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New representations of the Poincaré group describing two interacting bosons

by

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ABSTRACT. – New representations of the Poincaré group are given, which describe two bosons with interaction in four space-time dimensions. The quantum frame is the Schrödinger picture in momentum space. More precisely we start from the relativistic free model with Hilbert space $L^2(\mathbb{R}^3 \times \mathbb{R}^3, \sigma_2)$, where σ_2 is the Lorentz invariant measure. We add to the free Hamiltonian and the free Lorentz generators new interaction terms, without changing the Poincaré algebra commutation rules, and such that the algebra representation can be integrated to give a unitary representation of the group on $L^2(\mathbb{R}^6, \sigma_2)$. The physics of these models can be investigated through the bound state equation (a *relativistic Schrödinger equation*) and through the scattering matrix, which is shown to be unitary. Finally we give an example for which a bound state exists and for which the scattering matrix can be written down explicitely. This example assures that interaction can effectively occur between the particles. © Elsevier, Paris

RÉSUMÉ. – Nous établissons de nouvelles représentations du groupe de Poincaré décrivant deux bosons en interaction dans un espace-temps à quatre dimensions. Le cadre quantique est celui de la représentation de Schrödinger dans l'espace des moments. Plus précisément, nous

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partons de la représentation décrivant deux bosons libres, dont l'espace de Hilbert des états est $L^2(\mathbb{R}^3 \times \mathbb{R}^3, \sigma_2)$, où σ_2 est la mesure invariante de Lorentz. Nous ajoutons à l'Hamiltonien et aux générateurs de Lorentz de nouveaux termes, dits d'interaction, sans briser les règles de commutation de l'algèbre de Poincaré, et de sorte que la représentation de l'algèbre puisse s'intégrer et donner une représentation unitaire du groupe, toujours dans $L^2(\mathbb{R}^6, \sigma_2)$. La physique de ces modèles est accessible grâce à une équation aux valeurs propres pour les états liés (une sorte d'*équation de Schrödinger relativiste*) et grâce à l'existence de la matrice de diffusion, qui s'avère être unitaire. Finalement nous donnons un exemple pour lequel un état lié existe et pour lequel la matrice de diffusion peut être écrite explicitement. Cet exemple nous assure qu'une interaction entre particules peut effectivement avoir lieu dans ces modèles. © Elsevier, Paris

1. INTRODUCTION

A new family of unitary representations of the Poincaré group in four space-time dimensions is given, which describe two boson systems with interaction. The quantum frame is the Schrödinger picture of Quantum Mechanics in momentum space. The physics of these models can be obtained from the scattering operator and the bound state equation, an eigenvalue equation of the kind of the Schrödinger equation. The existence of a bound state and the non-triviality of the scattering matrix confirm that the interaction is effective. To our knowledge, this is the first example of relativistic quantum models for a finite number of interacting particles. Moreover they admit a rather simple mathematical construction.

The quantum frame of the Schrödinger picture means that the Hilbert state space is simply made of functions of the momenta. In particular the time variable does not appear explicitly. However, these models satisfy the relativistic principles, because they consist of unitary representations of the Poincaré group. An elementary example of such a model is given by the free model for two bosons ('free' means without interaction). Our models are obtained by perturbations of the Hamiltonian and the Lorentz generators of the free model, in such a way that the commutation rules of the Poincaré algebra remain satisfied. The existence of solutions to this problem is rather surprising. Indeed, such a perturbation of the Hamiltonian and the Lorentz generators has already been proposed by Dirac in the late 40's [1], in the framework of classical mechanics. Dirac thought that the classical theory should be established first, and that the quantum version would be obtained afterwards by applying the canonical quantization. Therefore his paper concerns only classical physics, and it does not really obtain an interesting conclusion. In the early 60's, Currie continued this approach, still in the classical frame. He obtained in [2] that, under additional natural assumptions, such modifications cannot generate an interaction (this is known as the 'non-interaction theorem'). The present paper shows that by going directly to the quantum frame, this difficulty can be overcome.

The result given here is the generalization of similar results in 2-d (two space-time dimensions) [3] and in 3-d [4]. Although the family of models in [3] is very general (the kernel of the interaction part of the Hamiltonian can be chosen almost arbitrarily in the centre-of-mass frame), in 3-d and 4-d we have only managed to construct models with special interaction terms (excuding local interactions by a potential). Moreover, the generalization to more than two particles [5] and to bosons with different masses [6], which is easy in 2-d, seems difficult in higher space-time dimensions.

Originally, the result in 2-d was suggested by a new approch of the bound state problem in 2-d Quantum Field Theory, see [4].

The paper is organized as follows. Section 2 presents a family of operators which formally satisfy the commutation rules of the Poincaré algebra, provided the 'interaction kernel' h (the kernel of the interaction term of the Hamiltonian) satisfies a 'fundamental equation'. In Section 3 the complete set of solutions to this equation is given, in the weak coupling regime. Then we focus on a subset of solutions which satisfy all the properties we need for the following proofs. Section 4 shows that the formal representations of the Poincaré algebra of Section 2 can be integrated to give unitary representations of the Poincaré group. Section 5 presents the eigenvalue equation for the bound states (the 'relativistic Schrödinger equation'). Section 6 establishes the existence of two-particle asymptotic states and of the scattering operator, which appears to be unitary. Section 7 gives a simple example (one-dimensional perturbation) for which the bound state equation admits a solution and for which the scattering matrix can be written down explicitly and is non-trivial. Finally, Section 8 collects the physical results of the paper.

This paper follows closely [4]. For a complete, detailed, and self-contained version see the preprint, available at mp-arc 98-305.

2. THE MODELS

The Poincaré group $\mathcal{P}_+^\uparrow$ is generated by the ten operators H (Hamiltonian), $\vec{P} = (P_1, P_2, P_3)$ (momentum), $\vec{J} = (J_1, J_2, J_3)$ (angular momentum) and $\vec{L} = (L_1, L_2, L_3)$ (generators of the Lorentz transformations), satisfying the standard commutation rules of the Lie algebra $\mathcal{G}$ of $\mathcal{P}_+^\uparrow$. This algebra admits two Casimir operators, the mass operator M and the Pauli–Lubanski operator W, the squares of which are given by

$$M^2 := H^2 - \vec{P}^2, \qquad W^2 := (\vec{P} \cdot \vec{J})^2 - (-H\vec{J} + \vec{P} \wedge \vec{L})^2, \tag{1}$$

where the dot "$\cdot$" and the wedge "$\wedge$" denote the ordinary scalar and vector products, respectively.

We start from the representation describing two free, identical, and spinless particles of mass $m > 0$, in the Schrödinger picture. If $\vec{p}_1$, $\vec{p}_2 \in \mathbb{R}^3$ are the momenta of the two particles, the total and relative momenta $\vec{P}$, $\vec{Q}$ are defined by

$$\begin{aligned} \vec{P} &= \vec{p}_1 + \vec{p}_2, \\ \vec{Q} &:= \frac{1}{2}(\vec{p}_1 - \vec{p}_2) - \frac{1}{2}\vec{P}\frac{\omega(\vec{p}_1) - \omega(\vec{p}_2)}{M_0(\vec{p}_1, \vec{p}_2) + \omega(\vec{p}_1) + \omega(\vec{p}_2)}, \end{aligned} \tag{2}$$

where $\omega(\vec{p}_i) := \sqrt{\vec{p}_i^{\,2} + m^2}$, $i \in \{1, 2\}$ and

$$M_0(\vec{p}_1, \vec{p}_2) = \left[(\omega(\vec{p}_1) + \omega(\vec{p}_2))^2 - \vec{P}^2\right]^{1/2}.$$

(The variable $\vec{Q}$ is defined as the momentum in the centre-of-mass frame.) In these variables, the operators of the free representation are the following

$$\begin{aligned} &\vec{P}\phi(\vec{P}, \vec{Q}) = \vec{P}\phi(\vec{P}, \vec{Q}), \qquad H_0\phi(\vec{P}, \vec{Q}) = \Omega(\vec{P}, \vec{Q})\phi(\vec{P}, \vec{Q}), \\ &\vec{J}\phi(\vec{P}, \vec{Q}) = (\vec{J}^{\vec{P}} + \vec{J}^{\vec{Q}})\phi(\vec{P}, \vec{Q}), \\ &\vec{L}_0\phi(\vec{P}, \vec{Q}) = \left(-i\Omega(\vec{P}, \vec{Q})\vec{\nabla}_{\vec{P}} - \frac{\vec{P} \wedge \vec{J}^{\vec{Q}}}{2\omega(\vec{Q}) + \Omega(\vec{P}, \vec{Q})}\right)\phi(\vec{P}, \vec{Q}), \end{aligned} \tag{3}$$

for almost all $(\vec{P}, \vec{Q}) \in \mathbb{R}^6$ and suitable functions ϕ, where we have put

$$\Omega(\vec{P},\vec{Q}) := \sqrt{\vec{P}^2 + 4\omega(\vec{Q})^2} = \sqrt{\vec{P}^2 + 4\vec{Q}^2 + 4m^2},$$
$$\vec{J}^P := -\mathrm{i}\vec{P}\wedge\vec{\nabla}_{\vec{P}}, \qquad \vec{J}^Q := -\mathrm{i}\vec{Q}\wedge\vec{\nabla}_{\vec{Q}}. \tag{4}$$

These operators define a representation of the algebra $\mathcal{G}$ which can be integrated to give a unitary and continuous representation of the group $\mathcal{P}_+^\uparrow$, in the Hilbert space

$$\mathcal{H} = L^2_{ev.\vec{Q}}(\mathbb{R}^3\times\mathbb{R}^3,\mu),$$
$$d\mu(\vec{P},\vec{Q}) := \frac{d\vec{Q}\,d\vec{P}}{2\omega(\vec{Q})\Omega(\vec{P},\vec{Q})} = \frac{d\sigma(\vec{Q})\,d\vec{P}}{\Omega(\vec{P},\vec{Q})} \tag{5}$$

made of even functions with respect to $\vec{Q}$ (i.e., $\phi(\vec{P},\vec{Q}) = \phi(\vec{P},-\vec{Q})$), where $d\sigma(\vec{Q}) = d\vec{Q}/2\omega(\vec{Q})$.

In the non-relativistic case, it is possible to find relative variables which are invariant under the proper Galilean transformations. Such variables cannot be found in the relativistic case. In (3) the Lorentz generators contain a term involving $\vec{J}^{\vec{Q}}$, i.e., acting on $\vec{Q}$. Note that only the angular variables of $\vec{Q}$ are concerned, because

$$\vec{Q}^2 = \frac{1}{4}\left[(\vec{p}_1-\vec{p}_2)^2 - (\omega(\vec{p}_1)-\omega(\vec{p}_2))^2\right]$$

is Lorentz invariant. This can also be seen from the Casimir operators, which are only functions of $\vec{Q}^2$ and $(\vec{J}^{\vec{Q}})^2$

$$M_0^2\,\phi(\vec{P},\vec{Q}) = 4\omega(\vec{Q})^2\,\phi(\vec{P},\vec{Q}),$$
$$W_0^2\,\phi(\vec{P},\vec{Q}) = 4\omega(\vec{Q})^2\,(\vec{J}^{\vec{Q}})^2\,\phi(\vec{P},\vec{Q}). \tag{6}$$

To introduce an interaction we take a symmetric operator $\mathcal{O}$, the *interaction operator*, on which we impose only, for the moment, the formal commutation properties

$$[\mathcal{O},\vec{P}] = 0, \qquad [\mathcal{O},\vec{J}] = 0. \tag{7}$$

The interaction representation is defined as follows

$$\vec{P},\vec{J}\quad\text{as in (3)},\qquad H := H_0 + \{\mathcal{O},H_0\},\qquad \vec{L} := \vec{L}_0 + \{\mathcal{O},\vec{L}_0\}, \tag{8}$$

where we have used the notation $\{A,B\} = AB + BA$. The commutation rules of the Poincaré algebra formally hold if $\mathcal{O}$ satisfies the following conditions

$$\begin{aligned}0&=[\{\mathcal{O},H_0\},L_{0,j}]+[H_0,\{\mathcal{O},L_{0,j}\}]+[\{\mathcal{O},H_0\},\{\mathcal{O},L_{0,j}\}]\\0&=[\{\mathcal{O},L_{0,j}\},L_{0,k}]+[L_{0,j},\{\mathcal{O},L_{0,k}\}]+[\{\mathcal{O},L_{0,j}\},\{\mathcal{O},L_{0,k}\}]\end{aligned}\tag{9}$$

for all $j \neq k \in \{1, 2, 3\}$. Thus we have to find non-trivial solutions $\mathcal{O} \neq 0$ to this system.

Let us write $\mathcal{O}$ as a kernel operator which commutes with $\vec{P}$

$$\mathcal{O}\phi(\vec{P},\vec{Q}') := \int_{\mathbb{R}^3} \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\phi(\vec{P},\vec{Q}')\frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q})+\Omega(\vec{P},\vec{Q}')}, \tag{10}$$

where $h(\vec{P}, \vec{Q}, \vec{Q}')$ is an arbitrary kernel (for the moment) which satisfies

$$h(\vec{P},\vec{Q},\vec{Q}') = h(\vec{P},-\vec{Q},\vec{Q}') = h(\vec{P},\vec{Q},-\vec{Q}'),$$

and the symmetry condition

$$h(\vec{P},\vec{Q},\vec{Q}')^* = h(\vec{P},\vec{Q}',\vec{Q}), \tag{11}$$

where the star '*' means complex conjugation, all these conditions being necessary for $\mathcal{O}$ to be a symmetric operator. The last condition of (7) imposes (formally)

$$(\vec{J}^{\vec{P}} + \vec{J}^{\vec{Q}} + \vec{J}^{\vec{Q}'})h(\vec{P},\vec{Q},\vec{Q}') = 0. \tag{12}$$

The substitution of (10) in (9) leads to six non-linear, integro-differential equations for h. To solve this system we restrict ourselves to solutions satisfying the following condition

$$\vec{J}^{\vec{Q}}h(\vec{P},\vec{Q},\vec{Q}') = 0 \tag{13}$$

which leads to $\vec{J}^{\vec{P}}h = \vec{J}^{\vec{Q}'}h = 0$ because of (11) and (12). Thus (13) is equivalent to the requirement that $h(\vec{P}, \vec{Q}, \vec{Q}')$ is function of the norms $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$ only. With this condition, the six equations (9) coincide. Introducing the differential operator D

$$Dh(\vec{P},\vec{Q},\vec{Q}') := \frac{\Omega(\vec{P},\vec{Q})\Omega(\vec{P},\vec{Q}')}{\Omega(\vec{P},\vec{Q})+\Omega(\vec{P},\vec{Q}')}\partial_{\|\vec{P}\|}h(\vec{P},\vec{Q},\vec{Q}'), \tag{14}$$

the conditions (9) and (13) lead to the single equation

$$\begin{aligned}0 = 2Dh(\vec{P},\vec{Q},\vec{Q}') &+ \int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P},\vec{Q}'')^2}\\ &\times\left\{-\frac{\|\vec{P}\|}{\Omega(\vec{P},\vec{Q}'')}h(\vec{P},\vec{Q},\vec{Q}'')h(\vec{P},\vec{Q}'',\vec{Q}') + Dh(\vec{P},\vec{Q},\vec{Q}'')\right.\end{aligned}$$

$$\times h(\vec{P},\vec{Q}'',\vec{Q}')+h(\vec{P},\vec{Q},\vec{Q}'')Dh(\vec{P},\vec{Q}'',\vec{Q}')\Big\}. \tag{15}$$

This is the fundamental equation which guarantees the relativistic structure of the theory. Let us sum up what we have found.

PROPOSITION 1. – *Let $h(\vec{P},\vec{Q},\vec{Q}')$ be a function of $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$ only which satisfies* (11) *and* (15). *Then the operators* (8), *with $\mathcal{O}$ given by* (10), *are symmetric and satisfy formally the commutation rules of the algebra $\mathcal{G}$. The total Hamiltonian and Lorentz generators H and $\vec{L}$ are represented in terms of h by the following integral operators*:

$$H\phi(\vec{P},\vec{Q}')=\Omega(\vec{P},\vec{Q})\phi(\vec{P},\vec{Q})+\int_{\mathbb{R}^3}\frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\phi(\vec{P},\vec{Q}')h(\vec{P},\vec{Q},\vec{Q}'),$$

$$\begin{aligned}\vec{L}\phi(\vec{P},\vec{Q}')\\ &=\vec{L}_0\phi(\vec{P},\vec{Q})+\int\frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}[\vec{L}_0\phi(\vec{P},\vec{Q}')]\frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\\ &\quad-\mathrm{i}\vec{P}\int\frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\phi(\vec{P},\vec{Q}')\left[\frac{Dh(\vec{P},\vec{Q},\vec{Q}')}{\|\vec{P}\|\Omega(\vec{P},\vec{Q}')}-\frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q}')^2}\right]\\ &\quad+\int\frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')}\phi(\vec{P},\vec{Q}')\frac{\vec{P}\wedge\left[\frac{\vec{J}_{\vec{Q}}}{2\omega(\vec{Q})+\Omega(\vec{P},\vec{Q})}+\frac{\vec{J}_{\vec{Q}'}}{2\omega(\vec{Q}')+\Omega(\vec{P},\vec{Q}')}\right]}{\Omega(\vec{P},\vec{Q})+\Omega(\vec{P},\vec{Q}')}\\ &\quad\times h(\vec{P},\vec{Q},\vec{Q}').\end{aligned}$$

3. EXISTENCE OF SOLUTIONS OF THE FUNDAMENTAL EQUATION

We give the complete set of solutions of (15) lying in a ball of a Banach space. Then we introduce a subset of solutions, which satisfy also (11) and (13), and we establish their useful properties.

DEFINITION 1. – *Let $\mathcal{K}$ be a compact set of $\mathbb{R}^3$ with Lebesgue-measure $|\mathcal{K}|\neq 0$. We denote by $\mathcal{B}$ the Banach space of continuous and bounded functions $h(\vec{P},\vec{Q},\vec{Q}')$ with support in $(\vec{P},\vec{Q},\vec{Q}')\in\mathbb{R}^3\times\mathcal{K}\times\mathcal{K}$ for which $Dh(\vec{P},\vec{Q},\vec{Q}')$ exists and is also bounded and continuous, given the norm*

$$|h|_{\mathcal{B}}:=\|h\|_\infty+\|Dh\|_\infty,$$

where the differential operator D is given by (14).

The following result assures the existence of a large class of solutions of (15).

PROPOSITION 2. – *There exists $K_1 \in (0, \infty)$ such that, for all $c \in C_0^0(\mathcal{K} \times \mathcal{K})$ satisfying $\|c\|_\infty < K_1$, there exists one and only one $h \in \mathcal{B}$ satisfying*:

(1) $|h|_{\mathcal{B}} < 2K_1$;

(2) $h(0, \vec{Q}, \vec{Q}') = c(\vec{Q}, \vec{Q}')$ *for all* $\vec{Q}, \vec{Q}' \in \mathbb{R}^3$;

(3) $h(\vec{P}, \vec{Q}, \vec{Q}')$ *satisfies* (15) *for all* $\vec{P}, \vec{Q}, \vec{Q}' \in \mathbb{R}^3$.

The proposition gives the complete set of solutions h of (15) lying in a certain ball of $\mathcal{B}$; they are uniquely defined by their restriction to $\vec{P} = 0$, which can be chosen arbitrarily in a certain ball of $C_0^0(\mathcal{K} \times \mathcal{K})$. Thus, apart from this analytic condition, the restriction of h for $\vec{P} = 0$ (centre-of-mass frame) is arbitrary.

The proof is an easy adaptation of Proposition 3 of [4] to the 4-d case and to the bilinear operator b defined on functions $f, g \in \mathcal{B}$ by

$$
\begin{aligned}
& b(g,h)(\vec{P}, \vec{Q}, \vec{Q}') \\
& := -\frac{1}{2} \int_0^{\|\vec{P}\|} d\|\vec{\zeta}\| \left(\frac{1}{\Omega(\vec{\zeta}, \vec{Q})} + \frac{1}{\Omega(\vec{\zeta}, \vec{Q}')} \right) \int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{\zeta}, \vec{Q}'')^2} \\
& \quad \times \Big\{ Dg(\vec{\zeta}, \vec{Q}, \vec{Q}'') h(\vec{\zeta}, \vec{Q}'', \vec{Q}') + g(\vec{\zeta}, \vec{Q}, \vec{Q}'') Dh(\vec{\zeta}, \vec{Q}'', \vec{Q}') \\
& \qquad - \frac{\|\vec{\zeta}\|}{\Omega(\vec{\zeta}, \vec{Q}'')} g(\vec{\zeta}, \vec{Q}, \vec{Q}'') h(\vec{\zeta}, \vec{Q}'', \vec{Q}') \Big\}
\end{aligned}
\tag{16}
$$

for all $(\vec{P}, \vec{Q}, \vec{Q}') \in \mathbb{R}^9$. Obviously $b : \mathcal{B} \to \mathcal{B}$ and satisfies

$$
|b(g,h)|_{\mathcal{B}} < K_1' |g|_{\mathcal{B}} |h|_{\mathcal{B}}, \tag{17}
$$

where

$$
\begin{aligned}
K_1' &= \frac{1}{2} \int_0^\infty \frac{2d\zeta}{\sqrt{\zeta^2 + 4m^2}} \int_{\mathcal{K}} \frac{d\sigma(\vec{Q}'')}{\zeta^2 + 4\vec{Q}''^2 + 4m^2} + \frac{1}{2} \int_{\mathcal{K}} \frac{d\sigma(\vec{Q}'')}{4(\vec{Q}''^2 + m^2)} \\
&< \frac{|\mathcal{K}|}{4m^3},
\end{aligned}
$$

and thus we can take $K_1 = m^3 / |\mathcal{K}|$.

Let us introduce the following set of functions.

DEFINITION 2. – *Let $R > 0$ and let $\mathcal{K}$ be the ball of $\mathbb{R}^3$ of centre 0 and radius R. Let $\mathcal{V}$ be the set of functions $c(\vec{Q}, \vec{Q}') \in C_0^3(\mathcal{K} \times \mathcal{K})$ which* *satisfy* $c(\vec{Q}, \vec{Q}')^* = c(\vec{Q}', \vec{Q})$, *which depend only on the norms* $\|\vec{Q}\|$ *and* $\|\vec{Q}'\|$, *and such that*

$$\|c\|_\infty \leqslant \frac{m^3}{2|\mathcal{K}|}. \tag{18}$$

Note that $\mathcal{V}$ depends on the two positive parameters m and R.

$\mathcal{V}$ is made of kernels $c(\vec{Q}, \vec{Q}')$ with compact support, bounded by a constant depending on the support, and depending on $\|\vec{Q}\|$ and $\|\vec{Q}'\|$ only. These requirements take us away from physical applications.

Each $c \in \mathcal{V}$ satisfies the hypothesis of Proposition 2 and

$$\|c\|_{L^2(\mathbb{R}^6)} \leqslant \frac{m^3}{2}, \tag{19}$$

where $L^2(\mathbb{R}^6)$ refers to the ordinary Lebesgue measure (not the σ_2 measure). The following properties hold on $\mathcal{V}$.

PROPOSITION 3. – *Let* $c \in \mathcal{V}$. *The function* h *deduced from* c *by Proposition* 2 *has the following properties*:

(1) h *satisfies* (15),

(2) $h(\vec{P}, \vec{Q}, \vec{Q}')$ *depends only on the norms* $\|\vec{P}\|$, $\|\vec{Q}\|$, $\|\vec{Q}'\|$,

(3) *for all* $\vec{Q}, \vec{Q}' \in \mathbb{R}^3$, *the function* $\vec{P} \to h(\vec{P}, \vec{Q}, \vec{Q}')$ *is a function of* $\vec{P}^2$ *analytic in a* $\mathbb{C}$-*neighbourhood of* $\mathbb{R}_+$,

(4) *for all* $\vec{P}^2$ *in some* $\mathbb{C}$-*neighbourhood of* $\mathbb{R}_+$, *the function* $\vec{Q}, \vec{Q}' \to h(\vec{P}, \vec{Q}, \vec{Q}')$ *has compact support and belongs to* $C^3(\mathbb{R}^6)$,

(5) *the partial derivatives of* $h(\vec{P}, \vec{Q}, \vec{Q}')$, *of any order with respect to* $\vec{P}^2$ *and of order* $\leqslant 3$ *with respect to* $\vec{Q}, \vec{Q}'$, *belong to* $\mathcal{B}$,

(6) h *and* Dh *satisfy the estimates*

$$\sup_{\vec{P}\in\mathbb{R}^3} \int_{\mathbb{R}^6} d\vec{Q}\, d\vec{Q}' |h(\vec{P}, \vec{Q}, \vec{Q}')|^2 \leqslant \left(\frac{2}{3}\right)^2 m^6,$$

$$\sup_{\vec{P}\in\mathbb{R}^3} \int_{\mathbb{R}^6} d\vec{Q}\, d\vec{Q}' |Dh(\vec{P}, \vec{Q}, \vec{Q}')|^2 \leqslant \left(\frac{1}{20}\right)^2 m^6. \tag{20}$$

Proof. – (1)–(5) are easy adaptations of the corresponding proofs in [4]. For (6) we introduce the norm

$$\|f\|_* := m^{-3} \sup_{\vec{P}\in\mathbb{R}^3} \sqrt{\int_{\mathbb{R}^6} d\vec{Q} d\vec{Q}'\ |f(\vec{P}, \vec{Q}, \vec{Q}')|^2} \tag{21}$$

for all $f \in \mathcal{B}$. From an easy estimate involving the Cauchy–Schwartz inequality we get

$$\begin{aligned}\|b(f,f)\|_* &< \tfrac{\sqrt{3}}{8}(\|f\|_*^2+\|Df\|_*^2),\\ \|Db(f,f)\|_* &< \tfrac{\sqrt{3}}{16}(\|f\|_*^2+\|Df\|_*^2)\end{aligned} \tag{22}$$

for all $f \in \mathcal{B}$. Now recall that $h = \lim_{n\to\infty} A^n$, where A^n are defined in [4]. Let us suppose that for some $n \in \mathbb{N}$ we know that $\|A^n\|_* \leqslant 2/3$ and $\|DA^n\|_* \leqslant 1/20$ (this is obvious for $n = 0$). Then we obtain

$$\begin{aligned}\|A^{n+1}\|_* &= \|c+b(A^n,A^n)\|_* \leqslant \|c\|_* + \|b(A^n,A^n)\|_*\\ &\leqslant \tfrac{1}{2} + \tfrac{\sqrt{3}}{8}(\|A^n\|_*^2+\|DA^n\|_*^2) \leqslant \tfrac{1}{2}+\tfrac{\sqrt{3}}{8}\left(\tfrac{4}{9}+\tfrac{1}{20^2}\right) < \tfrac{2}{3},\\ \|DA^{n+1}\|_* &= \|Db(A^n,A^n)\|_* \leqslant \tfrac{\sqrt{3}}{16}(\|A^n\|_*^2+\|DA^n\|_*^2)\\ &\leqslant \tfrac{\sqrt{3}}{16}\left(\tfrac{4}{9}+\tfrac{1}{20^2}\right) < \tfrac{1}{20}.\end{aligned}$$

It follows that these bounds hold for all n. Let us take the limit $n \to \infty$. Because the convergence of A^n and DA^n is uniform, the sequences $\|A^n\|_*$ and $\|DA^n\|_*$ converge too (taking the integral in (21) as a Riemann integral). Thus the above estimates still hold in the limit, which establishes (20). □

4. UNITARITY REPRESENTATIONS OF THE POINCARÉ GROUP

Until now, the representations (8) of the algebra $\mathcal{G}$ were only formal, in the sense that the domains of the operators and of the commutators were not specified.

THEOREM 4. – *Let $c \in \mathcal{V}$. Let $\mathcal{O}$ be the interaction operator* (10) *with the kernel h deduced from c in Proposition* 2. *Then $\{H, \vec{P}, \vec{J}, \vec{L}\}$, given by* (8), *are the generators of a unitary continuous representation of $\mathcal{P}_+^\uparrow$.*

Proof. – According to Theorem 5 of [7], H, $\vec{P}$, $\vec{J}$, $\vec{L}$ are the generators of a unitary continuous representation of $\mathcal{P}_+^\uparrow$ if the following three conditions are satisfied: (1) they are self-adjoint, (2) they admit a common invariant dense domain $\mathcal{D}$ on which the commutation rules of the algebra hold, (3) $\mathcal{D}$ is a domain of essential self-adjointness for the operator

$$\Delta = m^{-2}(H^2+\vec{P}^2)+\vec{J}^2+\vec{L}^2.$$

1*st step*: *Self-adjointness of H, $\vec{P}$, $\vec{J}$, $\vec{L}$.* It is clear for $\vec{P}$ and $\vec{J}$. To study H and $\vec{L}$ we need a general estimate of the norm of vectors like

$$\phi_\xi(\vec{P}, \vec{Q}) := \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P}, \vec{Q}')} \phi(\vec{P}, \vec{Q}') \xi(\vec{P}, \vec{Q}, \vec{Q}')$$

for all $\phi \in \mathcal{H}$ and for kernels ξ such that

$$|||\xi|||^2 := \sup_{\vec{P} \in \mathbb{R}^3} \int \frac{d\sigma(\vec{Q})}{\Omega(\vec{P}, \vec{Q})} \int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P}, \vec{Q}'')} |\xi(\vec{P}, \vec{Q}, \vec{Q}'')|^2 \tag{23}$$

is well defined. From the Cauchy–Schwarz inequality and the Fubini theorem we get

$$\|\phi_\xi\|^2 \leqslant |||\xi|||^2 \|\phi\|^2. \tag{24}$$

By using the expression of H given in Proposition 1 together with (20) we find the following bound on the operator norm of $H - H_0$:

$$\|H - H_0\|_{op} \leqslant |||h||| \leqslant m/6.$$

Thus $H - H_0$ is bounded and H is self-adjoint on the domain of H_0.

Using similarly the expression of $\vec{L}$ given by Proposition 1 together with the estimate (24) leads to

$$\left\|(L_j - L_{0,j})\phi\right\| \leqslant |||h/\Omega||| \|L_{0,j}\phi\| + \left(|||h/\Omega||| + |||Dh/\Omega|||\right) \|\phi\| \tag{25}$$

for all $j \in \{1, 2, 3\}$. From the Kato–Rellich theorem (Theorem X.12 of [8]) it follows that $\vec{L}$ is self-adjoint on the domain of $\vec{L}_0$ provided $|||h/\Omega||| < 1$. This condition is indeed satisfied because from (20)

$$|||h/\Omega||| \leqslant |||h|||/2m < 1/12.$$

2nd step: *Invariant domain and the commutation rules.* Let $\mathcal{D}$ be the domain

$$\mathcal{D} := \{\phi \in \mathcal{H} \mid \Omega(\vec{P}, \vec{Q})^\ell P_1^{n_1} P_2^{n_2} P_3^{n_3} L_{0,1}^{m_1} L_{0,2}^{m_2} L_{0,3}^{m_3} J_1^{j_1} J_2^{j_2} J_3^{j_3} \phi(\vec{P}, \vec{Q}) \in \mathcal{H} \text{ for all } \ell, n_1, n_2, n_3, m_1, m_2, m_3, j_1, j_2, j_3 \in \mathbb{N}\}. \tag{26}$$

$\mathcal{D}$ is dense (because it contains $C_0^\infty(\mathbb{R}^6)$) and is clearly left invariant by H_0, $\vec{P}$, $\vec{J}$ and $\vec{L}_0$. To show that it is also invariant under H and $\vec{L}$ it is sufficient to show that it is invariant under $\mathcal{O}$. Let $\phi \in \mathcal{D}$ and let us apply an operator

$$\Omega(\vec{P}, \vec{Q})^\ell P_1^{n_1} P_2^{n_2} P_3^{n_3} L_{0,1}^{m_1} L_{0,2}^{m_2} L_{0,3}^{m_3}$$

to the vector $\mathcal{O}\phi$ (recall that $J_k\mathcal{O} = 0$). From the Leibniz rule we find

$$\begin{aligned}
&\Omega(\vec{P},\vec{Q})^{\ell} P_1^{n_1} P_2^{n_2} P_3^{n_3} L_{0,1}^{m_1} L_{0,2}^{m_2} L_{0,3}^{m_3} \mathcal{O}\phi(\vec{P},\vec{Q}) \\
&\quad = \sum_{\vec{\alpha}+\vec{\beta}=\vec{m}} C_{\vec{\alpha},\vec{\beta}} \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P},\vec{Q}')} \\
&\qquad \times \left[\Omega(\vec{P},\vec{Q}')^{\ell+\beta_1+\beta_2+\beta_3+1} P_1^{n_1} P_2^{n_2} P_3^{n_3} L_{0,1}^{\alpha_1} L_{0,2}^{\alpha_2} L_{0,3}^{\alpha_3} \phi(\vec{P},\vec{Q}')\right] \\
&\qquad \times \left\{ \left(\frac{\Omega(\vec{P},\vec{Q})}{\Omega(\vec{P},\vec{Q}')} \right)^{\ell+\beta_1+\beta_2+\beta_3} \right. \\
&\qquad \left. \times\, \partial_{P_1}^{\beta_1} \partial_{P_2}^{\beta_2} \partial_{P_3}^{\beta_3} \frac{h(\vec{P},\vec{Q},\vec{Q}')}{\Omega(\vec{P},\vec{Q}')(\Omega(\vec{P},\vec{Q})+\Omega(\vec{P},\vec{Q}'))} \right\} \qquad (27)
\end{aligned}$$

for some coefficients $C_{\vec{\alpha},\vec{\beta}}$. Because $\phi \in \mathcal{D}$ the expression in [ ] is a vector of $\mathcal{H}$. By using the estimate (24) we find that (27) belongs to $\mathcal{H}$ if the expression in { } has a well defined $|||.|||$ norm, which follows from Proposition 3 and some straightforward estimate.

By working on $\mathcal{D}$ the commutation rules for $\{H, \vec{P}, \vec{J}, \vec{L}\}$ can be deduced from those for $\{H_0, \vec{P}, \vec{J}, \vec{L}_0\}$ without having to take care of their domain. Thus we only have to do formal calculations, and thus to apply Proposition 1.

3*th step*: *$\mathcal{D}$ is a domain of essential self-adjointness for the operator* Δ. Because $\{\vec{P}, H_0, \vec{J}, \vec{L}_0\}$ are the infinitesimal generators of a unitary continuous representation of a Lie group it follows from Theorem 3 of [7] that these operators and

$$\Delta_0 = m^{-2}(H_0^2 + \vec{P}^2) + \vec{J}^2 + \vec{L}_0^2$$

are self-adjoint and that they admit a domain $\mathcal{D}_0$ as common invariant domain and common core. However, the largest common invariant domain is (26), thus $\mathcal{D}$ is also a common core. By the Kato–Rellich theorem (Theorem X.12 of [8]), Δ is essentially self-adjoint on $\mathcal{D}$ if

$$\|(\Delta - \Delta_0)\phi\| \leqslant k_1 \|\Delta_0 \phi\| + k_2 \|\phi\| \qquad (28)$$

for all $\phi \in \mathcal{D}$, for some $0 < k_1 < 1$ and $0 < k_2 < \infty$.

Now $\Delta - \Delta_0 = m^{-2}(H^2 - H_0^2) + \vec{L}^2 - \vec{L}_0^2$. Using (24) and (20) and some straightforward estimates we can show that $H^2 - H_0^2$ is bounded.

Let us put $\vec{L}_h := \vec{L} - \vec{L}_0$. Using the same estimates and (25) we find for all $\phi \in \mathcal{D}$

$$\|\vec{L}_h \cdot \vec{L}_0 \phi\| < |||h/\Omega||| \, \|\vec{L}_0^2 \phi\| + (|||h/\Omega||| + |||Dh/\Omega|||) \sum_{j=1}^{3} \|L_{0,j}\phi\|,$$

$$\|\vec{L}_0 \cdot \vec{L}_h\phi\| \leqslant |||\Omega h/\Omega'^2||| \, \|\vec{L}_0^2\phi\| + 3(|||\Omega h/\Omega'^2||| + |||\Omega Dh/\Omega'^2|||) \times \sum_{j=1}^{3} \|L_{0,j}\phi\| + k_3\|\phi\|,$$

$$\|\vec{L}_h^2\phi\| \leqslant |||\Omega h/\Omega'^2||| \, \|\vec{L}_0^2\phi\| + [3|||h/\Omega|||(|||\Omega h/\Omega'^2||| + |||\Omega Dh/\Omega'^2|||) + |||h/\Omega||| + |||Dh/\Omega|||] \sum_{j=1}^{3} \|L_{0,j}\phi\| + k_4\|\phi\|,$$

for some $k_3, k_4 \in (0, \infty)$. Finally by collecting these estimates we obtain

$$\begin{aligned} &\|(\Delta - \Delta_0)\phi\| \\ &\leqslant \left(\left|\left|\left|\frac{h}{\Omega}\right|\right|\right| + \left|\left|\left|\frac{Dh}{\Omega}\right|\right|\right|\right)\|\vec{L}_0^2\phi\| \\ &\quad + \left[3\left(1 + \left|\left|\left|\frac{h}{\Omega}\right|\right|\right|\right)\left(\left|\left|\left|\frac{\Omega h}{\Omega'^2}\right|\right|\right| + \left|\left|\left|\frac{\Omega Dh}{\Omega'^2}\right|\right|\right|\right) + 2\left|\left|\left|\frac{h}{\Omega}\right|\right|\right| + 2\left|\left|\left|\frac{Dh}{\Omega}\right|\right|\right|\right] \\ &\quad \times \sum_{j=1}^{3} \|L_{0,j}\phi\| + k_5\|\phi\|, \end{aligned}$$

for some $k_5 \in (0, \infty)$ and all $\phi \in \mathcal{D}$. Using some technics explained in [4] we find also

$$\sum_{j=1}^{3} \|L_{0,j}\phi\| \leqslant \sqrt{3/2}(\|\Delta_0\phi\| + \|\phi\|), \qquad \|\vec{L}_0^2\phi\| < \|\Delta_0\phi\| + 3\|\phi\|$$

and then the constant k_1 of (28) can be taken as

$$\begin{aligned} k_1 = &\left(\left|\left|\left|\frac{h}{\Omega}\right|\right|\right| + \left|\left|\left|\frac{Dh}{\Omega}\right|\right|\right|\right) + \sqrt{\frac{3}{2}}\left[2\left|\left|\left|\frac{h}{\Omega}\right|\right|\right| + 2\left|\left|\left|\frac{Dh}{\Omega}\right|\right|\right| \right. \\ &\left. + 3\left(1 + \left|\left|\left|\frac{h}{\Omega}\right|\right|\right|\right)\left(\left|\left|\left|\frac{\Omega h}{\Omega'^2}\right|\right|\right| + \left|\left|\left|\frac{\Omega Dh}{\Omega'^2}\right|\right|\right|\right)\right]. \end{aligned}$$

It follows from (20) that $k_1 < 1/2 < 1$. □

5. BOUND STATES AND RELATIVISTIC SCHRÖDINGER EQUATION

In a quantum theory, the study of the physics of a model consists essentially in two problems, namely the search for the bound states and the construction of the scattering operator. In a relativistic theory, the bound states are related to the discrete part of the spectrum of the mass operator M. More precisely the mass m_B of a bound state is a solution to the eigenvalue problem $M\phi = m_B\phi$, with $\phi \in D(M)$ and $\phi \neq 0$. In our models this equation takes the form

$$\begin{aligned} &M^2\phi(\vec{P}, \vec{Q}) \\ &= 4(\vec{Q}^2 + m^2)\phi(\vec{P}, \vec{Q}) + \int \frac{d\sigma(\vec{Q}')}{\Omega(\vec{P}, \vec{Q}')}\phi(\vec{P}, \vec{Q}')\Xi(\vec{P}, \vec{Q}, \vec{Q}') \\ &= m_B^2\phi(\vec{P}, \vec{Q}) \end{aligned} \tag{29}$$

with $\phi \in D(M^2)$, $\phi \neq 0$, and where Ξ is the interaction kernel of H^2 given by

$$\begin{aligned} \Xi(\vec{P}, \vec{Q}, \vec{Q}') := &(\Omega(\vec{P}, \vec{Q}) + \Omega(\vec{P}, \vec{Q}'))h(\vec{P}, \vec{Q}, \vec{Q}') \\ &+ \int \frac{d\sigma(\vec{Q}'')}{\Omega(\vec{P}, \vec{Q}'')} h(\vec{P}, \vec{Q}', \vec{Q}'')h(\vec{P}, \vec{Q}'', \vec{Q}'). \end{aligned}$$

The function ϕ, when it exists, is continuous on $\vec{P}$ (see below). Thus one can take $\vec{P} = 0$ (centre-of-mass frame) and this equation reduces to

$$\begin{aligned} &4(\vec{Q}^2 + m^2)\varphi(\vec{Q}) + \int \frac{d\vec{Q}'}{4(\vec{Q}'^2 + m^2)}\varphi(\vec{Q}')\Xi(0, \vec{Q}, \vec{Q}') \\ &= m_B^2\varphi(\vec{Q}), \end{aligned} \tag{30}$$

where $\varphi(\vec{Q}) = \phi(0, \vec{Q})$. Now for $\vec{P} = 0$, M^2 is just H^2. Thus the positive square root of the operator involved in (30) is easily taken and leads to

$$2\sqrt{\vec{Q}^2 + m^2}\,\varphi(\vec{Q}) + \int \frac{d\vec{Q}'}{4(\vec{Q}'^2 + m^2)}\varphi(\vec{Q}')c(\vec{Q}, \vec{Q}') = m_B\varphi(\vec{Q}). \tag{31}$$

The operator M, the positive square root of the operator M^2 appearing in (29), satisfies the following properties.

THEOREM 5. – *For all $c \in \mathcal{V}$ the operator M is essentially self-adjoint on the domain $\mathcal{D}$* (26). *Moreover, its spectrum is contained in* $(0, \infty)$ *where* $(0, 2m)$ *contains at most a finite number of eigenvalues m_B, which are solutions to the eigenvalue Eq.* (31).

Remark. – Eq. (29) (or its reduced forms (30), (31)) can be considered as a *relativistic Schrödinger equation* because it plays the role of the Schrödinger equation in Quantum Mechanics: it generates the discrete structure of the bound states.

Historical remark. Looking for a relativistic equation for the Hydrogen atom, Dirac rejected the equation:

$$(\sqrt{-\Delta + m^2} - e^2/r) f = Ef,$$

which is the first natural attempt to a relativistic generalization of the Schrödinger equation. The reason was that it is not invariant under Lorentz transformations (see the discussion in [9]). Now, the analogue of this equation in our models is (31), which cannot satisfy such an invariance, because it has been obtained in a given referential frame (moreover, $\vec{Q}$ is not an ordinary momentum, because $\|\vec{Q}\|$ is Lorentz invariant). In fact the relativistic equation is (29) and its invariance is not obvious, because it involves $\vec{L}$ given by (8).

Proof of Theorem 5. – The spectrum of M_0^2 given by (6) covers the interval $[4m^2, \infty)$ and is absolutely continuous. Then M_0^2 is essentially self-adjoint on $\mathcal{D}$. From (20) and (24) it follows that $M^2 - M_0^2$ is bounded. Then M^2 is essentially self-adjoint on $\mathcal{D}$, and so is M.

Let $d\mathcal{E}(\rho, \vec{P})$ be the simultaneous spectral measure of H and $\vec{P}$, where $\rho > 0$ is the spectral variable associated with H. In the spectral representation M is the multiplication by $(\rho^2 - \vec{P}^2)^{1/2}$. Because M is a Casimir operator, the invariant subspaces are limited in the $(\rho, \vec{P})$ space by half-hyperboloids $\rho = (\vec{P}^2 + K^2)^{1/2}$ with $K > 0$.

Let us introduce the family of Hilbert spaces

$$\mathcal{H}_{\vec{P}} := L^2(\mathbb{R}^3, d\sigma_{\vec{P}}) \quad \text{where } d\sigma_{\vec{P}}(\vec{Q}) := d\sigma(\vec{Q})\Omega(\vec{P}, \vec{Q})^{-1} \tag{32}$$

and $\vec{P} \in \mathbb{R}^3$ plays the role of parameters. The Hilbert space $\mathcal{H}$ is the direct integral

$$\mathcal{H} = \int_{\mathbb{R}^3} \bigoplus \mathcal{H}_{\vec{P}} d\vec{P} \tag{33}$$

(see Section 1.5 of [11]). Let $H_{0,\vec{P}}$ and $H_{\vec{P}}$ be the restrictions of H_0 and H to $\mathcal{H}_{\vec{P}}$. For instance $H_{\vec{P}}$ is given by

$$H_{\vec{P}}\varphi(\vec{Q}) = H_{0,\vec{P}}\varphi(\vec{Q}) + \int d\sigma_{\vec{P}}(\vec{Q}')\varphi(\vec{Q}')h(\vec{P}, \vec{Q}, \vec{Q}')$$

for suitable φ, where $H_{0,\vec{P}}\varphi(\vec{Q}) = \Omega(\vec{P}, \vec{Q})\varphi(\vec{Q})$.

From Proposition 3 the integral

$$\int |h(\vec{P}, \vec{Q}, \vec{Q}')|^2 d\sigma_{\vec{P}}(\vec{Q}) d\sigma_{\vec{P}}(\vec{Q}')$$

is well defined, even if we give a small imaginary value to $\vec{P}^2$. Thus (see [4]) the essential spectrum of $H_{\vec{P}}$ is the interval $[\Omega(\vec{P}, 0), \infty)$ and it may exist a finite number of eigenvalues in $(0, \Omega(\vec{P}, 0))$, which form continuous hypersurfaces in the $(\rho, \vec{P})$-space. These hypersurfaces have to be half-hyperboloids $\rho = (\vec{P}^2 + m_B^2)^{1/2}$ for $m_B \in (0, 2m)$ solutions to the eigenvalue equation for $\vec{P} = 0$, that is to (31). □

6. SCATTERING STATES AND SCATTERING MATRIX

A vector $\psi \in \mathcal{H}$ is said to be a "two particle scattering state" if its time evolution, for very large time (or very small time), is not distinguishable from the two free particle evolution, that is, if there exist ψ_{out} (or ψ_{in}) in $\mathcal{H}$ such that $\psi = U^- \psi_{out}$ (or $\psi = U^+ \psi_{in}$), where the 'wave operators' U^+, U^- are given by

$$U^{\pm} := s\text{-}\lim_{t \to \mp\infty} e^{itH} e^{-itH_0}. \tag{34}$$

Here H_0 is the two free particle Hamiltonian given in (3). When the operators (34) exist, they map any vector of $\mathcal{H}$ on a two-particle scattering state.

PROPOSITION 6. – *For all $c \in \mathcal{V}$ the operators $U^{\pm}$ exist on all $\mathcal{H}$.*

The proof can be easily adapted from the corresponding proof in [4].

Proposition 6 leads to the existence of two particle states, defined as the states contained in the range of the operators $U^{\pm}$. Now two particle scattering processes can be obtained by computing the matrix elements of the scattering operator $S := U^{-*} U^+$. However the question of the existence of other states (apart from the bound states) naturally arises, which is related to the asymptotic completeness problem. We will give a partial answer to this question. In Quantum Mechanics, the asymptotic completeness problem is stated once the separation of the centre-of-mass motion has been done. In the relativistic case, this separation cannot be performed, but we can restrict ourselves to a particular frame, given by the choice of a fixed $\vec{P} \in \mathbb{R}^3$. We consider again the Hilbert space decomposition (33) and the restrictions $H_{\vec{P}}$, $H_{0,\vec{P}}$, $\vec{L}_{\vec{P}}$, $\vec{J}_{\vec{P}}$ and $M_{\vec{P}}$, acting on suitable domains of $\mathcal{H}_{\vec{P}}$, still given by the same formulas, but with fixed $\vec{P}$. Now for each $\vec{P}$ the "wave operators"

$$U^{\pm}_{\vec{P}} = s\text{-}\lim_{t \to \mp\infty} e^{itH_{\vec{P}}} e^{-itH_{0,\vec{P}}} \tag{35}$$

can be constructed on $\mathcal{H}_{\vec{P}}$ (their existence follows from the same proof as Proposition 6, with fixed $\vec{P}$). The question of the asymptotic completeness will be discussed in $\mathcal{H}_{\vec{P}}$. According to [10], it consists of two statements: (1) the Hamiltonian $H_{\vec{P}}$ has no singular continuous spectrum and (2) the ranges of $U^{\pm}_{\vec{P}}$ coincide with the subspace of $\mathcal{H}_{\vec{P}}$ corresponding to the absolutely continuous part of the spectrum of $H_{\vec{P}}$. Only the second statement, called the "completeness of the wave operators", will be considered here. We note that it is sufficient to insure the unitarity of the scattering operator $S_{\vec{P}} := U^{-*}_{\vec{P}} U^{+}_{\vec{P}}$.

THEOREM 7. – *For all $c \in \mathcal{V}$ and for all $\vec{P} \in \mathbb{R}^3$, the ranges of the $U^{\pm}_{\vec{P}}$ coincide with the absolutely continuous subspace of $H_{\vec{P}}$ in $\mathcal{H}_{\vec{P}}$ (i.e., the wave operators are complete). Then the scattering operator S is unitary.*

Proof. – Let $\vec{P} \in \mathbb{R}^3$ be fixed. Appendix A and the point (4) of Proposition 3 imply that $H_{\vec{P}} - H_{0,\vec{P}}$ is trace class in $\mathcal{H}_{\vec{P}}$. Then the ranges of $U^{\pm}_{\vec{P}=0}$ coincide with the absolutely continuous subspace of $H_{\vec{P}=0}$ (Kato–Rosenblum theorem, Section 6.2, [11]). □

The full asymptotic completeness (with absence of singular continuous spectrum) needs probably more restrictions on c (see [6] for the 2-d case).

7. AN EXAMPLE AND THE NON-TRIVIALITY

To establish the non-triviality we consider an example for which c is the kernel of a one-dimensional range operator. This has the advantage of leading to explicit results.

Let $\zeta \in C^3_0([0,\infty))$ with support in $[0, R]$, with uniform norm bounded by

$$\|\zeta\|_\infty \leqslant \sqrt{\frac{3}{8\pi}}\left(\frac{m}{R}\right)^{3/2} \tag{36}$$

and bounded below by

$$\zeta(r) \geqslant \begin{cases} \sqrt{\frac{1}{4\pi}}\left(\frac{m}{R}\right)^{3/2} & \text{if } r < \frac{R}{2}, \\ 0 & \text{if } r \geqslant \frac{R}{2}, \end{cases} \tag{37}$$

where R and m are the parameters appearing in Definition 2.

PROPOSITION 8. – *Let c be the function given by*

$$c(\vec{Q}, \vec{Q}') = \lambda\zeta(\|\vec{Q}\|)\,\zeta(\|\vec{Q}'\|) \tag{38}$$

for all $(\vec{Q}, \vec{Q}') \in \mathbb{R}^6$, ζ as above and $\lambda \in \mathbb{R}$. For all $|\lambda| \leqslant 1$ the function c is the interaction kernel in the centre-of-mass frame of a continuous *unitary representation of the Poincaré group. This representation admits a mass operator M with absolutely continuous spectrum $[2m, \infty)$ and at most a finite number of eigenvalues in $(0, 2m)$, and a unitary scattering operator S. Moreover*

(i) $S \neq Id$,

(ii) *M has an eigenvalue $m_B \in (0, 2m)$ if $\lambda = -1$ and $m/R > 2$.*

Proof. – The function (38) belongs to $\mathcal{V}$. Thus the first part of the proposition follows from Theorems 4, 5, 7 and Proposition 6. Moreover the completeness of the wave operators implies that the absolutely continuous spectrum of H (of M) is the same as for H_0 (for M_0).

Proof of (i). Note that $\mathcal{H}$ carries two different unitary continuous representations of the Poincaré group, the free representation generated by $\{H_0, \vec{P}, \vec{J}, \vec{L}_0\}$ and the interaction representation generated by $\{H, \vec{P}, \vec{J}, \vec{L}\}$. Because H_0 and $\vec{P}$ commute they admit a simultaneous spectral measure $d\mathcal{E}_0(E, \vec{P})$ where $E > 0$ is the spectral variable associated with H_0. In the spectral representation M_0 becomes the multiplication operator by $(E^2 - \vec{P}^2)^{1/2}$. Because $M_0 \geqslant 2m$ the support of $d\mathcal{E}_0$ is

$$\{(E, \vec{P}) \mid E^2 - \vec{P}^2 \geqslant 4m^2\}.$$

For such $(E, \vec{P})$ let us consider the space $d\mathcal{E}_0(E, \vec{P})\mathcal{H} =: \mathcal{H}_{E,\vec{P}}$ which appears in the spectral decomposition of $\mathcal{H}_{\vec{P}}$ which diagonalizes $H_{0,\vec{P}}$

$$\mathcal{H}_{\vec{P}} = \int_{sp(H_{0,\vec{P}})} \bigoplus \mathcal{H}_{E,\vec{P}} dE,$$

where $sp(H_{0,\vec{P}})$ is the spectrum of $H_{0,\vec{P}}$, that is the interval $[\Omega(\vec{P}, 0), \infty)$. The operator $S_{\vec{P}}$ restricted to $\mathcal{H}_{E,\vec{P}}$ is denoted $S_{E,\vec{P}}$ and is called the 'scattering matrix'. The action of $\vec{L}_0$ on the variables $(E, \vec{P})$ is easily computed. For $\vec{\beta} \in \mathbb{R}^3$, $\|\vec{\beta}\| < 1$, the Lorentz boost $L_0(\vec{\beta})$ of speed $\vec{\beta}$ (from which $\vec{L}_0$ are the infinitesimal generators) acts as follows

$$L_0(\vec{\beta})\mathcal{H}_{E,\vec{P}} = \mathcal{H}_{\Lambda_{\vec{\beta}}(E,\vec{P})}, \tag{39}$$

where $\Lambda_{\vec{\beta}}$ is the Lorentz matrix of speed $\vec{\beta}$. It follows from (39) that the relation between scattering matrix is

$$L_0(\vec{\beta}) S_{E,\vec{P}} L_0(\vec{\beta})^{-1} = S_{\Lambda_{\vec{\beta}}(E,\vec{P})} \tag{40}$$

which shows that S is known once $S_{E,0}$ is given. Let us restrict ourselves to $\mathcal{H}_{\vec{P}=0}$ on which the interaction operator

$$V_0 = H_{\vec{P}=0} - H_{0,\vec{P}=0}$$

has one-dimensional range, that is $V_0\varphi = \lambda(\eta, \varphi)_0\eta$ where $(.,.)_0$ is the scalar product of $\mathcal{H}_{\vec{P}=0}$ and $\eta(\vec{Q}) = \zeta(\|\vec{Q}\|)$. According to [11], Theorem 3 of Section 6.7, the scattering operator gives for all $v \in \mathcal{H}_{E,\vec{P}=0}$

$$\left(v, (S_{E,0} - I)v\right)_{E,0} = -2\pi\, \mathrm{i}\lambda D(E + \mathrm{i}0)^{-1} \left|(\eta_E, v)_{E,0}\right|^2, \tag{41}$$

where

$$D(z) = 1 + \lambda\left(\eta, (H_{0,0} - z)^{-1}\eta\right)_0 \tag{42}$$

for all $z \in \mathbb{C}$, $Im z > 0$. In (41), $(.,.)_{E,0}$ is the scalar product of $\mathcal{H}_{E,\vec{P}=0}$ and η_E is the restriction of η to $\mathcal{H}_{E,\vec{P}=0}$. In fact

$$\eta_E = \zeta\left(\tfrac{1}{2}\sqrt{E^2 - 4m^2}\right)$$

is a constant in $\mathcal{H}_{E,\vec{P}=0}$ so that

$$(\eta_E, v)_{E,0} = \zeta\left(\tfrac{1}{2}\sqrt{E^2 - 4m^2}\right)(1, v)_{E,0},$$

which is non-zero for all $2m < E < \sqrt{R^2 + 4m^2}$ (see the definition of η) and for all v in a large set of $\mathcal{H}_{E,\vec{P}=0}$. On the other hand, for E as above let us compute the limit needed in (41)

$$D(E + \mathrm{i}0) = 1 - \lambda\, \mathrm{i}\pi^2 \frac{\sqrt{E^2 - 4m^2}}{2E} \left|\zeta\left(\frac{1}{2}\sqrt{E^2 - 4m^2}\right)\right|^2$$
$$+ \frac{\pi}{2} p.v. \int_{2m}^{2\sqrt{R^2+m^2}} \frac{du}{u} \frac{\sqrt{u^2 - 4m^2}}{u - E} \left|\zeta\left(\frac{1}{2}\sqrt{u^2 - 4m^2}\right)\right|^2$$

which has obviously a non-zero imaginary part. Then $S_{E,0} - I$ is non-zero for all such E. Because the representation in $\mathcal{H}$ generated by $\{H_0, \vec{P}, \vec{J}, \vec{L}_0\}$ is continuous, if follows from (40) that the scattering operator S differs from identity.

Proof of (ii). From Theorem 5 with (38) and $\lambda = -1$ Eq. (31) can be written as follows

$$\varphi(\vec{Q}) = K \frac{\zeta(\|\vec{Q}\|)}{2\omega(\vec{Q}) - m_B},$$

where K is a constant. This function belongs to $D(M_{\vec{P}=0}) \subset \mathcal{H}_{\vec{P}=0}$ and is an eigenfunction of $M_{\vec{P}=0}$ provided the implicit equation

$$1 = \int \frac{d\vec{Q}}{4(\vec{Q}^2 + m^2)} \frac{\zeta(\|\vec{Q}\|)^2}{2\sqrt{\vec{Q}^2 + m^2} - m_B} \tag{43}$$

admits a solution $0 < m_B < 2m$. To study this last question we consider the right hand side of (43) as a function of the variable m_B. For $m_B = 0$ we get from (36)

$$\int \frac{d\vec{Q}}{4(\vec{Q}^2 + m^2)} \frac{\zeta(\|\vec{Q}\|)^2}{2\sqrt{\vec{Q}^2 + m^2}} \leqslant \|\zeta\|_\infty^2 \int_{\mathcal{K}} \frac{d\vec{Q}}{8(\vec{Q}^2 + m^2)^{3/2}}$$

$$\leqslant \frac{m^3}{2|\mathcal{K}|} \frac{|\mathcal{K}|}{8m^3} = \frac{1}{16} < 1.$$

Now it follows from (37) that for $m_B = 2m$ the right hand side of (43) is majorized as follows

$$\int \frac{d\vec{Q}}{4(\vec{Q}^2 + m^2)} \frac{\zeta(\|\vec{Q}\|)^2}{2\sqrt{\vec{Q}^2 + m^2} - 2m}$$

$$= \frac{\pi}{2} \int_0^R dr\, d(r)^2 \frac{\sqrt{r^2 + m^2} + m}{r^2 + m^2} \geqslant \frac{m^3}{8R^3} \int_0^{R/2m} dr \frac{1 + \sqrt{r^2 + 1}}{r^2 + 1}$$

$$= \left(\frac{m}{2R}\right)^3 \left(\arctan \frac{R}{2m} + \arg\sinh \frac{R}{2m}\right)$$

which is > 1 for $m/R > 2$. Thus (43) has a (unique) solution $0 < m_B < 2m$, which is an eigenvalue of (31) and then an eigenvalue of M. □

8. CONCLUSION

Let us sum up what we have found. We have constructed a family of unitary, continuous representations of the Poincaré group in four space-time dimensions, as perturbations of the two free boson model. The scattering operator S can be performed and is unitary and non-trivial in some cases.

The physical content of this mathematical construction is based on Wigner's famous interpretation [13], according to which an elementary particle is described by an irreducible, unitary, and continuous representation of $\mathcal{P}_+^\uparrow$. In the construction of S we have verified the existence of scattering states, i.e., states which for large (or small) time approach the tensor product of two irreducible unitary representations of $\mathcal{P}_+^\uparrow$, interpreted as describing two free particles. This allows us to assert that our models really describe relativistic quantum systems of two particles. Moreover the non-triviality of S allows us to claim that the interaction between the particles can be effective.

The absence of particle creation or annihilation makes these models interesting for low energy physics, in particular for the bound state problem. The existence of a bound state equation similar to the ordinary Schrödinger equation of Quantum Mechanics (and which can be called a "relativistic Schrödinger equation") is probably the strongest result of the paper.

However, these results have been obtained for interaction operators $\mathcal{O}$ with ranges contained in the subspace of zero angular momentum (the so-called "s-wave subspace"). To get a better characterization of this limitation, let us compute the change of the second Casimir operator, the Pauli–Lubanski operator, given by (1). In the representation (8) it becomes

$$W^2 = W_0^2 + \{\{\mathcal{O}, \vec{W}_0\}, \vec{W}_0\} + \{\mathcal{O}, \vec{W}_0\}^2$$

where W_0^2 is given by (6) and $\vec{W}_0 = -H_0\vec{J} + \vec{P} \wedge \vec{L}_0$. Now condition (13) implies $\{\mathcal{O}, \vec{W}_0\} = 0$ and thus

$$W^2 = W_0^2.$$

Our interactions have no effect on the Pauli–Lubanski operator.

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APPENDIX A. A CRITERIUM FOR TRACE-CLASS OPERATORS

Let $\mathcal{H}_s = L^2(\mathbb{R}^3, s)$ be a Hilbert space where s is a positive σ-finite measure, and let F be an integral operator on $\mathcal{H}_s$ of kernel $f : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{C}$, that is

$$(Fu)(x) = \int_{\mathbb{R}^3} f(x, y)u(y)ds(y) \tag{44}$$

for all suitable $u \in \mathcal{H}_s$.

LEMMA 9. – *Let K be a compact subset of $\mathbb{R}^3$ and $f \in L^2(\mathbb{R}^3 \times \mathbb{R}^3, s \otimes s)$ such that for almost all $x \in \mathbb{R}^3$ the function $y \mapsto f(x, y)$ has support in K. Let us suppose that the three derivatives*

$$\partial_{y_1} f(x, y), \quad \partial_{y_2}\partial_{y_1} f(x, y), \quad \partial_{y_3}\partial_{y_2}\partial_{y_1} f(x, y)$$

exist in $L^2(\mathbb{R}^3 \times \mathbb{R}^3, s \otimes s)$. Then F, defined by (44), *is trace class.*

Proof. – We follow [12, Section XI.9.32]. We use that a product of two Hilbert–Schmidt operators is a trace class operator. Let $k > 0$ be such that $K \subset [-k, k]^3$. Let $u \in \mathcal{H}_s$. From three integrations by parts we get

$$(Fu)(x) = \int_{[-k,k]^3} \left[\partial_{y_3}\partial_{y_2}\partial_{y_1} f(x, y)\right] \int_{-k}^{y_1}\int_{-k}^{y_2}\int_{-k}^{y_3} u(t)\, dt\, ds(y)$$

$$= \int_{\mathbb{R}^3} \left[\partial_{y_3}\partial_{y_2}\partial_{y_1} f(x, y)\right] \int_{-k}^{y_1}\int_{-k}^{y_2}\int_{-k}^{y_3} \chi_{[-k,k]^3}(y)u(t)\, dt\, ds(y).$$

We have written F as a product of two Hilbert–Schmidt operators, the first one of kernel

$$(x, y) \mapsto \partial_{y_3}\partial_{y_2}\partial_{y_1} f(x, y)$$

and the second one of kernel

$$(x, y) \mapsto \chi_{[-k,k]^3}(x)\chi_{[-k,x_1]}(y_1)\chi_{[-k,x_2]}(y_2)\chi_{[-k,x_3]}(y_3).$$

Thus F is trace class. □

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