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On the classical and quantum evolution of Lagrangian half-forms in phase space

by

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ABSTRACT. – The local expressions of a Lagrangian half-form on a quantized Lagrangian submanifold of phase space are the wavefunctions of quantum mechanics. We show that one recovers Maslov's asymptotic formula for the solutions to Schrödinger's equation if one transports these half-forms by the flow associated with a Hamiltonian H. We then consider the case when the Hamiltonian flow is replaced by the flow associated with the Bohmian, and are led to the conclusion that the use of Lagrangian half-forms leads to a quantum mechanics on phase space. © Elsevier, Paris

RÉSUMÉ. – Les expressions locales d'une demi-forme Lagrangienne sur une sousvariété Lagrangienne quantifiée de l'espace des phases sont les fonctions d'onde de la mécanique quantique. Nous montrons que le transport de ces demi-formes par le flot associé à un Hamiltonien H permet de retrouver les solutions asymptotiques de l'équation de Schrödinger donnés par la formule de Maslov. Nous étudions ensuite le cas où l'on remplace ce flot par celui associé au Bohmien, et arrivons à la conclusion que l'emploi des demi-formes Lagrangiennes permet d'élaborer une mécanique quantique en espace de phase. © Elsevier, Paris

1. INTRODUCTION

The fundamental tenet of Quantum Mechanics is Schrödinger's equation

$$i\varepsilon\frac{\partial\Psi^{\varepsilon}}{\partial t}=\widehat{H}^{\varepsilon}\Psi^{\varepsilon}, \tag{1.1}$$

where $\widehat{H}^{\varepsilon}$ is a differential operator associated with a classical Hamiltonian function $H=H(x,p)$ by some "quantization rule" and the parameter ε is given a certain value $\hbar$ (Planck's constant). It is a very common belief that Quantum Mechanics cannot be derived from Classical Mechanics. However, this belief is, strictly speaking, not justified because Eq. (1.1) can be *rigorously* derived for every $\varepsilon>0$ from Hamilton's equations of motion

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\partial H}{\partial p}(x,p),\qquad \frac{\mathrm{d}p}{\mathrm{d}t}=-\frac{\partial H}{\partial x}(x,p) \tag{1.2}$$

when H is a quadratic function of the positions and the momenta. The argument is not very difficult; it goes as follows. Solving Hamilton's equations one sees that the associated flow $(s_t)_t$ consists of linear symplectic transformations and is hence a continuous one parameter subgroup of the symplectic group $\mathrm{Sp}(n)$. Now, $\pi_1(\mathrm{Sp}(n))\cong(\mathbf{Z},+))$ hence $\mathrm{Sp}(n)$ has covering groups $\mathrm{Sp}_q(n)$ of all orders $q=1,2,\ldots,\infty$ ($\mathrm{Sp}_\infty(n)$ being the universal covering). It turns out that one of these groups, the twofold covering $\mathrm{Sp}_2(n)$, has a true representation as a group of unitary operators acting on $L^2(\mathbf{R}^n_x)$; that group is denoted by $\mathrm{Mp}(n)$ and is called the *metaplectic group*. By a classical result in the theory of covering spaces, there exists a unique one parameter subgroup $(s_{\infty,t})_t$ of $\mathrm{Sp}_\infty(n)$ whose projection on $\mathrm{Sp}(n)$ is precisely $(s_t)_t$; it is commonly called the lift of $(s_t)_t$ to $\mathrm{Sp}_\infty(n)$. If we now project that lift on of $\mathrm{Sp}_2(n)\equiv\mathrm{Mp}(n)$ we get a one parameter group $(S_t)_t$ of unitary operators, whose datum is then mathematically equivalent with that of the classical flow $(s_t)_t$: the knowledge of the first unambiguously determines the second and vice versa. Setting for $\Psi_0\in S(\mathbf{R}^n_x)$:

$$\Psi^{\varepsilon}(x,t)=M_{\sqrt{\varepsilon}}S_tM_{\frac{1}{\sqrt{\varepsilon}}}\Psi_0(x), \tag{1.3}$$

where M_λ is, for $\lambda>0$, the unitary scaling operator defined by

$$M_\lambda\Psi_0(x)=\sqrt{\lambda}\Psi_0(\lambda x)$$

we find that the function Ψ^{ε} satisfies Eq. (1.1) when one chooses

$$\widehat{H}^{\varepsilon} = H\left(x, -\mathrm{i}\varepsilon\frac{\partial}{\partial x}\right) - \frac{\mathrm{i}}{2}\mathrm{Trace}\left(H''_{xp}\right) \tag{1.4}$$

(H''_{xp} the Hessian matrix of H). This can be proven either by invoking Stone's theorem and using a Poisson bracket argument as in [16], Chapter 1, or by a direct calculation using the integral representation of the elements of $\mathrm{Mp}(n)$ together with the Hamilton–Jacobi theory for generating functions. Either way, one sees that Eq. (1.1) essentially is a *classical* equation which is *equivalent* to Hamilton's Eq. (1.2). We now want to emphasize the following point. Mathematically speaking there is no reason, whatsoever, to assign a particular value to ε (say $\varepsilon = \hbar$) in Eq. (1.1); such a choice must be motivated by a *physical postulate*, giving a *physical significance* to the corresponding solution of these equations. A first guess would be to interpret the solution $\Psi^{\varepsilon}(q,t) = S_t^{\varepsilon}\Psi_0(q)$ "à la Max Born", by deciding that $|\Psi^{\varepsilon}(q,t)|^2$ is the probability density for finding the system in some region of configuration space at time t. However, this misses the point, because it turns out that this interpretation is again perfectly classical and in perfect accord with the predictions of Liouville's theorem, at least as long as quadratic Hamiltonians and Gaussian wave-functions are considered (this is a straightforward consequence of Ehrenfest's theorem; see, for instance, [17]). Thus the probabilistic interpretation of (1.1) and of its solutions is not *per se* convincingly and unquestionably defining a new physics. It is only because we know *a posteriori* that Ψ^{ε} has a very special (quantum) interpretation *after* we have assigned the value $\hbar$ to the parameter ε that we can claim that quantum mechanics is born. A *mathematical* justification for the need of Planck's constant is thus needed. The aim of this article is precisely to give such a justification by reinterpreting the solutions to (1.1) as the local expressions in a conveniently chosen frame of the "Lagrangian catalogues" we have introduced in our prior works [13,14]. These Lagrangian catalogues are phase space objects whose local expressions in a conveniently chosen frame are Leray's Lagrangian functions [18,19]. They are defined on the universal covering of a connected Lagrangian submanifold V of the phase space $\mathbf{R}_x^n \times \mathbf{R}_p^n$ and consist of "pages" which are "Lagrangian half-forms" on V, that is expressions of the type

$$U_{\ell_\alpha}^{\rho}(\check{z}) = \mathrm{e}^{\frac{\mathrm{i}}{\varepsilon}\varphi(z)}\mathrm{i}^{m_\alpha(z)}\rho(z), \tag{1.5}$$

where $\varepsilon > 0$, ρ is a half-density on V and φ, m_α are functions defined on the universal covering $\check{V}$ of V:

(1) φ is the *phase* of V: it is a function $\check{V} \to \mathbf{R}$, uniquely defined up to an additive constant, by the requirement that

$$d\varphi(\check{z}) = \pi^*(p\,dx) \tag{1.6a}$$

(π is here the covering mapping $\check{V} \to V$). After the choice of an origin in $\check{V}$ has been made, the phase can alternatively be defined by the formula

$$\varphi(\check{z}) = \int_{z_0}^{z} p\,dx, \tag{1.6b}$$

where the integration is performed along the path $[z_0, z]$ in V joining z_0 to z and whose homotopy class is $\check{z}$; since V is Lagrangian that integral only depends on the homotopy class of $[z_0, z]$, that is on the element $\check{z}$ of V whose projection is $\pi(\check{z}) = z$.

(2) m_α is a function $\check{V} \to \mathbf{Z}$ associated with an (arbitrary) Lagrangian plane ℓ_α in $\mathbf{R}_x^n \times \mathbf{R}_p^n$ and related to the ALM (Arnol'd–Leray–Maslov) index m which we review in Section 2, by the formula

$$m_\alpha(\check{z}) = m(\ell_{\alpha,\infty}, \ell_\infty(\check{z})), \tag{1.7}$$

where $\ell_{\alpha,\infty}$ is any element of the universal covering of the Lagrangian Grassmannian with projection ℓ_α, and $\check{z} \to \ell_\infty(\check{z})$ is an arbitrary lift of the mapping $z \to \ell(z) = T_z V$. That function m_α, which depends only on $\check{z}$ and ℓ_α (and not on $\ell_{\alpha,\infty}$) allows us to define the *argument* of the half-density ρ by $m_\alpha \pi/2$ (see (1.5)). By the properties of the ALM index m_α is a locally constant function on $\check{V}$ outside the caustic Σ_α relative to ℓ_α (Σ_α is the set of all $z \in V$ such that $T_z V = \ell_\alpha$); it is thus constant on the connected components of the complement of Σ_α in $\check{V}$. If for instance ℓ_α is the vertical plane $0 \times \mathbf{R}_p^n$, then Σ_α is the caustic of V in the usual sense, e.g., the set of all $z \in V$ outside which V locally projects diffeomorphically on the horizontal plane $\mathbf{R}_x^n \times 0$ and m_α has constant value on each connected component of projects on $\mathbf{R}_x^n \times 0$.

The interest of the notion of Lagrangian catalogue in Quantum Mechanics cones from the following observation: suppose that the Lagrangian manifold V is a graph, that is

$$(x, p) \in V \Leftrightarrow p = \frac{\partial \Phi}{\partial x}(x), \quad x \in D, \tag{1.8}$$

where D is an open subset in $\mathbf{R}_x^n$. If D is simply connected, so is V, and hence we may identify V with its universal covering, so that both φ and m_α are defined on V itself. Moreover, the half-density ρ may be identified with its local expression $a(x)|\mathrm{d}x^{1/2}|$ in D. The datum of the "page" (1.5) is thus equivalent with the datum of the usual wavefunction

$$\Psi(x) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Phi(x)} a(x) \tag{1.9}$$

familiar from elementary quantum mechanics, noting that the phase is here given by

$$\varphi(z) = \Phi(x), \quad z = (x, p).$$

Conversely, to every such wave-function (1.9) one can associate a Lagrangian catalogue on the graph (1.8) and whose pages are of the type $\mathrm{i}^k\Psi$, $k = 0, 1, 2, 3$, Ψ given by (1.9). If we now require that every page (1.5) has *single valued* local expressions, then we must impose to the Lagrangian manifold V the *quantum condition*

$$\frac{1}{\varepsilon}\varphi(\check{z}) + \frac{\pi}{2} m_\alpha(\check{z}) \text{ is defined modulo } 2\pi \text{ on } V \tag{1.10a}$$

which is equivalent to the condition

$$\frac{1}{2\pi\varepsilon}\int_\gamma p\,\mathrm{d}x - \frac{1}{4} m(\gamma) \in \mathbf{Z} \tag{1.10b}$$

for all closed paths in V; $m(\gamma)$ is here the integer $m_\alpha(\gamma\check{z}) - m_\alpha(\check{z})$; it is independent of the choice of the Lagrangian plane ℓ_α. We now *define* Quantum Mechanics by a postulate of universal nature:

[QM] *The only quantized Lagrangian manifolds are those for which the quantum condition* (1.10) *holds true for the value* $\varepsilon = \hbar$ *where* $\hbar$ *is Planck's constant.*

Postulate [QM] means that all Lagrangian catalogues are single-valued on the Lagrangian manifolds singled out by the condition

$$\frac{1}{2\pi\hbar}\int_\gamma p\,\mathrm{d}x - \frac{1}{4} m(\gamma) \in \mathbf{Z} \tag{1.11}$$

for all paths γ in V. Lagrangian catalogues thus constitute the natural generalizations of the usual wave-functions on quantized Lagrangian manifolds; this will be developed in Section 3. Of course condition (1.11) trivially holds (as does (1.10) for every $\varepsilon > 0$) when V is simply connected.

The aim of this article is to study the motion of Lagrangian catalogues under the flow associated with a classical Hamiltonian function H, and under the flow associated with the "Bohmian" associated with an initial wave-function. The first leads to quasi-classical mechanics in phase space and allows us to derive Maslov's asymptotic solutions [21–23] of Schrödinger's equations. The second leads to a Quantum Mechanics in phase space.

We have structured this article as follows:

In Section 2 we briefly review the properties of the ALM index proven in [7,10,14] and relate that index to the Maslov index on the metaplectic group defined in [9,11] (see also [8]);

In Section 3 we consider the flow $(F_{t,t'})$ associated with a time-dependent Hamiltonian. Defining a Lagrangian manifold V_t by the formula $V_t = F_{t,0}V$. We then show that one can define in a natural way a Lagrangian catalogue on V_t by using the properties of the ALM index. For example, when $U^{\rho}_{\ell_\alpha}$ is given by (1.5) then

$$F_{t,0}U^{\rho}_{\ell_\alpha}(x) = \mathrm{e}^{\frac{\mathrm{i}}{\varepsilon}\varphi(z,t)}\mathrm{i}^{m_\beta(z)}(F_{t,0})_*\rho(z), \tag{1.12}$$

where the phase is given by

$$\varphi(z,t) = \varphi(z) + \int_{z}^{z(t)} p\,\mathrm{d}x - H\,\mathrm{d}t \tag{1.13}$$

the integral being calculated along the phase-space trajectory leading from z to $z(t) = F_{t,0}z$. The integer m_β is associated with the Lagrangian plane ℓ_β obtained by applying the tangent mapping to $F_{t,0}$ (see (3.3)). The properties of the ALM index then allow us to show that one can define the action of $F_{t,t'}$ for arbitrary t, t'' on Lagrangian catalogues on $V_{t'} = F_{t',0}V$, and that this action satisfies the Chapman–Kolmogorov law

$$F_{t,t'}F_{t',t''} = F_{t,t''}.$$

We then proceed to investigate the local properties of Lagrangian catalogues and of their images under Hamiltonian flows. The crucial result is that when the Lagrangian manifold V satisfies the quantization condition (1.11) then one can patch together the local expressions of (1.5) outside the caustic, when ℓ_α is the vertical plane, to obtain functions of the usual type

$$\Psi(x,t) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Phi(x,t)} a(x,t).$$

In Section 4 we relate the local expressions of Lagrangian half-forms in the quantized case to Maslov's formula

$$\Psi_{\text{approx}}(x,t) = \sum_j a(x_j)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_j(x,t)}\mathrm{i}^{-\mu_j}\left|\det\frac{\partial x_j(t)}{\partial x}\right|^{-1/2} \tag{1.14}$$

for asymptotic solutions to Schrödinger's equation. We show that these asymptotic solutions are exactly sums of local expressions of Lagrangian half-forms provided that the half-density ρ on V has a support that is compact and does not intersect the caustic: Lagrangian half-forms and their evolution under Hamiltonian flows are thus the *natural* phase space objects one should consider when studying "quasi-classical approximation".

In Section 5 we apply the results of Section 3, not to the Hamiltonian flow $(F_{t,t'})$ itself, but rather to the "Bohmian flow" associated with the Hamiltonian H and a solution $\Psi(x,t)$ of the Schrödinger equation. Recall [3,17] that to such a Ψ one can associate a "quantum potential", given in the one-dimensional case by the formula

$$Q = -\frac{\hbar^2}{2m}|\Psi|^{-1}\frac{\mathrm{d}^2}{\mathrm{d}x^2}|\Psi|,$$

when $|\Psi| > 0$ (see the discussion at the beginning of Section 5). Adding that term to the Hamiltonian H one obtains a modified Hamiltonian function, depending on H and $|\Psi|$, the "Bohmian"

$$H^{|\Psi|} = H + Q.$$

The Bohmian can be viewed, for given Ψ, as a new Hamiltonian function, usually time-dependent even when H is not. Writing Ψ in the form

$$\Psi(x,t) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(x,t)}R(x,t) \tag{1.15}$$

one sees that S satisfies the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H^{|\Psi|}\left(x, \frac{\partial S}{\partial x}, t\right) = 0 \tag{1.16}$$

and R the continuity equation

$$\frac{\partial R}{\partial t} + \frac{\partial}{\partial x}\left(R\frac{1}{m}\frac{\partial S}{\partial x}\right) = 0 \tag{1.17}$$

familiar from Fluid Mechanics. Using these equations we prove (Theorem 2) that if one applies to a Lagrangian half-form the flow $(F_{t,0})_t$ associated with the Bohmian $H^{|\Psi|}$, then the local expressions $F_{t,0}U^{\rho}_{\ell_\alpha}$ are *exactly* those obtained from the true solution Ψ of Schrödinger's equation. This result is striking because it shows that the concepts we have introduced in this article might be a starting point for a theory of quantum mechanics in phase space.

We finally want to emphasize the fact that the distinction we have made throughout this article between Planck's *constant* $\hbar$, and the *variable* parameter ε, is essential. That distinction was first introduced in Leray's book [18] (see also [19]). It is unavoidable not only if one wants to use quasi-classical approximation arguments, but also if one wants to justify quantization arguments, or the Bohmian theory of motion. Arguments invoking a variable Planck's constant, which one can let at will tend to zero are not only intellectually repellent; they also lead to inconsistencies (see Leray's discussion in the preface to [18]).

2. REVIEW OF THE PROPERTIES OF THE ALM AND MASLOV INDICES

For proofs and details we refer to our previous works [7,9,10,13] and to the references therein; see also Dazord [4] for related results in the category of symplectic fiber bundles. Cappell et al. [2] compare our constructions to other indices appearing in the literature. We will use the following notations: $Z(n)$ is the phase space $\mathbf{R}^n_x \times \mathbf{R}^n_p$; it will be equipped with the standard symplectic form $\omega = \mathrm{d}p \wedge \mathrm{d}x$. In the coordinates $x = (x_1, \ldots, x_n)$, $p = (p_1, \ldots, p_n)$ this is

$$\omega = \mathrm{d}p_1 \wedge \mathrm{d}x_1 + \cdots + \mathrm{d}p \wedge \mathrm{d}x_n.$$

We denote by $\mathrm{Lag}(n)$ the Lagrangian Grassmannian of $(Z(n), \omega)$: it is the set of all n-planes ℓ, ℓ', etc. through the origin in $Z(n)$ on which the symplectic form ω vanishes identically; as a manifold $\mathrm{Lag}(n)$ is the compact and connected coset space $U(n)/O(n)$. As usual $\mathrm{Sp}(n)$ stands for the symplectic group of $(Z(n), \omega)$: it is the group of all linear automorphisms of $Z(n)$ which preserve ω. Let $\sigma : \mathrm{Lag}^3(n) \to \mathbf{Z}$ be the Demazure–Kashiwara index [5,20]: for a triple ℓ, ℓ', ℓ'' of Lagrangian planes in $Z(n)$, $\sigma(\ell, \ell', \ell'')$ is the integer $a - b$, where (a, b) is the signature of the quadratic form

$$Q(z, z', z'') = \omega(z, z') + \omega(z', z'') + \omega(z'', z)$$

on $\ell \otimes \ell' \otimes \ell''$. An essential property is then that the function

$$\sigma : \mathrm{Lag}^3(n) \to \mathbf{Z}, \quad (\ell, \ell', \ell'') \to \sigma(\ell, \ell', \ell''),$$

is a *cocycle*, that is

$$\sigma(\ell, \ell', \ell'') - \sigma(\ell', \ell'', \ell''') + \sigma(\ell'', \ell''', \ell) - \sigma(\ell''', \ell, \ell') = 0$$

for all $\ell, \ell', \ell'', \ell'''$ in $\mathrm{Lag}(n)$. Notice that σ is obviously $\mathrm{Sp}(n)$-invariant, that is:

$$\sigma(s\ell, s\ell', s\ell'') = \sigma(\ell, \ell', \ell'')$$

for all $s \in \mathrm{Sp}(n)$. It is convenient to introduce the following associated function, called the *index of inertia*:

$$\mathrm{Inert}(\ell, \ell', \ell'') = \frac{1}{2}\left(n + \partial\mathrm{dim}(\ell, \ell', \ell'') + \sigma(\ell, \ell', \ell'')\right), \tag{2.1}$$

where $\partial\mathrm{dim}$ is the coboundary of the 1-cochain dim on $\mathrm{Lag}(n)$ defined by:

$$\dim(\ell, \ell') = \dim(\ell \cap \ell').$$

By the properties of the signature σ established in [13,20], $\mathrm{Inert}(\ell, \ell', \ell'')$ is an integer. Since

$$\partial\mathrm{Inert} = \frac{1}{2}(\partial n + \partial^2\mathrm{dim} + \partial\sigma) = 0$$

it follows that the index of inertia also is a $\mathrm{Sp}(n)$-invariant cocycle. In fact, Inert coincides with the index of inertia defined by Leray in [18], Chapter I, §§2, 4, in the transversal case: see [7] for a proof.

Let now $\mathrm{Lag}_\infty(n)$ denote the universal covering of $\mathrm{Lag}(n)$ and π the natural projection $\mathrm{Lag}_\infty(n) \to \mathrm{Lag}(n)$. Using chain intersection theory Leray (ibid. §§2, 5) has constructed an integer-valued function m defined on all pairs $(\ell_\infty, \ell'_\infty) \in \mathrm{Lag}^2_\infty(n)$ such that $\pi(\ell_\infty) \cap \pi(\ell'_\infty) = 0$.

He calls m the "Maslov index on $\mathrm{Lag}_\infty(n)$" and shows that m is characterized (together with a topological condition: see (2.5) below) by the property that $\partial m = \pi^*\mathrm{Inert}$, that is:

$$m(\ell_\infty, \ell'_\infty) - m(\ell_\infty, \ell''_\infty) + m(\ell'_\infty, \ell''_\infty) = \mathrm{Inert}(\ell, \ell', \ell''), \qquad (2.2)$$

where (ℓ, ℓ', ℓ'') is the projection $\pi(\ell_\infty, \ell'_\infty, \ell''_\infty)$. It turns out that one can extend Leray's definition to all pairs $(\ell_\infty, \ell'_\infty) \in \mathrm{Lag}^2_\infty(n)$, without any transversality assumption: using the cocycle property of the index of inertia it suffices to define $m(\ell_\infty, \ell'_\infty)$ for arbitrary $\ell_\infty, \ell'_\infty$ by formula (2.2) choosing ℓ''_∞ in such a way that $\ell'' \cap \ell = \ell'' \cap \ell' = 0$. One then verifies, by repeated use of the cocycle property of Inert that the value of $m(\ell_\infty, \ell'_\infty)$ is independent of the choice of ℓ''_∞ (see [7]). We will call the function $m : \mathrm{Lag}^2_\infty(n) \to \mathbf{Z}$ thus defined the ALM (Arnol'd–Leray–Maslov) index. The fundamental group $\pi_1(\mathrm{Lag}(n) \equiv (\mathbf{Z}, +)$ acts on $\mathrm{Lag}_\infty(n)$; denoting by β the generator of $\pi_1(\mathrm{Lag}(n))$ whose image in $\mathbf{Z}$ is $+1$ we have

$$m(\beta^k \ell_\infty, \beta^{k'} \ell'_\infty) = m(\ell_\infty, \ell'_\infty) + k - k' \qquad (2.3)$$

for all integers k, k'. This shows in particular that the range of m is $\mathbf{Z}$; i.e. $m(\ell_\infty, \ell'_\infty)$ takes all integer values when one of its arguments describes $\mathrm{Lag}^2_\infty(n)$. Moreover, the ALM index is invariant under the action of the universal covering $\mathrm{Sp}_\infty(n)$ of the symplectic group $\mathrm{Sp}(n)$:

$$m(s_\infty \ell_\infty, s_\infty \ell'_\infty) = m(\ell_\infty, \ell'_\infty) \qquad (2.4)$$

for all $s_\infty \in \mathrm{Sp}_\infty(n)$. This is easily seen using the fact that m is characterized by (2.2) together with (2.5) below, and taking into account the $\mathrm{Sp}(n)$-invariance of the index of inertia (2.1). We finally notice that m has the following topological property:

(2.5) *m is locally constant on the set $\{(\ell_\infty, \ell'_\infty) \colon \ell \cap \ell' = 0\}$, and is hence constant when ℓ_∞, ℓ'_∞ move continuously in such a way that their projections ℓ, ℓ' remain transverse.*

Let now V be a connected Lagrangian submanifold of $Z(n)$. We denote by

$$\ell(.) : V \to \mathrm{Lag}(n)$$

the mapping which to every $z \in V$ associates the tangent Lagrangian plane $\ell(z) = T_z V$. This mapping can be lifted (in infinitely many ways) to a mapping

$$\ell_\infty(.) : \check{V} \to \mathrm{Lag}_\infty(n).$$

Supposing $\ell_\infty(.)$ chosen once for all, we define for every $\ell_{\alpha,\infty} \in \mathrm{Lag}_\infty(n)$ a function

$$m_{\alpha,\infty} : \check{V} \to \mathbf{Z}$$

by the formula

$$m_{\alpha,\infty}(\check{z}) = m(\ell_{\alpha,\infty}, \ell_\infty(\check{z})). \tag{2.6}$$

This is the term appearing as the exponent of $\mathrm{i} = \sqrt{-1}$ in definition (1.5) of a Lagrangian half-form. Whenever there is no risk of confusion with the Maslov indices on $\mathrm{Mp}(n)$ defined below we will use the shorter notation $m_\alpha(\check{z})$ in lieu of $m_{\alpha,\infty}(\check{z})$. In view of (2.2) these functions transform following the law

$$m_\alpha(\check{z}) - m_\beta(\check{z}) = m(\ell_{\alpha,\infty}) - \mathrm{Inert}(\ell_\alpha, \ell_\beta, \ell(z)) \tag{2.7}$$

which shows that the "pages" of a Lagrangian catalogue are deduced from one another by the formula

$$U^\rho_{\ell_\alpha} = \mathrm{i}^{m(\alpha,\beta)} U^\rho_{\ell_\beta}, \tag{2.8}$$

where the integer-valued function $m(\alpha, \beta)$ is given by the right-hand side of Eq. (2.8).

Let now $\gamma \in \pi_1(V, z_0)$, the base point $z_0 \in V$ being chosen once for all. We claim that there exists an integer $m(\gamma) \in \mathbf{Z}$ such that

$$\ell_\infty(\gamma\check{z}) = \beta^{m(\gamma)} \ell_\infty(\check{z}), \quad \forall \check{z} \in \check{V}.$$

That integer is defined as follows: for all γ and all $\check{z}$, the elements $\ell_\infty(\gamma\check{z})$ and $\ell_\infty(\check{z})$ of $\mathrm{Lag}_\infty(n)$ have same projection $\ell(n)$ on $\mathrm{Lag}(n)$ hence there exists an integer k such that $\ell_\infty(\gamma\check{z}) = \beta^k \ell_\infty(\check{z})$. That integer k is denoted by $m(\gamma)$ because it is independent of $\check{z}$: if $\check{z}$ is the homotopy class of the path $\gamma_{z_0 z}$ joining z_0 to z and γ is the homotopy class of the closed loop $\gamma_{z_0 z_0}$ then $\gamma\check{z}$ is by definition the homotopy class of the path $\gamma_{z_0 z_0}\ \gamma_{z_0 z}$; from this follows that $\ell_\infty(\gamma\check{z})$ is the homotopy class of $\ell(\gamma_{z_0 z_0})\ell(\gamma_{z_0 z})$ so that $\beta^k \in \pi_1(\mathrm{Lag}, \ell(z_0))$ can be identified with the homotopy class of the closed loop $\ell(\gamma_{z_0 z_0})$ in $\mathrm{Lag}(n)$; this class is of course independent of z, and hence of $\check{z}$. By property (2.3) of the ALM index we immediately get the important identity

$$m_\alpha(\gamma\check{z}) = m_\alpha(\check{z}) - m(\gamma) \tag{2.9}$$

which holds for all γ and $\check{z}$. Notice that $m(\gamma)$ is the integer appearing in the quantum conditions (1.10)–(1.11). Clearly $m(\gamma) = 0$ when V is simply connected. One can prove that

$$m(\gamma) \text{ is even } \Leftrightarrow V \text{ is oriented}$$

(see [24]), more generally [4,13]:

$$m(\gamma) \equiv 0, \bmod 2q \Leftrightarrow V \text{ is } q\text{-oriented}$$

(see [4,18] for the definition of q-orientability; that notion is, intuitively, a measure of "how far" V is from being simply connected).

Let us now briefly discuss the *metaplectic group*. It can be constructed as follows: let s be a free linear symplectic transform. This means that if we set $(x, p) = s(x', p')$, then there exists a unique quadratic form $A = A(x, x')$ such that $\det(A_{xx'}) \neq 0$ ($A_{xx'}$ the Hessian matrix) and

$$p\,\mathrm{d}x = p'\,\mathrm{d}x' + \mathrm{d}A(x, x').$$

For f in the Schwartz space $S(\mathbf{R}^n_x)$ we then define

$$S^m_A f(x) = (2\pi\mathrm{i})^{-n/2}\Delta(A) \int \mathrm{e}^{\mathrm{i}A(x,x')} f(x')\,\mathrm{d}x', \tag{2.10}$$

where the complex number $\Delta(A)$ is given by

$$\Delta(A) = \mathrm{i}^m\sqrt{|\det(A_{xx'})|}, \quad m\pi = \arg\det(A_{xx'}).$$

The operators S^m_A are essentially Fourier transforms, and thus continuous automorphisms of $S(\mathbf{R}^n_x)$ which extend into unitary automorphisms of $L^2(\mathbf{R}^n_x)$. The inverse of the operator S^m_A is easily seen to be

$$(S^m_A)^{-1} = S^{m^*}_{A^*}, \quad A^*(x, x') = -A(x', x), \quad m^* = n - m.$$

The group generated by the S^m_A is by definition the metaplectic group $\mathrm{Mp}(n)$; it is a connected (nonclassical) Lie group with same dimension $n(2n+1)$ as $\mathrm{Sp}(n)$. In fact, $\mathrm{Mp}(n)$ is a realization as a group of unitary operators of the twofold cover $\mathrm{Sp}_2(n)$ of $\mathrm{Sp}(n)$; the covering projection

$$\pi : \mathrm{Mp}(n) \to \mathrm{Sp}(n)$$

is simply the mapping which to each S_A^m associates the free symplectic transformation s; this property suffices to define the projection $\pi(S)$ of any element S of $\mathrm{Mp}(n)$ since π is an epimorphism and S is a product of operators S_A^m (in fact exactly *two*; see [18], Chapter I). In [9] we proved the following results:

(1) For every $\ell_\alpha \in \mathrm{Lag}(n)$ there exists a unique mapping $m_\alpha : \mathrm{Mp}(n) \to \mathbf{Z}_4$ having the two following properties which uniquely characterize m_α:

$$m_\alpha(SS') = m_\alpha(S) + m_\alpha(S') - \mathrm{Inert}(\ell_\alpha, s\ell_\alpha, s'\ell_\alpha), \tag{2.11}$$

where s and s' are the projections on $\mathrm{Sp}(n)$ of $S, S' \in \mathrm{Mp}(n)$, and

$$m_\alpha \text{ is locally constant on the set } \{S\colon s\ell_\alpha \cap \ell_\alpha = 0\}. \tag{2.12}$$

We have called m_α the Maslov index *relative to the Lagrangian plane* ℓ_α.

(2) Moreover, for the choice $\ell_\alpha = 0 \times \mathbf{R}_p^n$ we have

$$m_0(S_A^m) = m \pmod 4. \tag{2.13}$$

It turns out that the *Maslov indices* m_α are related to the ALM index by the following crucial formula:

$$m_\alpha(S) = m(s_\infty \ell_{\alpha,\infty}, \ell_{\alpha,\infty}) \pmod 4, \tag{2.14}$$

where $s_\infty \in \mathrm{Sp}_\infty(n)$ (respectively, $\ell_{\alpha,\infty} \in \mathrm{Lag}_\infty(n)$) has projection $s \in \mathrm{Sp}(n)$ (respectively, $\ell_\alpha \in \mathrm{Lag}(n)$). Moreover the ALM index modulo 4 can be reconstructed using only the properties of the Maslov indices on $\mathrm{Mp}(n)$. Property (2.14) shows that if we define the pull-back by $S \in \mathrm{Mp}(n)$ of a Lagrangian half-form (1.5) via the formula

$$S^* U_{\ell_\alpha}^\rho = \mathrm{i}^{m(s_\infty \ell_{\alpha,\infty}, \ell(\check z))} e^{\frac{\mathrm{i}}{\varepsilon}\varphi(\check z)} s^* \rho(z), \tag{2.15}$$

where s_∞ is any element of $\mathrm{Sp}_\infty(n)$ with same projection s on $\mathrm{Sp}(n)$ as S, then

$$S^* U_{\ell_\alpha}^\rho = \mathrm{i}^{m_\alpha(S)} U_{s\ell_\alpha}^{s^*\rho}. \tag{2.16}$$

This formula is crucial because it is the key to the study of the action of Hamiltonian flows on Lagrangian catalogues of Sections 3 and 5. We notice that in view of the properties of the ALM and Maslov indices the pull-back satisfies

$$(SS'^{*})U^{\rho}_{\ell_\alpha} = S'^{*}S^{*}U^{\rho}_{\ell_\alpha} \tag{2.17}$$

for all S, S' in $\mathrm{Mp}(n)$; if one defines the action of a one-parameter subgroup (S_t) of $\mathrm{Mp}(n)$ on a Lagrangian half-form by

$$S_t U^{\rho}_{\ell_\alpha} = (S_{-t})^{*}U^{\rho}_{\ell_\alpha}, \tag{2.18}$$

then (2.17) will guarantee that we have

$$S_{t+t'}U^{\rho}_{\ell_\alpha} = S_t\left(S_{t'}U^{\rho}_{\ell_\alpha}\right) \tag{2.19}$$

for all t, t'.

3. THE CLASSICAL EVOLUTION OF LAGRANGIAN HALF-FORMS

This section precises and complements the results in [14], §6.

Let H be a smooth function $Z(n) \times \mathbf{R}_t \to \mathbf{R}$; we assume for simplicity that the time-dependent flow $(F_{t,t'})$ associated with H is defined for all values of t and t'. Let U^ρ be a Lagrangian catalogue (LC for short) on a Lagrangian manifold V; we define $F_t U^\rho = F_{t,0}U^\rho$ by:

$$F_t U^{\rho}_{\alpha}(\check{z}\varepsilon) = \mathrm{e}^{\frac{i}{\varepsilon}\varphi(\check{z},t)}\mathrm{i}^{m_\alpha(\check{z},t)}(F_t)_{*}\rho, \tag{3.1}$$

where the phase $\varphi(.,t)$ of $V_t = F_t V$ is given the mapping $\check{V} \to V$ ($\check{V}$ the universal covering of V) defined by the formula

$$\varphi(\check{z},t) = \varphi(\check{z}) + \int_{z}^{z(t)} p\,\mathrm{d}q - H\,\mathrm{d}s, \tag{3.2}$$

where the integration is performed along the trajectory going from the point $z \in V$ to $z(t) = F_t z \in V_t$. One verifies, using the properties [1] of the Poincaré invariant $p\,\mathrm{d}x - H\,\mathrm{d}t$ that $\varphi(.,t)$ is a phase of V_t in the sense of (1.6); the universal covering of V_t is identified with that of V by the projection

$$\pi_t : \check{V} \xrightarrow{\pi} V \xrightarrow{F_t} V_t.$$

The integer $m_\alpha(\check{z}, t)$ is expressed in terms of the ALM index by:

$$m_\alpha(\check{z}, t) = m(\ell_{\alpha,\infty}, s_{t,\infty}(z)\ell_\infty(\check{z})), \tag{3.3}$$

where $s_{t,\infty}(z) \in \mathrm{Sp}_\infty(n)$ is defined as follows: for every $z \in V$ the tangent mapping $s_t(z) = \mathrm{d}_z F_t$ is a symplectic mapping; $t \to s_{t,\infty}(z)$ is then the unique continuous mapping through the identity of $\mathrm{Sp}_\infty(n)$ at time $t = 0$ that covers the mapping $t \to s_{t,\infty}(z)$. Notice that

$$m_\alpha(\check{z}, 0) = m_\alpha(\check{z}),$$

where the integer $m_\alpha(\check{z})$ is defined by (2.6). Clearly formula (3.1) defines $F_t U_\alpha^\rho$ as a LC on V_t equipped with the phase (3.2), for in view of the $\mathrm{Sp}_\infty(n)$-invariance property (2.4) of the ALM index we have

$$m_\alpha(\check{z}, t) = m(s_{t,\infty}^{-1}(z)\ell_{\alpha,\infty}, \ell_\infty(\check{z})) \tag{3.4}$$

so that (3.1) actually defines a Lagrangian half-form (and hence a LC) on V_t; we will denote that page by $U_\alpha^\rho(., t)$ and the corresponding LC by the symbol $U^\rho(., t)$. It turns out that the mapping

$$F_t : \mathrm{Cat}(V) \to \mathrm{Cat}(V_t)$$

defined by (3.1) is an isomorphism in the following sense: suppose we *choose* the phase $\varphi(., t)$ of V_t as being given by (3.2). Then, if

$$U_\beta^\mu(\check{z}, t) = \mathrm{e}^{\frac{\mathrm{i}}{\varepsilon}\varphi(\check{z},t)}\mathrm{i}^{m_\beta(\check{z})}\mu(z)$$

for some half-density μ on V_t, we can determine *uniquely* U_α^ρ such that $F_t U_\alpha^\rho = U$ by choosing $\ell_{\alpha,\infty} = s_{t,\infty}\ell_{\beta,\infty}$ and $\rho = (F_t)^*\mu$. We will denote the LC on V thus defined by $(F_t)^{-1}U^\mu$. For $U \in \mathrm{Cat}(V_{t'})$ and arbitrary t, t'' we define:

$$F_{t,t'}U(\check{z}, t') = F_t(F_{t'})^{-1}U(\check{z}, t').$$

In view of the discussion here above it is clear that the law of Chapman–Kolmogorov

$$F_{t,t'}F_{t',t''}U = F_{t,t''}U \tag{3.5}$$

holds for all t, t', t'' and $U \in \mathrm{Cat}(V_{t''})$. In fact, we have a somewhat stronger result which shows that (3.5) remains valid for the pages of U, namely that

$$F_{t,t'}F_{t',t''}U_\alpha^\rho = F_{t',t''}U_\alpha^\rho. \tag{3.6}$$

This property is an immediate extension of the corresponding result for time-independent Hamiltonian flows (see (2.16)) and is proven in the same way.

Recall from the Introduction that the Lagrangian manifold V is *quantized* if it satisfies the "quantum condition":

$$\frac{1}{2\pi\hbar}\int_{\gamma} p\,\mathrm{d}x - \frac{1}{4}m(\gamma) \in \mathbf{Z} \tag{3.7}$$

for all $\gamma \in \pi_1(V, z_0)$; the integer $m(\gamma)$ was defined in Section 2. The interest of condition (3.7) is that if it is satisfied by V then it allows us to define Lagrangian half-forms (and hence Lagrangian catalogues) as objects on the manifold V itself. We begin with a local study. Assume that $\mathrm{Supp}(\rho)$ is compact and does not intersect the caustic Σ. There exists an open set Ω in V containing $\mathrm{Supp}(\rho)$ and such that $\Omega \cap \Sigma = \emptyset$. Then (Ω, f), f the projection $(x, p) \to x$, is a local chart of V. Choose now $\ell_\alpha = \ell_0$ (the vertical plane $0 \times \mathbf{R}^n_p$); we have

$$U_0^{\rho}(\check{z}) = \mathrm{e}^{\frac{\mathrm{i}}{\varepsilon}\varphi(\check{z})}\mathrm{i}^{m_\alpha(\check{z})} f^*\big(a|\mathrm{d}x|^{1/2}\big)$$

for some $a \in C_0^\infty(f(\Omega), \mathbf{R})$. Suppose now V is quantized. Then, by (2.3) together with the obvious identity

$$\varphi(\gamma\check{z}) = \varphi(\check{z}) + \int_{\gamma} p\,\mathrm{d}x, \qquad \gamma \in \pi_1(V, z_0),$$

it follows that

$$U_\alpha^{\rho}(\gamma\check{z}, \varepsilon) = U_\alpha^{\rho}(\check{z}, \varepsilon) \quad \text{for all } \gamma \in \pi_1(V, z_0)$$

and hence $U_\alpha^{\rho}(\check{z}, \hbar)$ is defined on V itself rather than on $\check{V}$. In view of (2.5), $m_\alpha(\check{z})$ is moreover constant for $\check{z}$ in any of the decks of $\pi^{-1}(\Omega)$ (assuming Ω small enough). It follows that if σ is *any* section of $\pi : \check{V} \to V$ over Ω, then $U_\alpha^{\rho}(\check{z}, \hbar) = U_\alpha^{\rho}(z, \hbar)$ has the local expression

$$\Psi_\Omega^\hbar(x) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Phi_\sigma(x)}\mathrm{i}^{m_{0,\sigma}} a(x)|\mathrm{d}x|^{1/2}, \quad x \in f(\Omega) \tag{3.8}$$

for *any* pair (Φ_σ, m_σ) defined by

$$\Phi_\sigma = f_*\sigma^*\varphi, \qquad m_{0,\sigma} = f_*\sigma^* m_0. \tag{3.9}$$

Let Σ_t be the caustic of $V_t = F_t V$. It is the set defined by

$$\begin{aligned}\Sigma_t &= \{z(t) \in V_t\colon\ \ell(z(t)) \cap \ell_0 = 0\}\\ &= \{z(t) \in V_t\colon\ \ell(z) \cap s_{-t}\ell_0 = 0\}\end{aligned}$$

and suppose that

$$F_{t'}\Omega \cap \Sigma_{t'} = \emptyset \quad \text{for } |t'| \leqslant t. \tag{3.10}$$

Then the mapping $x \to x(t)$ is a diffeomorphism $f(\Omega) \to f(\Omega_t)$; call it X_t. Let us calculate the local expression of $F_t U_0^\rho$ in the chart $(\Omega_t, f_t) = (F_t\Omega, f)$. In view of the expression (3.2) of the phase of $F_t U_0^\rho$ we can write, recalling that $\check{V}$ is identified with the universal covering of V_t using the projection $\pi_t = F_t \circ \pi$:

$$\varphi(\check{z}, t) = \varphi\left(\sigma_t \circ f_t^{-1}(x(t)), t\right), \quad x(t) \in \Omega_t,$$

where σ_t is the section $\Omega_t \to \check{V}$, of π_t associated with the section $\sigma : \Omega \to \check{V}$, that is $\sigma_t = \sigma \circ F_{-t}$. Thus:

$$\varphi(\check{z}, t) = \varphi\left(\sigma \circ F_{-t} \circ f_t^{-1}(x(t)), t\right) = \Phi_\sigma(x) + \int_z^{z(t)} p\,\mathrm{d}x - H\,\mathrm{d}t$$

the integration being performed along the unique trajectory arriving at $z(t)$ from $z = (x, p) \in \Omega$. Notice that the latter formula can be rewritten, if one wants, in the familiar form

$$\varphi(\check{z}, t) = \Phi_\sigma(x) + \int_0^t L(x, \dot{x}, t')\,\mathrm{d}t' = \Phi_\sigma(x(t), t), \tag{3.11}$$

where $L = p\dot{x} - H$ is the Lagrangian function associated to H by the Legendre transform. Consider next the term

$$m_0(\check{z}, t) = m\left(\ell_{0,\infty}, s_{t,\infty}(z)\ell_\infty(\check{z})\right).$$

Condition (3.10) implies that for $t \neq 0$ small enough the Lagrangian plane $s_t(z)\ell(z) = \ell(F_t z)$ has not crossed $\ell_0 = 0 \times \mathbf{R}_p^n$; in view of (2.5) we thus have

$$m_0(\check{z}, t) = m_0(\check{z}) = m_{0,\sigma}. \tag{3.12}$$

Let us finally determine the local expression $(f_t)_*(F_t)_*\rho = (f_t \circ F_t)_*\rho$ of $(F_t)_*\rho$ at $x(t)$. A straightforward calculation shows that we have

$$(f_t \circ F_t)_*\rho\big(x(t)\big) = \rho(z) = f_*\rho(x) = a(x)|\mathrm{d}x|^{1/2}$$

which reads, in the local coordinate $x(t)$:

$$(f_t \circ F_t)_*\rho\big(x(t)\big) = a(x)\left|\frac{\partial x(t)}{\partial x}\right|^{-1/2} |\mathrm{d}x(t)|^{1/2}. \tag{3.13}$$

Summarizing, we have thus proven:

PROPOSITION 1. – *Let Ω be an open subset of the quantized Lagrangian manifold V; we assume that Ω does not intersect the caustic: such that $\Omega \cap \Sigma = \emptyset$. Let ρ be a half-density on V with support in Ω. Then, for Ω and t sufficiently small, the local expression of $F_t U_0^\rho|_{\varepsilon=\hbar}$ in the chart (Ω_t, f_t), $\Omega = F_t\Omega$, $f_t = f|_{\Omega_t}$ is*

$$(f_t \circ F_t)_* U_0^\rho(x(t), t, \hbar) = \Psi^\hbar(x(t), t)|\mathrm{d}x(t)|^{1/2}, \tag{3.14}$$

where the function $\Psi^\hbar : \mathbf{R}_x^n \times \mathbf{R}_t^n \to \mathbf{C}$ is given by

$$\Psi_\Omega^\hbar(x(t), t) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Phi_\sigma(x(t),t)}\mathrm{i}^{m_{0,\sigma}} a(x)\left|\det\frac{\partial x(t)}{\partial x}\right|^{-1/2}, \tag{3.15}$$

if the local expression ρ in (Ω, f) is $a(x)|\mathrm{d}x|^{1/2}$; the phase $\Phi_\sigma(\cdot, t)$ appearing in (3.15) *is given by formula* (3.11) *and the integer $m_{0,\sigma}$ by formulas* (3.9), (3.12).

To deal with the general case need the following elementary algebraic result:

LEMMA 1. – *There exists an atlas $(\Omega_j, f_j)_j$ of V having the following properties: for all indices j for which $\Omega_j \cap \Sigma = \emptyset$ the diffeomorphisms f_j are given by $f_j = f|_{\Omega_j}$, and:*

(i) *If σ_j, σ_k are two sections of $\pi : \check{V} \to V$ over Ω_j, Ω_k respectively, and if $\Omega_j \cap \Omega_k \neq \emptyset$, then there exists $\gamma_{jk} \in \pi_1(V, z_0)$ such that $\sigma_j z = \gamma_{jk}\sigma_k z$ for $z \in \Omega_j \in \Omega_k$;*

(ii) *if σ_i, σ_j, σ_k are three such sections with $\Omega_i \cap \Omega_j \cap \Omega_k \neq \emptyset$, then we have the relation*

$$\sigma_i z = \gamma_{ij}\gamma_{jk}\sigma_k z = \gamma_{ik}\sigma_k z \tag{3.16}$$

for all $z \in \Omega_i \cap \Omega_j \cap \Omega_k$.

Proof. – It is straightforward; see, for instance, [14], Lemma 5.1 and Theorem 5.3. □

Lemma 1 allows us to prove the following sheaf property in the quantized case:

PROPOSITION 2. – *Let V be a quantized Lagrangian manifold and $(\Omega_j, f_j)_j$ an atlas satisfying the conditions* (i), (ii) *in Lemma* 1. *Let U^ρ be a Lagrangian catalogue on V. Then the local expressions* (3.14) *of $F_t U_0^\rho|_{\varepsilon=\hbar}$ in two overlapping charts $(\Omega_{j,t}, f_{j,t})$, $(\Omega_{k,t}, f_{k,t})$ of the Lagrangian manifold $V_t = F_t\Omega$ coincide; in fact*

$$\Psi_j^\hbar(x,t) = \Psi_k^\hbar(x,t) \quad \textit{for } x \in \Omega_j \cap \Omega_k, \tag{3.17}$$

where $\Psi_j^\hbar$, $\Psi_k^\hbar$ are the functions (3.15) *on, respectively, Ω_j and Ω_k.*

Proof. – We have, by formulas (3.11) and (3.9)

$$\Phi_{\sigma_j}(x(t),t) = \varphi(\sigma_j z) + \int_0^t L(x,\dot{x},t')\,\mathrm{d}t$$

for $x \in \Omega_j \cap \Omega_k$, and hence in view of Lemma 1(i):

$$\Phi_{\sigma_j}(x,t) = \Phi_{\sigma_k}(x,t) + \int_{\gamma_{jk}} p\,\mathrm{d}x.$$

By a similar argument we have the equality

$$m_{0,\sigma_j} = m_{0,\sigma_k} - m(\gamma_{jk})$$

and hence,

$$\Psi_j^\hbar(x,t) = \Psi_k^\hbar(x,t) \quad \text{for } x \in f(\Omega_j \cap \Omega_k),$$

since V satisfies the quantization condition (3.7). This proves Proposition 2. □

4. THE RELATION WITH SEMI-CLASSICAL APPROXIMATION

We briefly recall the main result of the theory of asymptotic solutions to the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi = H\left(x, -\mathrm{i}\hbar\frac{\partial}{\partial x}\right)\Psi \tag{4.1}$$

associated with a *time-independent* Hamiltonian

$$H(x, p) = \frac{p^2}{2} + U(x). \tag{4.2}$$

Here $U(x)$ is a smooth real valued function of $x = (x_1, \ldots, x_n)$ and p^2 is the square of the length of $p = (p_1, \ldots, p_n)$. Maslov and Fedoriuk [22], §12, prove that if one chooses an initial condition

$$\Psi_0(x) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Phi_0(x)}a(x), \tag{4.3}$$

where Φ_0 and a are smooth and real valued, a with compact support, then an approximate "modulo $\hbar$" solution Ψ to (4.1) is given by the formula

$$\Psi_{\text{approx}}(x, t) = \sum_j a(x_j)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Phi_j(x,t)}\mathrm{i}^{-\mu_j}\left|\det\frac{\partial x_j(t)}{\partial x}\right|^{-1/2} \tag{4.4}$$

as long as (x, t) is not a "focal point" (i.e., x is not the projection of a point of the caustic). The functions $S_j(x, t)$ are given by

$$S_j(x, t) = S_0(x_j) + \int_0^t L\big(x, \dot{x}, t'\big)\,\mathrm{d}t' \tag{4.5}$$

(L the Lagrangian function associated with H) and the x_j are defined as follows: consider a trajectory $s \to (x(s), p(s))$ of the Hamilton system (1.2). Since (x, t) is non-focal there exists a *finite* number of points (x_j, p_j) such that

$$\big(x(0), p(0)\big) = (x_j, p_j), \qquad \big(x(t), p(t)\big) = (x, p)$$

for some p. The μ_j appearing as exponents in (4.4) are the so-called "Morse indices" of the trajectories from x_j to x; see [1], App. 11. We claim that:

THEOREM 1. – *Let V be the Lagrangian manifold with equation $p = \partial\Phi_0/\partial x$ and ρ the half-density on V whose local expression in $\mathbf{R}^n_x$ is $a(x)|\mathrm{d}x|^{1/2}$. Then:*

(1) *For small t, i.e., as long as V_t remains a graph, the approximate solution is given by*

$$\Psi_{\text{approx}}(x(t),t) = \Psi^{\hbar}_V(x(t),t), \tag{4.6}$$

where $\Psi^{\hbar}_V$ is given by (3.15) *with $\Omega = V$;*

(2) *For arbitrary t such that $x(t)$ is not the projection of a point on the caustic Σ_t of V_t,* (4.6) *must be replaced by the formula*

$$\Psi_{\text{approx}}(x(t),t) = \sum_j \Psi^{\hbar}_{\Omega_j}(x(t),t), \tag{4.7}$$

where the sum on the right is calculated for all disjoint neighborhoods Ω_j of points $z_j = (x, p_j) \in V$ for some p_j.

Proof. – Everything readily follows from Propositions 1 and 2 noting that the Morse indices $-\mu_j$ in Maslov's formula (4.4) can be identified with the integers m_{0,σ_j} associated with sections σ_j over Ω_j in view of [22], §7, and properties (2.5) and [18], Theorem 6, §2.6. □

5. THE QUANTUM EVOLUTION OF LAGRANGIAN HALF-FORMS

In all what follows ε will again be a *variable* parameter > 0. We assume that the Hamiltonian H is of the type (4.2), but we now allow the potential U to depend on time:

$$H(x,p,t) = \frac{p^2}{2} + U(x,t). \tag{5.1}$$

We also assume that the solutions (x, p) to Hamilton's equations of motion (1.2) exist for all time and all initial data $(x(0), p(0))$. We consider the partial differential equation

$$\mathrm{i}\varepsilon\frac{\partial}{\partial t}\Psi = -\frac{\varepsilon^2}{2}\Delta\Psi + U\Psi \tag{5.2}$$

and look for solutions of the type

$$\Psi(x,t) = \mathrm{e}^{\frac{\mathrm{i}}{\varepsilon}\Phi(x,t)}a(x,t).$$

If we impose, as is usual in quasiclassical approximation methods, that the phase Φ is a real C^∞ function satisfying Hamilton–Jacobi's equation

$$\frac{\partial \Phi}{\partial t} + H\left(x, \frac{\partial \Phi}{\partial x}\right) = 0, \quad \Phi(x,0) = \Phi_0(x)$$

then a straightforward calculation shows that the amplitude $a(x,t)$ must satisfy the "transport equation"

$$\frac{\partial a}{\partial t} + \left\langle \frac{\partial a}{\partial x}, \frac{\partial \Phi}{\partial x} \right\rangle = \frac{\mathrm{i}\varepsilon}{2}\left[a\left\langle \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial x} \right\rangle + \Delta a\right],$$

which is then asymptotically solved by expanding a as a formal series

$$a(x,t,\varepsilon) = \sum_{j \geqslant 0} a_j(x,t)\varepsilon^j,$$

where the coefficients a_j are independent of ε, and to write down the equations satisfied by these functions. Of course, this method only yields asymptotic solutions for small t, more precisely as long as the Lagrangian manifolds $V_t = F_t V$ ($F_t = F_{t,0}$, the time-dependent flow associated with H) admit diffeomorphic projections on $\mathbf{R}^n_q$. For large t, V_t will eventually "bend" so that the projection $(x,p) \to x$ no longer in general is a diffeomorphism, i.e., caustics will appear. Global asymptotic solutions can, however, still be obtained by Leray's methods [18,19] or by Maslov's canonical operator method [21–23].

Suppose now we instead look for *exact* solutions Ψ to the Eq. (5.2); we write Ψ in polar form

$$\Psi(x,t) = \mathrm{e}^{\frac{\mathrm{i}}{\varepsilon}S(x,t)}R(x,t), \quad R \geqslant 0, \tag{5.3}$$

where *both* real functions S and R are now allowed to depend on the variable ε. If we assume that R vanishes to infinite order when it vanishes, one shows [3,17] that S and R must satisfy the system of partial differential equations

$$\begin{cases} \dfrac{\partial S}{\partial t} + H\left(x, \dfrac{\partial S}{\partial x}, t\right) - \dfrac{\varepsilon^2}{2}\dfrac{\Delta R}{R} = 0, \\ \dfrac{\partial R^2}{\partial t} + \dfrac{\partial}{\partial x}\left(R^2 \dfrac{\partial S}{\partial x}\right) = 0, \end{cases} \tag{5.4}$$

which is rigorously equivalent to Eq. (5.2). Introducing, following Bohm [3,17], the "quantum potential"

$$Q = -\frac{\varepsilon}{2}\frac{\Delta R}{R} \tag{5.5}$$

the first Eq. (5.4) can be viewed as the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H^\rho\left(x, \frac{\partial S}{\partial x}, t\right) = 0 \tag{5.6}$$

for the modified Hamiltonian function H^ρ (sometimes called "Bohmian" in the literature) associated with H and the quantity $\rho = R^2$:

$$H^\rho = H + Q. \tag{5.7}$$

Notice that all questions relative to the *existence* of the solution to the first Eq. (5.4) are here eliminated since by hypothesis S is already defined *globally*!

It is important to notice that Eq. (5.6) *does not* in general reduce to the usual Hamilton–Jacobi Eq. (5.3) "at the limit $\varepsilon \to 0$". This is because ρ itself, and hence also $\Delta\rho/\rho$, usually depends on ε and does not in general tend to 0 with ε (see [17] for examples); we will return to this point at the end of this section. We now apply the results of Section 3, not to H, but rather to the Bohmian H^ρ. For given Ψ, we can consider the time-dependent flow $F_t^\rho = F_{t,0}^\rho$ defined by the Hamilton equations

$$\frac{\mathrm{d}x^\rho}{\mathrm{d}t} = \frac{\partial H^\rho}{\partial p}, \qquad \frac{\mathrm{d}p^\rho}{\mathrm{d}t} = -\frac{\partial H^\rho}{\partial x}. \tag{5.8}$$

We denote by V the Lagrangian manifold with equation

$$p = \frac{\partial S_0}{\partial x}(x), \qquad x \in \mathbf{R}^n, \tag{5.9}$$

where we have set $S_0(x) = S(x, 0)$. Define the half-density ρ on V by

$$\rho = f^*\left(R_0|\mathrm{d}x|^{1/2}\right) \tag{5.10}$$

with $R_0(x) = R(x, 0)$ and consider the Lagrangian manifold $V_t^\rho = F_t^\rho V$. To prove Theorem 2 below we need the two following lemmas:

LEMMA 2. – *The phase of* V_t^ρ *is given by the formula*

$$\varphi(\check{z},t) = S(x^\rho(t),t) \quad \text{if } \pi(\check{z}) = (x,p)$$

and $(x^\rho(t), p^\rho(t)) = F_t^\rho(x,p)$.

Proof. – Set $X = x^\rho(t)$, $P = p^\rho(t)$; in these variables we have in view of (5.6)

$$P = \frac{\partial S}{\partial X}(x,t), \qquad \frac{\partial S}{\partial t}(X,t) + H^\rho(X,P,t) = 0,$$

and hence

$$\mathrm{d}S(X,t) = P\,\mathrm{d}X - H^\rho(X,P,t)\,\mathrm{d}t.$$

On, the other hand, differentiating the expression (3.2) of $\varphi(\check{z},t)$ with respect to X and Y one also gets

$$\mathrm{d}\varphi(\check{z},t) = P\,\mathrm{d}X - H^\rho(X,P,t).$$

The lemma follows since $\varphi(\check{z},t) = S(x,t)$. □

LEMMA 3. – *The index* $m_0(\check{z},t)$ *defined in* (3.3)–(3.4) *is constant and equal to* 0.

Proof. – V_t^ρ is a graph for all t and there are thus no caustics. This implies that $m_0(\check{z},t)$ is constant and equal to $m_0(\check{z}) = 0$. □

We are now able to prove the following essential result which clearly shows the relationship between wave-functions of quantum mechanics, half-densities, and "Bohmian trajectories":

THEOREM 2. – *Let* V *and* ρ *be given by* (5.9) *and* (5.10), *respectively. Then the local expression* (3.14) *of* $F_t^\rho U_0^\rho$ *in the global chart* (V, f_t) *where* $f_t : (x,p) \to x$, *is given by the formula*

$$\Psi(x^\rho(t),t)\left|\mathrm{d}x^\rho(t)\right|^{1/2} = \mathrm{e}^{\frac{\mathrm{i}}{\varepsilon}W^\rho(x,w^\rho(t))}\Psi_0(x)|\mathrm{d}x|^{1/2}, \tag{5.11}$$

where W *is the integral*

$$W^\rho(x,x^\rho(t')) = \int_z^{z^\rho(t)} p\,\mathrm{d}q - H^\rho\,\mathrm{d}t \tag{5.12}$$

calculated from the point $z = (x,p) = (x, \partial S_0/\partial x)$ *to* $z^\rho(t) = F_t^\rho z$, *and* Ψ *is the solution to* (5.2) *at time* t.

Proof. – We must show that (3.15) reduces to

$$\Psi_V^\hbar\big(x^\rho(t),t\big)=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(x^\rho(t),t)}R\big(x^\rho(t),t\big). \tag{5.13}$$

S and R solutions to (5.4). In view of (3.11) and Lemma 2 hereabove we have $\Phi_\sigma(x,t)=S(x,t)$ for all x and t; taking Lemma 3 into account we thus only have to prove that

$$R(x(t),t)=R_0(x)\left|\det\left(\frac{\partial x^\rho(t)}{\partial x}\right)\right|^{-1/2}. \tag{5.14}$$

Set $P=R^2$; Eq. (5.4_2) becomes the classical continuity equation of Fluid Mechanics:

$$\frac{\partial P}{\partial t}+\frac{\partial}{\partial x}\left(P\frac{\partial S}{\partial x}\right)=0.$$

It follows from the general theory of that equation that we have

$$P\big(x^\rho(t),t\big)\det\left(\frac{\partial x^\rho(t)}{\partial x}\right)=P(x,0),$$

hence (5.13). □

Let us finally briefly discuss the relationship between classical and quantum evolution of Lagrangian catalogues. It is often claimed, both in the physical and in the mathematical literature, that one obtains an approximate solution

$$\Psi=\mathrm{e}^{\frac{\mathrm{i}}{\varepsilon}\varphi}a$$

to Schrödinger's equation by replacing Eqs. (5.4) by their classical counterparts

$$\begin{cases}\dfrac{\partial\varphi}{\partial t}+H\left(x,\dfrac{\partial\varphi}{\partial x}\right)=0,\\[2mm] \dfrac{\partial a}{\partial t}+\dfrac{\partial}{\partial x}\left(a\dfrac{\partial\varphi}{\partial x}\right)=0.\end{cases} \tag{5.15}$$

Since this procedure of "resolution" of Schrödinger's equation amounts to neglecting the "quantum potential" (5.5), it is clear that one will obtain good approximations only if $Q\to 0$ when $\varepsilon\to 0$ and this is in fact usually *not* the case (recall that R is allowed to depend on ε). Consider for instance a solution to Schrödinger's equation of the type

$$\Psi(x,t) = \mathrm{e}^{-\frac{\mathrm{i}}{\varepsilon}Et} R_0(x), \tag{5.16}$$

i.e., $S(x,t) = -Et$. The amplitude R then satisfies the usual equation

$$-\frac{\varepsilon^2}{2}\Delta R_0 + U R_0 = E R_0,$$

so that $Q = E - U$; the "Bohmian" is in that case $H^\rho = p^2/2 + E$. If for instance $H = (p^2 + x^2)/2$ (the one-dimensional harmonic oscillator) and R_0 is the Gaussian centered at the origin

$$R_0(x) = \exp\left(-\frac{x^2}{2\varepsilon}\right),$$

then Ψ will be a solution for $E = \varepsilon/2$, given by

$$\Psi(x,t) = \mathrm{e}^{-\mathrm{i}t/2} \exp\left(-\frac{x^2}{2\varepsilon}\right). \tag{5.17}$$

However, a straightforward calculation using (3.15) (respectively, (4.6)) leads to an incorrect result. The reason for this failure is the following: the amplitude R_0 depends on ε; the domain of validity of Maslov's method is that of the stationary phase method, which does not apply in the considered case; see the example (12.22) in [22] for an illustration of this fact.

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