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ERIC DELABAERE
FRÉDÉRIC PHAM

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Resurgent methods in semi-classical asymptotics

by

Eric DELABAERE[1], **Frédéric PHAM**[2]

UMR No 6621 du CNRS, laboratoire J.A. Dieudonné,
Université de Nice-Sophia Antipolis, 06108 Nice Cedex 2, France

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ABSTRACT. – The present paper is a self-contained introduction to resurgent methods in semi-classical asymptotics. The first two sections explain how such well-known notions as Stokes phenomena for one dimensional WKB expansions fit in the framework of resurgence theory, thus making WKB analysis a tool for exactly handling wave functions. The remaining sections are devoted to a thorough study of quadratical confluence, i.e., what happens near values of the parameters where two simple turning points coalesce. © Elsevier, Paris

Key words: Borel resummation, resurgence theory, semi-classical asymptotics, Stokes phenomena, confluence

RÉSUMÉ. – Cet article présente une introduction aux méthodes résurgentes en asymptotique semi-classique. Les deux premiers chapitres expliquent comment des notions bien connues comme les phénomènes de Stokes pour les développements BKW à une dimension s'interprétent dans le cadre de la théorie de la résurgence. La suite de l'article est consacrée à une étude complète de la confluence quadratique, i.e., lorsqu'on se place près des valeurs des paramètres ou deux points tournants simples viennent en coïncidence. © Elsevier, Paris

[1] Member of CNRS. E-mail: delab@math.unice.fr.

[2] E-mail: fpham@math.unice.fr.

INTRODUCTION

In our common work [1] with Hervé Dillinger we gave a set of rules for dealing with WKB expansions in the one dimensional analytic case, whereby such expansions are not considered as approximations but as *exact encodings* of wave functions, thus allowing for analytic continuation with respect to whichever parameters the potential function depends on, with an exact control of small exponential effects.

Among various illustrations of these rules, we investigated the case of the symmetrical anharmonic oscillator, proving the Zinn-Justin quantization condition [2] (and giving a rigorous meaning to its solutions in terms of "multi-instanton expansions). In another article [3] we also proved in this way the long-standing conjecture of Bender and Wu [4] on the ramification of the energy levels in the complex plane of the coupling constant.

The present paper is a self-contained exposition of the mathematical apparatus on which this set of rules is based.

Section 0 is an overview of the theory of resurgent functions, following rather closely the ideas and notations of Ecalle, and supplementing them with considerations on the dependence of resurgent functions on parameters (introducing a general notion of *analytic dependence*, and a much stronger one which we call *regular dependence*).

Sections 1 and 2 explain how such "well-known" notions as Stokes phenomena for one dimensional WKB expansions fit in the framework of resurgence theory, thus making WKB analysis a tool for *exactly* handling wave functions. The main results of [5] (inspired by ideas of Voros [6,7] and Ecalle [8,9]) are re-exposed in this spirit, and made more precise and general, investigating to which extent they do not depend on the "generic" hypotheses made in [5] (e.g., the hypothesis that all turning points are *simple*).

The remaining sections are devoted to a thorough study of "quadratic confluence", i.e., what happens near values of the parameters where two simple turning points coalesce.

Section 3 describes the geometry of quadratic confluence, and the corresponding "local resurgence relations".

Section 4 uses this description to give "universal models" for quadratic confluence: a universal expression for Stokes multipliers in terms of the Euler Gamma function (Section 4.1); and a universal expression for the wave functions in terms of Weber parabolic cylinder functions (Section 4.2).

Section 5 shows how these universal models can be used to perform explicit computations on what we call the "rescaled" Schrödinger equation, where the energy parameter is considered to be infinitely close to a quadratic critical value of the potential function (so that a *double* turning point occurs).

Bibliographical comments. Our universal expression for wave functions in Section 4.2 looks very similar to the one given by Ahmedou ould Jidoumou ([10] Section 4). The difference is that we presently deal with two parameters (q, E), whereas Ahmedou ould Jidoumou dealt only with the single [3] parameter q; his result can be seen as a "specialization" of ours, after "rescaling" the energy as we do in Section 5.

Of course expressing wave functions near a double turning point in terms of parabolic cylinder functions is not a new idea. But unlike the so-called "uniform approximations" of traditional asymptotics [11] [4] our Theorem 4.2.1 provides us with an *exact* representation of wave functions. It bears close analogy with a recent result of Kawai and Takei [13] (see also [14]), which uses microdifferential operators. Our approach differs from theirs in making no use of (micro-) differential operators of any sort, only using (resurgent) functions. But like theirs, it uses a canonical transformation, thus making a further step towards understanding *quantized canonical transformations of resurgent functions.* [5]

The problem of extending the results of this second part to turning points of higher multiplicities is studied theoretically in [16].

0. A BIRD'S EYE VIEW ON RESURGENCE

0.0. The Borel–Laplace correspondence for power series

In what follows we denote by

- $\mathbb{C}[[x^{-1}]]$ the ring of formal integral power series in x^{-1}, with complex coefficients.
- $\mathcal{O}_0 = \mathbb{C}\{\xi\}$ the ring of complex holomorphic functions near the origin.

[3] Working with *one* parameter only is the right thing to do for analyzing confluence near a *simple* turning point (Airy-type confluence, studied in Chapters 1 and 2 of [10]); but *double* turning points are objects *of codimension* 2, and this is why the corresponding "universal models" must depend on *two* parameters.

[4] And their more sophisticated versions using pseudo-differential operators (cf. [12]).

[5] First steps in this direction were taken in [15].

A formal integral power series

$$\varphi = \sum_{n=0}^{\infty} a_n x^{-n} \in \mathbb{C}[[x^{-1}]] \tag{0.1}$$

is said to be *of Gevrey class* 1 if

$$|a_n| < n! C_n,$$

where (C_n) is a geometrical sequence.

Forgetting in (0.1) the a_0 term, and replacing x^{-n} by $\frac{\xi^{n-1}}{(n-1)!}$, we get a formal power series

$$\widehat{\varphi} = \sum_{n=1}^{\infty} a_n \frac{\xi^{n-1}}{(n-1)!} \tag{0.2}$$

which is called the *Borel transform* of $\varphi = \sum_{n=1}^{\infty} a_n x^{-n}$. Of course demanding φ to be of Gevrey class 1 is equivalent to demanding $\widehat{\varphi}$ to have a non-zero radius of convergence. We can thus identify $\widehat{\varphi}$ with a holomorphic function $\widehat{\varphi} \in \mathcal{O}_0$, which (following Ecalle) we shall call the *minor* of φ.

By the simple formula

$$x^{-n} = \int_0^{\infty} \mathrm{e}^{-x\xi} \frac{\xi^{n-1}}{(n-1)!} d\xi \tag{0.3}$$

we can consider that apart from the a_0 term (which has been dropped in the process of taking Borel transform), the formal power series φ is given by the "formal Laplace integral"

$$\fint_0^{\infty} \mathrm{e}^{-x\xi} \widehat{\varphi}(\xi)\, d\xi, \tag{0.4}$$

where the "formal integral" $\fint$ means that we replace the function $\widehat{\varphi}$ by its Taylor series and exchange the $\sum$ and $\int$ symbols.

But of course if the minor $\widehat{\varphi}$ is defined only in a neighbourhood of 0 formula (0.4) cannot be understood as an integral in the usual sense. Replacing it by a *truncated* integral

$$\int_0^{\kappa} \mathrm{e}^{-x\xi} \widehat{\varphi}(\xi)\, d\xi \tag{0.4'}$$

we get a function of x having $\sum_{n=1}^{\infty} a_n x^{-n}$ as its asymptotic expansion when $\Re(x) \to +\infty$, and depending on the cut-off κ modulo exponentially small terms only.

Extension to non-integral power series. Noticing that formula (0.3) still holds with n replaced by any complex number ν such that $\Re(\nu) > 0$ (with $(n-1)!$ replaced by $\Gamma(\nu)$), we get an obvious generalization of the above constructions to non-integral power series. Now the minor $\widehat{\varphi}$ is no longer holomorphic in a disc around 0, but multivalued analytic over a punctured disc $0 < |\xi| < \rho$.

The Borel–Laplace correspondence for majors. One can get rid of the restriction $\Re(\nu) > 0$ by replacing the integration axis $[0, \infty[$ in (0.4) by a contour γ as indicated on Fig. 1, and the minor $\widehat{\varphi}$ by what Ecalle calls a *major* $\check{\varphi}$. Instead of formula (0.3) we take, as the building stone of our construction, the following formula

$$x^{-\nu} = \int_{\gamma} \mathrm{e}^{-x\xi} \check{l}_{\nu}(\xi)\, d\xi, \tag{0.5}$$

where

$$\check{l}_{\nu}(\xi) = \begin{cases} \dfrac{-1}{2\mathrm{i}\sin(\pi\nu)} \dfrac{(-\xi)^{\nu-1}}{\Gamma(\nu)} & \text{if } \nu \neq 1, 2, 3, \ldots, \\ \xi^{\nu-1} \dfrac{\ln \xi}{2\pi\mathrm{i}\Gamma(\nu)} & \text{if } \nu - 1 \in \mathbf{N}. \end{cases}$$

This formula allows us to work with power series

$$\varphi = \sum_{\nu} a_{\nu} x^{-\nu},$$

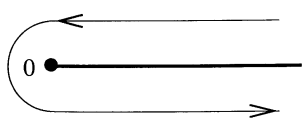

Fig. 1. The integration contour γ for majors.

where $\Re(\nu)$ need not be bounded from below. With suitable growth conditions on the a_ν's, the major of φ

$$\check{\varphi} = \sum_\nu a_\nu \check{l}_\nu$$

will be multivalued analytic over a punctured disc $0 < |\xi| < \rho$. By formula (0.5) we can recover φ from $\check{\varphi}$ by *formal* Laplace integration

$$\varphi(x) = \fint_\gamma \mathrm{e}^{-x\xi} \check{\varphi}(\xi)\, d\xi. \tag{0.6}$$

To give integral (0.6) a meaning other than formal, all we have to do is to replace the endless path γ by a truncated path γ_κ = that part of γ for which $\Re(\xi) \leqslant \kappa$, where κ is a positive number, small enough so that $\check{\varphi}$ is holomorphic on γ_κ. The resulting function φ_κ depends on the cut-off κ modulo exponentially small terms only, and it has φ as its asymptotic expansion.

0.1. The general Borel–Laplace correspondence

The above construction can obviously be generalized to formal series of functions other than power functions of x (e.g., series including logarithms, etc.), provided we know a corresponding generalization of (0.5) (for instance we may look in books giving Laplace transforms of remarkable functions!). In every case, growth conditions will be imposed so that the resulting series $\check{\varphi}$ should converge in a suitable domain close to 0. If we are not keen on explicit formulas we need not even bother about tables of "explicit" Laplace transform: forgetting about formal series of "explicit" functions, we may just start from the following general definition of a major.

DEFINITION 0.1.1. – *A major $\check{\varphi}$ is any germ at* 0 *of a holomorphic function in a split disc* $|\xi| < \varepsilon$, $\arg(\xi) \neq 0$, *analytically continuable along any path of the punctured disc* $0 < |\xi| < \varepsilon$.

Given a major $\check{\varphi}$, truncated integrals

$$\varphi_\kappa(x) = \int_{\gamma_\kappa} \mathrm{e}^{-x\xi} \check{\varphi}(\xi)\, d\xi \tag{0.7}$$

define functions of x which are of exponential type 0 (i.e., for every $\tau > 0$ they satisfy the inequalities

$$|\varphi_\tau(x)| < C\,\mathrm{e}^{\tau x}$$

far away in some sector containing the positive real axis). Furthermore they depend on the cut-off κ modulo exponentially small terms only. Denoting by $\mathcal{E}^0$ (respectively, $\mathcal{E}^{<0}$) the space of functions of exponential type 0 (respectively, of negative exponential type), we can thus define the formal Laplace transform as follows.

DEFINITION 0.1.2. – *The formal Laplace transform of a major* $\check{\varphi}$, *expressed symbolically by formula* (0.6), *is the element of* $\mathcal{E}^0/\mathcal{E}^{<0}$ *defined as the equivalence class modulo* $\mathcal{E}^{<0}$ *of the truncated integrals* (0.7).

Those elements of $\mathcal{E}^0/\mathcal{E}^{<0}$ *which are obtained in this manner will be called* 0-*symbols.*

With this definition, one can prove that two majors have the same formal Laplace transform *iff* their difference is holomorphic at $\xi = 0$. This suggests replacing majors $\check{\varphi}$ by their equivalence class modulo $\mathcal{O}_0 = \mathbb{C}\{\xi\}$.

DEFINITION 0.1.3. – *A microfunction at* 0 *is the equivalence class of a major modulo* $\mathcal{O}_0$. *Given a major* $\check{\varphi}$, *the corresponding microfunction is denoted by* $\overset{\triangledown}{\varphi}$, *and called the singularity of* $\check{\varphi}$ *at* 0.

In this way, the formal Laplace transformation defines a 1–1 *correspondence between microfunctions at* 0 *and* 0-*symbols.*

Small 0-symbols, and the Borel–Laplace correspondence for minors. Analytically continuing a major $\check{\varphi}$ on both sides of the cut as indicated on Fig. 2, and taking the difference

$$\widehat{\varphi}(\xi) = \check{\varphi}(\xi - \mathrm{i}0) - \check{\varphi}(\xi + \mathrm{i}0)$$

we get a function which is holomorphic in a neighbourhood of the segment $]0, \varepsilon[$, and continuable along any path of the punctured disc.

Of course this function $\widehat{\varphi}$ depends only on $\overset{\triangledown}{\varphi}$, and therefore on the 0-symbol φ. Its is called the *minor* of φ (or the *variation* of $\overset{\triangledown}{\varphi}$).

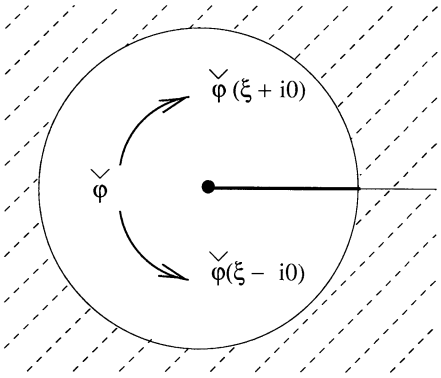

Fig. 2.

The minor of φ does not carry complete information on φ, because it vanishes whenever $\check{\varphi}$ is a Laurent series (with negative integral powers of ξ, corresponding to non-negative integral powers of x). A noteworthy exception is the case of *small* 0-*symbols*, which we now proceed to define.

Integrable majors and small 0-symbols. A major $\check{\varphi}$ is called *integrable* if all its analytic continuations over the punctured disc behave like $o(1/|\xi|)$ when $\xi \to 0$, uniformly in every angular sector of finite aperture. Of course this property depends only on $\overset{\nabla}{\varphi}$, and therefore on φ. We then say that $\overset{\nabla}{\varphi}$ is an *integrable microfunction*, or that φ is a *small* 0-*symbol*. The reason for the latter denomination is that φ is small *iff* it is represented (modulo $\mathcal{E}^{<0}$) by a function which tends to zero uniformly at infinity in a sector containing the positive real axis.

PROPOSITION 0.1.1 (The Borel–Laplace correspondence between small 0-symbols and integrable minors). – *Let g be a function, holomorphic in a neighbourhood of the segment* $]0, \varepsilon[$, *analytically continuable along every path of the punctured disc* $0 < |\xi| < \varepsilon$. *Then the following two properties are equivalent*:

(i) *g is the minor of a small* 0-*symbol φ.*

(ii) *$g = dG/d\xi$, where all analytic continuations of G over the punctured disc tend to some constant* (*say* 0) *when ξ tends to* 0, *uniformly on every sector of finite aperture.*

We summarize property (ii) by saying that g is an *integrable minor*.

The 1–1 correspondence between integrable minors and small 0-symbols is given by the formal Laplace integral

$$^{\flat}g(x) = \fint_0^{\infty} \mathrm{e}^{-x\xi} g(\xi)\, d\xi, \tag{0.8}$$

where $\fint_0^{\infty}$ must again be understood as the equivalence class modulo $\mathcal{E}^{<0}$ of truncated integrals $\int_0^{\kappa}$, and $^{\flat}g$ is a notation for the small 0-symbols with minor g.

The algebra of 0-symbols. A convolution operation can be defined on microfunctions, such that the corresponding operation on 0-symbols is the usual product of functions (or rather their classes modulo the ideal $\mathcal{E}^{<0} \subset \mathcal{E}^0$). The space of 0-symbols can thus be considered as a subalgebra of $\mathcal{E}^0/\mathcal{E}^{<0}$.

The convolution operation is defined on majors by suitable convolution integrals on truncated paths, whose class modulo $\mathcal{O}_0$ is known not to depend on the truncation. In the case of *small* 0-symbols the corresponding *convolution of* (*integrable*) *minors* is very easy to define: it is

$$(\widehat{\varphi} * \widehat{\psi})(\xi) = \int_0^{\xi} \widehat{\varphi}(\xi - \eta)\widehat{\psi}(\eta)\, d\eta,$$

where the integral is taken over the straight segment $]0, \xi[$.

Simply ramified and simple microfunctions. Analytically continuing a major across the cut, in the anticlockwise direction, we get another major. The corresponding operation on microfunctions is the "monodromy operator". When the monodromy operator acts on $\overset{\nabla}{\varphi}$ as the identity operator we say that $\overset{\nabla}{\varphi}$ is a *simply ramified* microfunction. This is equivalent to saying that the minor $\widehat{\varphi}$ is holomorphic at 0

$$\widehat{\varphi} \in \mathcal{O}_0$$

and this implies that a major reads

$$\overset{\vee}{\varphi} = \widehat{\varphi}\frac{\ln \xi}{2\pi \mathrm{i}} + f(\xi),$$

where f is a convergent Laurent series.

An interesting special case is the case when f reduces to a simple pole

$$f(\xi) = \frac{a_0}{2\pi \mathrm{i}\xi}.$$

In that case we say that $\overset{\nabla}{\varphi}$ is a *simple microfunction*. Formal Laplace transforms of simple microfunctions are exactly the Gevrey-1 series φ which have been considered at the beginning of Section 0.0. We shall call them *simple* 0-*symbols*. They make up a subalgebra of the algebra of 0-symbols.

0.2. Ecalle's generalization of Borel resummable series: formal resurgent functions

Borel resummation. A formal power series φ is *Borel resummable* if its minor $\widehat{\varphi}$ is holomorphic in a neighbourhood of the positive real axis, growing at most exponentially at infinity. The integral (0.4) then converges for every x with large enough real part, and it defines a holomorphic function of x called the *Borel sum* of the series (0.1) (or rather, of the series (0.1) with the constant term a_0 deleted).

Dropping the growth condition: Borel presummation. When nothing is known on how the minor $\widehat{\varphi}$ grows at infinity, the *truncated* integral (0.4′) still makes sense for every cut-off κ, defining a family of functions (φ_κ). Of course this family does not converge when $\kappa \to \infty$, but one can prove the following

PROPOSITION 0.2.1. – *There exists a function φ such that*

$$\forall \kappa > 0, \exists c_\kappa, \quad |\varphi(x) - \varphi_\kappa(x)| < c_\kappa \, \mathrm{e}^{-\kappa \Re(x)}.$$

Of course this function φ is not unique: one can add to it any function with *hyperexponential decrease*, i.e., any function satisfying bounds of the form $c_\kappa \, \mathrm{e}^{-\kappa \Re(x)}$ for every κ. Denoting by $\mathcal{E}^{-\infty}$ the ideal in $\mathcal{E}^0$ consisting of functions with hyperexponential decrease, we can consider the integral (0.4) as defining an element of $\mathcal{E}^0/\mathcal{E}^{-\infty}$, which is called the *Borel presum* of the series (0.1).

Endless continuability. Ecalle's idea for extending Borel resummation is to replace the analyticity assumption near the real axis by an "endless continuability" assumption in the whole ξ-plane.

DEFINITION 0.2.1 (Adapted from [17], cf. also [18]). – *A holomorphic function f, given in some open subset of $\mathbb{C}$, is called endlessly continuable if for all $L > 0$ there is a finite subset $\Omega_L \subset \mathbb{C}$ such that f can be analytically continued along every path λ of length $< L$ which avoids Ω_L.*

Intuitively, this definition means that the set of singular points is *discrete on the Riemann surface of f*; but it may very well be everywhere dense *when projected on* $\mathbb{C}$!

Right and left re-(or pre-)summations. Assuming the minor $\widehat{\varphi}$ of φ to be endlessly continuable, we no longer can define its Borel resummation because the integrand in (0.4) can have singularities along the positive real axis. But we can define its *right* (respectively, *left*) resummation by a formula similar to (0.4), with the integration axis distorted away from the singularities as shown on Fig. 3.

Of course exponential growth at infinity must again be assumed for $\widehat{\varphi}$ if we want the Laplace integrals to converge at infinity. If no such assumption is made, a proposition analogous to Proposition 0.2.1 allows us to consider that these integrals are defined *modulo* $\mathcal{E}^{-\infty}$, the ideal of functions with hyperexponential decrease. In that case we speak of right and left *presummation* of the formal power series φ. Right and left presummations of φ will be denoted respectively by $\mathrm{S}_+\varphi$ and $\mathrm{S}_-\varphi$.

The above constructions only made use of the analytic continuations of $\widehat{\varphi}$ "in the first sheet". But in order to compare right and left (pre)summations, Fig. 4 shows that we shall also need informations on the singularity structure of $\widehat{\varphi}$ in other sheets.

We shall come back to this problem in Section 0.4.

Of course all these constructions involving minors have a "major" counterpart (whose definition is left to the reader). Using the general

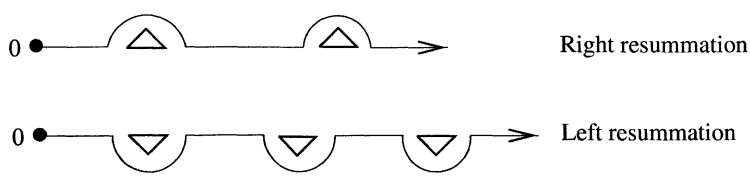

Fig. 3. The integration paths for *right*, respectively, *left* resummation: the singularities to be avoided lie on the *right*, respectively, *left* of the integration path.

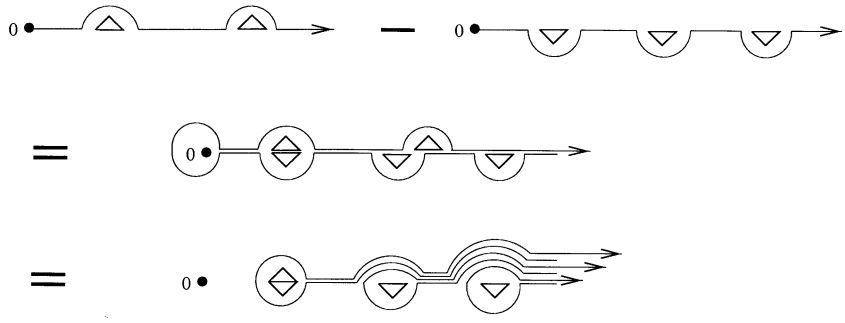

Fig. 4. The difference between right and left (pre)summations (figure above) is described as a Laplace integral along a contour. This integral can be understood as a right (pre)summation by distorting this contour in the complex plane, thus involving to explore other sheets (figure below).

Borel–Laplace correspondence explained in Section 0.1, we are thus led to the following definitions.

DEFINITION 0.2.2. – *A microfunction* $\overset{\nabla}{\varphi}$ *is called endlessly continuable if it has an endlessly continuable major. This is equivalent to saying that the corresponding minor* $\widehat{\varphi}$ *(the variation of* $\overset{\nabla}{\varphi}$*) is endlessly continuable. The formal Laplace transform* φ *of an endlessly continuable microfunction is called a formal resurgent function.*

THEOREM 0.2.1 (The algebra $\mathcal{R}$ of formal resurgent functions). – *Endless continuability is preserved by the convolution operations, so that formal resurgent functions make up a subalgebra* $\mathcal{R}$ *of the algebra of* 0*-symbols. Furthermore, right and left presummations are algebra homomorphisms of* $\mathcal{R}$ *into the algebra* $\mathcal{E}^0/\mathcal{E}^{-\infty}$.

Changing the presummation direction. The endless continuability hypothesis does not pay special attention to the positive real direction, and when we defined right and left presummation we could as well have chosen our integration path along some half-line 0α ($\alpha =$ an arbitrary direction) instead of the positive real axis. The corresponding right and left presummation operators will be denoted by $\mathrm{S}_{\alpha+}$ and $\mathrm{S}_{\alpha-}$. The details of their definition is left to the reader, with a warning: changing the presummation direction should lead naturally to

- changing the direction of the cut in the definition of majors and microfunctions;

- changing the direction of the sectors at infinity in the definition of $\mathcal{E}^0/\mathcal{E}^{-\infty}$ (so that the exponential $e^{-x\xi}$ should still decrease along the integration path).

For various directions α the homomorphisms $s_{\alpha+}$ and $s_{\alpha-}$ are therefore defined between algebras *which depend on* α (they are sheaves of algebras on the circle of all directions). Concerning $\mathcal{E}^0/\mathcal{E}^{-\infty}$ there is no way out of this complication (speaking of exponential type at infinity is meaningless if one does not precise which sectors one is working on). Concerning $\mathcal{R}$ an alternative point of view (the one used by Ecalle) consists in noticing that analytic continuation of majors around the punctured disc induce isomorphisms between $\mathcal{R}_\alpha$ (the algebra of "formal resurgent functions in the direction α") and $\mathcal{R}$. Unfortunately, there are infinitely many such isomorphisms (except in the "simply ramified" case the result of analytic continuation depends on the homotopy class of the path in the punctured disc). If we want $s_{\alpha+}$ and $s_{\alpha-}$ to be unambiguously defined on $\mathcal{R}$ we must therefore consider α as being a direction in $\mathbb{C}^\infty$, *the universal covering of* $\mathbb{C}^\star$ (i.e., the Riemann surface of the logarithm).

This discussion can be summarized by the following formulation: formal resurgent functions make up a *locally constant sheaf* of algebras on the circle of directions, so that on the universal covering of this circle it becomes a constant sheaf.

0.3. Main operations on formal resurgent functions

Substituting a small formal resurgent function into a convergent power series. Let $f \in \mathbb{C}\{u\}$ be a convergent integral power series in one indeterminate u. Then *substituting to u a small formal resurgent function ${}^\flat g$ yields a formal resurgent function $f({}^\flat g)$*: for instance, $\exp({}^\flat g)$, $\ln({}^\flat g)$, $(1 - {}^\flat g)^{-1}$ are formal resurgent functions. As a consequence, $1 + \varphi$ is invertible in $\mathcal{R}$ whenever φ is a small formal resurgent function. More generally, invertible formal resurgent functions may be characterized by the following

THEOREM 0.3.1. –

(i) *A formal resurgent function φ is exponentiable (i.e., e^φ is well defined as a formal resurgent function) iff $x^{-1}\varphi$ is small.*

(ii) *A formal resurgent function ψ is invertible in $\mathcal{R}$ iff $\psi = e^\varphi$ where φ is exponentiable.*

Usual derivative. Given a formal resurgent function φ, the derivative $\varphi' = d/dx\varphi$ is defined as the formal resurgent function with major

$$\overset{\vee}{\varphi}' = -\xi\overset{\vee}{\varphi},$$

where $\overset{\vee}{\varphi}$ is a major of φ.

Composition of formal resurgent functions.

THEOREM 0.3.2. – *Let φ and ψ be two formal resurgent functions such that*

$$\varphi(x) = x\big(1 + \varepsilon(x)\big),$$

where ε is a small formal resurgent function. Then the following series converges in $\mathcal{R}$, defining the composed formal resurgent function $\psi \circ \varphi$

$$(\psi \circ \varphi)(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{d^n}{dx^n}\psi(x)\right)\big(x\varepsilon(x)\big)^n.$$

Alien derivatives. The algebra $\mathcal{R}$ is endowed with a family of so-called *alien derivation operators*

$$(\Delta_\omega : \mathcal{R} \to \mathcal{R})_{\omega\in\mathbb{C}^\infty},$$

where $\mathbb{C}^\infty$ stands for the universal covering of $\mathbb{C}^\star$. Here the word "derivation" is taken in the usual algebraic sense of a linear operator satisfying the Leibniz rule

$$\Delta_\omega(\varphi\psi) = (\Delta_\omega\varphi)\psi + \varphi(\Delta_\omega\psi).$$

The explicit definition of Δ_ω will be given in Section 0.5. The construction of a major of $\Delta_\omega\varphi$ involves analytic continuations of $\widehat{\varphi}$ from 0 to ω along suitable paths, and we completely known the Riemann surface of $\widehat{\varphi}$ when we know which iterated alien derivatives $\Delta_{\omega_n}\cdots\Delta_{\omega_2}\Delta_{\omega_1}\varphi$ are non-zero. It turns out that formal resurgent functions arising from natural problems satisfy simple *alien differential equations*, i.e., simple systems of relations between their iterated alien derivatives. When translated in the ξ-plane, these relations induce remarkable "self-reproducing" properties of the singularities of the minor, and this is the reason why Ecalle coined the word "resurgence".

Alien derivative of a composed formal resurgent function.

$$\Delta_\omega(\psi \circ \varphi) = \mathrm{e}^{-\omega(\varphi - x)}(\Delta_\omega \psi) \circ \varphi + (\Delta_\omega \varphi)(\psi' \circ \varphi).$$

0.4. Extended resurgent functions and resurgent symbolic calculus

Consider a Laplace integral

$$\phi(x) = \int_\Gamma \mathrm{e}^{-x\xi} \check{\phi}(\xi)\, d\xi, \tag{0.9}$$

where:

- Γ is an endless path, running along the boundary of a domain D as indicated on Fig. 5 (D should be such that $\Re(\xi) \to \infty$ when $\xi \to \infty$ outside D);
- $\check{\phi}$ is a holomorphic function in D, endlessly continuable.

If we do not assume $\check{\phi}$ to be of exponential type at infinity, integral (0.9) does not converge, and we must truncate it. By a similar trick as the one alluded to in the beginning of Section 0.2, we can consider it as defining an element of $\mathcal{E}/\mathcal{E}^{-\infty}$, where $\mathcal{E}$ stands for the algebra of functions of (arbitrary) exponential type in sectors containing the positive real axis (the exponential type of ϕ is the greatest lower bound of τ's such that the half-plane $\Re(\xi) < -\tau$ lies in D).

Such an equivalence class of function we shall call an *extended resurgent function*, and $\check{\phi}$ will be called a *major* of ϕ (it is uniquely defined modulo entire functions). Extended resurgent functions make up a subalgebra of $\mathcal{E}/\mathcal{E}^{-\infty}$, which we shall denote by $\widehat{\dot{\mathcal{R}}}$. As a

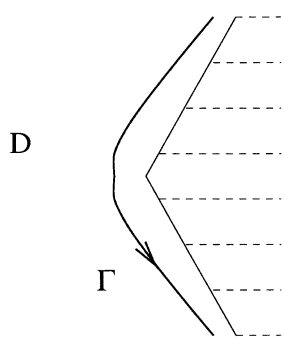

Fig. 5.

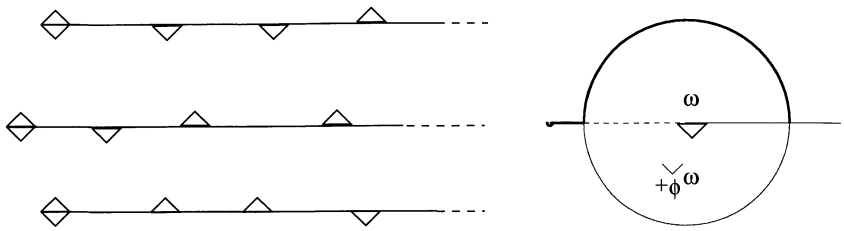

Fig. 6. The right (▽) and the left (△) singular points of $\check{\phi}$.

simple consequence of the endless continuability hypothesis, $\check{\phi}$ can be analytically continued in the whole complex plane with some horizontal "cuts" deleted (cf. Fig. 6(a)). The obstructions to analytically continuing $\check{\phi}$ through these cuts lie on two discrete subsets Ω_+, Ω_- of the cuts, the *right* and *left singular supports* of $\check{\phi}$: a point ω belongs to Ω_+ (the "right singular support") *iff* $sing^{\omega+}\check{\phi} \neq 0$, where by $sing^{\omega+}\check{\phi}$ we mean the singularity of $\check{\phi}$ at ω *seen from the right*, i.e., the class modulo $\mathcal{O}_\omega = \mathbb{C}\{\xi - \omega\}$ of the function ${}^+\check{\phi}^{\omega}$ thus defined (cf. Fig. 6(b)): ${}^+\check{\phi}^{\omega}$ is holomorphic in a split disc $|\xi - \omega| < \varepsilon$, $\arg(\xi - \omega) \neq 0$; its restriction to the lower half-disc coincides with $\check{\phi}$.

The left singular support of $\check{\phi}$ is defined in similar fashion.

Any germ $\check{\varphi}^{\omega}$ (e.g., ${}^+\check{\phi}^{\omega}$ or ${}^-\check{\phi}^{\omega}$), holomorphic in a split disc $|\xi - \omega| < \varepsilon$, $\arg(\xi - \omega) \neq 0$, and endlessly continuable, may be considered as defining (modulo $\mathcal{O}_\omega$) what we shall call a *resurgent microfunction at* ω. Denoting it by $\overset{\nabla}{\varphi}{}^{\omega}$, and denoting by $\overset{\nabla}{\varphi}_\omega = trsl_{-\omega}\, \overset{\nabla}{\varphi}{}^{\omega}$ the resurgent microfunction at 0 deduced from it by translation (e.g., $\check{\varphi}_\omega(\xi) = \check{\varphi}^{\omega}(\xi + \omega)$), we thus define a formal resurgent function φ_ω. Applying this construction to $\check{\varphi}^{\omega} = {}^+\check{\phi}^{\omega}$ (respectively, $\check{\varphi}^{\omega} = {}^-\check{\phi}^{\omega}$), let us denote by ${}^+\varphi_\omega$ (respectively, ${}^-\varphi_\omega$) the resulting formal resurgent function.

PROPOSITION 0.4.1. –

$$\phi = \sum_{\omega \in \Omega_+} (\mathrm{s}_+ {}^+\varphi_\omega)\, \mathrm{e}^{-x\omega} = \sum_{\omega \in \Omega_-} (\mathrm{s}_- {}^-\varphi_\omega)\, \mathrm{e}^{-x\omega}.$$

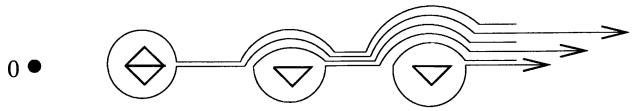

Fig. 7.

Proof. – This is an obvious consequence of the definitions (cf. Fig. 7). □

This proposition suggests the following definitions.

Resurgent symbols. A resurgent symbol is a formal sum

$$\varphi^\bullet = \sum_{\omega\in\Omega} \varphi_\omega \, \mathrm{e}^{-x\omega}, \tag{0.10}$$

where φ_ω are formal resurgent functions, and Ω is a discrete subset of $\mathbb{C}$, *the essential support* of $\varphi^\bullet$, such that for every c the set $\Omega \cap \{\Re(\xi) < c\}$ is finite.

The sum and product of two resurgent symbols are defined in an obvious fashion, so that resurgent symbols make up an algebra which we denote by $\overset{\bullet}{\mathcal{R}}$.

The *right* (respectively, *left*) presummation operations of formal resurgent functions are extended to resurgent symbols in an obvious fashion:

$$\mathrm{s}_+ \varphi^\bullet = \sum_{\omega\in\Omega} (\mathrm{s}_+ \varphi_\omega) \, \mathrm{e}^{-x\omega}$$

and, respectively,

$$\mathrm{s}_- \varphi^\bullet = \sum_{\omega\in\Omega} (\mathrm{s}_- \varphi_\omega) \, \mathrm{e}^{-x\omega}.$$

These operations s_+ and s_- are *isomorphisms of the algebra* $\overset{\bullet}{\mathcal{R}}$ *of resurgent symbols into the algebra* $\widehat{\overset{\bullet}{\mathcal{R}}}$ *of extended resurgent functions.*

The inverse isomorphisms have been described in the statement of above proposition: to every extended resurgent function ϕ they associate its *right*, respectively, *left symbol* which we denoted by

$$\sum_{\omega\in\Omega_+} {}^+\varphi_\omega \, \mathrm{e}^{-x\omega}, \quad \text{respectively,} \quad \sum_{\omega\in\Omega_-} {}^-\varphi_\omega \, \mathrm{e}^{-x\omega}$$

in the statement of the proposition.

Stokes automorphism. We denote by

$$\mathfrak{S} = \mathrm{s}_{+}^{-1} \circ \mathrm{s}_{-} : \overset{\bullet}{\mathcal{R}} \to \overset{\bullet}{\mathcal{R}}$$

that automorphism of the algebra of resurgent symbols which transforms the left symbol of an extended resurgent function into its right symbol.

Symbolic calculus in an arbitrary direction α. We leave it to the reader to transpose the above considerations by replacing the positive real direction by an arbitrary direction α, thus defining the *Stokes automorphism in the direction* α:

$$\mathfrak{S}_{\alpha} = \mathrm{s}_{\alpha+}^{-1} \circ \mathrm{s}_{\alpha-} : \overset{\bullet}{\mathcal{R}}_{\alpha} \to \overset{\bullet}{\mathcal{R}}_{\alpha}.$$

The warning are the same as in the end of Section 0.2, with an additional delicate feature: in contradistinction with the algebras $(\mathcal{R}_{\alpha})$, the algebras $(\overset{\bullet}{\mathcal{R}}_{\alpha})$ *no longer make up a locally constant sheaf*, so that nothing is gained by working on the universal covering of the circle of directions.

0.5. From Stokes automorphisms to alien differential calculus

It immediately follows from the definitions that the Stokes automorphism acts trivially on exponentials $\mathrm{e}^{-x\omega}$, and that its action on a formal resurgent function φ reads

$$\mathfrak{S}\varphi = \varphi + \sum_{\omega} {}^{+}\varphi_{\omega}\, \mathrm{e}^{-x\omega},$$

where the sum runs over those positive real numbers ω such that $sing_{\omega}^{+}\,\widehat{\varphi} \neq 0$, and ${}^{+}\varphi_{\omega} \in \mathcal{R}$ is the formal Laplace transform of the microfunction

$${}^{+}\overset{\nabla}{\varphi}_{\omega} = trsl_{-\omega}\, sing_{\omega}^{+}\,\widehat{\varphi}.$$

The important thing to notice is that

$$\mathfrak{S} = \mathbb{1} + {}^{+}\mathbb{S}^{n},$$

where the operator ${}^{+}\mathbb{S}$ commutes with multiplication by exponentials, and transforms formal resurgent functions into "exponentially small resurgent symbols".

This implies that the operator

$$\underline{\dot{\Delta}} := \ln \mathfrak{S} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} {}^{+}\mathbb{S}^{n}$$

is well defined on $\dot{\mathcal{R}}$. Since $\mathfrak{S}$ is an automorphism of $\dot{\mathcal{R}}$, $\underline{\dot{\Delta}}$ is a derivation of $\dot{\mathcal{R}}$, which we call the *alien derivation in the positive real direction.* It commutes with multiplication by exponentials, and its action on a formal resurgent function φ has the following explicit form

$$\underline{\dot{\Delta}}\varphi = \sum_{\omega \in \Omega(\hat{\varphi})} \mathrm{e}^{-x\omega} \Delta_{\omega}\varphi,$$

where $\Omega(\hat{\varphi})$ is a discrete subset of the positive real line, the set of so-called *glimpsed singularities* of $\hat{\varphi}$, to be defined below; Δ_{ω} is a derivation of $\mathcal{R}$, the so-called *alien derivation at* ω.

DEFINITION 0.5.1. – *The set $\Omega(\hat{\varphi})$ of "glimpsed singularities (singularités entrevues) of $\hat{\varphi}$ in the positive real direction" is the smallest subset Ω of the positive real line such that $\hat{\varphi}$ can be analytically continued along any path which moves away from* 0 *along the positive real axis, avoiding the points of Ω on either side without ever turning back. The first singularity to be avoided along such a path is called the "seen singularity of $\hat{\varphi}$ in the positive real direction".*

Explicit definition of Δ_{ω}. Let $\omega = \omega_n$ be the nth point of $\Omega(\varphi)$ met along the positive real axis, and let us encode the different paths of analytic continuations of $\hat{\varphi}$ along the positive real axis by their "signature"

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{n-1}),$$

where $\sigma_i = +$ (respectively, $-$) if the ith point is avoided from the right (respectively, left). For any such σ, we denote by $\overset{\nabla}{\varphi}{}^{\sigma}$ the singularity at ω of the analytic continuation of $\hat{\varphi}$ along the path of signature σ. Setting $\overset{\nabla}{\varphi}_{\sigma} = trsl_{-\omega}\, \overset{\nabla}{\varphi}{}^{\sigma}$, and denoting by φ_{σ} the corresponding elementary formal resurgent function, we have

$$\Delta_{\omega}\varphi = \sum_{\sigma} \varepsilon(\sigma)\varphi_{\sigma},$$

where the $\varepsilon(\sigma)$ are averaging weights which depend only on the number p of $+$ signs and $q = n - 1 - p$ of $-$ signs:

$$\varepsilon(\sigma) = \frac{p!q!}{n!}.$$

In particular, if ω is the first singular point to be met, one has

$$\Delta_\omega \varphi = \varphi_\omega,$$

the formal Laplace transform of the microfunction

$$\overset{\triangledown}{\varphi}_\omega = trsl_{-\omega}\, \overset{\triangledown}{\varphi}{}^{\omega},$$

where $\overset{\triangledown}{\varphi}{}^{\omega} = sing_\omega \widehat{\varphi}$ (the singularity of the analytic continuation of $\widehat{\varphi}$ along the segment $]0, \omega[$).

Looking in other directions than the positive real one. Replacing $\mathfrak{S}$ by $\mathfrak{S}_\alpha$, the Stokes automorphism in the direction α, we define the *alien derivation operator in the direction* α

$$\dot{\underline{\Delta}}_\alpha : \overset{\bullet}{\mathcal{R}}_\alpha \to \overset{\bullet}{\mathcal{R}}_\alpha$$

and the alien derivation operator at ω

$$\Delta_\omega : \mathcal{R}_\alpha \to \mathcal{R}_\alpha,$$

where ω belongs to the half line $]0\alpha$.

By the same trick as explained at the end of Section 0.2 we can also consider Δ_ω as an operator from $\mathcal{R}$ to $\mathcal{R}$, provided ω no longer stands for a point in $\mathbb{C}^\star$ but for a point in $\mathbb{C}^\infty$ (the Riemann surface of the logarithm). A noteworthy exception where we can still consider ω as living in $\mathbb{C}^\star$ is the case when *all microfunctions involved are simply ramified* (cf. end of Section 0.1). In particular the following subalgebra of $\mathcal{R}$ is of very frequent use.

DEFINITION 0.5.2. – *A formal resurgent function φ is called simple if its major defines a simple microfunction at* 0, *and all analytic continuations of its minor have only simple singularities, i.e., singularities of the form*

$$\frac{a_0}{2\pi \mathrm{i}(\xi - \omega)} + g(\xi)\frac{\ln \xi}{2\pi \mathrm{i}} + \text{hol.fct.}, \quad g \in \mathcal{O}_\omega.$$

This is equivalent to saying that φ is a simple 0-symbol *as well as all its iterated alien derivatives.*

Simple formal resurgent functions make up a subalgebra of $\mathcal{R}$, denoted by $\mathcal{R}^+(1)$, *stable under all alien derivation operators* (such algebras are called by Ecalle *resurgence algebras*). Alien derivations Δ_ω on that algebra can be indexed by $\omega \in \mathbb{C}^\star$.

Pointed alien derivations. Instead of working with the Δ_ω's, it is sometimes convenient to work with the so-called *pointed alien derivations*

$$\dot{\Delta}_\omega := \mathrm{e}^{-x\omega}\,\Delta_\omega.$$

They have the disadvantage of not being operators on $\mathcal{R}$, but rather on spaces of "homogeneous symbols"

$$\overset{\bullet}{\mathcal{R}}{}^{\omega'} = \mathcal{R}\,\mathrm{e}^{-x\omega'}, \qquad \dot{\Delta}_\omega : \overset{\bullet}{\mathcal{R}}{}^{\omega'} \to \overset{\bullet}{\mathcal{R}}{}^{\omega'+\omega}.$$

But they have the advantage of *commuting with d/dx.*

Non-integral powers of the Stokes automorphism. Median presummation. Since the logarithm of $\mathfrak{S}$ is well defined ($\ln \mathfrak{S} = \underline{\Delta}$), one can also define arbitrary powers of $\mathfrak{S}$

$$\mathfrak{S}^\nu = \exp(\nu\underline{\Delta}) \quad (\nu \in \mathbb{C}).$$

In particular, $\mathfrak{S}^{\pm 1/2}$ is used to define the so-called *median presummation* s_{med}: multiplying the relation $\mathrm{s}_- = \mathrm{s}_+ \circ \mathfrak{S}$ on the right by $\mathfrak{S}^{-1/2}$ one gets

$$\mathrm{s}_- \circ \mathfrak{S}^{-1/2} = \mathrm{s}_+ \circ \mathfrak{S}^{1/2} =: \mathrm{s}_{med}.$$

The virtue of the median presummation is that *it transforms power series with real coefficients into real analytic functions of x*: if $\mathcal{C}$ denotes the involutive ($\mathcal{C}^2 = \mathbb{1}$) operator of complex conjugation

$$(\mathcal{C}f)(x) = \overline{f(\overline{x})}.$$

$\mathcal{C}$ clearly exchanges right and left presummations

$$\mathcal{C} \circ \mathrm{s}_+ = \mathrm{s}_- \circ \mathcal{C},$$

so that

$$\mathcal{C} \circ \mathfrak{S}^{-1} = \mathfrak{S} \circ \mathcal{C}.$$

We deduce that $\underline{\Delta}$ anticommutes with $\mathcal{C}$

$$\underline{\Delta} \circ \mathcal{C} = -\mathcal{C} \circ \underline{\Delta},$$

so that s_{med} and $\mathcal{C}$ commute.

Just as right and left presummation, median presummation can be explicitly expressed in terms of Laplace integrals over (a suitable combination of) contours in the complex plane. Since we shall not use these expressions here we refer the reader to [19,20].

0.6. Analytic and regular dependence on parameters

0.6.1. The "Balian and Bloch" viewpoint

Analytic dependence on parameters is most easily defined for what we called "extended resurgent functions" (cf. Section 0.4): for $\Re(x) \gg 0$, and u in a small neighbourhood U of a point $u_0 \in \mathbb{C}^n$, what we shall call an *extended resurgent function of x, depending analytically on u*, is a function given by a Laplace integral

$$\Phi(x, u) = \int_{\Gamma} \mathrm{e}^{-x\xi} \check{\phi}(\xi, u)\, d\xi,$$

where the integration path Γ runs along the boundary of a domain D as represented on Fig. 5, whereas $\check{\phi}(\xi, u)$ is a holomorphic function in $D \times U$, endlessly continuable with respect to ξ for all $u \in U$. The relevance of such objects in semi-classical asymptotics has first been pointed out by Balian and Bloch in [21,22], and mainly in [23]. As emphasized by them, special attention should be given to *confluence of singularities* of $\check{\phi}$, i.e., the meeting in ξ-plane, for special values of u, of two or more singular points.

To make this idea more precise, the following terminology will be useful.

DEFINITION 0.6.1. – *A resurgent microfunction at (ω_0, u_0) is the class, modulo $\mathcal{O}_{\mathbb{C}\times\mathbb{C}^n, \omega_0 \times u_0}$ of a function $\check{\varphi}$ holomorphic in a sectorial neighbourhood of (ω_0, u_0), and endlessly continuable.*

Here again "endlessly continuability" must be understood with respect to ξ, for every fixed u in a neighbourhood of u_0. By a "sectorial neighbourhood" of (ω_0, u_0) we mean an open set $V \subset \mathbb{C} \times \mathbb{C}^n$ intersecting $\mathbb{C} \times \{u_0\}$ as shown on Fig. 8.

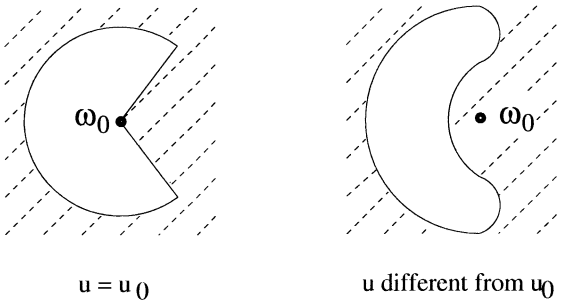

Fig. 8. A sectorial neighbourhood V of (ω_0, u_0) represented by its trace for given u.

The above function $\check{\varphi}$ is called a *major*, and the corresponding microfunction is called the *singularity of* $\check{\varphi}$ *at* (ω_0, u_0).

DEFINITION 0.6.2. – *A resurgent microfunction at* (ω_0, u_0) *is called non-confluent if there is a full neighbourhood* $\Omega \times U_0$ *of* (ω_0, u_0) *such that, for every fixed* $u \in U_0$, *the major* $\check{\varphi}$ *can be analytically continued along any path of* $\Omega \backslash \{\omega(u)\}$, *where* $\omega(u)$ *is one point of* Ω *(which of course depends holomorphically on* u, *with* $\omega(u_0) = \omega_0$*).*

With this definition we can give an obvious condition for the right or left symbol of Φ (say, in the positive real direction) to depend holomorphically on u. Fixing u to the value u_0, let ω_0 be some point in $\Omega_+(u_0)$ (respectively, $\Omega_-(u_0)$), the right (respectively, left) support of $\check{\phi}(\cdot, u_0)$ (cf. Section 0.4). Then, working again in $\mathbb{C} \times \mathbb{C}^n$, let ${}^+\check{\varphi}_{\omega_0,u_0}$ (respectively, ${}^-\check{\varphi}_{\omega_0,u_0}$) be the singularity at (ω_0, u_0) of that determination of $\check{\phi}$ obtained by approaching the singular point from the right (respectively, left) of the cut (cf. Fig. 6).

1st assumption (no confluence). *For all such* ω_0*'s,* ${}^+\check{\varphi}_{\omega_0,u_0}$ *and* ${}^-\check{\varphi}_{\omega_0,u_0}$ *are non-confluent microfunctions.*

2nd assumption (no Stokes phenomena). There is a neighbourhood U of u_0 such that, for every $\omega_0 \in \Omega_+(u_0)$ (respectively, $\Omega_-(u_0)$), and for every $u \in U$, the corresponding singular point $\omega(u)$ of Definition 0.6.2 still belongs to the right (respectively, left) singular support of $\check{\phi}(\cdot, u)$.

This second assumption forbids singularities to escape across the cut in other sheets, or to come into sight from other sheets. It can be summarized

by saying that *the topological patterns of right and left singular support does not depend on u.*

Remark. – This implies that if two singular points lie on the same cut, their relative position should be independent on u (a non-constant holomorphic function cannot have an identically zero imaginary part).

Continuity principle. Under the above assumptions the (right or left) symbol of Φ (in the positive real direction) depends holomorphically on u, i.e., it reads

$$\sum_{\omega\in\Omega} \varphi_\omega \, \mathrm{e}^{-x\omega(u)},$$

where the coefficients of the formal power series φ_ω are holomorphic functions of u.

0.6.2. The symbolic viewpoint

The continuity principle gave us conditions under which the decomposition of an extended resurgent function into elementary components depends regularly on u.

For each corresponding elementary symbol $\varphi_\omega \mathrm{e}^{-x\omega(u)}$, φ_ω is an example of what we shall call a *formal resurgent function depending regularly on u.*

DEFINITION 0.6.3. – *A formal resurgent function φ is said to depend regularly on u near u_0 if the corresponding resurgent microfunction at $(0, u_0)$ is non-confluent in the sense of Definition* 0.6.2. *In other words, its major has the origin as its only singularity in a small disc $|\xi| < \varepsilon$, independent of u near u_0.*

PROPOSITION 0.6.1. – *Let $\varphi = \varphi(x, u)$ be a formal resurgent function depending regularly on one parameter $u \in \mathbb{C}$ near* 0.

Then substituting to u a small formal resurgent function $\psi = \psi(x)$ yields a formal resurgent function $\varphi(x, \psi(x))$.

Addendum. Assume further that for every u close enough to 0 the minor of φ undergoes no Stokes phenomena in the positive real direction, i.e., its pattern of seen and glimpsed singularities on either side is topologically independent on u. Then substitution of a small formal resurgent function commutes with (right or left) resummation of φ:

$$\mathrm{S}_+\{\varphi(x, \psi(x))\} = \mathrm{S}_+\{\varphi\}(x, \mathrm{S}_+\psi)$$

(and similarly for S_-).

In all cases, the alien derivative of $g(x) = \varphi(x, \psi(x))$ is given by the following formula (compare with the last formula of Section 0.3): for all $\omega \in \mathbb{C}^\infty$,

$$\Delta_\omega g(x) = \{\Delta_\omega \varphi\}(x, \psi(x)) + \Delta_\omega \psi(x) \frac{\partial}{\partial u} \varphi(x, \psi(x)).$$

THEOREM 0.6.1 [24,18]. – *Let $\varphi = \varphi(x, u)$ be as in the above proposition. Assume further that φ is small, and $\frac{\partial \varphi}{\partial u}(x, 0)$ is an invertible formal resurgent function (cf. Theorem* 0.3.1). *Then the implicit equation*

$$\varphi(x, \psi(x)) = 0,$$

where ψ is the unknown, can be formally solved in a canonical way, yielding a small formal resurgent function ψ.

0.7. Historical note

In exposing the ideas of this section we stuck rather closely to the terminology of Ecalle, and to the notations he finally adopted in the years 1990 (cf., e.g., [19]). Considerations on dependence of formal resurgent functions on parameters are implicit in the works of Ecalle, but he did not develop a systematic terminology (the terminology of Section 0.6 is our own); for exposing these notions, we found it convenient to follow a point of view introduced in [18] under the name of "extended resurgent functions" (cf. Section 0.4), which is strongly inspired by the ideas of Balian and Bloch [21–23].

Notice that Sternin and Shatalov recently introduced a notion of "resurgent function of several variables" ([25]). This notion is completely (although not in a unique manner) translatable in terms of resurgent functions of one variable depending analytically of parameters.

1. STOKES PHENOMENA FOR WKB EXPANSIONS: THE RESURGENT VIEWPOINT

1.0. Reminder on complex WKB expansions

For consistency with Section 0 our notations throughout this paper will be slightly different from those common in physics, which we used in [1]. Instead of Planck's constant $\hbar$ we shall work with $x = 1/\hbar$ (which

will be our "resurgence variable"), writing the Schrödinger equation as follows

$$-x^{-2}\frac{d^2\Phi}{dq^2}+(V(q)-E)\Phi=0, \tag{1.0}$$

where the potential V is assumed to be a (possibly complex) polynomial function.

A *turning point* is a zero of V and its order of multiplicity will be called the *order* of the turning point.

Our "momentum" p and "action function" S will be $-\mathrm{i}$ *times those of physicists*:

$$p(q)=(V(q)-E)^{1/2}, \qquad S(q)=\int_{q_0}^{q} p(q')\,dq'.$$

Following Voros, we shall denote by $\dot{\mathbb{C}}$ the punctured complex q-plane (with the turning points deleted) and by $\dot{\mathbb{C}}_2$ its two-fold covering (i.e., the Riemann surface of $p(q)$). Locally on that covering, complex WKB expansions thus read

$$\varphi(q)=\left(\varphi_0(q)+\varphi_1(q)x^{-1}+\varphi_2(q)x^{-2}+\cdots\right)\mathrm{e}^{-xS(q)}. \tag{1.1}$$

The fact that (1.1) should be a formal solution of the Schrödinger equation characterizes this formal power series *up to an arbitrary normalization factor* (an invertible formal power series in x^{-1}, with constant coefficients). A possible choice of normalization is

$$\varphi(q)=P\left(q,x^{-2}\right)^{-1/2}\mathrm{e}^{-x\int_{q_0}^{q}P(q',x^{-2})\,dq'}, \tag{1.2$_{q_0}$}$$

where the formal power series

$$P\left(q,x^{-2}\right)=p(q)+p_1(q)x^{-2}+p_2(q)x^{-4}+\cdots$$

is defined as the even part (in x^{-1}) of the solution of the Riccati equation [6]

$$Y^2+x^{-1}Y'=E-V(q).$$

[6] This Riccati equation is deduced from Eq. (1.0) by the change of unknown function $\phi=\mathrm{e}^{-x\int Y}$. Formally solving it, and separating even and odd parts (in x^{-1}), one checks that $Y_{odd}=-\frac{x^{-1}}{2}\frac{Y'_{even}}{Y_{even}}$.

Such expansions will be called *well normalized at* q_0 ($q_0 \in \dot{\mathbb{C}}_2$); of course they are multivalued analytic on $\dot{\mathbb{C}}_2$, because the integral in the exponential depends on the homotopy class of the integration path. For our purposes it will be more convenient to work with slightly different normalization conventions, which read

$$\varphi(q) = P\left(q, x^{-2}\right)^{-1/2} \mathrm{e}^{-x \int_{\infty}^{q} (P(q', x^{-2}) - p(q'))\, dq'} \mathrm{e}^{-xS(q)}, \qquad (1.2)_{\infty}$$

where S is as in (1.1) (we have used the fact that $P - p$ is integrable at infinity); such expansions will be called *well normalized at infinity*; here again the precise meaning of this expression depends on the homotopy class of the integration path.

Analyticity with respect to q. It is easy to find conditions under which the above well normalized WKB expansions will depend analytically on E.[7] Since the function P in (1.2) is analytic outside the turning points, the integral $\int_{q_0}^{q} P(q', x^{-2})\, dq'$ will depend analytically of (q, E) as long as the path of integration from q_0 to q can be deformed continuously without ever meeting turning points;[8] besides the obvious requirement that q should never be a turning point, it is therefore enough to assume that when E varies

(A1) no turning point ever meets q_0;

(A2) no couple of turning points ever "pinches" the integration path.

Both these conditions are obviously satisfied when the endpoint q_0 is at infinity and the energy E is not critical, so that all turning points remain *simple*.

1.1. Geometry of the complex action function

1.1.1. Fading lines

Starting in some small neighbourhood of a point q_0 in $\dot{\mathbb{C}}$, let as above $S(q)$ be a primitive of $p(q)$ (one of the two determinations of $(V(q) - E)^{1/2}$). The lines along which $\mathrm{e}^{-xS(q)}$ decreases fastest (for *positive real* x) are the integral curves of the vector field $\operatorname{grad} \Re S(q) = \overline{p}(q)$. We shall call them *fading lines*.

Since we are free to choose between two determinations of $p(q)$, it is convenient to see fading lines as *oriented lines in* $\dot{\mathbb{C}}_2$.

[7] Similarly, we could also make the coefficients of the polynomial V vary.

[8] We assume here for simplicity that the "base point" q_0 is independent on E, but we might as well assume it to be a holomorphic function of E.

Since the vector field $\overline{p}(q)$ is non-singular on $\dot{\mathbb{C}}_2$, we thus get a *foliation of* $\dot{\mathbb{C}}_2$ *by* (*oriented*) *fading lines*, and of course changing sheet on the two-fold covering simply results in changing the orientation (the "fading orientation").

Given some starting point (q_0, p_0) in $\dot{\mathbb{C}}_2$ (i.e., $q_0 \in \dot{\mathbb{C}}$, $p_0 =$ some determination of $(V(q) - E)^{1/2}$), the fading line starting at this point is therefore the solution of the differential equation

$$\frac{dq}{dt} = \overline{p}(q(t)) \tag{1.3}$$

with the initial condition $q(0) = q_0$.

Example. – Assuming E and V to be *real*, those segments of the real axis where $E < V(q)$ (i.e., the classically forbidden regions) carry fading lines (oriented towards right, *respectively*, left for positive, *respectively*, negative p).

Fading lines can also be seen as level lines of $\Im S(q)$ (suitably oriented). Near a turning point $q_\star$ of order ν, where (say) $p(q) \sim (q - q_\star)^{\nu/2}$, they look like the level lines of $\Im(q - q_\star)^{\nu/2+1}$.

In the $\nu = 1$ case (*simple turning point*) the configuration is well known (cf. Fig. 9): it can be thought of as a "monkey saddle" (Fig. 9(b)) folded in two.

The $\nu = 2$ case (*double turning point*) is the easiest: it is just the usual saddle configuration (Fig. 9(c)), or rather two copies of it with opposite orientations.

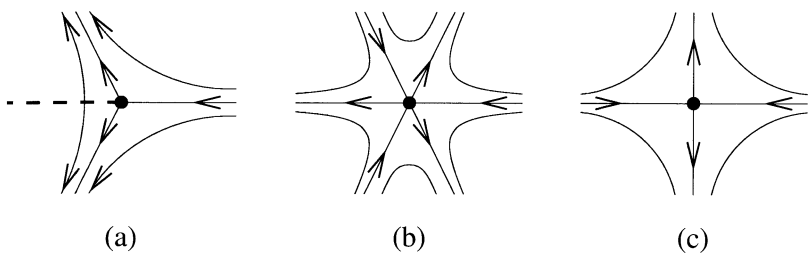

Fig. 9. The behaviour of a simple turning point has been drawn on Fig. 9(a). The dotted line on the left is a cut, and one has only drawn that part of the picture "lying in the first sheet". Fig. 9(b) is a monkey saddle: this is Fig. 9(a) drawn in the $(q - q_\star)^{1/2}$ plane. Fig. 9(c) for a double turning point: (usual) saddle for one of the two possible determinations of p (for the other determination, please inverse the arrows!).

The higher ν case can be drawn analogously, giving generalized saddles (folded in two or duplicated, depending on the parity of ν).

Notice that *only a finite number of fading lines are attached to the turning point*: forgetting about their orientation we have exactly $\nu + 2$ of them, distributed at angular distances $\frac{2\pi}{\nu+2}$ around the turning point. Along such "singular" fading lines the solution $q(t)$ of the differential Eq. (1.3) behaves differently for simple and multiple turning points: if $q_\star$ is simple $q(t)$ reaches it at a finite time, whereas if $q_\star$ is multiple it takes an infinite (> 0 or < 0) time for $q(t)$ to reach it.

The following proposition describes the global geometry of fading lines.

PROPOSITION 1.1.1 (cf. [26,27]). – *The only two kinds of fading lines are*:

(1) *the non-singular fading lines, which start from* ∞ *and get to* ∞ (*in a finite time if deg* $V > 2$);
(2) *the singular fading lines, which leave or reach a turning point* (*or do both*): *there are only a finite number of them.*

Remark. – This proposition forbids the existence of "non-singular fading cycles" where $q(t)$ would run cyclically along a closed curve in $\dot{\mathbb{C}}$. But as far as *singular* fading lines are concerned, it does not exclude the possibility for a fading line to leave one turning point q_1 and reach another turning point q_2, allowing us to build a *fading cycle* between q_1 and q_2 (going first from q_1 to q_2, then back to q_1 along the same path with the opposite momentum): this is represented on Fig. 10.

1.1.2. Stokes lines

Stokes lines[9] are those (a priori) unoriented lines in $\dot{\mathbb{C}}$ which carry singular fading lines (cf. Proposition 1.1.1). They divide $\dot{\mathbb{C}}$ into a finite number of connected open regions called *Stokes regions. Stokes regions are simply connected and can never be bounded* (cf. [27,26]).

Fig. 11 shows examples of Stokes lines and the Stokes regions between them. These pictures illustrate the fact that one can distinguish between two kinds of Stokes lines.

Unbounded Stokes lines are those which connect ∞ to a turning point.

We shall orient them *towards the turning point*. WKB expansions

[9] For the "Stokes-anti Stokes" controversy, cf. [28,17].

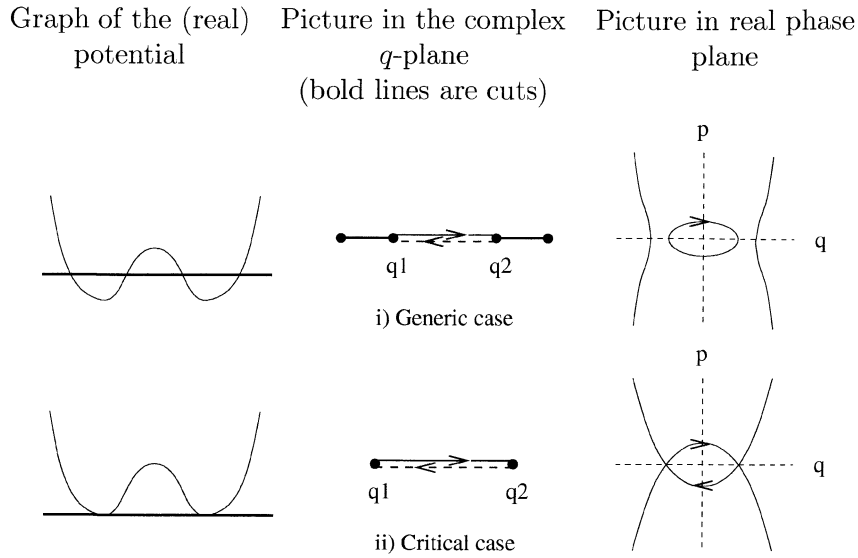

Fig. 10. A fading cycle for the anharmonic oscillator.

for which this orientation is the "fading orientation" will be called *dominant on* L, and their determination of p will be called the *dominant determination*. The opposite determination of p is called the *recessive determination*.

Bounded Stokes lines are those which connect two turning points. They carry the "fading cycles" considered in Remark 1.1.3 (cf. Fig. 10), and have no natural orientation.

Remark. – The fact that Stokes regions cannot be bounded implies that bounded Stokes lines cannot be cyclically connected to each other as on the left picture on Fig. 12, although they can be attached to each other as on the right picture on Fig. 12.

1.1.3. Steepest descent for other determinations of $\arg x$

Instead of considering steepest descent lines of $e^{-xS(q)}$ for *positive real* x, we can do the same job for arbitrary values of $\arg x$. This is the idea of what Voros calls the "radar method" for exploring singularities of minors of WKB expansions in the ξ-plane. The result of that exploration, which will be given in the next subsection, will use the following terminology (introduced in [5]).

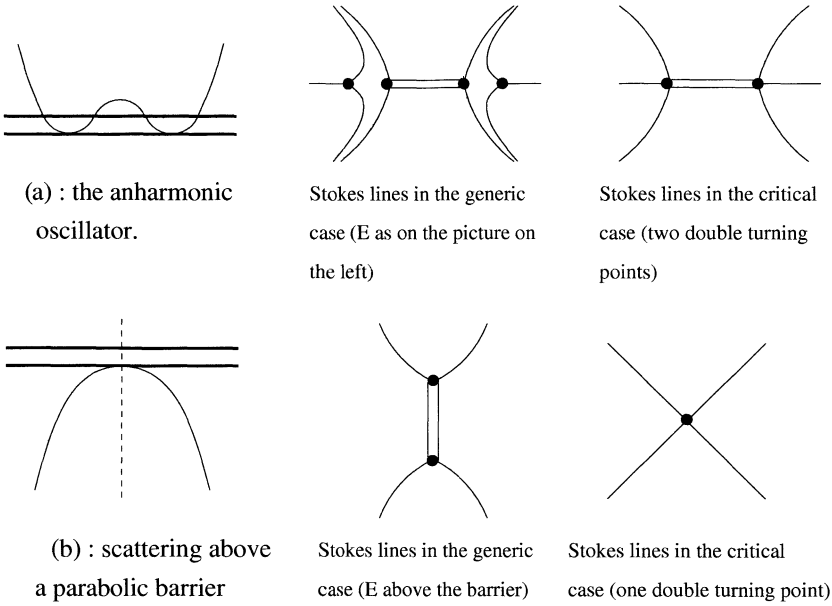

Fig. 11. Bounded and unbounded Stokes lines.

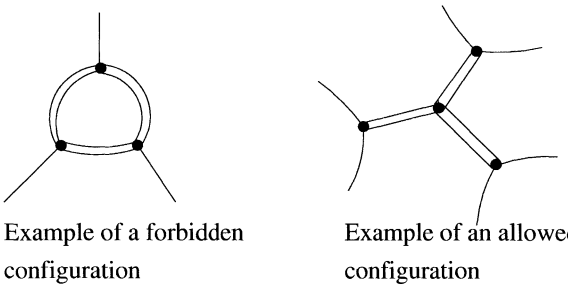

Fig. 12.

A geodesic path is a path in $\dot{\mathbb{C}}_2$ such that the value of the indefinite integral $S = \int p\,dq$ along that path moves monotonically along a straight line in the complex S-plane; the direction of that straight line is called the direction of the geodesic (in particular what we called "fading lines" are the geodesics of real positive direction). A geodesic path is called *singular* if it passes through a turning point.

A geodesic loop with origin $q \in \dot{\mathbb{C}}_2$ is a singular geodesic of the following type: it starts from q, reaches a turning point, then goes all

the way backwards on the other sheet until it reaches the point facing q on the other sheet.

A geodesic cycle is a singular geodesic going from one turning point to another one, and then coming all the way back on the other sheet (in particular what we called "fading cycles" are the geodesic cycles of real positive direction)

Remark. – The important role played by paths *going through turning points* makes it natural to complete the two-fold covering $\dot{\mathbb{C}}_2$ by adding its ramification points. The Riemann surface thus obtained is nothing but the complex hyperelliptic curve

$$\mathcal{L}_E := \{(p,q) \in \mathbb{C}^2 \mid p^2 = V(q) - E\}.$$

1.2. Resurgence properties of well normalized WKB expansions

The starting point of all our study is the following theorem, due to Ecalle.

THEOREM 1.2.1. – *Well normalized WKB expansions are (resummable) resurgent symbols of $x = 1/\hbar$. They depend regularly* [10] *on (q, E) (and on the coefficients of the polynomial V) as long as conditions* (A1) *and* (A2) *of Section* 1.0 *are satisfied.*

Comment. The proof of Ecalle (sketched in [9], see also [29]) also yields a detailed description of the analytic structure and behaviour at infinity of the minors "in the first sheet of the Borel plane" (cf. parts (a) and (c) of Theorem 1.2.2 hereafter). To make it complete, all that would remain to prove is the "endless continuability" in other sheets. How this can be done still remains a bit unclear to us (perhaps it might be easier to do with the alternative construction proposed by Shatalov and Sternin [25], but they do not make that point precise either).

Notice that in practice the knowledge of the first sheet structure *and of the resurgence equations* (as given by Theorem 1.2.2 hereafter) implies complete knowledge of the singularity structure *on all sheets*. But the very notion of "resurgence equation" is in principle meaningless if one cannot prolong a little bit beyond the first sheet.

We now proceed to describe the "first sheet singularities" of $\widehat{\varphi}$ (the minor of a well normalized WKB expansion φ), and the resurgence equations they satisfy.

[10] Cf. Definition 0.6.3.

There are two kinds of singularities:

the moving singularities, whose positions depend on q;

the fixed singularities, whose positions are independent on q (but may depend on the energy E).

THEOREM 1.2.2. – *Assume that the energy is "generic", in the sense that all turning points are simple. Then*

(a) *The positions of the moving singularities of $\widehat{\varphi}$ are the action integrals $\omega_l = \int_l p\,dq$ over all geodesic loops l with origin q.*

(b) *Let α_l be the direction of such a geodesic loop. Assume that in that direction, $\widehat{\varphi}$ glimpses no other singularity than ω_l (this is a generic hypothesis on q). Then*

$$\mathfrak{S}_{\alpha_l}\varphi = \varphi + \ell\varphi,$$

where $\ell\varphi$ is the analytic continuation of φ along the loop ℓ, deduced from l by slightly distorting it in $\dot{\mathbb{C}}_2$ so as to turn anticlockwise around the turning point.

(c) *Suppose that φ is well normalized at infinity, along a path which crosses no geodesic cycle. Then $\widehat{\varphi}$ has no fixed singularities.*

For the proof of that theorem, cf. [5]. An extension of part (c) will be given in Section 2 (Theorem 2.5.1).

1.3. Stokes phenomena

From Theorem 1.2.2 we easily get a sufficient condition for Borel resummability of WKB expansions.

A point (q_0, p_0) in $\dot{\mathbb{C}}_2$ will be said to *fade away at infinity* if the fading line starting at this point goes to infinity as time increases: this means that q_0 is either inside a Stokes region, or on an *unbounded* Stokes line such that p_0 is the *recessive* determination.

PROPOSITION 1.3.1. – *Assuming that (q_0, p_0) fades away at infinity, let φ_0 be a WKB expansion defined in a neighbourhood of (q_0, p_0),* well normalized at infinity along the fading line starting from (q_0, p_0). *Then φ_0 is Borel resummable.*

Proof. – By Theorem 1.2.2 our hypothesis implies that $\widehat{\varphi}_0$ sees no singularity in the positive real direction. □

Let now Φ_0 be the Borel sum of such a WKB expansion φ_0. Being a solution of the Schrödinger equation, Φ_0 extends to a holomorphic function Φ in the whole q-plane. But it is not true that the analytic continuations φ of φ_0 everywhere in $\dot{\mathbb{C}}$ are Borel resummable. Nor is it true that in the regions where they are Borel resummable, their Borel sum coincide with Φ (remember that φ is multivalued over $\dot{\mathbb{C}}$, whereas Φ is not !). What is true is the following

PROPOSITION 1.3.2. –

(a) *All analytic continuations of φ_0 are right and left resummable.*

(b) *Right resummation S_+ (respectively, left resummation S_-) commutes with analytic continuation as long as one does not cross a Stokes line, or when one reaches a Stokes line from the left (respectively, right) with respect to the fading direction of φ (cf. Fig.* 13).

(c) *In every Stokes region R, Φ can be written (at will) as the right (respectively, left) resummation of a formal object called its right symbol (respectively, left symbol), which for generic values of E (i.e., when all turning points are simple) is a linear combination of different determinations of φ.*

Proof. – Part (a) directly follows from Theorem 1.2.1, and part (b) from Theorem 1.2.2(a) (by the "continuity principle" in Section 0.6.1, right, respectively, left resummation commutes with analytic continuation as long as no singularity is seen to approach the integration axis from the right, respectively, left). Part (c) follows from Theorem 1.2.2(b) (cf. [5] for the details). □

Remark. – Parts (a) and (b) of the above proposition require no assumption on the "genericity" of E: for critical energies, they can be deduced from the generic case by a limiting procedure, chosen so as to depend *regularly* on E (cf. Theorem 1.2.1).

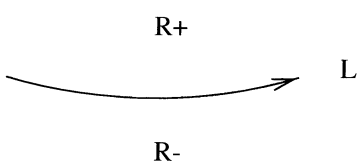

Fig. 13. Reaching a Stokes line from one side. The Stokes line L has been oriented as the fading line of φ. One has $\mathrm{S}_+\varphi|L = \mathrm{S}_+\varphi|R_+$ (left side of L) and $\mathrm{S}_-\varphi|L = \mathrm{S}_-\varphi|R_-$ (right side of L).

As regards part (c), we shall see in the second part of this paper how the lack of regularity stemming from *confluence of singularities for critical E* forces special prefactors into WKB expansions, with a more complicated dependence on x^{-1} than just integral power series (cf. Section 3).

Our main problem will now be to calculate, in each Stokes region, the right and left symbols of Φ: this is the *connection problem*, which will be the subject of the whole following section.

2. WKB SYMBOLS AND THE CONNECTION PROBLEM

We collect here some general facts on the connection problem, true not only for generic energies but also in the critical cases, with turning points of arbitrary order. These results will hold not only when the energy is a given number E_0, but also when it is a formal resurgent function of x, of the form

$$E = E_0 + (\text{small formal resurgent function of } x)$$

(more generally, all the coefficients of the Schrödinger operator could be added small formal resurgent functions of x).

2.1. General WKB symbols, and operations on them

2.1.1. From local symbolic objects ...

All objects in this subsection should be considered *locally on* $\dot{\mathbb{C}}$, the complex q-plane without the turning points.

DEFINITION 2.1.1. – *By a* general WKB symbol *we mean any resurgent symbol depending analytically on q, and satisfying the Schrödinger equation.*

Model example. Given two well-normalized WKB expansions $\varphi^{(p)}$ and $\varphi^{(-p)}$, with opposite determinations of the momentum, the following expression is a general WKB symbol:

$$\psi = a^+\varphi^{(p)} + a^-\varphi^{(-p)} \tag{2.1}$$

for any two $a^+, a^- \in \mathcal{A}$, the ring of *scalar resurgent symbols* (resurgent symbols not depending on q).

PROPOSITION 2.1.1. – *Locally on* $\dot{\mathbb{C}}$, any *general WKB symbol* ψ *is of the form* (2.1), *with coefficients* a^+ *and* a^- *which are uniquely determined once* $\varphi^{(p)}$ *and* $\varphi^{(-p)}$ *have been chosen. In other words, general WKB symbols make up a local system of* $\mathcal{A}$*-modules, locally free of rank* 2, *which splits locally into a direct sum*

$$\mathbf{WKB} \overset{loc}{=} \mathbf{WKB}^p \oplus \mathbf{WKB}^{-p},$$

where each direct summand is freely generated over $\mathcal{A}$ *by any well normalized WKB expansion with the corresponding determination* (p *or* $(-p)$) *of the momentum.*

Proof. – Let

$$W\left(\varphi^{(p)}, \varphi^{(-p)}\right) = \frac{d}{dq}\varphi^{(p)}\varphi^{(-p)} - \varphi^{(p)}\frac{d}{dq}\varphi^{(-p)}$$

be the Wronskian of $\varphi^{(p)}$ and $\varphi^{(-p)}$. Since $\varphi^{(p)}$ and $\varphi^{(-p)}$ satisfy the Schrödinger equation this Wronskian does not depend on q, and therefore belongs to $\mathcal{A}$. Since it reads $-x(2p + \mathrm{O}(x^{-1}))$, it is an *invertible* element of $\mathcal{A}$. The conclusion easily follows, with

$$a^+ = W\left(\psi, \varphi^{(-p)}\right) / W\left(\varphi^{(p)}, \varphi^{(-p)}\right)$$

and

$$a^- = W\left(\varphi^{(p)}, \psi\right) / W\left(\varphi^{(p)}, \varphi^{(-p)}\right). \qquad \square$$

2.1.2. ... to global solutions

Remind that when true functions of q are concerned (instead of symbolic objects) any solution of the Schrödinger equation, once constructed locally, prolongs analytically to an entire function in the whole q-plane.

DEFINITION 2.1.2. – *By a* resurgent solution *of the Schrödinger equation we mean any extended resurgent function of* x *depending analytically on* q *in the whole* q*-plane and satisfying the Schrödinger equation. Resurgent solutions of the Schrödinger equation make up a vector space which we shall denote by* $\mathcal{S}$.

Let now $\mathcal{A}$ be the ring of all *scalar resurgent functions*, i.e., extended resurgent functions of x which do not depend on q. Obviously $\mathcal{S}$ is a $\mathcal{A}$-module.

By definition of an extended resurgent function right and left resummations $\mathrm{s}_\pm$ send the ring $\mathcal{A}$ of scalar resurgent symbols isomorphically on the ring $\mathcal{A}$. Furthermore, near any point $q \in \dot{\mathbb{C}}$ where no Stokes phenomenon occurs, they send the space of general WKB symbols into the space $\mathcal{S}$ of solutions, and this mapping is a *module homomorphism*: for every ψ of the form (2.1) one has

$$\mathrm{s}_\pm \psi = (\mathrm{s}_\pm a_+)(\mathrm{s}_\pm \varphi^p) + (\mathrm{s}_\pm a_-)(\mathrm{s}_\pm \varphi^{(-p)}).$$

PROPOSITION 2.1.2. – *In every Stokes region R right resummation s_+ and left resummation s_- are* isomorphisms *between the $\mathcal{A}$-module* $\mathbf{WKB}(R)$ *and the $\mathcal{A}$-module $\mathcal{S}$, so that the latter is* free of rank 2.

Proof. – It is an immediate consequence of Proposition 1.3.2 that no Stokes phenomenon occurs as long as q stays in a Stokes region R. To prove that s_+ and s_- are module isomorphisms, we only have to prove that $\mathcal{S}$ is free of rank 2 over $\mathcal{A}$, with $\mathrm{s}_\pm \varphi^p$ and $\mathrm{s}_\pm \varphi^{-p}$ for its generators. The proof is exactly the same as the proof of Proposition 2.1.1, except for the fact that the reasoning deals with *extended resurgent functions* instead of resurgent symbols (since $W(\varphi^{(p)}, \varphi^{(-p)})$ is invertible in $\mathcal{A}$, $W(\mathrm{s}_\pm \varphi^{(p)}, \mathrm{s}_\pm \varphi^{(-p)}) = \mathrm{s}_\pm W(\varphi^{(p)}, \varphi^{(-p)})$ is invertible in $\mathcal{A}$). □

2.1.3. Connection isomorphisms and connection operators

Let now R and R' be two Stokes regions. The above proposition leads naturally to the following definition.

DEFINITION 2.1.3. – *The right-connection isomorphism $\mathcal{C}^+_{R'R}$ from R to R' is the $\mathcal{A}$-linear isomorphism which makes the following diagram commute*

$$\begin{tikzcd} \mathbf{WKB}(R) \arrow[rr, "\mathcal{C}^+_{R'R}"] \arrow[dr, "\mathrm{s}_+"'] & & \mathbf{WKB}(R') \arrow[dl, "\mathrm{s}_+"] \\ & \mathcal{S} & \end{tikzcd}$$

The left-connection isomorphism $\mathcal{C}^-_{R'R}$ is defined in a similar fashion, replacing s_+ by s_-.

Decomposing $\mathbf{WKB}(R)$ and $\mathbf{WKB}(R')$ into their direct summands corresponding to the two possible determinations of the momentum, we can write $\mathcal{C}^\pm_{R'R}$ as a 2×2 matrix of operators, whose entries will be called the "connection operators" from R to R': more precisely the *connection operator from* (R, p) *to* (R', p') will be the entry

corresponding to the initial determination p and the final determination p' of the momentum.

With this terminology, "solving the connection problem" means computing all connection operators. In the special case when all turning points are *simple* we know from [5] (see also [1]) that the connection operators are just linear combinations of *analytic continuation operators* (along suitable paths in $\dot{\mathbb{C}}_2$).

2.1.4. Comparing right and left symbols: the Stokes automorphism

DEFINITION 2.1.4. – *The Stokes automorphism $\mathfrak{S}$ of WKB symbols (say at a given point $q \in \dot{\mathbb{C}}$) is defined by the commutativity of the following diagram.*

$$\begin{tikzcd} \mathbf{WKB}_q \arrow[rr, "\mathfrak{S}"] \arrow[dr, "\mathrm{S}_-"'] & & \mathbf{WKB}_q \arrow[dl, "\mathrm{S}_+"] \\ & \mathcal{S} & \end{tikzcd}$$

Notice that this automorphism is not $\mathcal{A}$-linear, but satisfies instead

$$\mathfrak{S}(a\psi) = \mathfrak{S}(a)\mathfrak{S}(\psi).$$

Of course a necessary and sufficient condition for a WKB symbol ψ to be Borel resummable is

$$\mathfrak{S}\psi = \psi.$$

The following is an easy consequence of Propositions 1.3.2(b) and 2.1.1.

PROPOSITION 2.1.3. – *The action of $\mathfrak{S}$ on WKB symbols commutes with analytic continuation inside a Stokes region, so that $\mathfrak{S}$ can be considered as an automorphism of* $\mathbf{WKB}(R)$.

This automorphism is diagonal with respect to the direct sum decomposition

$$\mathbf{WKB}(R) = \mathbf{WKB}(R)^p \oplus \mathbf{WKB}(R)^{-p},$$

so that both isomorphisms S_+ and S_- induce the same direct sum decomposition of $\mathcal{S}$, depending only on R:

$$\mathcal{S} = \mathcal{S}_R^p \oplus \mathcal{S}_R^{-p}.$$

It will be shown in Section 2.5 that $\mathfrak{S}$ also commutes with analytic continuation from one Stokes region to another *provided the path of analytic continuation does not cross any bounded Stokes line.*

2.2. Crossing Stokes lines

2.2.1. Crossing simple Stokes lines: elementary connection operators

DEFINITION 2.2.1. – *A simple Stokes line is an unbounded Stokes line L fading into a turning point $q_\star$ such that no bounded Stokes line ends at $q_\star$.*

PROPOSITION 2.2.1. – *Let the Stokes regions R, R' be separated by a simple Stokes line L as indicated on Fig.* 14 *(i.e., one goes from R to R' crossing L clockwise around $q_\star$).*

In **WKB**(R) *and* **WKB**(R'), *take the first (respectively, second) summand as that one corresponding to WKB symbols which are dominant (respectively, recessive) on L. Then*

$$\mathcal{C}^+_{R'R} = \mathcal{C}^-_{R'R} = \begin{pmatrix} \mathbf{1} & 0 \\ \delta_L & \mathbf{1} \end{pmatrix} := \mathcal{C}_L,$$

where $\mathbf{1}$ *means analytic continuation across L, whereas δ_L, the so-called elementary connection operator across L, is homogeneous with weight*

$$\omega(q) = 2\int_q^{q_\star} p(q')\,dq',$$

i.e., it transforms $\varphi = \varphi_\bullet \mathrm{e}^{-xS(q)}$ into $\delta_L\varphi = \psi = \psi_\bullet \mathrm{e}^{-x(\omega(q)-S(q))}$; note that the integral is taken with the dominant determination of the momentum, so that $\omega(q)$ is positive real for $q \in L$.

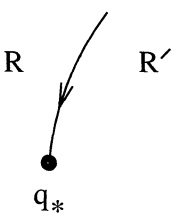

Fig. 14.

Proof. – Due to the $\mathcal{A}$-linearity of the connection isomorphisms, using Proposition 2.1.1 we can of course analyze the action of these connection isomorphisms on well-normalized WKB symbols satisfying the hypothesis of Theorem 1.2.2(c). Since such a symbols have no fixed singularity, $\mathcal{C}^{+}_{RR'}$ and $\mathcal{C}^{-}_{RR'}$ act on them in the same way as $\mathfrak{S}$ (cf. Proposition 1.3.2(b)). For such a symbol $\varphi^{(-p)}$ which is *recessive*, no singularity is seen in the positive real direction, so that

$$\mathfrak{S}\varphi^{(-p)} = \varphi^{(-p)}.$$

For such a symbol $\varphi^{(p)}$ which is *dominant*, only *one* singularity is glimpsed in the positive real direction, namely $\omega(q)$ (for generic E this is just what Theorem 1.2.2(a) tells; the general case follows by continuity). Therefore

$$\mathfrak{S}\varphi^{(p)} = \varphi^{(p)} + \dot{\Delta}_{\omega(q)}\varphi^{(p)}$$

thus proving the proposition, with

$$\delta_L\varphi^{(p)} = \dot{\Delta}_{\omega(q)}\varphi^{(p)} \qquad \square$$

A detailled analysis of the case where $q_\star$ is a double turning point will be given in Sections 3–5.

Remark. – Of course the connection operator in the reverse direction is given by the inverse matrix

$$\mathcal{C}^{\pm}_{RR'} = (\mathcal{C}_L)^{-1} = \begin{pmatrix} \mathbf{1} & 0 \\ -\delta_L & \mathbf{1} \end{pmatrix}$$

crossing L in the reverse direction changes the elementary connection operator into its opposite.

COROLLARY 2.2.1. – *Assume there are no bounded Stokes line, so that all Stokes lines are simple. Then*

$$\mathcal{C}^{+}_{R'R} = \mathcal{C}^{-}_{R'R}$$

for all couples of Stokes regions R, R'.

2.2.2. Splitting multiple line configurations into simple ones

In order to "split" multiple line configurations into simple ones, one just has to slightly rotate the resummation direction α away from the $\arg\alpha = 0$ direction.

Given any resummation direction α, the *Stokes lines relative to* α are defined by replacing in Section 1.1.2 the gradient trajectories of $\Re S$ by those of $\Re(S\,\mathrm{e}^{-\mathrm{i}\arg\alpha})$ (cf. Section 1.1.3), so that Eq. (1.3) becomes $dq/dt = \exp(\mathrm{i}\arg\alpha)\overline{p}(q(t))$. When $\arg\alpha$ is a small (positive or negative) number, the corresponding configuration of Stokes lines is very close to the $\arg\alpha = 0$ case, *but it has only unbounded Stokes lines* as illustrated in [1] §II and III (each Stokes line which was not simple splits into a number of simple lines).

A very simple algorithm enables one to draw the topology of the "split" configurations by just looking at the $\arg\alpha = 0$ configuration.

Splitting algorithm. (We describe the algorithm in the $\arg\alpha > 0$ case. The $\arg\alpha < 0$ case would be similar, replacing "right" by "left" everywhere in the instructions below.) Looking at the multiple line configuration, draw a copy of all unbounded Stokes lines.

Each of them can be considered as the route of a car following a one-way road and stopping at a croosroad (the turning points); if taking the first road on the right is allowed by trafic regulations (i.e., if it is a bounded Stokes line) draw another unbounded Stokes line, defining it as the route of another car (cf. Fig. 15): this second car starts all the way side by side on the right of the first one, until it gets near the turning point; there it takes the right turn along the above mentioned road, and stops at the next turning point.

If right turn is again allowed at this turning point, application of the same procedure to the route of this second car defines a third one, etc. . . . .

The procedure stops when the first road on the right is a no-entry road.

Key remark. – The topology of the split configuration is not the same in the $\arg\alpha > 0$ and the $\arg\alpha < 0$ case.

Let us show how this remark accounts for the difference between right and left connection isomorphisms in the multiple line case.

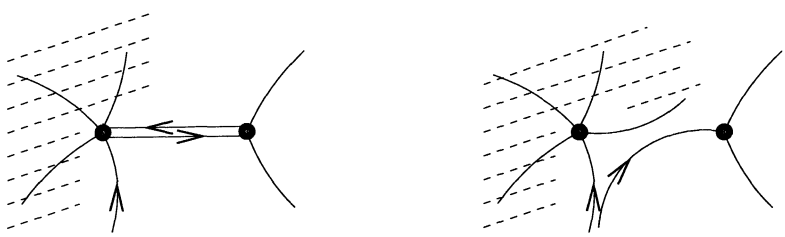

Fig. 15.

Let R, R' be two Stokes regions in the $\arg \alpha = 0$ configuration. By an obvious semi-continuity argument, the split configuration with $\arg \alpha$ small (whether > 0 or < 0) has two unambiguously defined Stokes regions $R(\alpha)$, $R'(\alpha)$ which tend to R, R' when $\arg \alpha \to 0$. Since the split configuration has only simple lines, we can define $\mathcal{C}^{\alpha}_{R'(\alpha)R(\alpha)}$, the *connection isomorphism relative to the resummation direction* α, with no need to specify whether we mean the *right* or *left* connection isomorphism (by Corollary 2.2.1 they are equal).

With these notations we have the

PROPOSITION 2.2.2. –

$$\mathcal{C}^{+}_{R'R} = \mathcal{C}^{\alpha}_{R'(\alpha)R(\alpha)} \quad \textit{for } \arg \alpha \textit{ small enough, } > 0,$$
$$\mathcal{C}^{-}_{R'R} = \mathcal{C}^{\alpha}_{R'(\alpha)R(\alpha)} \quad \textit{for } \arg \alpha \textit{ small enough, } < 0.$$

Of course these equalities must be interpreted via the obvious isomorphisms $\mathrm{WKB}(R) \approx \mathrm{WKB}(R(\alpha))$, $\mathrm{WKB}(R') \approx \mathrm{WKB}(R'(\alpha))$.

Proof. – This is again an application of the "continuity principle" of Section 0. □

In the next two subsections we show this proposition at work in explicit computations of connection isomorphisms.

2.2.3. Crossing a multiple unbounded Stokes line

Let L be an unbounded Stokes line fading into a turning point $q_\star$. Its right (respectively, left) splitting consists of an array of "parallel" simple Stokes lines L, L^+, L^{++}, ... (respectively, L, L^-, L^{--}, ...) as represented on Fig. 16.

Since the fading orientations of these lines are parallel, the connection isomorphism $\mathcal{C}^{\pm}_{R'R}$ will be given by a product of lower triangular matrices of the type considered in Section 2.2.1, thus yielding as a final result

$$\mathcal{C}^{\pm}_{R'R} = \begin{pmatrix} \mathbf{1} & 0 \\ \delta^{\pm}_{L} & \mathbf{1} \end{pmatrix} := \mathcal{C}^{\pm}_{L},$$

where

$$\delta^{+}_{L} = \delta_{L} + \delta_{L^+} + \delta_{L^{++}} + \cdots,$$
$$\delta^{-}_{L} = \delta_{L} + \delta_{L^-} + \delta_{L^{--}} + \cdots$$

are the corresponding sums of elementary connection operators.

In the above formulas we assumed that the Stokes line was crossed from right to left, according to our conventional orientation for Stokes

lines (cf. Fig. 14). When it is crossed in the opposite direction the operators $\mathcal{C}_L^{\pm}$ must be changed into their opposites.

2.2.4. Crossing a bounded Stokes line

Let L be a bounded Stokes line connecting two turning points q_1 and q_2. Looking at its right (respectively, left) splitting as represented on Fig. 17, we see that crossing from R to R' means crossing in succession two arrays of simple Stokes lines:

first an array of lines oriented towards q_1 (respectively, q_2),

then an array of lines oriented towards q_2 (respectively, q_1).

Taking as the first (respectively, second) summand of **WKB** in the neighbourhood of L that one which fades towards q_1 (respectively, q_2), the connection matrices across *either one* of these arrays (crossed from right to left) the same form as in Section 2.2.3:

$$\mathcal{C}_1^{\pm} := \begin{pmatrix} \mathbf{1} & 0 \\ \delta_1^{\pm} & \mathbf{1} \end{pmatrix}, \qquad \mathcal{C}_2^{\pm} := \begin{pmatrix} \mathbf{1} & \delta_2^{\pm} \\ 0 & \mathbf{1} \end{pmatrix},$$

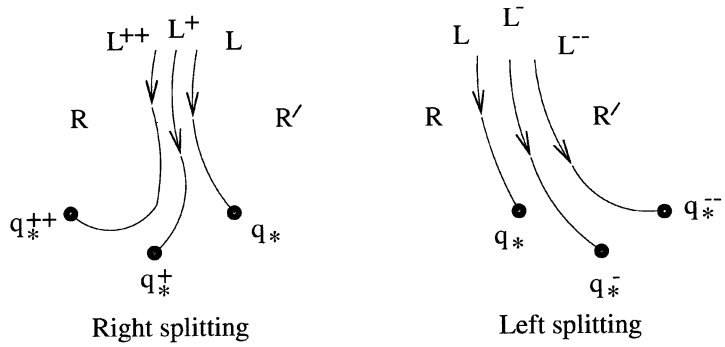

Fig. 16.

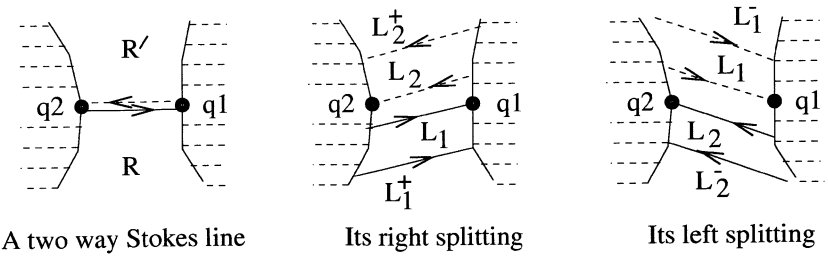

Fig. 17.

where for $i = 1, 2$,

$$\delta_i^+ = \delta_i + \delta_{i+} + \delta_{i++} + \cdots,$$
$$\delta_i^- = \delta_i + \delta_{i-} + \delta_{i--} + \cdots$$

are the sums of the elementary connection operators across the corresponding simple Stokes lines.

The connection matrices across *both* arrays are therefore the following product matrices

$$\mathcal{C}^+_{R'R} = (\mathcal{C}_2^+)^{-1}\mathcal{C}_1^+ = \begin{pmatrix} \mathbf{1} - \delta_2^+\delta_1^+ & -\delta_2^+ \\ \delta_1^+ & \mathbf{1} \end{pmatrix},$$
$$\mathcal{C}^-_{R'R} = \mathcal{C}_1^-(\mathcal{C}_2^-)^{-1} = \begin{pmatrix} \mathbf{1} & -\delta_2^- \\ \delta_1^- & \mathbf{1} - \delta_1^-\delta_2^- \end{pmatrix}.$$

2.3. Connection paths and Voros symbols

As a general consequence of Section 2.2 the connection operator from (R, p) to (R', p') is given by a finite sum

$$\varphi \to \sum_\lambda \lambda\varphi,$$

where each λ acts as a sequence of *analytic continuation operators and elementary connection operators* $\pm\delta$ (with the $(+)$—respectively, $(-)$—sign when the corresponding fading line is crossed from right to left—respectively, left to right).

DEFINITION 2.3.1. – *Any sequence λ of the above type is called a connection path from (R, p) to (R', p').*

Any finite (formal) combination of such connection paths

$$\sum_\lambda n_\lambda \lambda \quad (n_\lambda = \textit{relative integers})$$

is called a connection chain from (R, p) to (R', p').

Connection chains starting and ending in the same (R, p) are called connection cycles based in (R, p).

Of course two connection paths λ, λ' can be composed whenever λ' starts where λ ends. Denoting by $\lambda'\lambda$ the composed connection path, we see that *connection cycles with a given base can be multiplied as well as added, thus forming a ring.*

Any connection path (or connection chain) from (R, p) to (R', p') defines in an obvious way an operator from $\mathbf{WKB}^p(R)$ to $\mathbf{WKB}^{p'}(R')$. Two connection chains are called *equivalent* if they define the same operator.

Remark. – For generic energies we know from [5,1] that any connection path is equivalent to a path of analytic continuation (and the latter can be chosen in a canonical way); furthermore it is generically true that all equivalences between connection chains are generated by equivalences between the connection paths they are made from. This is no longer true for critical energies (cf. for instance the relations in [1] §III.4).

Equivalence of connection cycles. In the case of connection cycles the above equivalence relation was defined only for cycles *with a specified base* (R, p). It is convenient to broaden it, defining an equivalence relation on the set of all connection cycles (with unspecified base); this equivalence relation will be generated (via the product law) by the previous one, and by the following *commutation relation*:

$$\lambda_1\lambda_2 = \lambda_2\lambda_1$$

whenever both sides make sense, i.e., λ_1 starts where λ_2 ends and conversely.

An important example of this equivalence is the *change of base of cycles*: given a cycle γ and a path of analytic continuation λ starting at the base of γ, one has the equivalences

$$\lambda\gamma\lambda^{-1} \equiv \gamma\lambda^{-1}\lambda \equiv \gamma.$$

Assuming the Riemann surface of the momentum to be connected we can altogether forget specifying the base, and speak of the *ring of* (*equivalence classes of*) *connection cycles*—an obviously *commutative* ring.

When the Riemann surface of the momentum is not connected (i.e., when there are only turning points of even order), one should distinguish between *two rings of connection cycles*, depending on which sheet of determinations the "basic" momentum p has been chosen on.

Voros symbols and Voros multipliers

LEMMA 2.3.1. – *The action of a connection cycle γ on any WKB symbol $\varphi \in \mathbf{WKB}^p(R)$ consists in multiplying it by some scalar resurgent symbol a^γ depending only on the equivalence class of γ.*

Proof. – This immediately follows from Proposition 2.1.1. □

DEFINITION 2.3.2. – *This symbol a^γ will be called the Voros symbol of the connection cycle γ. Elementary Voros symbols (i.e., those the essential support of which consists of just one point) will be called* Voros multipliers.

Example: The Voros multipliers $a^{[L]^+}$, $a^{[L]^-}$ of a bounded Stokes line L. With notation 2.2.4, they are defined as the Voros symbols of the connection cycles

$$[L]^+ :\equiv -\delta_2^+\delta_1^+ \equiv -\delta_1^+\delta_2^+,$$
$$[L]^- :\equiv -\delta_1^-\delta_2^- \equiv -\delta_2^-\delta_1^-.$$

Notice that these Voros symbols are canonically attached to L, with no need to choose a "crossing direction" (changing this crossing direction just exchanges the indices 1 and 2 in Section 2.2.4).

In the special case when L is an *isolated* bounded Stokes line (i.e., is not attached to any other bounded Stokes line), these two connection cycles coincide, and will simply be denoted by

$$[L] :\equiv -\delta_2\delta_1 \equiv -\delta_1\delta_2.$$

The Voros ring(s). Like the cycles they represent, Voros symbols make up a commutative ring $\mho$, or two rings $\mho_p$, $\mho_{-p}$ when the Riemann surface of the momentum is disconnected.

Special case when there are only simple turning points. We know that in that case all connection cycles are equivalent to sums of "geometric cycles" in $\dot{\mathbb{C}}_2$ (homology classes of closed paths in $\dot{\mathbb{C}}_2$). Because of the $p^{-1/2}$ factor in (1.2) (cf. (1.0)), the Voros multiplier of any such geometric cycle is of the form $a_\bullet\, \mathrm{e}^{-\omega x}$, with

$$a_\bullet = a_0 + a_1 x^{-1} + a_2 x^{-2} + \cdots \in \mathbb{C}[[x^{-1}]],$$

where the leading coefficient a_0 in $a_\bullet$ is ± 1 depending on whether the closed path has index 0 or $2 \bmod 4$ with respect to the set of turning points).

The Voros multipliers of geometric cycles of index $0 \bmod 4$ make up a multiplicative subgroup $\mathcal{V}$ of $\mathcal{A}$, which we call the *Voros group*. The Voros ring is the corresponding group ring

$$\mho = \mathbb{Z}\mathcal{V}$$

consisting of all linear combinations of elements of $\mathcal{V}$ with integral coefficients. It follows from this that each Voros symbol can be written as a finite sum

$$a = \sum_{\omega} a_{\omega} \mathrm{e}^{-\omega x},$$

where the leading coefficient of every a_{ω} is a relative integer n_{ω}. We call *positive Voros symbols* those elements $a \in \mathcal{V}$ whose all leading coefficients n_{ω} are natural integers: they are linear combinations of elements of $\mathcal{V}$ with natural integers as coefficients.

2.4. Restricting the scalars

2.4.1. Restricted notion of WKB symbols

The Definition 2.1.1 of WKB symbols is unnecessarily general for most practical purposes. Among the many possible ways of restricting it, a minimal requirement is that they should form a *local system of abelian groups, stable by all connection operators*. The simplest natural way of fulfilling that requirement is to start from a so-called "primitive" WKB expansion φ_0 defined in some region R_0, with momentum p_0, and transform it under the action of *all possible connection chains starting at* (R_0, p_0). When the Riemann surface of the momentum is connected we thus get a locally free $\mathcal{V}$-module of rank 2

$$\mathrm{WKB} \overset{loc.}{=} \mathcal{V}\varphi_p \oplus \mathcal{V}\varphi_{-p}$$

(where φ_p and φ_{-p} are any two analytic continuations of φ_0 with opposite momenta).

When the Riemann surface of the momentum has two connected components, one has instead

$$\mathrm{WKB} \overset{loc.}{=} \mathcal{V}_{p_0}\varphi \oplus \sum_{\delta} \mathcal{V}_{-p_0}\delta\varphi,$$

where φ is any analytic continuation of φ_0, and δ runs over the set of all elementary operators with p_0 as the starting determination of the momentum.

2.4.2. "Relative" WKB and Voros symbols

Let $\psi = \lambda\varphi$ and $\psi' = \lambda'\varphi$ be obtained from φ by the action of two connection paths λ, λ' with a common origin and a common end. One

should like to write

$$\psi' = a_{\lambda'\lambda^{-1}}\psi,$$

but even if there exists a resurgent scalar symbol $a_{\lambda'\lambda^{-1}}$ with this property, there is no reason why it should belong to the Voros ring, because what we would like to call the "cycle" $\lambda'\lambda^{-1}$ is not built according to the rules 2.3, where we did not allow for *inverses* of elementary connection operators (an exception is of course the case when there are only simple turning points). Allowing for such inverses (of those elementary connection operators which are indeed invertible) gives what we shall call *relative* connection paths, relative connection chains and cycles, relative WKB expansions, relative Voros symbols, ...

An advantage of working with relative objects is that the case when the Riemann surface of the momentum is disconnected need not be considered separately, since the two opposite determinations of the momentum can (presumably) always be connected to each other by at least one invertible elementary connection operator, allowing us *in all cases* to consider WKB^{rel} as locally free of rank 2 on the ring $\mathcal{V}^{rel}$ of relative Voros symbols.

For instance near a *double* turning point, where the Riemann surface of p splits into two connected components, we shall see in Section 5.1 that the elementary connection operator has a $1/\Gamma(-s)$ factor, which can only fail to be invertible when the "monodromy exponent" s is close to a natural integer n (more precisely, when $s-n$ is small and non-invertible). But we shall see that starting with the opposite determination of p changes the monodromy exponent s into $s^\star = -s-1$, and invertibility then holds for $1/\Gamma(-s^\star)$.

2.4.3. Rational Voros (or WKB) symbols

By their very construction, the above spaces of restricted WKB symbols correspond to each other through the connection isomorphisms (i.e., $\mathcal{C}^{\pm}_{R'R}$ sends $\mathrm{WKB}(R)$ onto $\mathrm{WKB}(R')$). But knowing whether the Stokes automorphism $\mathfrak{S}$ sends $\mathrm{WKB}(R)$ into itself is another question.

Consider first the case when there are no bounded Stokes lines, and the initial WKB expansion φ_0 has been chosen to be Borel resummable (as in Proposition 1.3.1). Then it follows from Proposition 1.3.2 that all analytic continuations of φ_0, and therefore all restricted WKB symbols, are Borel resummable inside each Stokes region (so that $\mathfrak{S} \mid \mathrm{WKB}(R) = \mathbf{1}$).

In all other cases, $\mathfrak{S}$ fails to send $\mathrm{WKB}(R)$ *(and the Voros ring $\mathcal{V}$) into itself.* Notice that by the remark at the end of Section 1.1.2 the two

generators of WKB(R) over the Voros ring can always be chosen so as to be Borel resummable, so that the question of how $\mathfrak{S}$ acts on WKB(R) is reduced to how it acts on $\mathcal{V}$. We shall show in Section 2.5 that when there are bounded Stokes lines a ring extension is needed to make the Voros ring globally invariant under $\mathfrak{S}$: a very natural one is the *rational Voros ring* $\mathcal{V}^{rat}$.

DEFINITION 2.4.1. – *The rational Voros ring $\mathcal{V}^{rat}$ is the ring extension of $\mathcal{V}$ generated by all elements of the form*

$$(1-\varepsilon)^{-1} = 1 + \varepsilon + \varepsilon^2 + \cdots$$

with $\varepsilon \in \mathcal{V}$ exponentially small.

From this extension of Voros symbols to "rational Voros symbols", we define the group of rational WKB symbols by

$$\mathrm{WKB}^{rat} = \mathcal{V}^{rat} \underset{\mathcal{V}}{\otimes} \mathrm{WKB}.$$

One defines in a similar fashion the *rational relative* Voros ring, and rational relative WKB symbols (starting from $\mathcal{V}^{rat}$ instead of $\mathcal{V}$).

2.4.4. Analytic (WKB or Voros) symbols

At the very root of Ecalle's "alien calculus" lies the idea that one can freely use not only the Stokes automorphism $\mathfrak{S}$ but also its logarithm (which leads to the notion of "alien derivation"), its non integral powers $\mathfrak{S}^r = \exp(r \log \mathfrak{S})$, etc. .... For instance in [1] we compute concrete examples of "median symbols", which involve the use of $\mathfrak{S}^{1/2}$. Further extension of our spaces of WKB (and Voros) symbols is needed to make them stable by such more general automorphisms.

DEFINITION 2.4.2. – *An analytic Voros symbol is any resurgent symbol obtained by substituting Voros symbols in an analytic function (under some natural conditions making such substitution permissible: cf. Section 0). Analytic Voros symbols make up an algebra, the (analytic) Voros algebra $\mathcal{V}^{an}$. Analytic WKB symbols are deduced from usual WKB symbols by the corresponding "extension of scalars":*

$$\mathrm{WKB}^{an} = \mathcal{V}^{an} \underset{\mathcal{V}}{\otimes} \mathrm{WKB}.$$

Examples of analytic Voros symbols. For every invertible Voros symbol α (invertible not in $\mathcal{V}$ of course, but as a resurgent symbol),

$\log\alpha$, $\alpha^{1/2}, \ldots$, are analytic Voros symbols. An important example is the *monodromy factor* of a geodesic cycle γ, defined by

$$s = \frac{1}{2\pi \mathrm{i}} \ln a^{\gamma} - \frac{1}{2}$$

which is of special interest when γ is the "vanishing cycle" associated to a double turning point (cf. Section 3.3 hereafter): as we shall see, the connection formulas for a double turning point involve multipliers of the form

$$\frac{\sqrt{2\pi}\, x^{-s-\frac{1}{2}}}{\Gamma(-s)}$$

which are examples of analytic Voros symbols.

2.5. Computing the Stokes automorphism

PROPOSITION 2.5.1. – *Inside a Stokes region R, the action of $\mathfrak{S}$ on general WKB symbols commutes with analytic continuation, and it preserves the momentum, i.e., it sends* $\mathbf{WKB}^p(R)$ *into itself.*

Proof. – The first statement is just a reformulation of Proposition 2.1.3. Preservation of the momentum comes from the fact that inside a Stokes region none of the "moving" singularities (those corresponding to the opposite determination of the momentum) are seen in the real positive direction (cf. [5] for the details). □

COROLLARY 2.5.1. – *By Proposition* 2.1.1 *the action of $\mathfrak{S}$ and $\mathfrak{S}^{-1}$ inside a Stokes region R on any invertible $\varphi \in \mathbf{WKB}^p(R)$ can be written*

$$\mathfrak{S}\varphi = \sigma\varphi, \qquad \mathfrak{S}^{-1}\varphi = \sigma^{\star}\varphi$$

with $\sigma, \sigma^{\star} \in \mathcal{A}$.

We shall call σ (respectively, $\sigma^{\star}$) the *Stokes factor* (respectively, *inverse Stokes factor*) of φ.

Remarks. –

(i) σ and $\sigma^{\star}$ depend on φ, and are not inverse to one another (remember that $\mathfrak{S}$ is not $\mathcal{A}$-linear!): instead, one has

$$\sigma = \mathfrak{S}\left(\sigma^{\star -1}\right).$$

(ii) $\sigma = 1 + \varepsilon$, with ε an exponentially small factor (and similarly for $\sigma^\star$): this is because $\mathfrak{S} \equiv \mathbf{1}$ mod. small exponentials (cf. Section 0).

(iii) $\sigma = \sigma^\star = 1$ iff φ is Borel resummable.

2.5.1. How the Stokes automorphism changes after the action of a connection path

THEOREM 2.5.1. – *Let λ be a (relative) connection path starting and ending outside Stokes lines. Then*

$$\mathfrak{S}\lambda = \sigma_\lambda \lambda \mathfrak{S},$$

where $\sigma_\lambda = 1 + \varepsilon_\lambda$ (ε_λ exponentially small) is a scalar resurgent symbol depending only on λ, which we call the Stokes factor of λ, with the following properties.

(i) *$\sigma_\lambda = 1$ if there are no bounded Stokes lines.*

(ii) *If λ is a path of analytic continuation, σ_λ belongs to the rational Voros ring, and is given by the formula*

$$\sigma_\lambda = \prod_L \left(1 + a^{[L]^+}\right)^{\langle L, \lambda \rangle},$$

where the product is taken over all bounded Stokes lines L; $a^{[L]^+}$ is the right-Voros symbol of L, defined in Section 2.3; $\langle L, \lambda \rangle$ is the intersection index of λ with the fading cycle lying above L, i.e., the algebraic number of times λ crosses it, counted with $(+)$ sign whenever it crosses from right to left, and $(-)$ sign in the opposite case.

(iii) *If λ is a connection path, or a relative connection path, σ_λ belongs to the rational relative Voros ring.*

Remark. – A completely similar statement holds for $\mathfrak{S}_\alpha$ (the Stokes automorphism in any direction α), with the fading cycles replaced by the geodesic cycles of direction α.

Basic example. Replace assumption (c) in Theorem 1.2.2 by the following one:

φ is well normalized at infinity along a path λ which crosses only *one* geodesic cycle γ, with intersection index $\langle \gamma, \lambda \rangle = +1$.

Denoting by α the direction of that geodesic cycle, assume further (this is a generic assumption on q) that the minor of φ glimpses no moving

singularity in the direction α. Then

$$\mathfrak{S}_\alpha \varphi = \left(1 + a^\gamma\right) \varphi.$$

COROLLARY 2.5.2. – *The rational relative Voros ring is stable under* $\mathfrak{S}$.

(Applying the theorem to a closed connection path λ, one finds $\mathfrak{S}(a_\lambda) = (1 + \varepsilon_\lambda) a_\lambda)$, where a_λ is the Voros symbol of λ.)

Proof of the theorem. – It is enough to prove it when λ is an analytic continuation operator across a Stokes line, or an elementary connection operator. By definition of connection isomorphisms and Stokes automorphism one has $\mathfrak{S}_{R'} \mathcal{C}^-_{R'R} = \mathcal{C}^+_{R'R} \mathfrak{S}_R$ (where the index R or R' below $\mathfrak{S}$ is written to help us remember in which Stokes region the operator $\mathfrak{S}$ is considered).

Comparing the explicit forms of the right and left connection operators in Sections 2.2.3 and 2.2.4 easily gives the following results.
(i) $\mathfrak{S}\lambda = \lambda\mathfrak{S}$ if λ is an analytic continuation operator across an unbounded Stokes line (proof: λ is just one of the diagonal connection operators in Section 2.2.3, simply denoted there by **1**).
(ii)

$$\mathfrak{S}\lambda = \left(1 + a^{[L]^+}\right)\lambda\mathfrak{S}, \quad \text{respectively,} \quad \mathfrak{S}\lambda = \left(1 + a^{[L]^+}\right)^{-1}\lambda\mathfrak{S},$$

if λ is a path of analytic continuation across a bounded Stokes line L, crossing it from right to left (respectively, left to right), with respect to the fading orientation on that sheet of $\dot{\mathbb{C}}_2$ where λ lies (proof: compare the right and left diagonal connection operators in Section 2.2.4).
(iii) Comparing off-diagonal connection operators in Section 2.2.3 or 2.2.4 shows that

$$\begin{aligned} \mathfrak{S}_{R'}\delta^- &= \delta^+\mathfrak{S}_R, \\ \mathfrak{S}_R\delta^- &= \delta^+\mathfrak{S}_{R'}, \end{aligned} \tag{$\star$}$$

where $\delta^\pm$ are the connection operators denoted by $\delta^\pm_L$ in Section 2.2.3 and $\delta^\pm_i$ $(i = 1, 2)$ in Section 2.2.4.

This is not yet the kind of formulas we would like to get, because we want to know how $\mathfrak{S}$ changes after the action of an *elementary* connection operator δ. In the special case when δ^- is elementary $(\delta^- = \delta^\pm)$, such a formula can be obtained in the following way.

Write $\delta^+ = \delta + \delta_+ + \delta_{++} + \cdots$, where δ_+, δ_{++}, ... are the elementary operators "following δ on the right".

Then, introducing the relative connection cycle

$$[\delta]^+ = \delta_+\delta^{-1} + \delta_{++}\delta^{-1} + \cdots$$

formulas ($\star$) can be rewritten

$$\mathfrak{S}_{R'}\delta = (1 + a_{[\delta]^+})^{-1}\delta S_R$$

(and similarly with R and R' exchanged). □

Remark. – It is a straightforward exercise to generalize these formulas to

$$\delta^- = \delta + \delta_- + \delta_{--} + \cdots.$$

HINT: the sum $\delta_- + \delta_{--} + \cdots$ is "essentially" δ'^-, the left connection operator across the singular fading line "next to L on the left". But an analytic continuation is required to get from L to that other fading line, where formula ($\star$) can be applied to δ'^-. One should be careful to take into account (using (i) an (ii)) the changes in $\mathfrak{S}$ which may occur during that analytic continuation.

Remark. – It immediately follows from relation ($\star$) that the right and left Voros symbols of a bounded Stokes line L are related by

$$\mathfrak{S}(a^{[L]^-}) = a^{[L]^+}.$$

In particular if the bounded Stokes line is *isolated*, one has

$$\mathfrak{S}(a^{[L]}) = a^{[L]},$$

i.e., the Voros symbol of L is Borel resummable.

3. CONFLUENCE NEAR A QUADRATIC CRITICAL POINT

We want to investigate how the resurgent structure of WKB expansions depends on (q, E) in a neighbourhood of (q_{crit}, E_{crit}), a *quadratic* critical point of V and the corresponding critical value. Assuming for simplicity that $q_{crit} = 0$, $E_{crit} = 0$, we shall denote by $U = D_\eta \times D_\varepsilon$ a small polydisc in $\mathbb{C}^2$ ($|q| < \eta$, $|E| < \varepsilon$). WKB expansions can be chosen to depend

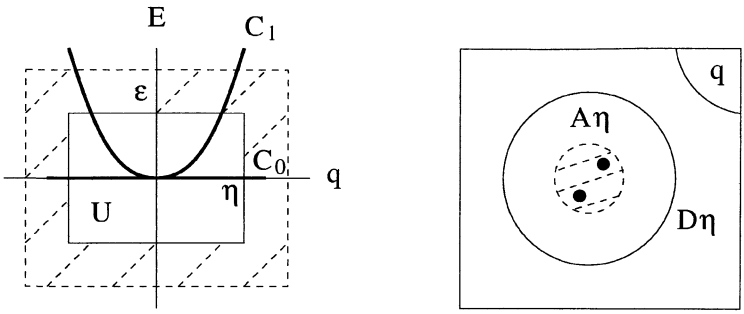

Fig. 18.

analytically on (q, E) in U outside the two "confluence curves" (cf. Fig. 18)

$$C_0 = \{(q, E) \in U \mid E = 0\}, \qquad C_1 = \{(q, E) \in U \mid V(q) = E\}.$$

Of course this dependence is multivalued, and our first task will be to describe this multivaluedness.

3.1. Local topological analysis

Choosing η small enough so that the origin is the only zero of $V(q)$ in the disc D_η: $|q| < \eta$, let ε be small enough so that for all $E \in D_\varepsilon$ (the disc $|E| < \varepsilon$) the function $V(q) - E$ has no zero in the annulus A_η: $\eta/2 < |q| < \eta$ (cf. Fig. 18). Choosing some base point q_0 in A_η and E_0 in D_ε, let us describe the first homotopy group $\pi_1(U \backslash C_0 \cup C_1, (q_0, E_0))$ of the open set $U \backslash C_0 \cup C_1$ with base point (q_0, E_0).

Projection $\mathrm{pr}: (q, E) \mapsto E$ makes $U \backslash C_0 \cup C_1$ a fibre bundle with base $D_\varepsilon^\star = D_\varepsilon \backslash \{0\}$, whose fibre above E reads $\mathrm{pr}^{-1}(E) = D_\eta \backslash \{q_1, q_2\}$. Let us consider in the fibre $\mathrm{pr}^{-1}(E_0)$ the loops ℓ_1 and ℓ_2 with base point (q_0, E_0), drawn on Fig. 19(a), and let β be the loop with base point (q_0, E_0) drawn on Fig. 19(b).

The fundamental group $\pi_1(U \backslash C_0 \cup C_1, (q_0, E_0))$ is generated by these three loops, subject to the following two relations

$$\begin{cases} \beta \ell_1 \beta^{-1} = \ell_2, \\ \beta \ell_2 \beta^{-1} = \ell_2^{-1} \ell_1 \ell_2 \end{cases} \tag{3.0}$$

(we denote by $\lambda'\lambda$ the composed path, consisting in first following λ then λ').

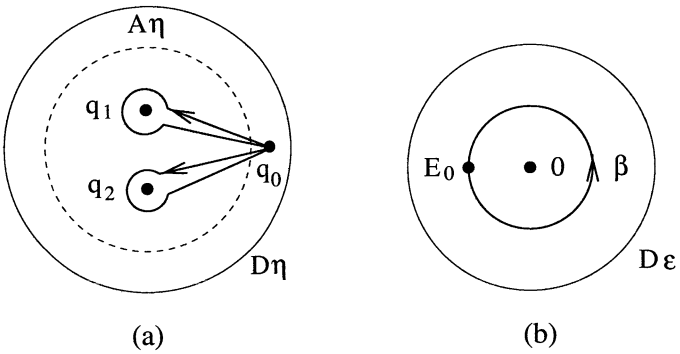

Fig. 19.

3.2. Confluent action integrals

Choosing some determination of $p = (V(q) - E)^{1/2}$ in $A_\eta \times D_\varepsilon$, we define the action function $S = S(q, E)$ by

$$\begin{cases} dS_{|\mathcal{L}_E} = p\,dq, \\ S(q_0, E) = 0 \qquad \text{for all } E \in D_\varepsilon. \end{cases}$$

In $A_\eta \times D_\varepsilon$ this is a multivalued analytic function of (q, E): analytic continuation around the annulus changes S into $S + \omega_\gamma$, where

$$\omega_\gamma = \omega_\gamma(E) = \int_\gamma p\,dq$$

is the *action integral of the vanishing cycle* γ (cf. Fig. 20). This action integral ω_γ is a *holomorphic function of* E *in* D_ε, with a simple zero at $E = 0$.

For any given $E \in D_\varepsilon^\star = D_\varepsilon \backslash \{0\}$ the above function S extends to a multivalued analytic function in $D_\eta \backslash \{q_1, q_2\}$, where $\{q_1, q_2\}$ is the set of turning points in D_η (cf. Fig. 19(a)). The various determinations of this multivalued function are deduced from the initial one by a larger group of transformation, namely the group generated by two symmetries

$$\sigma_1 : S \mapsto \omega_{l_1} - S, \qquad \sigma_2 : S \mapsto \omega_{l_2} - S.$$

Where ω_{l_1} (respectively, ω_{l_2}) is the action integral $\int p\,dq$ on the loop ℓ_1 (respectively, ℓ_2) drawn on Fig. 19(a), or equivalently its "geodesic" version l_1 (respectively, l_2) going through the turning point:

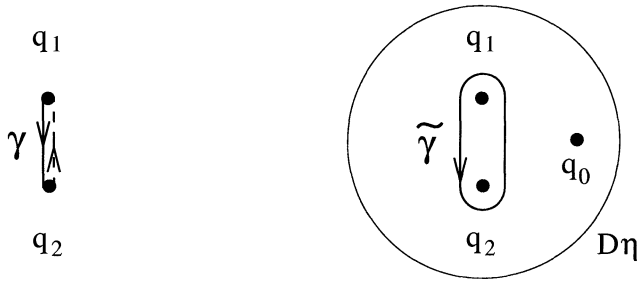

Fig. 20. The vanishing cycle γ, and its deformation $\widetilde{\gamma}$ avoiding the turning points.

our typographical ditinction between ℓ's and l's here is the same as in Theorem 1.2.2(b).

Notice that the loops on Fig. 19(a) have been chosen so that $\ell_1\ell_2 \equiv \gamma$ (where the equivalence symbol $\equiv$ stands for homology in $\mathcal{L}_E$). Therefore the composed transformation $\sigma_1 \circ \sigma_2$ transforms S into $S + \omega_\gamma$, as previously seen. Notice also that when E turns around 0 anticlockwise l_i is changed into $l_i + \gamma$ (for instance l_1 is changed into l_2). Since the function $\omega_{l_i}(E)$ is clearly bounded when E approaches 0, it follows that our "loop action integrals" ω_{l_i} are just two special determinations of one multivalued analytic function with a logarithmic singularity at $E = 0$:

$$\omega_l(E) = \frac{\omega_\gamma(E)}{2\mathrm{i}\pi} \ln \omega_\gamma(E) + h(E) \quad \text{(where } h(E) \text{ is holomorphic in } D_\varepsilon\text{)}.$$

3.3. Local monodromy of WKB expansions

We now describe the action of $\pi_1(U \backslash C_0 \cup C_1, (q_0, E_0))$ on a WKB expansion φ. Denoting by $\widetilde{\gamma}$ the loop of $U \backslash C_0 \cup C_1$ defined by $\widetilde{\gamma} = \ell_1\ell_2$ (cf. Fig. 20), relations (3.0) can be rewritten as follows:

$$\begin{cases} \beta\ell_1\,\varphi = \ell_2\beta\,\varphi, \\ \beta\widetilde{\gamma}\,\varphi = \widetilde{\gamma}\beta\,\varphi. \end{cases}$$

These relations, of purely topological origin, must be supplemented by the following ones, specifically concerned with analytic continuations of WKB expansions along *loops of* $\dot{\mathbb{C}}_2$, *the Riemann surface of* p.

$$\begin{cases} \widetilde{\gamma}\,\varphi = -a^\gamma\,\varphi, \\ \ell_1^2\,\varphi = \ell_2^2\,\varphi = -\varphi, \end{cases} \tag{3.1}$$

and

$$\beta\varphi = \varphi \tag{3.1$'$}$$

the latter being true if φ is well normalized along some path which does not intersect the vanishing cycle γ (under this assumption φ depends holomorphically on E throughout D_ε). Relations (3.1) are easily read on the expression (1.2) of WKB expansions: the *minus* sign comes from the change of determination in $P^{-1/2}$, whereas the Voros multiplier a^γ reads

$$a^\gamma = \mathrm{e}^{-x\int_\gamma P\,dq}.$$

For later use it will be convenient to rewrite the first line of formula (3.1) as follows

$$\widetilde{\gamma}\,\varphi = \mathrm{e}^{2\pi\mathrm{i}s}\varphi,$$

where $s = s_\gamma(x, E)$ is the *monodromy exponent* (of the vanishing cycle γ), defined by

$$s_\gamma = -\frac{x}{2\pi\mathrm{i}}\int_\gamma P\,dq - \frac{1}{2} = -\frac{\omega_\gamma}{2\pi\mathrm{i}}x - \frac{1}{2} + \mathrm{O}(x^{-1})$$

a resurgent series in x^{-1} depending regularly on E in the whole disc D_η. Given the local "confluence" situation of Fig. 18, the monodromy exponent depends only on which sheet of $\dot{\mathbb{C}}_2$ has been chosen for the determination of p in the annulus A_η: replacing p by $-p$ would result in replacing γ by $-\gamma$, and therefore s by $-s-1$. Whenever in the sequel we shall write such phrase as "the monodromy exponent of φ", it must be understood that the corresponding determination of p is that one which has been chosen for φ.

3.4. Local resurgent structure of WKB expansions

By the "local resurgent structure" we mean the resurgent structure with respect to the "confluent" singularities, i.e., those singularities in the ξ-plane which tend to zero as (q, E) tends to $(0, 0)$.

Let us consider first the "moving singularities", in the sense of Section 1.2. Depending on how E is chosen, there is *one* or *two* such singularities, lying at the position(s) ω_l shown on Fig. 21 (the action integral on the "vanishing loop" $l = l_1$ or l_2). For $(q, E) \in U_1$, the half-neighbourhood of $(0, 0)$ corresponding to the "one loop case" of Fig. 21,

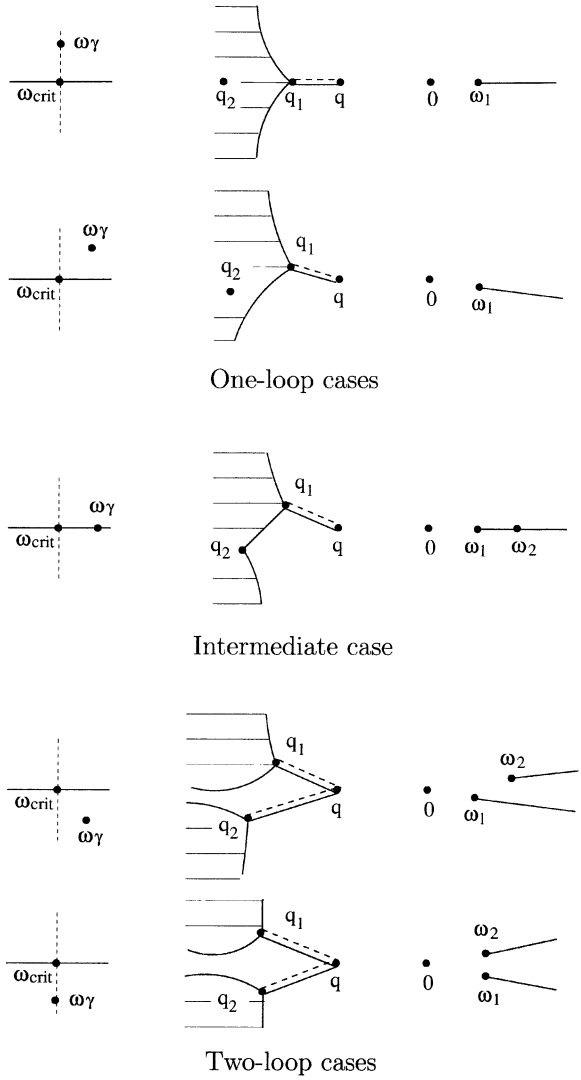

Fig. 21. The confluent "moving singularities", for a couple of confluent turning points. The shady regions are those regions in q-plane which cannot be reached by non-singular geodesics coming from q. When q_2 lies in the shady region the singularity ω_{l_2} is "hidden in the second sheet". Starting from the $\Im\omega_\gamma > 0$ region (one-loop case) and letting ω_γ move clockwise through the positive real axis to the $\Im\omega_\gamma < 0$ region, one gets the two-loop situation where the index 1 for q_1 (respectively, l_1) labels the turning point (respectively, the geodesic loop) deduced from the one-loop situation by continuity. Anticlockwise motion (through the negative real axis) would have led to the opposite convention.

let φ be a WKB expansion, well normalized at infinity along a path which does not intersect the vanishing cycle γ. Then its minor $\widehat{\varphi}$ has no other confluent singularity than ω_l ($= \omega_{l_1}$ of Fig. 21, top), and by Theorem 1.2.2(b) the corresponding alien derivative reads

$$\dot{\Delta}_{\omega_l}\varphi = \ell_1\varphi \tag{3.2}$$

(the analytic continuation of φ along ℓ_1). Now setting $\varphi^\star = \ell_1\varphi$, we get a WKB expansion whose minor $\widehat{\varphi}^\star$ has not only moving but also fixed confluent singularities in the first sheet, because the normalization path of $\varphi^\star$ intersects the vanishing cycle γ. The moving singularity is at $\omega_{l_1^\star} = -\omega_l$ (where $l_1^\star$ is the loop deduced from l_1 by changing the determination of p). The corresponding alien derivative reads

$$\dot{\Delta}_{-\omega_l}\varphi^\star = -\varphi \tag{3.3}$$

(we used the fact that $\ell_1^\star\ell_1\varphi = -\varphi$).

The fixed singularities are at $n\omega_\gamma$ ($n \in \mathbb{Z}^\star$), where ω_γ is the action integral over the vanishing cycle γ. Since the intersection index $\langle\gamma, \ell_1\rangle$ equals $+1$, $\varphi^\star$ satisfies the hypotheses in the "basic example" after Theorem 2.5.1; translating the resulting resurgence equations in terms of alien derivatives one easily gets (cf., e.g., [18]):

$$\dot{\Delta}_{n\omega_\gamma}\varphi^\star = \frac{(-1)^{n-1}}{n}a^{n\gamma}\varphi^\star \tag{3.4}$$

(of course $a^{n\gamma} = (a^\gamma)^n$). It is convenient to write the system of alien differential Eqs. (3.2)–(3.4) as a *linear system of order* 2,

$$\dot{\Delta}_\omega\begin{pmatrix}\varphi\\ \varphi^\star\end{pmatrix} = A_\omega\begin{pmatrix}\varphi\\ \varphi^\star\end{pmatrix}, \quad \omega \in \Omega, \tag{3.5}$$

where the A_ω's are the 2×2 matrices

$$A_{\omega_l} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \qquad A_{-\omega_l} = \begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix},$$
$$A_{n\omega_\gamma} = \begin{pmatrix}0 & 0\\ 0 & \frac{(-1)^{n-1}}{n}a^{n\gamma}\end{pmatrix}.$$

Notice that the cycle γ has self-intersection 0 (it is not "pinched" by the confluence of turning points), so that by Theorem 2.5.1 a^γ is a "local resurgence constant", i.e.,

$$\dot{\Delta}_\omega a^\gamma = 0 \quad \text{for all } \omega \in \Omega.$$

Thus the system (3.5) is the "alien" analog of a *linear differential system with constant coefficients.*

A similar study could be made when $(q, E) \in U_2$, the half-neighbourhood of $(0, 0)$ where two moving singularities ω_{l_1} and ω_{l_2} are seen (Fig. 21, "two-loop case"). The details are left to the reader.

4. UNIVERSAL MODELS FOR QUADRATIC CONFLUENCE

4.1. Confluence in Stokes multipliers

Keeping the hypotheses and notations of Section 3, Fig. 22 shows the local Stokes pattern near the origin of the q-plane (a double turning point when $E = 0$).

We concentrate our attention on two neighbouring Stokes regions R, R' which for $E = 0$ are separated by a Stokes line L as shown on Fig. 22(a). For $E \neq 0$ the "dividing wall" between them may still consist of *one* Stokes line L (as shown on Fig. 22(b)), or of *two* Stokes lines with a thin "intermediate Stokes region" between them (as shown on Fig. 22(c)): these two situations occur, respectively, when E belongs to the following "half-disc"

$$D_\varepsilon^{(1)} = \{E \in D_\varepsilon \mid \Im\omega_\gamma(E) > 0\} \qquad \text{(Fig. 22(b))},$$
$$D_\varepsilon^{(2)} = \{E \in D_\varepsilon \mid \Im\omega_\gamma(E) < 0\} \qquad \text{(Fig. 22(c))}.$$

Let $A'_\eta \subset A_\eta$ be a simply connected domain in the annulus A_η, such that for every $E \in D_\varepsilon$ it contains the intersection of A_η with the "dividing wall" between R and R', but contains no other Stokes line.

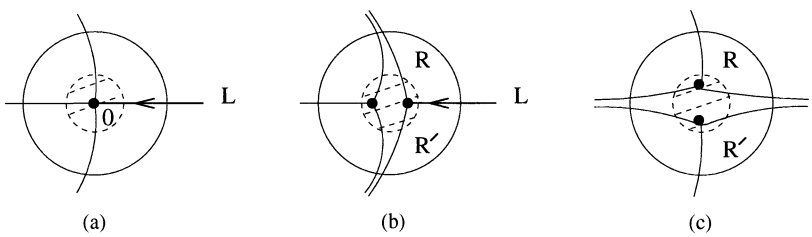

Fig. 22. Splitting of a Stokes line L ending at a double turning point. (a) Critical case $E = 0$. (b) Generic case E in $D_\varepsilon^{(1)}$, i.e., $\operatorname{Im}\omega_\gamma(E) > 0$. (c) Generic case E in $D_\varepsilon^{(2)}$, i.e., $\operatorname{Im}\omega_\gamma(E) < 0$.

Assuming that all Stokes line are unbounded, [11] let (φ, φ^r) be a basis of WKB expansions depending regularly on (q, E) in $A'_\eta \times D_\varepsilon$, such that φ, respectively, φ^r is *dominant*, respectively, *recessive* on L. It will be convenient [12] to assume that φ and φ^r are well normalized at infinity along paths *which do not cross any bounded Stokes line.* Then it follows from Theorem 1.2.2 (cf. also Section 1.3) that for every $E \in D_\varepsilon$

1. φ is Borel resummable in R and in R';
2. φ^r is Borel resummable in a simply connected domain containing R and R', and the dividing wall between them.

Denoting by Φ^r the Borel sum of φ^r in the latter domain, and by Φ_R, respectively, $\Phi_{R'}$ the Borel sum of φ in R, respectively, R', the difference between the latter two functions is *recessive* on L, so that

$$\Phi_{R'} - \Phi_R = C\Phi^r, \tag{4.1}$$

where $C = C(E, x)$ is the so-called *Stokes multiplier across* L, a *holomorphic function of* (E, x) for all E in D_ε and large enough x (it inherits that holomorphy property from Φ_R, $\Phi_{R'}$ and Φ^r, because the latter function is invertible).

Remark. – If we drop the assumption that the normalization paths of φ and φ^r cross no bounded Stokes line, the above discourse still holds with the Borel sum consistently replaced by the right-sum (or the left-sum, if one prefers). It follows from Theorem 2.5.1 that the resulting Stokes multiplier C does not depend on whether the right or left sum has been chosen for defining it.

THEOREM 4.1.1. – *One has*

$$C(E, x) = \frac{\sqrt{2\pi}\, x^{-s-\frac{1}{2}}}{\Gamma(-s)}\, C^{\mathrm{red}}(E, x) \tag{4.2}$$

where $s = s((E, x)$ *must be understood as the* Borel sum *of the monodromy exponent of* φ, [13] *whereas* $C^{\mathrm{red}}(E, x)$, *the* reduced Stokes multiplier, *is an invertible holomorphic function, equal to the Borel sum of an elementary simple resurgent symbol, depending regularly on* E *throughout* D_ε.

[11] The "splitting algorithm" of Section 2.2.2 shows that this assumption does not really restrict the generality.

[12] Although not essential, as shown in the remark hereafter.

[13] That the monodromy exponent is Borel resummable follows from our hypothesis that all Stokes lines are unbounded.

Proof. – *Let E belong to $D_\varepsilon^{(1)}$*, the "half-disc" corresponding to the situation shown on Fig. 22(b). It then follows from Theorem 1.2.2 that the left-hand side of (4.1) is the Borel sum of $\ell\varphi$, the analytic continuation of φ around the loop ℓ. By Lemma 2.1.1 one can write $\ell\varphi = c\varphi^r$, where c is an elementary simple resurgent symbol independent on q, which inherits from φ^r and $\ell\varphi$ the property of being Borel resummable. Of course C in formula (4.1) is nothing but the Borel sum of c.

LEMMA 4.1.1. – *c satisfies the following local resurgence equations*

$$\dot{\Delta}_{n\omega_\gamma} c = \frac{(-1)^{n-1}}{n} a^{n\gamma} c \quad (n \in \mathbb{Z}) \tag{4.3}$$

and its minor has no other confluent singularities that the integral multiples of ω_γ.

Proof. – This immediately follows from Eq. (3.4) in Section 3.4, using the Leibniz rule and the fact that φ^r is a "local resurgence constant", i.e., its minor has no confluent singularities. □

In order to solve the above alien differential equations it will be convenient to change our resurgence variable: taking s, the monodromy exponent, as our new resurgence variable, we can rewrite Eq. (4.3) as follows

$$\dot{\Delta}^{(s)}_{2\pi i n} c = \frac{1}{n} \mathrm{e}^{-2\pi i n s} c,$$

i.e.,

$$\Delta^{(s)}_{2\pi i n} c = \frac{1}{n} c \tag{4.4}$$

a linear homogeneous alien differential system, an explicit solution of which is known (cf. [6], Appendix A, [18] §4.4, [30]), namely $1/\Gamma(-s)^{\mathrm{Stir}}$, where Stir denotes the operation of replacing the Gamma function by its Stirling expansion.

Getting back to the x variable, and using the fact that the solution of a linear homogeneous first order (alien) differential equation is unique up to multiplication by a resurgence constant (a resurgent function with vanishing alien derivatives), we thus immediately get the following lemma.

LEMMA 4.1.2. – *For $E \in D_\varepsilon^{(1)}$ one has*

$$c(E,x) = \frac{\sqrt{2\pi}\, x^{-s-\frac{1}{2}}}{\Gamma(-s)^{\mathrm{Stir}}}\, c^{\mathrm{red}}(E,x), \tag{4.5}$$

where c^{red} is an elementary simple resurgent symbol which is a "local resurgence constant", i.e., $\dot{\Delta}_\omega c^{\mathrm{red}} = 0$ for every ω in some neighbourhood of 0, independent on E.

Comment. In the above formula it must be understood that s stands for the monodromy exponent *considered as a (resurgent) formal expansion in x^{-1}*, which one substitutes in the Stirling expansion.

The $x^{-s-\frac{1}{2}}$ factor in the formula inherits from s the property of being a local resurgence constant, so that omitting it would not change the conclusion that c^{red} is a local resurgence constant. But then c^{red} would no longer be a *simple* symbol, because the Stirling formula for $\Gamma(-s)$ contains a $(-s)^{-s-\frac{1}{2}} \sim (\frac{\omega}{2\pi \mathrm{i}} x)^{-s-\frac{1}{2}}$ factor, whose large $|x|$ expansion involves not only powers of x but also powers of $\ln x$.

The reason for inserting only $\sqrt{2\pi}\, x^{-s-\frac{1}{2}}$ in the numerator instead of $\sqrt{2\pi}(-s)^{-s-\frac{1}{2}}\, \mathrm{e}^s$ (the leading term of the Stirling expansion) is the *non-holomorphic dependence of $(-s)^{-s-\frac{1}{2}}$ on E near $E = 0$*, which would prevent us to pursue the reasoning as follows.

End of the proof of Theorem 4.1.1. Under our hypothesis that all the Stokes lines on Fig. 22 are unbounded, it is easily checked that every factor in the right-hand side of (4.5) is Borel resummable, so that for *$E \in D_\varepsilon^{(1)}$ one can define C^{red} in formula* (4.2) *as the Borel sum of c^{red}*. Since c^{red} is a local resurgence constant its regular dependence on E *in the whole disc D_ε* will be ensured if we can prove that C^{red} is a holomorphic function of (E,x) for every $E \in D_\varepsilon$ and large enough $|x|$ (cf. [10], Lemma 3.1 in Appendix 2).

From Eq. (4.2) we can only infer at first sight that C^{red} is *meromorphic*, with possible simple poles along the complex curves $s(x,E) = n$ ($n \in \mathbb{N}$).[14] The fact that C^{red} is *holomorphic and invertible* will be ensured by the following lemma.

LEMMA 4.1.3. – *The zeros of $C(x,E)$ for large $|x|$ are exactly given by $s(x,E) = n$ ($n \in \mathbb{N}$).*

[14] s now stands for the *Borel resummed* monodromy exponent.

Proof. – First notice that since

$$s = -\frac{\omega_\gamma}{2\pi \mathrm{i}} x - \frac{1}{2} + \mathrm{O}(x^{-1}),$$

where ω_γ is E times an invertible factor (cf. Section 3.3), the equation $s = n$ for large $|x|$ and $n \in \mathbb{N}$ can only be satisfied when $\Im(\omega_\gamma x) < 0$; assuming therefore $E \neq 0$, we can restrict to $\arg x = 0$, [15] and let only the argument of E vary.

For $E \in D_\varepsilon^{(1)}$, i.e., $\Im(\omega_\gamma) > 0$, we know that C is the Borel sum of an elementary resurgent symbol which is *invertible*; therefore C cannot vanish for large x, as expected from the fact that the equation $s = n$ with $n \in \mathbb{N}$ has no solution there.

Let now E move to the $D_\varepsilon^{(2)}$ region, where $\Im(\omega_\gamma x) < 0$. When ω_γ crosses the (positive or negative) real axis, a Stokes phenomenon occurs, because the lattice $n\omega_\gamma$ ($n \in \mathbb{Z}$) crosses the positive real axis. How the symbol of C changes because of such a crossing is easily computed from the resurgence Eqs. (3.4), or more conveniently from their "Stokes automorphism" version: when ω_γ crosses the positive real axis the situation is like in the "basic example" after Theorem 2.5.1, so that the symbol of C changes from c to $(1 + a^\gamma)c$.

Since $(1+a^\gamma) = 1 - \mathrm{e}^{2\pi \mathrm{i} s}$, the zeros of C occur exactly for those values of E in $D_\varepsilon^{(2)}$ for which the Borel resummation of $s + \frac{1}{2}$ yields an integer (a positive one since $\Re(s + \frac{1}{2}) > 0$ in $D_\varepsilon^{(2)}$). □

4.2. Universality of the Weber model

The simplest model for illustrating the results of the previous subsection is the *harmonic oscillator* $V(q) = q^2$: the Schrödinger equation then reduces to the *Weber equation*, whose solutions are known under the name of *parabolic cylinder functions*. After recalling this "Weber model" (Section 4.2.0), we shall show that it is "universal", in the sense that *the general case can be locally reduced to this special case by suitable changes of variables.*

4.2.0. The Weber model

In the case of the harmonic oscillator $V(q) = q^2$ it is easily shown (using a residue calculus at infinity) that $a^\gamma = \mathrm{e}^{-x\omega_\gamma}$, where $\omega_\gamma = 2\mathrm{i}\pi E$, so that the monodromy exponent simply reads $s = -Ex - \frac{1}{2}$.

[15] This assumption is just meant to simplify the notations.

The Schrödinger equation reads

$$\left(-x^{-2}\frac{d^2}{dq^2}+q^2-E\right)\Phi=0. \tag{4.6}$$

By the change of variable $z=\sqrt{x}q$ it becomes the Weber equation

$$\left(-\frac{d^2}{dz^2}+z^2+2s+1\right)\Phi=0 \tag{4.7}$$

a remarquable solution of which is the parabolic cylinder function

$$Y_s(z)=\frac{2^{(s+1)/2}}{\mathrm{i}\sqrt{2\pi}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{e}^{\frac{u^2}{2}+\frac{z^2}{2}-\sqrt{2}zu}u^{-s-1}\,du$$

characterized among all solutions of (4.7) by its recessive asymptotic behaviour when z goes to infinity along the positive real axis

$$Y_s(z)\sim z^{-s-1}\,\mathrm{e}^{-z^2/2}.$$

Setting

$$Y_s^+(z)=\mathrm{e}^{\mathrm{i}\pi s/2}Y_{-s-1}(-\mathrm{i}z)\quad\text{and}\quad Y_s^-(z)=\mathrm{e}^{-\mathrm{i}\pi s/2}Y_{-s-1}(\mathrm{i}z)$$

we get two other solutions of (4.7) which are recessive along the positive imaginary and negative imaginary axis, respectively, and with the same (dominant) asymptotic behaviour along the positive real axis, namely

$$Y_s^+(z)\sim Y_s^-(z)\sim z^s\,\mathrm{e}^{+z^2/2}.$$

Therefore $Y_s^+-Y_s^-$ is a multiple of Y_s:

$$Y_s^+(z)-Y_s^-(z)=C_sY_s(z). \tag{4.8}$$

The "Stokes multiplier" C_s can be explicitely computed from the Weber integral representations of cylinder parabolic functions (cf., e.g., [31]): it reads

$$C_s=-\mathrm{i}\sqrt{2\pi}\frac{2^{-s-1/2}}{\Gamma(-s)}. \tag{4.9}$$

Getting back to the (q,x) variables let us define

$$\begin{aligned}\Phi^r &= x^{\frac{s}{2}+\frac{1}{2}} Y_s(\sqrt{x}q),\\ \Phi_R &= x^{-\frac{s}{2}} Y_s^-(\sqrt{x}q),\\ \Phi_{R'} &= x^{-\frac{s}{2}} Y_s^+(\sqrt{x}q),\end{aligned} \tag{4.10}$$

it is readily checked that these functions satisfy the hypotheses of Section 4.1. Formulae (4.8) and (4.9) thus yield illustrative examples of our general formulae (4.1) and (4.2).

4.2.1. Classical reduction

What follows is a complex version of the well-known "action-angle coordinate" construction in classical mechanics (see, e.g., [32]).

Denote by $H(p,q) = V(q) - p^2$ the classical hamilton function (recall that our p is $-\mathrm{i}$ times the classical momentum). By our assumption the origin $p = q = 0$ is a non-degenerate quadratic critical point of this function, with critical value $E = 0$. For any value of E close enough to zero we denote by γ_E the (one-dimensional) "vanishing cycle" on the complex curve

$$\mathcal{L}_E = \{(p,q) \in \mathbb{C}^2 \mid H(p,q) = E\},$$

and by

$$I(E) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma_E} p\,dq = \frac{1}{2\mathrm{i}\pi}\omega_{\gamma_E}$$

the corresponding "action coordinate".

As already mentioned, $I(E)$ is a holomorphic function of E with a simple zero at the origin.

PROPOSITION 4.2.1. – *There exists, in a neighbourhood U of the origin in classical phase space $\mathbb{C}^2$, a holomorphic system of canonical coordinates P, Q such that*

$$Q^2 - P^2 = 2I(E).$$

(By a system of "canonical coordinates" we of course mean coordinates P, Q such that $dP \wedge dQ = dp \wedge dq$.)

Proof. – Set

$$\begin{cases} P = (2I)^{1/2}\sinh(\theta),\\ Q = (2I)^{1/2}\cosh(\theta),\end{cases}$$

where the "action" coordinate $I = I(H(p,q))$ has just been defined above, and θ is the "angle" coordinate which we shall now proceed to define.

On every level curve $\mathcal{L}_E$ ($E \neq 0$, $|E|$ small) define the *generating function* $\mathcal{S}$ by

$$\begin{cases} d\mathcal{S}_{|\mathcal{L}_E} = p\,dq, \\ \mathcal{S}(0, q_{crit}(E)) = 0, \end{cases}$$

where $q_{crit}(E)$ is any one of the two zeros of $V(q) - E$ close to the origin. This function $\mathcal{S}$ is of course multivalued analytic on $\mathcal{L}_E$, increasing by $2\pi \mathrm{i} I(E)$ when (p,q) runs around the vanishing cycle γ_E.

Now let (p,q) move in $U \backslash \mathcal{L}_0$ (U a small ball in $\mathbb{C}^2$ centered at $(0,0)$). Through the substitution $E = H(p,q)$, the above function $\mathcal{S}$ becomes a multivalued analytic function in $U \backslash \mathcal{L}_0$, whose differential reads

$$d\mathcal{S} = p\,dq - \theta\,dI,$$

thus defining the *angle coordinate* θ as a multivalued analytic function in $U \backslash \mathcal{L}_0$, increasing by $2\pi\mathrm{i}$ when (p,q) runs around the vanishing cycle.

Notice that $\mathcal{S}$ (and therefore θ) is *odd* under the involution $(p,q) \mapsto (-p,q)$.

Remark 4.2.1. – The above construction depended on two arbitrary choices:

(1) Which of the two determinations of p was chosen for defining the vanishing cycle.

(2) Which of the two turning points $q_{crit}(E)$ was chosen as the origin of the indefinite integral $\mathcal{S}$.
Making the opposite second choice (2) would result in changing (P, Q) into $(-P, -Q)$ ($\mathcal{S}$ is changed into $\mathcal{S} \pm \mathrm{i}\pi I$ mod. $2\mathrm{i}\pi I$, so that θ is changed into $\theta \pm \mathrm{i}\pi$ mod. $2\mathrm{i}\pi$).
Making the opposite choice (1) would result in changing (P, Q) into $(\pm\mathrm{i}P, \pm\mathrm{i}Q)$ (I is changed into $-I$).

In order to prove the proposition, all we have to do now is to prove the following.

LEMMA 4.2.1. – *$I^{1/2}\mathrm{e}^{\theta}$ is holomorphic in U, with a simple zero along one of the two components of $\mathcal{L}_0$ (so that $I^{1/2}\mathrm{e}^{-\theta}$ will be holomorphic in U with a simple zero along* the other *component of $\mathcal{L}_0$).*

Proof. – Let us analyse the singular behaviour of $\mathcal{S}$ near a *generic* point of either branch of $\mathcal{L}_0$. Turning around such a branch makes

$E = H(p, q)$ turn around 0, so that $q_{crit}(E)$ is changed into the other zero of $V(q) - E$. The effect on $\mathcal{S}$ is to change it into $\mathcal{S} + \mathrm{i}\pi I$, where the $\mathrm{i}\pi I$ term is the integral of $p\,dq$ over *half the vanishing cycle*. Since $\mathcal{S}$ is clearly bounded, it follows from the Riemann extension theorem that $\mathcal{S} - \frac{1}{2} I \ln(I)$ is holomorphic in a neighbourhood of the point under consideration. Deriving with respect to I, we find that $\theta = -\frac{1}{2}\ln(I) +$ hol.fct., i.e.,

$$\mathrm{e}^{\theta} = I^{-1/2} h,$$

where h is an invertible holomorphic function near the generic point under consideration.

A similar equality holds at generic points of the other branch of $\mathcal{L}_0$ (with $I^{-1/2}$ replaced by $I^{1/2}$). Therefore $I^{1/2}\mathrm{e}^{\theta}$ is holomorphic near every point of $\mathcal{L}_0$, "except perhaps the origin". But since no holomorphic function of two complex variables can have an *isolated* singular point, $I^{1/2}\mathrm{e}^{\theta}$ is holomorphic in the whole of U. $\square$

4.2.2. Quantum reduction

Every value of (q, E) close enough to the origin corresponds to two opposite values of p: $p = \pm(V(q) - E)^{1/2}$. Since the change of coordinates $(p, q) \mapsto (P, Q)$ defined in Section 4.2.1 commutes with the involution (i.e., $(-p, q) \mapsto (-P, Q)$) we can thus consider the coordinate Q as a holomorphic function of (q, E):

$$Q = Q(q, E).$$

Conversely, every value of (Q, I) close enough to the origin corresponds to two opposite values of P

$$P = \pm(Q^2 - 2I)^{1/2}$$

so that by the reciprocal change of coordinates we can consider q as a holomorphic function of (Q, I):

$$q = q(Q, I).$$

We thus get a biholomorphic correspondence $(q, E) \leftrightarrow (Q, I)$ between two neighbourhoods $U \leftrightarrow \underline{U}$ of $(0, 0) \in \mathbb{C}^2$. Any WKB expansion $\varphi^{q,E}$ depending analytically on the parameter (q, E) in some submanifold of U can thus be reparametrized by (Q, I), giving a WKB expansion $\varphi^{Q,I}$ depending analytically on the parameter (Q, I) in some submanifold

of $\underline{U}$. One should notice that since the transformation $(p, q) \mapsto (P, Q)$ transforms the family of curves $\mathcal{L}_E$: $V(q) - p^2 = E$ into the family of curves L_I: $Q^2 - P^2 = 2I$, and since it is *canonical*, it transforms all action integrals in $\mathcal{L}$ into the corresponding action integrals in L_I, i.e., *those of the harmonic oscillator.*

Therefore the essential support S of the WKB expansion $\varphi^{Q,I}$ satisfies the Hamilton–Jacobi equation of the harmonic oscillator. But the *monodromy* of $\varphi^{Q,I}$ still differs from that of the harmonic oscillator: after one turn along the vanishing cycle γ, $\varphi^{Q,I}$ gets multiplied by the Voros symbol

$$a^\gamma = a_\gamma \, \mathrm{e}^{-x\omega_\gamma},$$

where

$$\omega_\gamma = \int_\gamma p\, dq = \int_\gamma P\, dQ = 2\pi \mathrm{i} I$$

is indeed the period of the harmonic oscillator, but the extra factor

$$a_\gamma = 1 + \text{small. form. res. funct.}$$

is not exactly 1, as it should be in the case of the harmonic oscillator.

To get rid of this difference, all one has to do is to replace the variable x by

$$X = x\left(1 - \frac{\ln a_\gamma}{x\omega_\gamma}\right) = \frac{-s - 1/2}{I}.$$

Since

$$a_\gamma = 1 + \text{small. form. res. funct.},$$

one has

$$X = x + \text{small. form. res. funct.},$$

so that (by the implicit resurgent function theorem) the correspondence $x \leftrightarrow X$ is a *change of resurgence variable*. Since this change of variable is tangent to the identity, it does not change the essential support of symbols, and by the formula $a^\gamma = \mathrm{e}^{-X\omega_\gamma}$, it transforms the resurgent symbol $\varphi(x; q, E)$ into a resurgent symbol in X, depending regularly on (Q, I), *which has the same essential support S and the same monodromy (in the sense of Section* 3.3) *as some WKB solution $w(X; Q, I)$ of the Schrödinger equation for the harmonic oscillator* (Eq. (4.6) with (x, q, E) replaced by $(X, Q, 2I)$).

THEOREM 4.2.1. – *With the above notations, there is a unique decomposition*

$$\varphi(x;q,E)=a(x;q,E)w\big(X(x,E);Q(q,E),I(E)\big)$$
$$+b(x;q,E)\partial_Q w\big(X(x,E);Q(q,E),I(E)\big),$$

where a and b are formal resurgent functions of x with regular dependence on (q,E) when $(q,E)\to(0,0)$.

Proof. – The idea is the same as sketched in [33] and detailed in [10] (in a slightly different context): it is based on the fact that after our change of variables, *φ satisfies exactly the same system of local alien differential equations as w*, a *linear* system of the second order whose coefficients are *local resurgence constants*: by a "local resurgence constant" we mean (cf. Section 3.4) any resurgent function (or symbol) c such that $\dot{\Delta}_\omega c=0$ for all ω's inside a small disc around the origin, independent of (q,E) as (q,E) tends to $(0,0)$.

First recall that the resurgent function a^γ, from which we built our change of resurgence variable $x\mapsto X$, is a local resurgence constant (with respect to x). This implies that for all ω's inside a small disc as above, the action of the alien derivation operators $\dot{\Delta}_\omega$ does not depend on whether x or X has been chosen for our resurgence variable. Since $a^\gamma=\mathrm{e}^{-X\omega_\gamma}$, we thus see that *the local resurgence equations for φ are exactly the same as for w.*

As explained in Section 3.4, working for instance in the half-neighbourhood $\underline{U}_1$ (the image of U_1 by the biholomorphic correspondence $(q,E)\leftrightarrow(Q,I)$), they can be written as a linear system

$$\dot{\Delta}_\omega\begin{pmatrix}\varphi\\ \varphi^\star\end{pmatrix}=A_\omega\begin{pmatrix}\varphi\\ \varphi^\star\end{pmatrix},\quad \omega\in\Omega, \tag{4.11}$$

where:
- Ω stands for the set of "confluent action periods";
- $\varphi^\star$ is $\dot{\Delta}_{\omega_\star}\varphi$ for some suitably chosen $\omega_\star\in\Omega$;
- the A_ω's are 2×2 matrices whose entries are local resurgence constants.

Now consider the decomposition we are looking for:

$$\varphi=aw+b\partial_Q w. \tag{4.12}$$

Since we want a and b to depend regularly on (Q,I), they should be local resurgence constants. Applying the derivation operator $\dot{\Delta}_{\omega_\star}$ to (4.12) we

thus get

$$\varphi^\star = aw^\star + b\partial_Q w^\star, \tag{4.12'}$$

where we have used the fact that $\dot{\Delta}_{\omega_\star}$ commutes with ∂_Q. Eqs. (4.12) and (4.12′) can be considered as a system of two linear equations with two unknown a, b which can be solved by the Cramer formulas

$$a = \frac{\begin{vmatrix} \varphi & \partial_Q w \\ \varphi^\star & \partial_Q w^\star \end{vmatrix}}{\begin{vmatrix} w & \partial_Q w \\ w^\star & \partial_Q w^\star \end{vmatrix}}, \qquad b = \frac{\begin{vmatrix} w & \varphi \\ w^\star & \varphi^\star \end{vmatrix}}{\begin{vmatrix} w & \partial_Q w \\ w^\star & \partial_Q w^\star \end{vmatrix}}. \tag{4.13}$$

The denominator in these formulas (the wronskian of w and $w^\star$) does not depend on Q, and for every $I \neq 0$ it is an *invertible* resurgent function of X (depending regularly on I): this is so because w and $w^\star$ are linearly independent solutions of the Schrödinger equation (harmonic oscillator case). Therefore formulas (4.13) define a and b as resurgent functions of X, depending regularly on (Q, I) in $\underline{U}_1$ (or equivalently as resurgent functions of x, depending regularly on (q, E) in U_1). Since

$$\begin{pmatrix} \varphi \\ \varphi^\star \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w \\ w^\star \end{pmatrix}$$

both satisfy (4.11), where A_ω is a local resurgent constant, the Leibniz rule for alien derivatives implies that $\dot{\Delta}_\omega a = 0$ and $\dot{\Delta}_\omega b = 0$ for all $\omega \in \Omega$, i.e., *a and b are local resurgent constants.*

To prove that the dependence of a and b on (Q, I) is regular in a neighbourhood of $(0, 0)$ (i.e., their majors $\check{a}$, $\check{b}$ satisfy Definition 0.6.2 at $(\omega_0 = 0; (Q, I) = (0, 0))$), one can consider the true functions of $(X; Q, I)$ deduced from (a, b) by "local resummation" (i.e., truncated Laplace transforms of $\check{a}$, $\check{b}$): by Lemma 3.1, Appendix 2 of [10] it is enough to prove that these functions depend holomorphically on (X, Q, I) for large positive X, (Q, I) in some neighbourhood of $(0, 0)$. Replacing all the formal objects in the Cramer formulas (4.13) by their "local resummations", we observe that the numerators and denominators are indeed holomorphic in such a neighbourhood, and there only remains to check that they have the same zeros; now observe that the terms φ, $\partial_Q\varphi$, w, $\partial_Q w$ in (4.13) all yield invertible functions, whereas $\varphi^\star$, $\partial_Q\varphi^\star$, $w^\star$, $\partial_Q w^\star$ yield functions which only fail to be invertible by a factor C of the form (4.2) (cf. Theorem 4.1.1), with simple zeros at $s = n$ $(n \in \mathbb{N})$. □

Remark. – Instead of working in $\underline{U}_1$ we could as well have worked in the other half-neighbourhood $\underline{U}_2$ where two moving singularities ω_{l_1} and ω_{l_2} are seen. The two corresponding systems of linear equations are related by the formula $\ell_2^+\varphi = a^\gamma \ell_1^+\varphi$ (deduced from Section 3.1) so that each of these two system of course gives the same a and b.

For later use, it will be convenient to work with the "resummed" version of Theorem 4.2.1, which is most conveniently expressed under the hypotheses of Section 4.1.

Let the change of variable in Section 4.2.1 be so chosen that it transforms the Stokes line L of Section 4.1 into the positive real axis in the Q-plane when I is a positive real number (that such a choice is possible follows from Remark 4.2.1).

THEOREM 4.2.2. – *Denoting by W_s the "Weber" wave function*

$$W_s(X;Q) = X^{\frac{s}{2}+\frac{1}{2}} Y_s(\sqrt{X}Q)$$

(recessive along the positive real Q-axis), one has a unique decomposition of the recessive wave function Φ^r introduced in Section 4.1:

$$\Phi^r(x;q,E) = A(x;q,E)W_s(X;Q) + B(x;q,E)\partial_Q W_s(X;Q),$$

where A and B are extended resurgent functions of x, depending holomorphically on (q,E) near $(0,0)$. *More precisely, A and B are the Borel sums of elementary resurgent symbols a and b depending regularly on (q,E) near* $(0,0)$.

A similar decomposition holds (with the same A and B) for the dominant wave function Φ_R, $\Phi_{R'}$ of Section 4.1, *with W_s replaced, respectively, by*

$$W_s^-(X;Q) = X^{-\frac{s}{2}} Y_s^-(\sqrt{X}Q)$$

and

$$W_s^+(X;Q) = X^{-\frac{s}{2}} Y_s^+(\sqrt{X}Q).$$

5. FROM THE GENERIC TO THE CRITICAL CASE: THE "EXACT MATCHING" METHOD

We want to investigate here how the resurgent functions studied in Section 4 behave when E is "infinitely close" to a (quadratical) critical

energy E_{crit} (which can be assumed to be equal to zero) in the sense of [1] §III, that is after an "energy rescaling"

$$x^{-1}E_r \quad (E_r = \text{"rescaled energy"}),$$

which "shrinks" the Stokes configuration to the "critical" case of Fig. 22(a).

Under such a rescaling the Schrödinger Eq. (1.0) yields the so-called *rescaled Schrödinger equation*

$$-x^{-2}\frac{d^2\Phi}{dq^2} + (V(q) - E_{crit})\Phi = x^{-1}E_r\Phi. \tag{5.0}$$

As far as *regular* WKB symbols near E_{crit} are concerned, the action of the rescaling can be easily analysed. Let us assume for instance that

$$\varphi(q, E) \simeq p(q, E)^{-1/2}\,\mathrm{e}^{-xS(q,E)} \tag{5.1}$$

is well normalized along a path starting at q_0 and ending at q, so that it is not pinched by the confluence of turning points when E tends to E_{crit}. The effect of the rescaling $E = x^{-1}E_r$ (recall that $E_{crit} = 0$) on

$$S(q, E) = \int_{q_0}^{q} p(q', E)\,dq' \tag{5.2}$$

gives

$$S(q, x^{-1}E_r) = S_{crit}(q) + x^{-1}E_r\frac{\partial S}{\partial E}(q, 0) + \mathrm{O}(x^{-2}) \tag{5.3}$$

with

$$S_{crit}(q) := S(q, 0) = \int_{q_0}^{q} p_{crit}(q')\,dq', \tag{5.4}$$

where $p_{crit}(q) := p(q, 0)$ and

$$\frac{\partial S}{\partial E}(q, 0) = -\int_{q_0}^{q} \frac{dq'}{2p_{crit}(q')} := t(q, q_0). \tag{5.5}$$

The simple rescaled WKB expansion φ^{resc} solution of (5.0) thus reads

$$\varphi^{\mathrm{resc}} \simeq p_{crit}^{-1/2}\, \mathrm{e}^{E_r t}\, \mathrm{e}^{-x S_{crit}(q)} \tag{5.6}$$

and it is still resurgent and regular on E_r as a consequence of Proposition 0.6.1.

Such a rescaling can be also considered on a Voros coefficient a^γ for a cycle γ which is not pinched when E tends to E_{crit}. This is in particular true when γ is a vanishing cycle so that the monodromy exponent of φ around γ

$$s = -\frac{x}{2\pi \mathrm{i}} \int_\gamma P\, dq - \frac{1}{2} = -\frac{\omega_\gamma}{2\pi \mathrm{i}} x - \frac{1}{2} + \mathrm{O}(x^{-2}) \tag{5.7}$$

will becomes after rescaling the resurgent *monodromy exponent* s^{resc} *of the simple rescaled WKB expansion* φ^{resc} *at the double turning point.* A simple algorithm for computing such a s^{resc} is given in [1] §III.2 and will be used in Section 5.2 hereafter.

5.1. The "critical" connection formula

When L is a simple Stokes line fading into a *simple turning point*, the so-called elementary connection operator δ_L (see Section 2.2.1) is completely described in term of analytic continuation (Theorem 1.2.2). When L is a simple Stokes line fading into a *double turning point* (which will be assume to be zero in what follows), the elementary connection operator δ_L cannot be described in such a simple way, and factors like $\sqrt{2\pi}\, x^{-s-1/2}/\Gamma(-s)$ can be expected from the "confluence of Stokes multipliers" (Section 4.1).

Let us look at formula (4.1). Remembering that Φ_R (*respectively*, $\Phi_{R'}$) in the left-hand side is the Borel sum, for q *above* (respectively, *below*) the Stokes line, of a WKB expansion φ *depending regularly on* E *near* 0 (as (5.1) for instance), it follows that the "rescaled" function Φ_R^{resc} (*respectively*, $\Phi_{R'}^{\mathrm{resc}}$) is nothing but the *left* (respectively, *right*)-sum of φ^{resc}, the rescaled WKB expansion. In other words, rescaling the left-hand side of (4.1) yields

$$\Phi_{R'}^{\mathrm{resc}} - \Phi_R^{\mathrm{resc}} = \text{the Borel sum of } \left(\delta_L \varphi^{\mathrm{resc}}\right),$$

where δ_L *is the elementary connection operator across the Stokes line* L.

Now the right-hand side of (4.1) is, by Theorem 4.1.1, a product factors which, *apart for the* $\sqrt{2\pi}x^{-s-1/2}/\Gamma(-s)$ *factor in formula* (4.2), all depend regularly on E near 0. Recalling that s in formula (4.2) must be understood as *the Borel sum* of the monodromy exponent, which also depends regularly on E near 0, one has

$$\left(\frac{\sqrt{2\pi}x^{-s-1/2}}{\Gamma(-s)}\right)^{\text{resc}} = \frac{\sqrt{2\pi}x^{-s^{\text{resc}}-1/2}}{\Gamma(-s^{\text{resc}})}, \tag{5.8}$$

where s^{resc} is short for the *Borel sum* of the rescaled monodromy exponent. [16] We have thus proved the following

THEOREM 5.1.1. – *Denoting by* s^{resc} *the rescaled monodromy exponent of* φ^{resc} *at the double turning point, the connection formula reads*

$$\delta_L\varphi^{\text{resc}} = \frac{\sqrt{2\pi}x^{-s^{\text{resc}}-1/2}}{\Gamma(-s^{\text{resc}})}\delta_L^{\text{red}}\varphi^{\text{resc}},$$

where $\delta_L^{\text{red}}\varphi^{\text{resc}}$ *is another simple rescaled WKB expansion depending regularly on* E_r *throughout* $\mathbb{C}$.

5.1.1. Comparing leading terms I: the "exact matching" method

To make the above theorem more precise, let us tell more about our normalization conventions.

For generic E formula (4.1) reads

$$\Phi_{R'} - \Phi_R = \text{the Borel sum of } (\delta_L\varphi),$$

where δ_L *is the elementary connection operator across the Stokes line* L drawn on Fig. 22(b). This Stokes line fade into a *simple turning point* $q_c(E)$: we know from Theorem 1.2.2 that the action of δ_L on the WKB expansion φ just consists in an analytic continuation along a loop ℓ_q with base point q around the simple turning point $q_c(E)$.

Let φ be now this WKB symbol given by Eq. (5.1): analytic continuation along the loop ℓ_q multiplies $p^{-1/2}$ by $-\mathrm{i}$, and acts on the action

[16] In other words equality (5.8) must be understood as an equality between *true functions*. But the *symbol* of this (resurgent) function is not immediately readable on the formula (it is computable from the right-hand side by using the expanded Stirling formula).

exponent S of (5.2) as the symmetry of centre ΔS, where

$$\Delta S(E, q_o) = \int_{\ell_{q_o}} p(q', E)\, dq'.$$

We thus deduce the leading term of $\delta_L \varphi$:

$$\delta_L \varphi = -\mathrm{i} p(q, E)^{-1/2} \mathrm{e}^{-x(\Delta S - S(q,E))} \big(1 + \mathrm{O}(x^{-1})\big). \tag{5.9}$$

As a matter of fact Theorem 4.1.1 shows that this WKB symbol reads also

$$\delta_L \varphi = \frac{\sqrt{2\pi} x^{-s-1/2}}{\Gamma(-s)^{\mathrm{Stir}}} \varphi^r, \tag{5.10}$$

where φ^r is a *regular* WKB expansion whose leading term can be computed just by comparing Eqs. (5.9) and (5.10): this is what we call the "exact matching" method. Recalling that $\Im \omega_\gamma(E) > 0$, hence $-s \sim \frac{\omega_\gamma}{2\pi \mathrm{i}} x + \frac{1}{2}$ has a positive real part which goes to infinity as $x \to +\infty$, Stirling formula gives [17]

$$\frac{\sqrt{2\pi} x^{-s-1/2}}{\Gamma(-s)^{\mathrm{Stir}}} = \exp\left(x \left(\frac{\omega_\gamma}{2\pi \mathrm{i}} - \frac{\omega_\gamma}{2\pi \mathrm{i}} \ln \frac{\omega_\gamma}{2\pi \mathrm{i}} \right) \right) \big(1 + \mathrm{O}(x^{-1})\big).$$

We thus see that

$$\varphi^r = -\mathrm{i} p(q, E)^{-1/2} \mathrm{e}^{-x S^r} \big(1 + \mathrm{O}(x^{-1})\big),$$

where the action exponent reads

$$S^r = -S + \Delta S + \frac{\omega_\gamma}{2\pi \mathrm{i}} - \frac{\omega_\gamma}{2\pi \mathrm{i}} \ln \frac{\omega_\gamma}{2\pi \mathrm{i}}.$$

It is holomorphic in E near the origin, with

$$\begin{aligned} S^r(q, E) = {} & 2S_{crit}(0) - S_{crit}(q) - Et \\ & - \lim_{\substack{E \to 0 \\ \Im \omega_\gamma(E) > 0}} \left\{ \int_{\ell_{q_o}} \frac{dq}{2p(q, E)} + \frac{1}{2\mathrm{i}\pi} \frac{\partial \omega_\gamma}{\partial E}(0) \ln \frac{\omega_\gamma(E)}{2\mathrm{i}\pi} \right\} \\ & + \mathrm{O}(E^{-2}) \end{aligned}$$

[17] In all this section ln denotes the usual neperian logarithmic function, real on the positive real.

as its Taylor expansion, where $t = t(q, q_o)$ is the "time coordinate" of Eq. (5.5). It remains to substitute $x^{-1}E_r$ to E to get the leading term of the rescaled WKB expansion $\varphi^r_{\text{resc}} = \delta_L^{\text{red}}\varphi^{\text{resc}}$ and to conclude.

PROPOSITION 5.1.1. – *Considering the simple rescaled WKB expansion* (5.6), *one has*

$$\delta_L^{\text{red}}\varphi^{\text{resc}} = -\mathrm{i}p_{crit}^{-1/2}\,\mathrm{e}^{E_r t^\star}\,\mathrm{e}^{-xS^\star_{crit}(q)}\big(1+\mathrm{O}(x^{-1})\big),$$

where the action exponent $S^\star_{crit}$ is deduced from the action exponent $S_{crit}(q)$ of φ^{resc} by the symmetry of center $S_{crit}(0)$, and the "time coordinate" $t^\star$ of $\delta_L^{\text{red}}\varphi^{\text{resc}}$ is deduced from the "time coordinate" t of φ^{resc} (Eq. (5.5)) *by the rule*

$$t^\star + t = \lim_{\substack{E\to 0\\ \Im\omega_\gamma(E)>0}} \left\{ \int_{\ell_{q_o}} \frac{dq}{2p(q,E)} + \frac{1}{2\mathrm{i}\pi}\frac{\partial\omega_\gamma}{\partial E}(0)\ln\frac{\omega_\gamma(E)}{2\mathrm{i}\pi}\right\}. \tag{5.12}$$

5.1.2. Comparing leading terms II: reduction to the Weber model

Theorem 4.2.2 gives us an alternative tool to fill the gap between simple turning points and double turning points. Using what we know from the "Weber model" (Section 4.2.0), it allows computation by the "exact matching method".

We assume as previously that for $(q, E) \in U_1$ the WKB symbol φ given in (5.1) is dominant on the simple Stokes line L drawn on Fig. 22(b) separating the two Stoke regions R and R'. We keep the same notations Φ_R and $\Phi_{R'}$ for the Borel sums of φ in R and R', respectively. Introducing the change of variables $(q, E) \leftrightarrow (Q, I)$ defined by

$$\begin{cases} I = \dfrac{1}{2\mathrm{i}\pi}\displaystyle\int_{\gamma(E)} p\,dq = \dfrac{1}{2\mathrm{i}\pi}\omega_\gamma, \\[2ex] \displaystyle\int_{q_c(E)}^{q} p(q',E)\,dq' = \int_{Q_c(I)}^{Q} P(Q',I)\,dQ', \end{cases}$$

(cf. Section 4.2.2, here $\Im\omega_\gamma > 0$ and $Q_c(I) = \sqrt{2I}$) as well as the change of resurgence variable X such that $IX = -s - 1/2$, it follows from Theorem 4.2.2 that there exists a unique decomposition

$$\begin{cases} \Phi_R = AW_s^- + B\partial_Q W_s^-, \\ \Phi_{R'} = AW_s^+ + B\partial_Q W_s^+, \end{cases} \tag{5.13}$$

where A and B are extended resurgent functions depending holomorphically on (q, E) near $(0, 0)$.

It comes form their very definitions (Section 4.2.0) that the Weber wave functions W_s^- and W_s^+ in Eq. (5.13) can be considered as the Borel sums in R and R', respectively, of the following WKB symbol

$$w \sim P^{-1/2} \mathrm{e}^{-XS_w} \tag{5.14}$$

regular near $(0, 0)$, where $P = (Q^2 - 2I)^{1/2}$, while the action exponent S_w is simply related to the action exponent S of (5.2) by the rule

$$S_w = S - \frac{I}{2}(1 + \ln 2) - R \tag{5.15}$$

with

$$R = R(E, q_0) = \int_{q_0}^{q_c(E)} p(q', E)\, dq' - \frac{I(E)}{2} \ln I(E). \tag{5.16}$$

Now the extended resurgent functions A and B are the Borel sums of the two resurgent *regular* symbols a and b, given by the cramers formula (4.13). Expanding these formulas and using (5.1), (5.14) and (5.15), it comes

$$a = \frac{p^{-\frac{1}{2}} P^{\frac{1}{2}} \mathrm{e}^{-x(\frac{I}{2}(1+\ln 2)+R)}}{2} \left(\mathrm{e}^{(X-x)S_w} + \mathrm{e}^{(X-x)S_w^\star}\right)\left(1 + \mathrm{O}(X^{-1})\right), \tag{5.17}$$

while

$$b = \frac{p^{-\frac{1}{2}} P^{-\frac{1}{2}} \mathrm{e}^{-x(\frac{I}{2}(1+\ln 2)+R)}}{2X} \left(-\mathrm{e}^{(X-x)S_w} + \mathrm{e}^{(X-x)S_w^\star}\right) \times \left(1 + \mathrm{O}(X^{-1})\right), \tag{5.18}$$

where the action exponent $S_w^\star$ is deduced from the action exponent S_w by the symmetry of center $S_w(Q_c(I))$:

$$S_w^\star = -I(1 + \ln 2 - \ln I) - S_w. \tag{5.19}$$

The Weber model. Following Section 4.2.0 the difference between W_s^+ and W_s^- is a recessive wave function on L which can be written

$$W_s^+ - W_s^- = C W_s,$$

where

$$C = -\mathrm{i}\frac{\sqrt{2\pi}(2X)^{-s-1/2}}{\Gamma(-s)} \sim -\mathrm{i}\frac{\sqrt{2\pi}(2x)^{-s-1/2}}{\Gamma(-s)} \tag{5.20}$$

(because $X \sim x$) is the *Stokes multiplier across* L.[18] It follows from its very definition (Section 4.2.0) that W_s is the Borel sum of

$$w^r \sim P^{-1/2}\,\mathrm{e}^{XS_w}$$

a regular WKB symbol near $(0,0)$.

Considering now Eqs. (5.13) we can write

$$\Phi_{R'} - \Phi_R = A\big(W_s^+ - W_s^-\big) + B\partial_Q\big(W_s^+ - W_s^-\big),$$

so that

$$\Phi_{R'} - \Phi_R = C\Phi^r \tag{5.21}$$

with

$$\Phi^r = AW_s + B\partial_Q W_s$$

by the previous considerations. The extended resurgent function Φ^r is the Borel sum of a recessive symbol ψ^r which is *regular* near $(0,0)$, and Eqs. (5.17) and (5.18) can be used to state precisely the leading term of this symbol:

$$\begin{aligned}\psi^r = {} & \frac{p^{-\frac{1}{2}}P^{\frac{1}{2}}\,\mathrm{e}^{-x(\frac{I}{2}(1+\ln 2)+R)}}{2}\big(\mathrm{e}^{(X-x)S_w} + \mathrm{e}^{(X-x)S_w^\star}\big)P^{-1/2}\,\mathrm{e}^{XS_w} \\ & \times\big(1+\mathrm{O}(X^{-1})\big) + \frac{p^{-\frac{1}{2}}P^{-\frac{1}{2}}\,\mathrm{e}^{-x(\frac{I}{2}(1+\ln 2)+R)}}{2X} \\ & \times\big(-\mathrm{e}^{(X-x)S_w} + \mathrm{e}^{(X-x)S_w^\star}\big)P^{-1/2}(XP)\,\mathrm{e}^{XS_w}\big(1+\mathrm{O}(X^{-1})\big),\end{aligned}$$

that is

$$\psi^r = p^{-\frac{1}{2}}\,\mathrm{e}^{XS_w}\,\mathrm{e}^{(X-x)S_w^\star}\,\mathrm{e}^{-x(\frac{I}{2}(1+\ln 2)+R)}\big(1+\mathrm{O}(x^{-1})\big)$$

hence by (5.15) and (5.19)

$$\psi^r = p^{-\frac{1}{2}}\,\mathrm{e}^{xS}\,\mathrm{e}^{-x(I+2R+I\ln 2)}\big(1+\mathrm{O}(x^{-1})\big). \tag{5.22}$$

[18] As in Theorem 4.1.1, s denotes in formula (5.20) the Borel sum of the monodromy exponent of φ.

Rescaling. Describing the elementary "critical" connection operator δ_L across this Stokes lines just amount now to interpret the difference $\Phi_{R'}^{\text{resc}} - \Phi_R^{\text{resc}}$ in terms of Borel sums. Eq. (5.21) shows that this problem reduces to rescale both Φ^r and the Stokes multiplier C.

Rescaling Φ^r yields Φ_{resc}^r, the Borel sum of that WKB symbol ψ_{resc}^r deduced from ψ^r by rescaling, which we are going now to compute. Considering (5.22) one first remarks that

$$\begin{aligned} &I(E) + 2R(E) \\ &\quad = 2\int_{q_0}^{0} p_{crit}\, dq - E \lim_{\substack{E\to 0 \\ \Im\omega_\gamma(E)>0}} \left\{ \int_{\ell_{q_0}} \frac{dq}{2p(q,E)} + \frac{1}{2i\pi}\frac{\partial\omega_\gamma}{\partial E}(0) \ln \frac{\omega_\gamma(E)}{2i\pi} \right\} \\ &\qquad + \mathrm{O}(E^{-2}). \end{aligned}$$

Our second remark is that the monodromy exponent s reads (cf. (5.7)) $s + \frac{1}{2} = -xI(E) + \mathrm{O}(x^{-2})$ (cf. (5.7)). Writing down also the Taylor expansion of the action exponent S as in (5.3) and substituting $x^{-1}E_r$ to E, (5.22) thus yields (by regularity)

$$\psi_{resc}^r = 2^{s^{\text{resc}}+1/2} p_{crit}^{-1/2} \mathrm{e}^{E_r t^\star} \mathrm{e}^{-xS_{crit}^\star(q)} \left(1 + \mathrm{O}(x^{-1})\right)$$

with the same notation as in Section 5.1.1. Putting all pieces together give both Theorem 5.1.1 and Proposition 5.1.1.

5.2. Confluence analysis for a Voros coefficient

Confluence analysis can be performed for a Voros coefficient in the spirit of Section 5.1 as well, and it allows computations at all orders. We shall present here an example of such an analysis, considering the Schrödinger Eq. (1.0) with $V(q)$ being the following real quartic polynomial function

$$V(q) = \left(q^2 - 1\right)^2/16.$$

This potential presents two critical values, the "bottom of the double well" $E = 0$, and the "top of the double well" $E = 1/16$.

In what follows we shall denote by γ, γ_- and γ_+ the three cycles drawn on Fig. 23 when $0 < E < 1/16$. We specify the orientation of these cycles by assuming that the action integrals ω_{γ_-} and ω_{γ_+} are pure negative imaginary whereas the action integral ω_γ of the cycle γ is positive real.

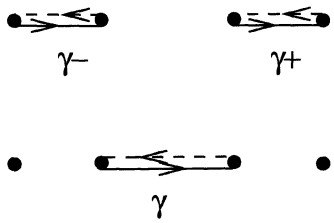

Fig. 23. The three cycles γ, γ_- and γ_+. The bullets drawn on this picture are the four real turning points.

This correspond to the following index of intersection

$$\langle \gamma_-, \gamma \rangle = \langle \gamma_+, \gamma \rangle = -1$$

which will be useful in the sequel.

5.2.1. Confluence near the bottom of the double well

We start our analysis with by considering the critical value $E_{crit} = 0$ corresponding to the two critical points -1 and $+1$.

Regular monodromy exponents. Since γ_- and γ_+ are vanishing cycles when $E \to 0$, the corresponding monodromy exponents

$$s_\pm(E, x) = -\frac{x}{2i\pi} \int_{\gamma_\pm} P\, dq - \frac{1}{2}$$

are resurgent functions with *regular dependence on E near zero*, so that substituting in them $E \to x^{-1}E_r$ is allowed, thus defining the monodromy exponents $s_\pm(x^{-1}E_r, x)$ of the double turning points -1 and $+1$, respectively.

The resurgent functions s_- and s_+ can be computed at all orders. We remark first that the symmetry of the potential V allows to write

$$s_- = s_+ := s.$$

We now use the algorithm described in [1] §III.2 to get

$$s(E, x) + \tfrac{1}{2} = t_{-1}(E)x + t_1(E)x^{-1} + \mathrm{O}(x^{-3}),$$

where

$$t_{-1}(E) = E + \tfrac{3}{2}E^2 + \tfrac{35}{4}E^3 + \mathrm{O}(E^4),$$
$$t_{+1}(E) = \tfrac{1}{8} + \tfrac{25}{16}E + \tfrac{735}{32}E^2 + \mathrm{O}(E^3).$$

We deduce that $s(x^{-1}E_r, x)$ reads

$$s(x^{-1}E_r, x) = \left(E_r - \tfrac{1}{2}\right) + \left(\tfrac{1}{8} + \tfrac{3}{2}E_r^2\right)x^{-1} + \left(\tfrac{25}{16}E_r + \tfrac{35}{4}E_r^3\right)x^{-2} + \mathrm{O}(x^{-3}).$$

Confluent Voros coefficient. We are now interested in the Voros coefficient a^γ which is *not regular* near $E = 0$: Theorem 2.5.1 (see also [5]) shows that the minor of a^γ has singularities at all points $n\omega_{\gamma_-}$ and $n\omega_{\gamma_+}$, $n \in \mathbb{Z}\backslash\{0\}$, and these points collapse to the origin when E tends to zero. Considering the value of the index of intersection of $\gamma_\pm$ with γ, it follows that the local resurgent equations for a^γ read

$$\dot{\Delta}_{n\omega_{\gamma_\pm}} a^\gamma = \frac{(-1)^n}{n} a^{n\gamma_\pm} a^\gamma, \quad n \in \mathbb{Z}^\star. \tag{5.23}$$

Assuming that $\Re(E) > 0$ with E near 0, and adapting now the result (4.5) of Lemma 4.1.2, it follows from the Leibniz rule that

$$c(E, x) = \frac{1}{\Gamma(s_- + 1)^{\mathrm{Stir}}\Gamma(s_+ + 1)^{\mathrm{Stir}}}$$

satisfy the previous local resurgent Eqs. (5.23) (we have dropped here the factors $\sqrt{2\pi}x^{s_\pm + 1/2}$ for a matter of convenience, see the comment of Section 4.1), so that a^γ reads

$$a^\gamma = c(E, x)a^{\mathrm{red}}, \tag{5.24}$$

where a^{red} is an elementary resurgent symbol which is a local resurgent constant.

We know from §2.2 of [5] that the Borel sum $A(E, x) = \mathrm{S}_{(0)}a^\gamma$ *depends holomorphically on E near E_{crit}*. Be aware now of fact that $s_\pm$ are *not* Borel resummable. Considering the previous equality we are therefore led to consider the right (say) resummation of the symbols $s_\pm$, thus getting two (extended) resurgent functions with holomorphic dependence on E near the origin. We denote for short these two functions by the same letters $s_\pm$. Observing that the right Borel sum

$$C(E, x) = \frac{1}{\Gamma(s_- + 1)\Gamma(s_+ + 1)}$$

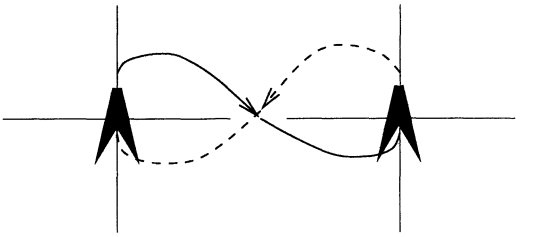

Fig. 24. The connection cycle $[L]$ where L is the isolated bounded Stokes line (with respect to the positive real direction).

of $c(x, E)$ is a holomorphic function near the origin, the same arguments used in the proof of Theorem 4.1.1 allows to conclude that a^{red} *depends regularly on E in a neighbourhood of the origin.*

From the previous consideration, substituting $E \to x^{-1}E_r$ in A or in a^{red} as well is allowed. Moreover it can be easily seen that $A^{crit}(E_r, x) = A(x^{-1}E_r, x)$ *is nothing but the (extended) resurgent function defined as the Borel sum of the Voros symbol $a^{[L]}$ associated with the connection cycle* $[L]$ (cf. Section 2.3) and pictographically represented on Fig. 24 (with the conventions of [1] §III). [19]

Our aim in what follows will be to show how equality (5.24) can be used in order to compute the Voros symbol $a^{[L]}$ theoretically to all orders.

How to compute a^{red}. It will be useful in the sequel to have numerical informations about our confluent factor $c(x, E)$. Remembering that $s_\pm = s$ and using the well known expanded Stirling formula, a straightforward computation gives

$$\begin{aligned}
\ln c(x, E) &= -2\ln\Gamma(s+1) \\
&= -\ln(2\pi) - 2\left(s + \tfrac{1}{2}\right)\ln(s) + 2s - \tfrac{1}{6s} + \mathrm{O}\big((s)^{-3}\big) \\
&= \big(-2t_{-1}x - 2t_1x^{-1} + \mathrm{O}(x^{-3})\big)\ln x + \big(2t_{-1} - 2t_{-1}\ln(t_{-1})\big)x \\
&\quad - \ln(2\pi) + \left(\tfrac{1}{12t_{-1}} - 2t_1\ln(t_{-1})\right)x^{-1} + \mathrm{O}(x^{-3}).
\end{aligned}$$

[19] For instance the basic example of Section 2.5.1 shows that $\mathfrak{S}\varphi = (1 + a^\gamma)\varphi$ for a WKB symbol φ well normalized at infinity along a path λ such that $\langle\gamma, \lambda\rangle = +1$. Assumming φ to be regular in E in a neighbourhood of zero, then $\mathfrak{S}\varphi^{\mathrm{resc}} = (1 + a^{[L]})\varphi^{\mathrm{resc}}$ for the same WKB symbol after rescaling.

We now want to describe a^{red}. It follows from its very definition (5.24) that

$$\ln a^{\mathrm{red}} = \ln a^{\gamma} - \ln c,$$

so that our task will be to get some information about $\ln a^{\gamma}$: one can write

$$\ln a^{\gamma} = -x \int_{\gamma} P\, dq,$$

where

$$P(q, E; x) = p(q, E) + x^{-2} p_1(q, E) x^{-2} + p_2(q, E) x^{-4} + \cdots.$$

We shall restrict our attention on the first two terms p and p_1,

$$\begin{cases} p(q, E) = (V(q) - E)^{1/2}, \\ p_1(q, E) = \frac{1}{4}(p'')^2 p^{-2} - \frac{3}{8}(p')^2 p^{-3} = \frac{1}{8} V'' p^{-3} - \frac{5}{32}(V')^2 p^{-5}, \end{cases}$$

where $'$ denotes the derivative with respect to q.

We begin with the computation to the zero'th order. Considering $\omega_{\gamma}(E) = \int_{\gamma} p\, dq$ for $E \gtrsim 0$ (say), we know from the Leray–Gelfand theory [34] that

$$\omega_{\gamma}(E) \sim \sum_{n \geqslant 0} \alpha_n E^n + \sum_{n \geqslant 1} \beta_n E^n \ln(E)$$

and our aim is to compute the (α_n) and (β_n). The trick will be to introduce the following Mellin transform [20]

$$M(t) = \int_0^{E_0} \omega_{\gamma}(E) E^{t-1}\, dE,$$

where it will be convenient to assume $E_0 = 1/16$ (this is the other critical value of the potential for which the cycle γ vanishes). $M(t)$ is a well defined holomorphic function in the $\Re(t) > 0$ half-plane, and its analytic continuation define a meromorphic function $M(t)$ in the complex t-plane such that

(*) $M(t)$ has a simple pole at $t = 0$ with residue α_0.

[20] We follow here an idea of Zinn-Justin [35].

(**) $M(t)$ has a double poles at $t=-n$ for $n\in\mathbb{N}^\star$, with

$$M(t)=\frac{-\beta_n}{(t+n)^2}+\frac{\alpha_n}{t+n}+\mathrm{O}(1)$$

as its corresponding Laurent expansion.

In order to compute $M(t)$ we first assume that $t\in\mathbb{N}\backslash\{0,1\}$ and we integrate by parts: we thus find

$$M(t)=2\frac{\Gamma(t)\Gamma(3/2)}{\Gamma(t+3/2)}\int_{-1}^{+1}p_{crit}^{2t+1}(q)\,dq,$$

where $p_{crit}(q)=p(q,0)=(1-q^2)/4$. The integral in the right-hand part of this equality is easy to compute for all non-zero positive integer t and we get:

$$M(t)-2^{-2t}\frac{\Gamma(3/2)}{t}\frac{\Gamma^2(t+1)}{\Gamma(2t+5/2)}=0.$$

Considering now the growth at infinity of the analytic function defined by the left-hand part of the previous equality for $\Re(t)>0$, it can be shown (we leave it to the reader) that this equality holds in fact for all t in the complex plane, as a consequence of a known theorem of Carlson in analytic functions theory, see for instance [36] p. 153.

Computing the Laurent expansions of $M(t)$ near $t=0,-1,-2,\dots$, we shall find $\alpha_0=\frac{2}{3}$, $\alpha_1=-2-4\ln 2$ and $\beta_1=2$, $\alpha_2=\frac{17}{2}-6\ln 2$ and $\beta_2=3$, $\alpha_3=59-35\ln 2$ and $\beta_3=\frac{35}{2},\dots$ and so on.

We now want to push the expansion to the next order, that is expand $\upsilon_\gamma(E)=\int_\gamma p_1\,dq$ for $E\gtrsim 0$. Here again the Leray–Gelfand theory allows us to expect the following asymptotic expansion:

$$\upsilon_\gamma(E)\sim\sum_{n\geqslant -1}\delta_n E^n+\sum_{n\geqslant 0}\tau_n E^n\ln(E).$$

Introducing the Mellin transform

$$N(t)=\int_0^{E_0}\upsilon_\gamma(E)E^{t-1}\,dE.$$

$N(t)$ is a well defined holomorphic function in the $\Re(t)>0$ half-plane and it extends as a a meromorphic function in the complex t-plane: it as a

simple pole at $t = 1$ with residue δ_{-1}, and a double pole at $t = -n$, $n \in \mathbb{N}$, with the following Laurent expansion:

$$N(t) = \frac{-\tau_n}{(t+n)^2} + \frac{\delta_n}{t+n} + \mathrm{O}(1).$$

Assuming that $t \in \mathbb{N}\backslash\{0, 1\}$ and integrating by parts, we find

$$N(t) = \Gamma(t) \int_{-1}^{+1} \frac{\Gamma(-1/2)}{4\Gamma(t-1/2)} V'' p_{crit}^{2t-3}(q) - \frac{5\Gamma(-3/2)}{16\Gamma(t-3/2)} (V')^2 p_{crit}^{2t-5}(q)\, dq.$$

A straightforward computation of the right-hand part integral of this equality allows us to write

$$N(t) - \frac{\Gamma^2(t)\Gamma(-3/2)}{2^{2t+2}\Gamma(2t-1/2)} \frac{t-3/2}{t-1} = 0$$

and the same arguments as those used for $M(t)$ show that this identity holds for all $t \in \mathbb{C}$.

We now compute the Laurent expansions of $N(t)$ near $t = 1, 0, -1, -2, \ldots$, thus getting $\delta_{-1} = -\frac{1}{12}$, $\delta_0 = \frac{11}{12} - \frac{1}{2}\ln 2$ and $\tau_0 = \frac{1}{4}$, $\delta_1 = \frac{605}{48} - \frac{25}{4}\ln 2$ and $\tau_1 = \frac{25}{8}$, $\delta_2 = \frac{2261}{12} - \frac{735}{8}\ln 2$ and $\tau_2 = \frac{735}{16}, \ldots$.

Putting pieces together we finally compute the regular resurgent symbol a^{red}:

$$\begin{aligned}
&-\ln a^{\mathrm{red}} \\
&= \ln(x)\Big\{-2x\Big(E + \tfrac{3}{2}E^2 + \tfrac{35}{4}E^3 + \mathrm{O}(E^4)\Big) \\
&\quad - 2x^{-1}\Big(\tfrac{1}{8} + \tfrac{25}{16}E + \tfrac{735}{32}E^2 + \mathrm{O}(E^3)\Big) + \mathrm{O}(x^{-3})\Big\} \\
&\quad + x\Big\{\tfrac{2}{3} - 4E\ln 2 + \Big(\tfrac{17}{2} - 6\ln 2\Big)E^2 + \Big(\tfrac{227}{4} - 35\ln 2\Big)E^3 + \mathrm{O}(E^4)\Big\} \\
&\quad - \ln 2\pi + x^{-1}\Big\{\tfrac{19}{24} - \tfrac{1}{2}\ln 2 + \Big(\tfrac{187}{16} - \tfrac{25}{4}\ln 2\Big)E + \mathrm{O}(E^2)\Big\} \\
&\quad + \mathrm{O}(x^{-3}).
\end{aligned}$$

The Voros coefficient $a^{[L]}$. Writing

$$a^{\mathrm{red}}(x, x^{-1}E_r) = a_{red}^{[L]}(x, E_r)$$

we find

$$\begin{aligned}
&\ln a_{red}^{[L]}(x, E_r) \\
&\quad = \Big\{2E_r + \Big(\tfrac{1}{4} + 3E_r^2\Big)x^{-1} + \Big(\tfrac{25}{8}E + \tfrac{35}{2}E^3\Big)x^{-2} + \mathrm{O}(x^{-3})\Big\}\ln(4x) - \tfrac{2}{3}x \\
&\qquad + \ln(2\pi) - \Big(\tfrac{19}{24} + \tfrac{17}{2}E_r^2\Big)x^{-1} - \Big(\tfrac{187}{16}E_r + \tfrac{227}{4}E_r^3\Big)x^{-2} + \mathrm{O}(x^{-3}).
\end{aligned}$$

We now conclude with the following result as already announced in [1] IV.2 (after an easy rescaling), see also [35]:

$$a^{[L]} = \frac{a_{red}^{[L]}}{\Gamma^2(s+1)} = \frac{2\pi(4x)^{2s+1}}{\Gamma^2(s+1)}\,\mathrm{e}^{-\frac{2}{3}x}\,\mathrm{e}^{-D},$$

where

$$D(x, E_r) = \Big(\tfrac{19}{24} + \tfrac{17}{2}E_r^2\Big)x^{-1} + \Big(\tfrac{187}{16}E_r + \tfrac{227}{4}E_r^3\Big)x^{-2} + \mathrm{o}(x^{-2})$$

and

$$s(E_r, x) = \Big(E_r - \tfrac{1}{2}\Big) + \Big(\tfrac{1}{8} + \tfrac{3}{2}E_r^2\Big)x^{-1} + \Big(\tfrac{25}{16}E_r + \tfrac{35}{4}E_r^3\Big)x^{-2} + \mathrm{O}(x^{-3}).$$

5.2.2. Confluence near the top of the double well

We now carry on our analysis by considering the critical value $E_{crit} = 1/16$ corresponding to the critical point zero. Since the reasoning is quite similar as the previous one, our presentation will be rough.

Regular monodromy exponent. Since the cycle γ is now the vanishing cycle when $E \to 1/16$, the monodromy exponent

$$s(E, x) = -\frac{x}{2\mathrm{i}\pi}\int_\gamma P\,dq - \frac{1}{2}$$

is a regular resurgent function near E_{crit} and substituting in it $E \to E_{crit} + x^{-1}E_r$ yields what we shall called "the" monodromy exponent $s(E_{crit} + x^{-1}E_r, x)$ of the double turning point zero.

Using the algorithm described in [1] §III.2, one easily gets

$$s(E, x) + \tfrac{1}{2} = t_{-1}(E)\big(\mathrm{e}^{-\mathrm{i}\pi/2}x\big) + t_1(E)\big(\mathrm{e}^{-\mathrm{i}\pi/2}x\big)^{-1} + \mathrm{O}(x^{-3}),$$

where

$$t_{-1}(E)=\sqrt{2}\Big((E_{crit}-E)+\tfrac{3}{2}(E_{crit}-E)^2+\tfrac{35}{4}(E_{crit}-E)^3 + \tfrac{1155}{16}(E_{crit}-E)^4+\mathrm{O}\big((E_{crit}-E)^5\big)\Big),$$

$$t_{+1}(E)=\sqrt{2}\Big(\tfrac{3}{16}+\tfrac{85}{32}(E_{crit}-E)+\tfrac{2625}{64}(E_{crit}-E)^2+\mathrm{O}\big((E_{crit}-E)^3\big)\Big),$$

while $s(E_{crit}+x^{-1}E_r,x)$ reads

$$s(E_r,x)+\tfrac{1}{2}=\mathrm{i}\sqrt{2}\Big(E_r+\Big(\tfrac{3}{16}-\tfrac{3}{2}E_r^2\Big)x^{-1}+\Big(-\tfrac{85}{32}E_r+\tfrac{35}{4}E_r^3\Big)x^{-2} + \Big(-\tfrac{1995}{1024}+\tfrac{2625}{64}E_r^2-\tfrac{1155}{16}E_r^4\Big)x^{-3}+\mathrm{O}(x^{-4})\Big).$$

Confluent Voros coefficients. The Voros coefficients a^{γ_+} and a^{γ_-} are *not regular near* $E_{crit}=1/16$. Let us for instance analyse a^{γ_+}: Theorem 2.5.1 shows that that the minor of a^{γ_+} has singularities at all points $n\omega_{-\gamma}$, $n\in\mathbb{Z}\backslash\{0\}$, which collapse to the origin when E tends to E_{crit}. Considering the value of the index of intersection $\langle-\gamma,\gamma_+\rangle=-1$ of $-\gamma$ with γ_+, it follows that the local resurgent equations for a^{γ} reads

$$\dot{\Delta}_{n\omega_{-\gamma}}a^{\gamma_+}=\frac{(-1)^n}{n}a^{-n\gamma}a^{\gamma_+},\quad n\in\mathbb{Z}^\star. \tag{5.25}$$

In order to avoid the difficulty due to the fact that ω_γ is real, it will be convenient in the sequel to rotate the direction of resummation so that our new direction of resummation will have $-\pi/2$ for its argument. Under the previous conditions a^{γ_+} is Borel resummable with respect to the $(-\pi/2)$ direction of resummation, and its sum $A^+(E,x)=\mathrm{S}_{(-\pi/2)}a^{\gamma_+}$ *depends holomorphically on E near E_{crit}*, cf. [5] §2.2; performing the substitution $E\to E_{crit}+x^{-1}E_r$ allows to define the (extended) resurgent function $A^{+crit}(E_r,x)=A^+(x^{-1}E_r,x)$ which also reads

$$A^{crit}=\mathrm{S}_{(-\pi/2)}a^{[\gamma_+]},$$

where $a^{[\gamma_+]}$ is the Voros symbol associated with the connection cycle $[\gamma_+]$ pictographically represented on Fig. 25 (with the conventions of [1] §III).

We asssume now that $\Re(E-E_{crit})<0$ and E near E_{crit}. Adapting again now the identity (4.5) of Lemma 4.1.2, it can be checked that:

- $c(E,x)=\frac{1}{\Gamma(s+1)^{\mathrm{Stir}}}$ satisfy Eq. (5.21) so that we can write

$$a^{\gamma_+}=c(E,x)a^{\mathrm{red}+}, \tag{5.26}$$

where $a^{\mathrm{red}+}$ is an *elementary local resurgent constant.*

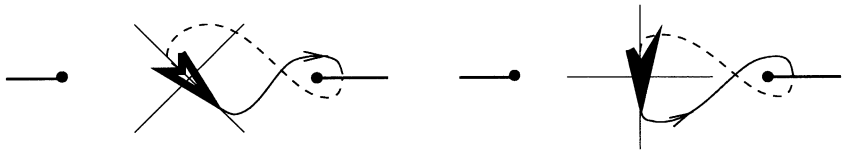

Fig. 25. On the left-hand side the connection cycle $[\gamma_+]$ for real positive direction of resummation, and on the right-hand side its deformation for direction of argument $-\pi/2$.

- s is not Borel resummable with respect to the $(-\pi/2)$ direction of resummation. We shall denote by the same letter s the extended resurgent function got from s by *right* Borel resummation. Then the right Borel sum

$$C(E,x) = \frac{1}{\Gamma(s+1)}$$

of $c(E,x)$ in the $(-\pi/2)$ direction of resummation depends holomorphically on E in a neighbourhood of E_{crit}.

As a consequence using the same arguments as in the proof of Theorem 4.1.1, we conclude that $a^{\mathrm{red}+}$ *depends regularly on E in a neighbourhood of the origin.*

Computing the Voros coefficient $a^{[\gamma_+]}$. It follows from the Stirling formula that

$$\begin{aligned}\ln c(x,E) &= -\ln\Gamma(s+1)\\ &= -\{t_{-1}(E)(\mathrm{e}^{-\mathrm{i}\pi/2}x) + t_1(E)(\mathrm{e}^{-\mathrm{i}\pi/2}x)^{-1} + \mathrm{O}(x^{-3})\}\\ &\quad\times \ln(\mathrm{e}^{-\mathrm{i}\pi/2}x) + \{t_{-1} - t_{-1}\ln(t_{-1})\}(\mathrm{e}^{-\mathrm{i}\pi/2}x) - \ln(\sqrt{2\pi})\\ &\quad + \left\{\tfrac{1}{24t_{-1}} - t_1\ln(t_{-1})\right\}(\mathrm{e}^{-\mathrm{i}\pi/2}x)^{-1} + \mathrm{O}(x^{-3}).\end{aligned}$$

In order to describe $a^{\mathrm{red}+}$ we write (5.26) as follows

$$\ln a^{\mathrm{red}+} = \ln a^{\gamma_+} - \ln c$$

and we now compute

$$\ln a^{\gamma_+} = -x\int_{\gamma_+} P\,dq.$$

We begin with the computation to the zero'th order, that is $\omega_{\gamma_+}(E) = \int_{\gamma_+} p\,dq$ for $E \lesssim E_{crit}$ (say). We expect the asymptotic expansion of ω_{γ_+}

when $E \to E_{crit}$ to be

$$\omega_{\gamma_+}(E) \sim \sum_{n \geqslant 0} \alpha_n (E_{crit} - E)^n + \sum_{n \geqslant 1} \beta_n (E_{crit} - E)^n \ln(E_{crit} - E).$$

In order to compute the (α_n) and (β_n) we introduce the Mellin transform

$$M(t) = \int_{1/16}^{0} \omega_{\gamma_+}(E)(E_{crit} - E)^{t-1}\, dE.$$

We first assume $t \in \mathbb{N}\backslash\{0, 1\}$ and we integrate by parts: noting that the two simple turning points are located at $-\sqrt{2}$ and $\sqrt{2}$, respectively, $M(t)$ reads

$$M(t) = 2\mathrm{i}\frac{\Gamma(t)\Gamma(3/2)}{\Gamma(t+3/2)} \int_{0}^{+\sqrt{2}} \underline{p}_{crit}^{2t+1}(q)\, dq,$$

where $\underline{p}_{crit}(q) = q\sqrt{2 - q^2}/4$. We then get the following identity

$$M(t) - \mathrm{i}\sqrt{2}\frac{\Gamma^2(t+1)\Gamma(3/2)}{2^{2t+1} t\Gamma(2t+5/2)} = 0$$

for all non-zero positive integer t, an equality which extends for all t in the complex plane.

Computing the Laurent expansions of the meromorphic function $M(t)$ near $t = 0, -1, -2, \ldots$, we shall find $\alpha_0 = -\frac{\sqrt{2}}{3}\mathrm{i}$, $\alpha_1 = \sqrt{2}\mathrm{i}(1 + 2\ln 2)$ and $\beta_1 = -\mathrm{i}\sqrt{2}$, $\alpha_2 = \frac{\sqrt{2}}{4}\mathrm{i}(-17 + 12\ln 2)$ and $\beta_2 = -\frac{3\sqrt{2}}{2}\mathrm{i}$, $\alpha_3 = \frac{\sqrt{2}}{2}\mathrm{i}(-59 + 35\ln 2)$ and $\beta_3 = -\frac{35\sqrt{2}}{4}\mathrm{i}, \ldots$ and so on.

We now compute to the next order, that is describe the asymptotic expansion

$$\upsilon_\gamma(E) \sim \sum_{n \geqslant -1} \delta_n (E_{crit} - E)^n + \sum_{n \geqslant 0} \tau_n (E_{crit} - E)^n \ln(E_{crit} - E)$$

of $\upsilon_{\gamma_+}(E) = \int_{\gamma_+} p_1\, dq$ for $E \lesssim E_{crit}$. Integrating by parts the following Mellin transform

$$N(t) = \int_{1/16}^{0} \upsilon_{\gamma_+}(E)(E_{crit} - E)^{t-1}\, dE$$

for $t \in \mathbb{N}\backslash\{0, 1\}$, we find

$$N(t) = \mathrm{i}\Gamma(t) \int_0^{\sqrt{2}} \frac{\Gamma(-1/2)}{4\Gamma(t-1/2)} V'' \underline{p}^{2t-3}(q) + \frac{5\Gamma(-3/2)}{16\Gamma(t-3/2)} (V')^2 \underline{p}^{2t-5}(q)\, dq$$

that is also

$$N(t) + \mathrm{i}\sqrt{2} \frac{\Gamma^2(t)\Gamma(-3/2)}{2^{2t+2}\Gamma(2t-1/2)} \frac{t-9/8}{t-1} = 0.$$

This identity extends for all $t \in \mathbb{C}$ and allows to compute the Laurent expansions of $N(t)$ near $t = 1, 0, -1, -2, \ldots$; we thus get $\delta_{-1} = -\frac{\sqrt{2}}{48}\mathrm{i}$, $\delta_0 = -\frac{\sqrt{2}}{48}\mathrm{i}(-35 + 18\ln 2)$ and $\tau_0 = \frac{3\sqrt{2}}{16}\mathrm{i}$, $\delta_1 = -\frac{\sqrt{2}}{192}\mathrm{i}(-2093 + 1020\ln 2)$ and $\tau_1 = \frac{85\sqrt{2}}{32}\mathrm{i}, \ldots$.

The previous results yield finally the regular resurgent symbol $a^{\mathrm{red}+}$:

$$\begin{aligned}
&-\ln a^{\mathrm{red}+} \\
&= -\sqrt{2}\ln\left(\mathrm{e}^{-\mathrm{i}\pi/2}x\right)\Big\{\Big((E_{crit}-E) + \tfrac{3}{2}(E_{crit}-E)^2 + \tfrac{35}{4}(E_{crit}-E)^3 \\
&\qquad + \mathrm{O}\big((E_{crit}-E)^4\big)\Big)\left(\mathrm{e}^{-\mathrm{i}\pi/2}x\right) \\
&\qquad + \Big(\tfrac{3}{16} + \tfrac{85}{32}(E_{crit}-E) + \tfrac{2625}{64}(E_{crit}-E)^2 \\
&\qquad + \mathrm{O}\big((E_{crit}-E)^3\big)\Big)\left(\mathrm{e}^{-\mathrm{i}\pi/2}x\right)^{-1} + \mathrm{O}\left(x^{-3}\right)\Big\} \\
&\quad + \sqrt{2}\left(\mathrm{e}^{-\mathrm{i}\pi/2}x\right)\Big\{\tfrac{1}{3} - \tfrac{5}{2}(E_{crit}-E)\ln 2 - \tfrac{1}{4}(-17 + 15\ln 2) \\
&\qquad \times (E_{crit}-E)^2 + \tfrac{1}{8}(-227 + 175\ln 2)(E_{crit}-E)^3 \\
&\qquad + \mathrm{O}\big((E_{crit}-E)^4\big)\Big\} - \ln\sqrt{2\pi} \\
&\quad - \sqrt{2}\left(\mathrm{e}^{-\mathrm{i}\pi/2}x\right)^{-1}\Big\{\tfrac{1}{96}(67 - 45\ln 2) + \tfrac{1}{64}(671 - 425\ln 2) \\
&\qquad \times (E_{crit}-E) + \mathrm{O}\big((E_{crit}-E)^2\big)\Big\} + \mathrm{O}\left(x^{-3}\right).
\end{aligned}$$

Defining now

$$a_{red}^{[\gamma_+]}(x, E_r) = a^{\mathrm{red}+}\left(x, E_{crit} + x^{-1}E_r\right)$$

we get

$$\begin{aligned}
&\ln a_{red}^{[\gamma_+]}(x, E_r) \\
&= \left\{ s(x, E_r) + \tfrac{1}{2} + \mathrm{O}(x^{-3}) \right\} \ln\left(4\sqrt{2}x\, \mathrm{e}^{-\mathrm{i}\pi/2}\right) + \mathrm{i}\tfrac{\sqrt{2}}{3}x + \ln\left(\sqrt{2\pi}\right) \\
&\quad + \mathrm{i}\sqrt{2}\left\{ \left(-\tfrac{67}{96} + \tfrac{17}{4}E_r^2\right)x^{-1} + \left(\tfrac{671}{64}E_r - \tfrac{227}{8}E_r^3\right)x^{-2} + \mathrm{O}(x^{-3}) \right\}.
\end{aligned}$$

We are now ready to conclude with the following result, as announced in [1] IV.3 (up to an easy rescaling):

$$a^{[\gamma_+]} = \frac{a_{red}^{[\gamma_+]}}{\Gamma(s+1)} = \frac{\sqrt{2\pi}(4\sqrt{2}x\, \mathrm{e}^{-\mathrm{i}\pi/2})^{s+1/2}}{\Gamma(s+1)}\, \mathrm{e}^{\mathrm{i}\frac{\sqrt{2}}{3}x}\, \mathrm{e}^{-D},$$

where

$$\begin{aligned}
D(x, E_r) = &-\mathrm{i}\sqrt{2}\left\{ \left(-\tfrac{67}{96} + \tfrac{17}{4}E_r^2\right)x^{-1} + \left(\tfrac{671}{64}E_r - \tfrac{227}{8}E_r^3\right)x^{-2} \right\} \\
&+ \mathrm{o}(x^{-2})
\end{aligned}$$

and

$$\begin{aligned}
s(E_r, x) + \tfrac{1}{2} = \mathrm{i}\sqrt{2}\Big\{ &E_r + \left(\tfrac{3}{16} - \tfrac{3}{2}E_r^2\right)x^{-1} + \left(-\tfrac{85}{32}E_r + \tfrac{35}{4}E_r^3\right)x^{-2} \\
&+ \mathrm{O}(x^{-3}) \Big\}.
\end{aligned}$$

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