \tag{5.35}$$

For such configurations and for F small enough, by construction, there exists $\lambda \in \Delta$ such that the complexified operator $H^{\mathrm{ext}}(\theta)$ contains $\nu = \nu(\lambda F^{3+2\varepsilon+\varepsilon'}, -F^{11/2+3\varepsilon+2\varepsilon'})$ in its resolvent set and there exists a spectral gap of size $F^{(4+2\varepsilon+2\varepsilon')}$ around λ for the operator H_{I_2}. On the other hand, by Lemma 5.3 there exists some gaps $G_+, G_- \subset \Delta$ around a spectral interval $\mathcal{N}(\lambda)$ for the operator H^{int} containing a cluster $Cl^+$of eigenvalues and we have obtained the uniform estimate $|G_+|, |G_-| = \mathrm{O}(F^{6+2\varepsilon+2\varepsilon'})$. There also exists some gaps $G_+^d, G_-^d \subset \Delta$ and $G_+ \subset G_+^d,\ G_- \subset G_-^d$ for the operator, $H_l^{\mathrm{int}} \oplus H_r^{\mathrm{int}}$, satisfying the same uniform estimate, so the same holds for the operator H_l^{int}. Formulas (5.33) and (5.34) together with (5.29), (5.31) and (5.32) imply that for F small enough, $z \in \nu$ and

$$\operatorname{dist}(z, \nu^c)\big(\operatorname{dist}(z, \mathcal{N}(\lambda))\big)^{2p} \geqslant \mathrm{const}\, F^{-(3+2\varepsilon+\varepsilon')} \exp(-2\gamma/F) \tag{5.36}$$

that $z \in \rho(H(\theta))$. In particular if $H(\theta)$ has eigenvalues near $\mathcal{N}(\lambda)$, then their imaginary part satisfy (5.28).

For the configuration chosen above and for a suitable contour $\mathcal{C}$ around $\mathcal{N}(\lambda)$, we define the spectral projector, $P(\theta)$ for the operator $H(\theta)$ associated with the spectrum inside $\mathcal{C}$, by

$$P(\theta) = -(2i\pi)^{-1} \oint_{\mathcal{C}} \big(H(\theta) - z\big)^{-1} \mathrm{d}z, \tag{5.37}$$

notice that by construction we also have $P^{\mathrm{int}} = -(2i\pi)^{-1} \oint_{\mathcal{C}} (H^{\mathrm{int}} - z)^{-1} \mathrm{d}z$ and since the complexified exterior operator has no spectrum inside $\mathcal{C}$ the corresponding spectral projector $P^{\mathrm{ext}} = 0$. Clearly the arguments of the proof of Lemma 5.3, applied to the projectors $P(\theta)$ and P^{int}, imply that $\dim P(\theta) > 0$. We now show that $\dim P(\theta) \leqslant \dim P^{\mathrm{int}}$, hence let $\mathcal{I}$ a map from $\operatorname{Ran} P(\theta) \to \operatorname{Ran} P^{\mathrm{int}}$, defined by

$$\mathcal{I} : \varphi \in \operatorname{Ran} P(\theta) \to P^{\mathrm{int}} \widetilde{J}^{\mathrm{int}} \varphi \in \operatorname{Ran} P^{\mathrm{int}} \tag{5.38}$$

our last statement is proven if in these conditions $\mathcal{I}$ is an injective map and to see this fact, we need some estimates on $\|(\mathcal{F}^{\mathrm{ext}})^{\star} R^{\mathrm{ext}}(\theta, z) \widetilde{J}^{\mathrm{ext}}\|$

and $\|(\mathcal{F}^{\text{int}})^{\star} R^{\text{int}}(z)\tilde{J}^{\text{int}}\|$. For $\operatorname{Re} z \in \Delta$, since support $\mathcal{F}^{\text{ext}} \cap$ support $\tilde{J}^{\text{ext}} = \emptyset$ (A.14) gives, for F small enough, uniformly on Γ'',

$$\|(\mathcal{F}^{\text{ext}})^{\star} R^{\text{ext}}(\theta, z)\tilde{J}^{\text{ext}}\| \leqslant \text{const}\, \|\mathbf{1}_{\chi_{\text{ext}}} R^{\text{ext}}(\theta, z)\tilde{J}^{\text{ext}}\|. \tag{5.39}$$

On the other hand, let $\bar{J}^{\text{ext}} \in \mathcal{C}^{\infty}(\mathbf{R}^{\text{ext}})$ such that $\bar{J}^{\text{ext}}\tilde{J}^{\text{ext}} = 0$ and $\bar{J}^{\text{ext}}(x) = 1$ if $\operatorname{dist}(x, \text{support}\, \tilde{J}^{\text{ext}}) > \eta$ for some $\eta > 0$ and F independent, we denote $\text{support}(\bar{J}^{\text{ext}})' = \bar{\chi}_{\text{ext}}$, then

$$\|\mathbf{1}_{\chi_{\text{ext}}} R^{\text{ext}}(\theta, z)\tilde{J}^{\text{ext}}\| \leqslant \|\mathbf{1}_{\chi_{\text{ext}}} R^{\text{ext}}(\theta, z)\bar{J}^{\text{ext}}\| \tag{5.40}$$

and by using the formula (4.52) with the function $\bar{J}^{\text{ext}}$ instead, for F small enough we get,

$$\|\mathbf{1}_{\chi_{\text{ext}}} R^{\text{ext}}(\theta, z)\bar{J}^{\text{ext}}\| \leqslant \text{const}\, \|\mathbf{1}_{\chi_{\text{ext}}} R_{\mathbf{I}_2}(z)\mathbf{1}_{\bar{\chi}_{\text{ext}}}\|\, \|R^{\text{ext}}(\theta, z)\|, \tag{5.41}$$

and then,

$$\|(\mathcal{F}^{\text{ext}})^{\star} R^{\text{ext}}(\theta, z)\tilde{J}^{\text{ext}}\| \leqslant \text{const}\, \|\mathbf{1}_{\chi_{\text{ext}}} R_{\mathbf{I}_2}(z)\mathbf{1}_{\bar{\chi}_{\text{ext}}}\|\, \|R^{\text{ext}}(\theta, z)\|. \tag{5.42}$$

Similarly, we have,

$$\|(\mathcal{F}^{\text{int}})^{\star} R^{\text{int}}(z)\tilde{J}^{\text{int}}\| \leqslant \text{const}\, \|\mathbf{1}_{\chi_{\text{int}}} R_{l}^{\text{int}}(z)\mathbf{1}_{\bar{\chi}_{\text{int}}}\|\, \|R^{\text{int}}(z)\| \tag{5.43}$$

which lead us to define a new event, consider the set of configurations ω for which,

$$\|\mathbf{1}_{\tilde{\chi}_{\text{ext}}} R_{\mathbf{I}_2}(z)\mathbf{1}_{\bar{\chi}_{\text{ext}}}\| \leqslant \text{const}\, \exp(-\gamma/F)\|R_{\mathbf{I}_2}(z)\|^{p} \tag{5.44}$$

and

$$\|\mathbf{1}_{\tilde{\chi}_{\text{int}}} R_{l}^{\text{int}}(z)\mathbf{1}_{\bar{\chi}_{\text{int}}}\| < \text{const}\, \exp(-\gamma/F)\|R_{l}^{\text{int}}(z)\|^{p}, \tag{5.45}$$

$\operatorname{Re} z \in \Delta$ for some $\gamma > 0$ and finally the event $\Gamma = \{\omega \in \Gamma''$ s.t. formulas (5.45) and (5.46) hold$\}$. By Theorem 2.1, for F small enough, we have the estimate

$$\mathbf{P}(\Gamma) \geqslant 1 - \mathrm{O}(F^{\varepsilon'}). \tag{5.46}$$

Then let $\omega \in \Gamma$, a straightforward calculation gives

$$\|(J^{\text{int}} R^{\text{int}}(z)\mathcal{F}^{\text{ext}}\sigma, J^{\text{ext}} R^{\text{ext}}(\theta, z)\mathcal{F}^{\text{int}}\sigma) \cdot \mathcal{A}(\theta, z) \times ((\mathcal{F}^{\text{ext}})^{\star} R^{\text{ext}}(\theta, z)\tilde{J}^{\text{ext}}, (\mathcal{F}^{\text{int}})^{\star} R^{\text{int}}(z)\tilde{J}^{\text{int}})\| \leqslant \text{const}\, F^{q} \exp(-\gamma/F) \tag{5.47}$$

for some $-\infty < q < 0$ and this estimate holds uniformly for $z \in \mathcal{C}$ and for $\omega \in \Gamma$. For all $\varphi \in \mathcal{H}$, we have $\|P^{\text{int}}\widetilde{J}^{\text{int}}\varphi\| \geqslant \|J^{\text{int}}P^{\text{int}}\widetilde{J}^{\text{int}}\varphi\|$, then by (5.29), (5.48) for F small enough,

$$\left\|P^{\text{int}}\widetilde{J}^{\text{int}}\varphi\right\| \geqslant \left(1 - F^q \exp(-\gamma/F)\right)\|\varphi\|, \tag{5.48}$$

which shows, that for F small enough, $\mathcal{I}$ is an injective map and this proves the theorem. □

By some standard arguments see, e.g., [27,21], an immediate corollary of this result is

THEOREM 5.2 (on the existence of spectral resonances). – *In the same conditions as in Theorem* 5.1, *there exists* ε', $F_0 > 0$ *and small enough, such that for* $0 < F < F_0$, *there exists* $\Gamma \subset \Omega$ *with a probability measure satisfying*

$$\mathbf{P}(\Gamma) \geqslant 1 - \mathrm{O}\left(F^{\varepsilon'}\right) \tag{5.49}$$

such that for all $\omega \in \Gamma$, *the operator* $H_\omega(F)$ *has at least one resonance whose real part is in* Δ *and the imaginary part satisfies* (5.28).

We now give the proof of Theorem 1.1: it follows from Theorem 5.2 and the Borel–Cantelli lemma passing to the complement of the event given in Theorem 5.2.

Notice that Theorem 1.1 is an improvement of Theorem 5.2 in the sense that it gives the uniformity with respect to F, for the result on the existence of resonances of $H_\omega(F)$.

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APPENDIX: THE GEOMETRIC PERTURBATION THEORY

In this section, we describe the perturbation framework adapted to our problem, for a more general version see, e.g., [5,6]. Let $H(F)$ be the Schrödinger operator on $\mathcal{H} = L^2(\mathbf{R})$,

$$H = H(F) = -\Delta + V(x) + Fx, \quad F > 0, \tag{A.1}$$

and suppose $V \in L^\infty(\mathbf{R})$. Then $H(F)$ is essentially selfadjoint on $\mathcal{C} = \mathcal{C}_0^\infty(\mathbf{R})$. The spectral analysis of H is done here from local information given, e.g. by the geometry of the potential. Hence let $H^a = H^a(F)$, $a = 1, 2$, the local operators, defined respectively on $\mathcal{H}^a = L^2(\mathcal{R}_a)$ as the restriction of H on $\mathcal{R}_a = (-\infty, x_a)$ (see the introduction for definitions). Here $x_1 = x_1(F)$, $x_2 = x_2(F)$, depend continuously on F, $-\infty < x_2 < x_1 < +\infty$ and

$$|x_1 - x_2| \to \infty \quad \text{as } F \to 0. \tag{A.2}$$

It must be noted that the set $\{\varphi \in \mathcal{C}_0^\infty(\mathcal{R}),\ \varphi = 0 \text{ on } \partial\mathcal{R}\}$ is a core for H, in the sequel we will denote by $\mathcal{C}_a$ the core for H^a. Let $H^d = H^d(F) = H_1 \oplus H_2$ acting on $\mathcal{H}_d = \mathcal{H}_1 \oplus \mathcal{H}_2$ be the decoupled operator, we want to compare the operators H and H^d. Let $F_0 > F > 0$ for some $F_0 > 0$, we introduce the identification operator $J : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}$, defined from the cut-off functions $J_a \in \mathcal{C}^\infty(\mathbf{R})$ which are positive, monotone and satisfy $J_a(x) = 1$if $\operatorname{dist}(x, \partial\mathcal{R}_a) > \eta$, for some $\eta > 0$ and F independent, we also denote by J_a the identification operators,

$$\forall \varphi_a \in \mathcal{H}_a \to J_a\varphi_a \in \mathcal{H} \quad \text{and} \tag{A.3}$$

$$J \bigoplus_{a=1,2} \varphi_a = J_1\varphi_1 + J_2\varphi_2. \tag{A.4}$$

On the other hand let x_m the middle point between x_1, x_2 and the two positive functions $\widetilde{J}_a$, $a = 1, 2$, on $\mathbf{R}$, such that $\widetilde{J}_1 + \widetilde{J}_2 = 1$ on $\mathbf{R}$ and

$$\widetilde{J}_1 = 1 \text{ on } \{x < x_m\} \quad \text{and} \quad \widetilde{J}_2 = 1 \text{ on } \{x \geqslant x_m\} \tag{A.5}$$

as above, let $\widetilde{J}$ be the associated identification operator, clearly, $J\widetilde{J}^* = \mathbf{1}_{\mathcal{H}}$, i.e., $\widetilde{J}^*$ is a right inverse of the operator J. Let now $z \in \rho(H) \cap \rho(H^d)$, then we have the Geometric Resolvent Equation, between $R(z) = (H - z)^{-1}$ and $R^d(z) = (H^d - z)^{-1}$ (G.R.E. in short),

$$R(z)J - JR^d(z) = R(z)JM^dR^d(z) \tag{A.6}$$

where $M^d : \bigoplus_a \mathcal{H}^1(\text{support } J'_a) \to \mathcal{H}^d$ (here $\mathcal{H}^1$ denoting the standard Sobolev space) is the differential operator of the first order, defined in the quadratic form sense on $\bigoplus_a \mathcal{C}_a$,

$$\begin{aligned}(M^d \oplus \varphi_a, \oplus\psi_a) &= \sum_{a=1,2} (M_a\varphi_a, \psi_{\bar a}) \\ &= \sum_{a=1,2} (J'_a\varphi_a, -\psi'_{\bar a}) + (\varphi'_a, J'_a\psi_{\bar a})\end{aligned} \tag{A.7}$$

where $\bar{a} = 1(2)$ if $a = 2(1)$. Since the potential $V(x) + Fx$ is bounded on the support of the function J'_a, by the arguments of Section 4 and the estimates which we will give below, formula (A.6) is valid in the bounded operator sense from $\mathcal{H}$ to $\mathcal{H}^d$. To solve the G.R.E. (A.6), we use the following differential operators, $\mathcal{F}_a(l) : \mathcal{H}^1(\text{support } J'_a) \oplus \mathcal{H}_a \to \mathcal{H}_a$, $b > 0$,

$$\mathcal{F}_a(l)u \oplus v = -l^{-1/2}\mathbf{1}_{\text{support } J'_a} u' + l^{1/2} J'_a v \tag{A.8}$$

denoting by

$$\sigma_a : \mathcal{H}_a \oplus \mathcal{H}_a \to \mathcal{H}_a \oplus \mathcal{H}_a \quad \text{and} \quad \sigma_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

it holds in the quadratic form sense on $\mathcal{C}_a$

$$M_a = \mathcal{F}_a \sigma \mathcal{F}_a^\star, \quad a = 1, 2. \tag{A.9}$$

In Section 5, these formulas are used with $l = 1$, whereas in Section 4, we want to take in account the smallness of the $\|J'_a\|_\infty$ and then in this case, we choose $l = F^{(5+3\varepsilon)/2}$. For $a = 1, 2$, $z \in \rho(H_1) \cap \rho(H_2)$, let $K_a(z) : \mathcal{H}_a \oplus \mathcal{H}_a \to \mathcal{H}_a \oplus \mathcal{H}_a$ be the operators,

$$K_a(z) = \mathcal{F}_a^\star R_a(z) \mathcal{F}_{\bar{a}} \sigma_{\bar{a}}, \tag{A.10}$$

we will show that these operators are bounded, moreover for $z \in \rho(H^d) \cap \rho(H)$ and $\|K_a(z)\|\,\|K_{\bar{a}}(z)\| < 1$, the formal iteration of the G.R.E. (A.8) together with (A.1) and (A.12) lead to the following Geometric Perturbation Formula (G.P.F. in short),

$$\begin{aligned} R(z) = J R_d(z) \widetilde{J}^\star &+ \big(J_1 R_1(z) \mathcal{F}_2 \sigma_2, J_2 R_2(z) \mathcal{F}_1 \sigma_1\big) \cdot \mathcal{A}_\theta(z) \\ &\times \big(\mathcal{F}_2^\star R_2(z) \widetilde{J}_2, \mathcal{F}_1^\star R_1(z, \theta) \widetilde{J}_1\big) \end{aligned} \tag{A.11}$$

here "·" denotes the usual scalar product in $\mathbf{R}^2$ and $\mathcal{A}(z)$ the 2×2 matrix:

$$\mathcal{A}(z) = \begin{pmatrix} (1 - K_2(z)K_1(z))^{-1} & K_2(z)(1 - K_1(z)K_2(z))^{-1} \\ K_1(z)(1 - K_2(z)K_1(z))^{-1} & (1 - K_1(z)K_2(z))^{-1} \end{pmatrix}. \tag{A.12}$$

Under the conditions stated above, the G.P.F. is valid in the bounded operator sense on $\mathcal{H}$.

It must be also noted that this theory is valid if: one or both regions $\mathcal{R}_1, \mathcal{R}_2$ are open and bounded intervals or if we consider H_1 as the distorted operator instead, if the potential V satisfies suitable analyticity conditions, since the support of the distortion is such that support $f \cap \Lambda = \emptyset$, $\Lambda = \mathcal{R}_1 \cap \mathcal{R}_2$ and in the other hand the distortion preserves the core of H_1. In this last case, the G.P.F. has the same explicit form as (A.11) where now $R(z)$, $R^d(z)$, $R_1(z)$ are respectively the resolvent of the operators $H(\theta)$, $H^d(\theta)$, $H_1(\theta)$, $|\theta|$ small enough and for a suitable choice of $z \in \mathbf{C}$. We give now some useful estimates which we often use, let for $a = 1, 2$, $\zeta_k \in \mathbf{C}^\infty(\mathbf{R})$, $a = 1, 2$, such that,

$$\begin{aligned} \zeta_a(x) &= 0 \quad \text{if } \operatorname{dist}(x, \text{support } J_a') \geqslant \eta, \\ \zeta_a(x) &= 1 \quad \text{if } x \in \text{support } J_a', \end{aligned} \tag{A.13}$$

for some $\eta > 0$ and F independent and denote by χ_a the support of the function ζ_a, then for $0 < F < F_0$, $\operatorname{Im} z > 0$ big enough and $u \in \mathcal{H}_a$,

$$\begin{aligned} \left\|\mathcal{F}_a^\star R_b(z)u\right\|^2 \leqslant c\big(&\operatorname{Re}(\zeta_a^2 R_b(z)u, u) + \kappa(z)\left\|\zeta_a R_b(z)u\right\|^2 \\ &+ \left\|\zeta_a' R_b(z)u\right\|^2\big), \end{aligned} \tag{A.14}$$

$b = 1, 2$, where $\kappa(z) = 1 + \sup\{|(V - \operatorname{Re} z)|;\ x \in \chi_k\}$ is then uniformly bounded in all bounded neighbourhood of $z = 0$. In the one hand (A.14) implies, in the norm operator sense,

$$\left\|\mathcal{F}_a^\star r_b(z)\right\|^2 \leqslant c\kappa(z)\big(\left\|R_b(z)\right\| + \kappa(z)\left\|R_b(z)\right\|^2\big) \tag{A.15}$$

for another uniform constant $c > 0$. On the other hand, in the same conditions as for (A.14) and for a constant $\tilde{c} > 0$,

$$\left\|\mathcal{F}_a^\star R_a(z)\mathcal{F}_{\bar{a}}\sigma\right\| \leqslant \tilde{c}\kappa(z)\left\|\mathbf{1}_{\chi_a} R_k(z)\mathbf{1}_{\chi_{\bar{a}}}\right\|. \tag{A.16}$$

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