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Band edge localization and the density of states for acoustic and electromagnetic waves in random media

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ABSTRACT. – The propagation of sound waves and electromagnetic radiation in a background medium is strongly influenced by the addition of random scatterers. We study random perturbations of a background medium which has a spectral gap in its permissible energy spectrum. We prove that the randomness localizes waves at energies near the band edges of the spectral gap of the background medium. The perturbations of the dielectric function or the sound speed are described by Anderson-type potentials, which include random displacements from the equilibrium positions which model thermal vibrations. The waves with energies near the band edges are almost-surely exponentially localized. We prove a Wegner estimate valid at

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all energies in the spectral gap of the unperturbed operator. It follows that the integrated density of states is Lipschitz continuous at energies in the unperturbed spectral gaps. © Elsevier, Paris

Key words: localization, random operators, pure point spectrum, Schrödinger operators, band gaps.

RÉSUMÉ. – Nous étudions la propagation des ondes acoustiques et électromagnétiques dans les milieux aléatoires. Le modèle est décrit par un opérateur non-perturbé avec un gap dans le spectre. Ajoutant une perturbation aléatoire, nous montrons l'existence des ondes localisées aux énergies aux bords du spectre de l'opérateur non-perturbé presque sûrement. En plus, nous donnons une démonstration de l'existence et la continuité lipschitzienne de la densité d'états intégrée, suivant une nouvelle estimation de type Wegner. © Elsevier, Paris

1. INTRODUCTION, THE MODELS, AND MAIN RESULTS

Properties of wave propagation in perturbed structures has been extensively studied for both deterministic and random perturbations. In this paper, we apply the methods developed in [4], [8], and [10] to prove localization at energies near the band edges of the unperturbed, background medium, and to study the density of states for families of random self-adjoint operators which describe the propagation of acoustic or electromagnetic waves in randomly perturbed media. These families of operators have the form

$$H_\omega = A_\omega^{-1/2} H_0 A_\omega^{-1/2}, \tag{1.1}$$

where $\{A_\omega(x)\}$ belongs to a permissible class of stochastic processes and H_0 is a (deterministic) self-adjoint operator. We assume that the spectrum of H_0 has an open spectral gap $G \equiv (B_-, B_+) \subset I\!R$ in its resolvent set. We study the propagation properties of waves in the perturbed medium with energies near the band edges $B_\pm$ of this gap. Our two main examples come from the study of wave propagation in random media:

(1) Acoustic waves. The wave equation for acoustic waves propagating in a medium with sound speed C and density ρ is

$$\partial_t^2 \psi + \hat{H}\psi = 0, \tag{1.2}$$

where the propagation operator $\hat{H}$ is given by

$$\hat{H} \equiv -C^2 \rho \nabla \rho^{-1} \nabla. \tag{1.3}$$

This operator is self–adjoint (under appropriate conditions on the coefficients C and ρ) on the Hilbert space $\mathcal{H}_0 = L^2(\boldsymbol{R}^d, (C^2\rho)^{-1} d^d x)$. We are interested in the spectral properties of this operator. It is convenient to change Hilbert spaces; by a standard unitary transformation, we consider the operator H, unitarily equivalent to $\hat{H}$, given by

$$\begin{aligned} H &\equiv -C\Delta C - \frac{1}{2}\left\{ \frac{C^2 \Delta\rho}{\rho} - \frac{3}{2}\frac{C^2 |\nabla \rho|^2}{\rho^2} \right\} \\ &= -(C^2\rho)^{\frac{1}{2}} \nabla \cdot \rho^{-1} \nabla (C^2 \rho)^{\frac{1}{2}}, \end{aligned} \tag{1.4}$$

acting on the Hilbert space $\mathcal{H} = L^2(\boldsymbol{R}^d)$, $d \geq 1$. In this paper, we concentrate on the case of media characterized by a randomly perturbed sound speed and a positive, bounded density ρ satisfying $0 < \kappa_0 < \rho^{-1} < \kappa_1 < \infty$, for some positive constants $\kappa_i, i = 0, 1$.

We consider perturbed sound speeds of the form

$$C_\omega(g) \equiv (1 + g\tilde{C_\omega})^{-1/2} C_0, \tag{1.5}$$

for $g \geq 0$. The role of the coupling constant g is to insure that for a given process $\tilde{C_\omega}$, the quantity $(1+g\tilde{C_\omega})$ is boundedly invertible. The unperturbed sound speed C_0 is assumed to be a bounded, nonnegative function. The perturbation $\tilde{C_\omega}$ is given by a bounded, Anderson-type stochastic processes, constructed below. To relate this to (1.1), we factor out the unperturbed sound speed C_0 and define the unperturbed acoustic wave propagation operator H_0 by

$$H_0 \equiv -C_0 \rho^{\frac{1}{2}} \nabla \cdot \rho^{-1} \nabla \rho^{\frac{1}{2}} C_0. \tag{1.6}$$

The coefficient A_ω appearing in (1.1) is given by

$$A_\omega \equiv (1 + g\tilde{C_\omega}). \tag{1.7}$$

(2) Electromagnetic waves. The wave equation for electro- magnetic waves can be written in the form of equation (1.2) for vector-valued functions ψ. In this case, the operator $\hat{H}$ describing the propagation of electromagnetic waves in a medium characterized by a dielectric function ϵ and a magnetic permeability $\mu = 1$ is given by

$$\hat{H} \equiv -\epsilon^{-1/2} \Delta \Pi \epsilon^{-1/2}, \tag{1.8}$$

acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^3)$. The matrix-valued operator Π is the orthogonal projection onto the subspace of transverse modes. This representation of Maxwell's equations is derived in Appendix 4. We consider random perturbations of a background medium described by a dielectric function ϵ_0 and given by

$$\epsilon_\omega(g) \equiv 1 + \epsilon_0 + g\tilde{\epsilon_\omega}, \tag{1.9}$$

where $\tilde{\epsilon_\omega}$ is a stochastic process. The unperturbed operator describing the background medium is defined by

$$H_0 \equiv -(1 + \epsilon_0)^{-1/2} \Delta \Pi (1 + \epsilon_0)^{-1/2}, \tag{1.10}$$

and the coefficient A_ω in (1.1) is given by

$$A_\omega \equiv (1 + g(1 + \epsilon_0)^{-1} \tilde{\epsilon_\omega}). \tag{1.11}$$

We note that A_ω is the velocity of light for the realization ω. The electromagnetic case is of interest because of the intense research going on in this field with the aim of finding experimental evidence for localization. In the case of electrons, the repulsion between electrons can cause effects which tend to obscure disorder-induced localization. These effects are not present for photons. For a review of these questions, see [6].

Whereas many of the results below hold for a general, non-negative, self-adjoint operator H_0, we will take H_0 to be of either of the two forms given above. Note that H_0 need not be a periodic operator. We simply require that its spectrum contain an open spectral gap (there are other technical assumptions on H_0 given below). Unlike the Schrödinger case, zero is always the bottom of the spectrum for multiplicatively perturbed operators and, as we discuss below, no localization is possible near this band edge.

Our processes will be of Anderson-type which are constructed as follows. As in [8], we introduce two classes of stochastic processes indexed by $\mathbb{Z}^d$. Let $\{\lambda_n(\omega) | n \in \mathbb{Z}^d\}$ be a family of independent, identically distributed random variables (*iid*) with a common distribution $h(\lambda)\, d\lambda$. We will assume that the density h has bounded support (see, however, the remark after Theorem 1.2 below). The precise conditions are given below. Let $\{\xi_i(\omega) | i \in \mathbb{Z}^d\}$ be another family of vector-valued, *iid*, random variables with $\xi_i(\omega) \in B_R(0)$, $0 < R < \frac{1}{2}$. We assume that the random variable ξ_i has an absolutely continuous distribution, for example, a uniform distribution. These random variables will model thermal fluctuations of the scatterers with random strengths about the lattice points $\mathbb{Z}^d$. We will denote by $\Lambda_l(x)$ a cube of side l centered at x,

$$\Lambda_l(x) \equiv \{y \in \boldsymbol{R}^d | \forall 1 \leq i \leq d, \ |y_i - x_i| \leq l/2\},$$

and by $\chi_{\Lambda_l(x)}$ the characteristic function for $\Lambda_l(x)$. When $x = 0$, we will write Λ_ℓ for simplicity. We denote by $u \geq 0$ a single-site potential function.

The random perturbation of the sound speed $\tilde{C}_\omega$ and the dielectric function $\tilde{\epsilon}_\omega$ are taken to be Anderson-type. The Anderson-type perturbation is defined by

$$\tilde{C}_{\omega,\omega'}(x), \quad \tilde{\epsilon}_{\omega,\omega'}(x) = \sum_{i \in \boldsymbol{Z}^d} \lambda_i(\omega) u(x - i - \xi_i(\omega')), \tag{1.12}$$

for a single-site potential $u > 0$ satisfying the conditions below.

In consideration of the form of the modification to the sound speed given in (1.7), and the modification of the dielectric function given in (1.11), we unify the notation as follows. We define an effective "potential" $V^X_{\omega,\omega'}$, for $X = A$ (acoustic) and $X = M$ (Maxwell) by

$$V^A_{\omega,\omega'} \equiv \tilde{C}_{\omega,\omega'}, \tag{1.13}$$

in the acoustic case, and by

$$V^M_{\omega,\omega'} \equiv (1 + \epsilon_0)^{-1} \tilde{\epsilon}_{\omega,\omega'}, \tag{1.14}$$

in the Maxwell case. We define an operator $A^X_{\omega,\omega'}$ by

$$A^X_{\omega,\omega'} \equiv \{1 + g V^X_{\omega,\omega'}\}. \tag{1.15}$$

The family of random operators can now be written

$$H^X_{\omega,\omega'} = (A^X_{\omega,\omega'})^{-1/2} H_0 (A^X_{\omega,\omega'})^{-1/2}, \tag{1.16}$$

where the unperturbed operator H_0 is given in (1.6) for $X = A$ and in (1.10) for $X = M$.

We next turn to a description of the assumptions on the random processes and on the unperturbed operator H_0. Let us recall that the unperturbed operator H_0 has the form

$$H^A_0 = -C_0 \rho^{1/2} \nabla \cdot \rho^{-1} \nabla \rho^{1/2} C_0, \tag{1.17}$$

in the acoustic case, and

$$H_0^M = -(1+\epsilon_0)^{-1/2}\Pi\Delta(1+\epsilon_0)^{-1/2}, \tag{1.18}$$

in the electromagnetic case. When a result concerning H_0^X does not depend on whether it is for the acoustic case or Maxwell case, we will not distinguish between the unperturbed operators and simply write H_0.

(H1) The operator H_0^X is essentially self-adjoint on $C_0^\infty(I\!R^d)$, for $X = A$, and on $C_0^\infty(I\!R^3, C\!\!\!\!I^3)$, for $X = M$. This is a condition on the unperturbed medium described by the functions C_0 and ϵ_0.

(H2) The spectrum of H_0, $\sigma(H_0)$, is semibounded and has an open gap G. That is, there exist finite constants Σ_0, B_-, B_+ such that $\inf \sigma(H_0) = \Sigma_0$ and $\Sigma_0 \leq B_- < B_+ \leq \infty$ such that

$$\sigma(H_0) \subset [\Sigma_0, B_-] \cup [B_+, \infty).$$

Remark. – For H_0 of the form (1.17) or (1.18), we show below that $\Sigma_0 = 0$. However, this is not essential for the results.

(H3) The operator H_0 is strongly locally compact in the sense that for any $f \in L^\infty(I\!R^d)$, for $X = A$, or for any $f \in L^\infty(I\!R^3, C\!\!\!\!I^3)$, for $X = M$, with compact support, the operator $f(H_0+M_0+1)^{-1} \in \mathcal{J}_q$, for some integer q, $1 \leq q < \infty$, and for some constant M_0.

(H4) Let $\rho(x) \equiv (1 + \|x\|^2)^{1/2}$. The operator

$$H_0(\alpha) \equiv e^{i\alpha\rho} H_0 e^{-i\alpha\rho},$$

defined for $\alpha \in I\!R$, admits an analytic continuation as a type-A analytic family to a strip

$$S(\alpha_0) \equiv \{x + iy \in C\!\!\!\!I \mid |y| < \alpha_0\},$$

for some $\alpha_0 > 0$.

Remark. – We note that if the functions C_0 and ϵ_0 are bounded and in $C^2(I\!R^d)$ ($d = 3$ for $X = M$), then the conditions (H1) and (H3)-(H4) hold. Condition (H4) is verified in Appendix 3, for a very general class of unperturbed operators of the form (1.15)–(1.16), as in [4].

Statements regarding relative compactness in the Maxwell case are to be understood for the unperturbed Hamiltonian $(1 - P_0)H_0^M$, where P_0 is the projection onto the zero eigenvalue subspace.

We now enumerate the conditions on the random coupling constants λ_i.

(H5) The coupling constants $\{\lambda_i(\omega) \mid i \in \mathbb{Z}^d\}$ form a family of independent, identically distributed (iid) random variables. The common distribution has a density h satisfying $0 \leq h \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$. There exist finite, positive constants $0 < m, M$ such that $\operatorname{supp} h \subset [-m, M]$ and $h(0) > 0$.

(H6) The density h decays sufficiently rapidly near $-m$ and near M in the following sense:

$$0 < \mathbb{P}\{|\lambda + m| < \varepsilon\} \leq \varepsilon^{3d/2+\beta},$$

$$0 < \mathbb{P}\{|\lambda - M| < \varepsilon\} \leq \varepsilon^{3d/2+\beta},$$

for some $\beta > 0$.

Remarks.

1. It is important to note that for a given density h as in Condition (H6), we can take the coupling constant g small enough so that $A^X_{\omega,\omega'}$, given in (1.15), is invertible. This is the only reason we insert a coupling constant g. In particular, once g is chosen so that $A^X_{\omega,\omega'}$ is invertible, it does not have to be adjusted again to prove localization.
2. The hypothesis (H6) can be modified to allow densities with support $[-m, 0]$ or $[0, M]$. In these cases, we only require decay of the density h as $\lambda \to -m$ or $\lambda \to M$, respectively. In the former case, we will prove localization near the lower gap edge B_- (provided it is not equal to 0), whereas in the latter case, localization will be established near the upper gap edge B_+.

We take $\Omega \equiv [\operatorname{supp} h]^{\mathbb{Z}^d}$ to be the probability space equipped with the probability measure $\mathbb{P}$ induced by the finite product measure. The single-site potential u in (1.12) is assumed to satisfy the following condition.

(H7) The single-site potential u has compact support and $0 \leq u \leq 1$.

The fixed sign assumption on the single-site potential u is necessary in the proof of spectral averaging given in section 4. Specifically, it guarantees that the real part of the derivative in (4.6) has a fixed sign.

We need to assume the existence of deterministic spectrum Σ for families of multiplicatively perturbed operators as in (1.1) with H_0 satisfying (H1)-(H7). If H_0 is periodic with respect to the translation group $\mathbb{Z}^d$, and for Anderson-type perturbations described above, the random families of operators $H^X_{\omega,\omega'}$ are measurable, self-adjoint, and $\mathbb{Z}^d$-ergodic (see Appendix 1). In this case, it is known (cf. [34]) that the spectrum of the family is deterministic. That is, there exists a closed subset $\Sigma \subset \mathbb{R}$ such that $\sigma(H^X_{\omega,\omega'}) = \Sigma$ almost surely. We also need to consider the nature of Σ near the unperturbed spectral gap $G = (B_-, B_+)$.

(H8) There exists constants $B'_\pm$ satisfying $B_- < B'_- < B'_+ < B_+$ such that

$$\Sigma \cap \{(B_-, B'_-) \cup (B'_+, B_+)\} \neq \emptyset .$$

In light of hypothesis (H8), we define the band edges of the almost sure spectrum Σ near the gap G, as follows:

$$\tilde{B_-} \equiv \sup\{E \leq B'_- \mid E \in \Sigma\},$$

and

$$\tilde{B_+} \equiv \inf\{E \geq B'_+ \mid E \in \Sigma\},$$

Let us give a simple example for which these hypotheses hold. We will give further examples at the end of section 4 of operators H_0 for which hypothesis (H8) holds. The question of the existence of open gaps for the background operator H_0, hypothesis (H2), is discussed at the end of Appendix 1. Let us assume that H_0 is a $\mathbb{Z}^d$ periodic operator with an open spectral gap $G = (B_-, B_+)$, as in (H2). Let the single-site potential $u = \chi_{\Lambda_1(0)}$ be the characteristic function on the unit cube. It is easy to check that $E \in \sigma(H_\omega)$ if and only if $0 \in \sigma(H_0 - EA_\omega)$. The almost sure spectrum $\tilde{\Sigma}(E)$ of the $\mathbb{Z}^d$–ergodic Schrödinger operator $H_0 - EA_\omega$ can be described precisely:

$$\tilde{\Sigma}(E) = \cup_{\lambda \in [-m,M]} \sigma(H_0 - E(1+\lambda)).$$

(The proof of this result is basically the same as for Schrödinger operators, cf. [28] and section 1 of [29].) It is now easy to check that the almost sure spectrum Σ of H_ω fills the intervals $[B_-, (1-m)^{-1}B_-]$ and $[(1+M)^{-1}B_+, B_+]$. By adjusting m and M, we see that Σ has an open spectral gap as in (H8) with $\tilde{B}_- \equiv (1-m)^{-1}B_-$ and $\tilde{B}_+ \equiv (1+M)^{-1}B_+$.

The main results are the following two theorems.

THEOREM 1.1. – *Assume (H1)-(H8). There exist constants $E_\pm$ satisfying $B_- \leq E_- < \widetilde{B}_-$ and $\widetilde{B}_+ < E_+ \leq B_+$ such that $\Sigma \cap (E_-, E_+)$ is pure point with exponentially decaying eigenfunctions.*

THEOREM 1.2. – *Assume (H1)-(H8). The integrated density of states is Lipschitz continuous on the interval (B_-, B_+).*

Remarks.

1. It is easy to modify the proofs of Theorems 1.1–1.2 for the case when the random variables $\lambda_i(\omega)$ are correlated without changing the results (see the discussion in [9, 11]).

2. The proofs easily apply to the situation when the disorder is such that $\tilde{B}_- = \tilde{B}_+$ so that the unperturbed spectral gap is no longer open. To control the location of the eigenvalues in this case, we must adjust the coupling constant g. We refer the reader to [4] for a discussion. The proofs here can also be extended to the case when the density h has unbounded support. We refer the reader to section 6 of [4] and [5].
3. Using the methods of this paper, one can also prove the absence of diffusion, in the sense that

$$\limsup_{t\to\infty} t^{-1}\langle x^2(t)\rangle_\psi = 0, \tag{1.19}$$

using an argument of Barbaroux [3].

Let us make some comments about the localization of classical waves. We note that for the unperturbed operators H_0^X given in (1.17)–(1.18), we always have the bottom of the spectrum $\Sigma_0 = 0$. This is easily proved using a Weyl sequence argument making use of the bounded invertibility of the coefficients. (In the Maxwell case, one uses a smooth localization function such that $\Pi\chi \neq 0$.) In fact, one can verify by the same type of argument that the above hypotheses imply that $\inf \sigma(H_\omega) = 0$ for any realization ω. Hence, there is no fluctuation boundary near zero energy and we do not expect Lifshitz tail behavior for the integrated density of states as $E \to 0$. This indicates that we do not expect localized states near zero energy. Physically, the absence of localized states near zero energy can also be seen from the fact that the Rayleigh scattering cross-section for waves with wavelength λ scales like λ^{-4}. At very low energies (long wavelengths), the Rayleigh scattering cross-section vanishes. This provides some indication that wave propagation at low energies is similar to free wave propagation. This is in contrast to the case of a Schrödinger operator with a positive perturbation which pushes the bottom of the essential spectrum upwards. Hence, the bottom of the spectrum is a fluctuation boundary. On the other hand, near nonzero band edges, the results of Theorem 1.1 are similar to those for band edge localization for Schrödinger operators ([4], [29]). If a background operator H_0 does have $\Sigma_0 \neq 0$, then Σ_0 is a fluctuation boundary and we expect to have localized states near Σ_0. For example, we can take $H_0 = -\Delta + V_{per}$, where V_{per} is a positive, periodic potential.

There is another phenomenon concerning the localization of waves which we would like to mention which is not treated in this paper. Consider the model constructed above with C_0 or ϵ_0 equal to constants. One can then verify that $\Sigma = I\!R^+$. The expected localization range in this case is quite different from electron localization. Electron localization is caused primarily by the trapping of the particle in a potential well, whereas localization of light is caused primarily by backscattering. By considering the Helmholtz equation for a wave of frequency ω, one sees that localization at low energy is not expected at any disorder since the influence of the randomness vanishes with the frequency. Similarly, localization is not expected at high energy (for fixed disorder) since the influence of the disorder is very small. Hence, for fixed disorder, one expects (if at all) *localization for waves in a fixed, positive energy interval bounded away from zero and infinity.* For further analysis of light localization and the Ioffe-Regel criteria for localization, we refer to the review article by John [27]. The proof of this conjecture remains an open problem.

There are several recent, related papers on this topic, some of which appeared while this work was in progress. The lattice version of the Anderson-type model with $\xi_i = 0$ was treated by Faris [16, 17] in the acoustic case with $H_0 = -\Delta$, the discrete Laplacian. Figotin and Klein proved localization for the lattice case of acoustic and electromagnetic waves in randomly perturbed periodic media in [19]. Most recently, Figotin and Klein [20, 21] studied a continuous version of these models for which the background medium is periodic. Stollmann [38] studied a related family of perturbations of the form $H_\omega = -\sum_{i,j=1}^{d} \partial_i \epsilon_{i,j} \partial_j$, for a family of random, matrix valued functions $(\epsilon_{i,j})$. These models describe anisotropic media. Band edge localization for Schrödinger operators on the lattice was proved in [2, 18], and for continuum models in [4, 29].

Figotin and Klein [20, 21] studied localization near the band edges for Anderson-type perturbations of *periodic* operators H_0 of the following form. For acoustic waves, they considered operators of the form

$$H_\omega = -\nabla \cdot \rho^{-1} \nabla, \tag{1.20}$$

where $\rho = \rho_{per}(1 + g\rho_\omega) > 0$ describes the random perturbation, for $g > 0$, of a periodic background medium characterized by the strictly positive, bounded density ρ_{per}. Electromagnetic waves are described by the operator of the form

$$H_\omega = \nabla \times \epsilon^{-1} \times \nabla, \tag{1.21}$$

where ϵ is a random perturbation of a background, periodic dielectric function similar to the acoustic case. For both models, they assume that the background periodic operator has a gap in its spectrum and prove exponential localization near the band edges.

To see the relation between multiplicatively perturbed operators H_ω in (1.1) and (1.20)–(1.21), let us define an operator $Q^* \equiv A_\omega^{-1/2} H_0^{-1/2}$, where we have used the fact that $H_0 \geq 0$. Then, the operators in (1.1) have the form Q^*Q. It is easy to check that the operators studied by Figotin and Klein are of the form QQ^* (see Appendix 4 for a detailed discussion of this in the electromagnetic case). It is well-known (cf. [7, 25]) that Q^*Q, restricted to the closure of the range of Q^*, and QQ^*, restricted to the closure of the range of Q, are unitarily equivalent. Hence the localization results proved here apply to the models studied by Figotin and Klein [20, 21] in the periodic case. In addition to the periodicity assumption, we mention that the results of [20, 21] require that the coupling constant g be taken sufficiently small for localization. They also require that the single-site potential u satisfies $\sum_{i \in \mathbf{Z}^d} u(x-i) > C_0 > 0$. These conditions, and the periodicity for the background operator H_0, allow them to locate Σ precisely.

Both our work, and that of Figotin and Klein, require a certain rate of decay of the density of the distribution of the random variables near the edge of its support (see hypothesis [H6]). The recent work of Klopp [30] on internal Lifshitz tails may allow removal of this hypothesis.

This paper is one of a series of papers concerning localization for disordered systems, cf. [4, 8, 9, 10, 11]. Some of the results of this paper were announced in [9].

There is an extremely large literature on random Schrödinger operators. We refer to the monographs by Carmona and Lacroix [12] and by Pastur and Figotin [34] for references to earlier papers on random operators. The key background papers related to this work are [24, 26, 31, 37, 36, 40], which deal primarily with the lattice case.

2. THE WEGNER ESTIMATE

In this section, we prove a Wegner estimate for local Hamiltonians valid at all energies in the spectral gap G of H_0. For ease of notation, we continue to write

$$A_\omega = 1 + gV_\omega, \tag{2.1}$$

where the potential V_ω is given in (1.13) and (1.14). We will need one result on spectral averaging in this section, Lemma 4.1, which is proven in section 4.

Let $\Lambda \subset I\!R^d$ be a bounded region and denote by $V_\Lambda \equiv V_\omega|\Lambda$, so that $V_\Lambda(x) = 0$ for $x \notin \Lambda$, and define $A_\Lambda \equiv 1 + gV_\Lambda$. The local Hamiltonian associated with the region Λ, and acting on the Hilbert space $\mathcal{H}$, has the form $H_\Lambda = A_\Lambda^{-1/2} H_0 A_\Lambda^{-1/2}$, where for the acoustic case $d \geq 1$ and $\mathcal{H} = L^2(I\!R^d)$, and for the Maxwell case $d = 3$ and $\mathcal{H} = L^2(I\!R^3, \mathbb{C}^3)$. In the Maxwell case, we use the identity $A_\Lambda^{-1/2} H_0 A_\Lambda^{-1/2} = A_\Lambda^{-1/2} P_0^\perp H_0 A_\Lambda^{-1/2}$, where P_0 is the projection onto the zero eigenvalue subspace of H_0, and $P_0^\perp \equiv 1 - P_0$. This results in an additional term in (2.6) that can be absorbed into the $\mathrm{Tr}(E_\Lambda(I_0))$-term provided $g < 1/2$. We drop the subscript ω in this section. It is easy to check that V_Λ is relatively H_0 compact. Consequently, the spectrum $\sigma(H_\Lambda) \cap (B_-, B_+)$ is discrete.

Since the single-site potential u has compact support, if two regions Λ_1 and Λ_2 are sufficiently far apart, the local operators H_{Λ_i}, for $i = 1, 2$, are independent random variables. In particular, if we define another local perturbation $\tilde{A}_\Lambda \equiv 1 + g \sum_{j \in \tilde{\Lambda}} \lambda_j u_j$, where $\tilde{\Lambda} \equiv \Lambda \cap \mathbb{Z}^d$, then the difference $A_\Lambda - \tilde{A}_\Lambda$ has compact support near the boundary of the region Λ. The in-radius of this region depends only on supp u. This overlap region, and the consequent lack of independence, can be controlled in the multi-scale analysis as described in [11]. Consequently, we will assume independence for local operators associated with disjoint regions and refer the reader to [11] for the details.

Let $I\!P_\Lambda$ and $I\!E_\Lambda$ denote the probability and expectation with respect to the random variables associated with $\Lambda \cap \mathbb{Z}^d \equiv \widetilde{\Lambda}$. We denote by Tr the trace on $L^2(I\!R^d)$. Let $R_\Lambda(z)$ and $E_\Lambda(.)$ denote the resolvent and the spectral projection for $H_{\Lambda,\omega}$ respectively. We write $R_0(z)$ for $(H_0 - z)^{-1}$.

The proof of the following theorem follows the lines of that given in [4]. We will give all the modifications necessary for the case of multiplicative perturbations. We simply sketch those parts of the argument which are common with the Schrödinger case given in [4].

THEOREM 2.1. – *Assume (H1)-(H3), (H5) and (H7)-(H8). For any $E_0 \in (B_-, B_+)$ and for any $\eta < \frac{1}{2}\mathrm{dist}(E_0, \sigma(H_0))$, there exists finite constant $C_{E_0} > 0$, depending on E_0, the dimension d, the integer q in (H3), the coupling constant g, and $[\mathrm{dist}(\sigma(H_0), E_0)]^{-1}$, such that:*

$$I\!P_\Lambda\{\mathrm{dist}(\sigma(H_{\Lambda,\omega}), E_0) < \eta\} \leq C_{E_0} \eta \, |\Lambda|. \tag{2.2}$$

Proof.

1. Let $I_\eta = [E_0 - \eta, E_0 + \eta]$; we write R_0 for $R_0(E_0)$. By Chebyshev's inequality the left hand side of (2.2) is bounded above by

$$I\!E_\Lambda(Tr(E_\Lambda(I_\eta))). \tag{2.3}$$

We control the trace by writing a perturbation-type formula for the projection using the boundedness of R_0 and the relative compactness of V_Λ. We define an operator K_0 by

$$K_0 \equiv A_\Lambda^{1/2} R_0 V_\Lambda A_\Lambda^{-1/2}. \tag{2.4}$$

Suppose that ψ_E is an eigenfunction of H_Λ satisfying $H_\Lambda \psi_E = E \psi_E$, $E \in I_\eta$. In a manner similar to the derivation of the Birman-Schwinger kernel, the eigenvalue equation can be written as

$$\psi_E = gK_0 H_\Lambda \psi_E + A_\Lambda^{1/2} R_0 A_\Lambda^{-1/2} (H_\Lambda - E_0)\psi_E. \tag{2.5}$$

Since the spectrum of H_Λ in the gap G is discrete, it follows easily from (2.5) that,

$$E_\Lambda(I_\eta) = gK_0 H_\Lambda E_\Lambda(I_\eta) + A_\Lambda^{1/2} R_0 A_\Lambda^{-1/2} (H_\Lambda - E_0) E_\Lambda(I_\eta). \tag{2.6}$$

Let $\| \ \|_q$ denote the norm in the Schatten class $\mathcal{I}_q$ (see Simon [37] for the results concerning these ideals that we use here). Noting that $E_\Lambda(I_\eta)$ is a positive, trace class operator,

$$\begin{aligned} Tr(E_\Lambda(I_\eta)) &= \|E_\Lambda(I_\eta)\|_1 \\ &\leq g|Tr(K_0 H_\Lambda E_\Lambda(I_\eta))| + \eta \|R_0\| \, \|E_\Lambda(I_\eta)\|_1, \end{aligned} \tag{2.7}$$

and by the assumption on η:

$$\begin{aligned} Tr(E_\Lambda(I_\eta)) &\leq 2g|Tr(K_0 H_\Lambda E_\Lambda(I_\eta))| \\ &\leq 2g(E_0 + \eta)\|K_0 E_\Lambda(I_\eta)\|_1. \end{aligned} \tag{2.8}$$

Remark. – A first result of (2.8) is the following. We take the expectation of (2.8) and then apply Hölder's inequality twice, first on the trace norm (cf. [35]) and then on the expectation, to obtain,

$$\begin{aligned} \mathbb{E}_\Lambda(\|E_\Lambda(I_\eta)\|_1) &\leq 2g(E_0+\eta)\mathbb{E}_\Lambda(\|K_0 E_\Lambda(I_\eta)\|_1) \\ &\leq 2g(E_0+\eta)\mathbb{E}_\Lambda(\|K_0\|_q \|E_\Lambda(I_\eta)\|_p) \quad \left(\frac{1}{q} + \frac{1}{p} = 1\right) \\ &\leq 2g(E_0+\eta)\mathbb{E}_\Lambda(\|K_0\|_q{}^q)^{(1/q)} \mathbb{E}_\Lambda(\|E_\Lambda(I_\eta)\|_p{}^p)^{(1/p)}. \end{aligned} \tag{2.9}$$

Since all of the nonzero eigenvalues of an orthogonal projector are one, we have that $\|E_\Lambda(I_\eta)\|_p^p = \|E_\Lambda(I_\eta)\|_1$. In Appendix 2, we summarize various results on the Schatten class properties of the operator K_0.

From this remark concerning projections and Proposition 8.4 of Appendix 2, we obtain from (2.9),

$$\mathbb{E}_\Lambda(\|E_\Lambda(I_\eta)\|_1) \leq 2g(E_0+\eta)C^{1/q}|\Lambda|^{1/q}\mathbb{E}_\Lambda(\|E_\Lambda(I_\eta)\|_1)^{1/p}. \quad (2.10)$$

Upon solving this inequality, we obtain,

$$\mathbb{E}_\Lambda(\|E_\Lambda(I_\eta)\|_1) \leq (2g)^q C(E_0+\eta)^q|\Lambda|. \quad (2.11)$$

The existence of the integrated density of states for the multiplicatively perturbed models at energies in the unperturbed gap follows from this result.

2. We return to (2.8). We must continue to iterate the procedure (2.6)–(2.8) until we get q factors of K_0 or $K_0^\star$ in the trace on the right side of (2.8), where q is determined by (H3). We take the adjoint of (2.6) and multiply on the left by K_0 to derive the equation,

$$K_0E_\Lambda(I_\eta) = gK_0H_\Lambda E_\Lambda(I_\eta)K_0^\star + K_0E_\Lambda(I_\eta)(H_\Lambda - E_0)A^{-1/2}R_0A^{1/2}. \quad (2.12)$$

The trace norm of the second term on the right of (2.12) can be estimated as above. Following that argument, we obtain,

$$\|K_0E_\Lambda(I_\eta))\|_1 \leq 2g|Tr(K_0H_\Lambda E_\Lambda(I_\eta)K_0^\star)|. \quad (2.13)$$

Hence, by (2.8), we have the estimate.

$$\mathbb{E}_\Lambda(Tr(E_\Lambda(I_\eta))) \leq (2g)^2(E_0+\eta)\mathbb{E}_\Lambda(Tr(K_0H_\Lambda E_\Lambda(I_\eta)K_0^\star). \quad (2.14)$$

If $q > 2$, one continues this procedure. Returning to (2.6), one finds an expression for $K_0^{q-2}H_\Lambda^{q-2}E_\Lambda(I_\eta)K_0^\star$ by multiplying (2.6) on the left by K_0^{q-2} and on the right by $H_\Lambda^{q-2}K_0^\star$,

$$\begin{aligned} K_0^{q-2}H_\Lambda^{q-2}E_\Lambda(I_\eta)K_0^\star &= gK_0^{q-1}H_\Lambda^{q-1}E_\Lambda(I_\eta)K_0^\star \\ &\quad + K_0^{q-2}A^{1/2}R_0A^{-1/2}E_\Lambda(I_\eta)(H_\Lambda - E_0)H_\Lambda^{q-2}K_0^\star. \quad (2.15) \end{aligned}$$

We first bound the second term on the right. One has by cyclicity of the trace and Hölder's inequality,

$$\begin{aligned} &|Tr(K_0^{q-2}A^{1/2}R_0A^{-1/2}E_\Lambda(I_\eta)(H_\Lambda - E_0)H_\Lambda^{q-2}K_0^\star)| \\ &= |Tr(E_\Lambda(I_\eta)K_0^\star K_0^{q-2}A^{1/2}R_0A^{-1/2}E_\Lambda(I_\eta)(H_\Lambda - E_0)H_\Lambda^{q-2})| \\ &\leq \eta(E_0+\eta)^{q-2}\,\|R_0\|\,\|K_0^\star K_0^{q-2}\|_{q/q-1}\|E_\Lambda(I_\eta)\|_q \\ &\leq \eta(E_0+\eta)^{q-2}\|R_0\|\,\|K_0\|_q^{q-1}\|E_\Lambda(I_\eta)\|_q. \quad (2.16) \end{aligned}$$

Taking the expectation and then applying Hölder's inequalities and Proposition 8.2, one can bound the expectation of the left hand side of (2.16) by:

$$C_0\,(E_0+\eta)^{q-2}\eta\,\|R_0\|\,|\Lambda|, \qquad (2.17)$$

for some constant C_0. Consequently, results (2.14)–(2.17) imply the bound,

$$\begin{aligned} \mathbb{E}_\Lambda(Tr(E_\Lambda(I_\eta))) \leq (2g)^2(E_0+\eta) & \\ \{g^{q-2}\mathbb{E}_\Lambda(|Tr(K_0^{q-1}H_\Lambda^{q-1}E_\Lambda(I_\eta)K_0^\star)|) & \\ +\,C(q,E_0)\eta\,\mathrm{dist}(E_0,\sigma(H_0))^{-1}|\Lambda|\}, & \qquad (2.18) \end{aligned}$$

for some constant depending on q and E_0. The second term on the right in (2.18) exhibits the correct volume dependence, so we concentrate on the first term.

3. We now estimate the expectation on the right hand side of (2.18). From the definition of K_0 given in (2.4), we find that

$$K_0^\star K_0^{q-1} = A^{-1/2}(K^{(1)}+K^{(2)})A^{-1/2}, \qquad (2.19)$$

where

$$K^{(1)} \equiv V_\Lambda R_0^2 V_\Lambda (R_0V_\Lambda)^{q-2}, \qquad (2.20)$$

and

$$K^{(2)} \equiv V_\Lambda (R_0 V_\Lambda)^q. \qquad (2.21)$$

We expand the potential $V_\Lambda = \sum_{i\in\tilde{\Lambda}}\lambda_i u_i$, where $u_i(x)\equiv u(x-i)$ and $\tilde{\Lambda}\equiv\Lambda\cap\mathbb{Z}^d$. For each q-tuple of indices $\{i\}\equiv(i_1,\ldots,i_q)\in\widetilde{\Lambda}^q$, we define:

$$K^{(1)}_{i_1\ldots i_q} \equiv u_{i_1}^{\frac{1}{2}} R_0^2 u_{i_2} R_0 u_{i_3}\ldots u_{i_{q-1}} R_0 u_{i_q}^{\frac{1}{2}}. \qquad (2.22)$$

Similarly, for each $(q+1)$-tuple of indices $\{i\}\equiv(i_1,\ldots,i_{q+1})\in\widetilde{\Lambda}^{q+1}$, we define

$$K^{(2)}_{i_1\ldots i_{q+1}} \equiv u_{i_1}^{\frac{1}{2}} R_0 u_{i_2} R_0\ldots R_0 u_{i_{q+1}}^{\frac{1}{2}}. \qquad (2.23)$$

We prove in the Appendix 2 that (H3) implies that $K^{(j)}_{\{i\}}\equiv K^{(j)}_{i_1\ldots i_{q'}}\in\mathcal{J}_1$, for either $j=1$ and $q'=q$ or $j=2$ and $q'=q+1$. In terms of these operators, the first term on the right side of (2.18) becomes the sum of two expressions. We first look at the expression involving $K^{(1)}_{\{i\}}$:

$$\mathbb{E}_\Lambda \left\{ \sum_{i_1,\dots i_q \in \widetilde{\Lambda}} |\lambda_{i_1}(\omega)\dots\lambda_{i_q}(\omega)| \, |Tr\left\{ K^{(1)}_{\{i\}} (A_\Lambda^{-\frac{1}{2}} u_{i_q}^{\frac{1}{2}} E_\Lambda(I_\eta) u_{i_1}^{\frac{1}{2}} A_\Lambda^{-\frac{1}{2}}) \right\}| \right\}. \tag{2.24}$$

Since $K^{(1)}_{\{i\}}$ is compact, we write it in terms of its singular value decomposition. For each multi-index $\{i\}$, there exists a pair of orthonormal bases, $\left\{\phi_k^{\{i\}}\right\}$ and $\left\{\psi_k^{\{i\}}\right\}$, and nonnegative numbers $\left\{\mu_k^{\{i\}}\right\}$, all independent of ω, such that

$$K^{(1)}_{\{i\}} = \sum_{k=1}^{\infty} \mu_k^{\{i\}} \left|\phi_k^{\{i\}}\right\rangle \left\langle \psi_k^{\{i\}}\right| \tag{2.25}$$

Inserting the representation (2.25) into (2.24) and expanding the trace in $\{\phi_k^{\{i\}}\}$, we obtain

$$\mathbb{E}_\Lambda \left\{ \sum_{\{i\} \in \widetilde{\Lambda}^q} \sum_{k \geq 1} |\lambda_{\{i\}}(\omega)| \mu_k^{\{i\}} \langle \psi_k^{\{i\}}, (A_\Lambda^{-\frac{1}{2}} u_{i_q}^{\frac{1}{2}} E_\Lambda(I_\eta) u_{i_1}^{\frac{1}{2}} A_\Lambda^{-\frac{1}{2}}) \phi_k^{\{i\}} \rangle \right\}, \tag{2.26}$$

where $\lambda_{\{i\}}(\omega) \equiv \lambda_{i_1}(\omega)\dots\lambda_{i_q}(\omega)$. Recalling that $E_\Lambda(I_\eta) \geq 0$, we bound the k-sum above by:

$$\frac{1}{2} \sum_{k \geq 1} \mu_k^{\{i\}} \mathbb{E}_\Lambda \Big\{ |\lambda_{\{i\}}(\omega)| \langle \psi_k^{\{i\}}, (A_\Lambda^{-\frac{1}{2}} u_{i_q}^{\frac{1}{2}} E_\Lambda(I_\eta) u_{i_q}^{\frac{1}{2}}) A_\Lambda^{-\frac{1}{2}} \psi_k^{\{i\}} \rangle + |\lambda_{\{i\}}(\omega)| \langle \phi_k^{\{i\}}, (A_\Lambda^{-\frac{1}{2}} u_{i_1}^{\frac{1}{2}} E_\Lambda(I_\eta) u_{i_1}^{\frac{1}{2}}) A_\Lambda^{-\frac{1}{2}} \phi_k^{\{i\}} \rangle \Big\} \tag{2.27}$$

4. We now apply the spectral averaging theorem (see section 4, [8] or [10]) to each term in (2.27), using the independence of the λ_i's, to define a new density $\lambda g(\lambda)$. For example, the first term of (2.27) can be bounded as:

$$\mathbb{E}_\Lambda \left\{ |\lambda_{\{i\}}(\omega)| \langle \psi_k^{\{i\}}, (A_\Lambda^{-\frac{1}{2}} u_{i_q}^{\frac{1}{2}} E_\Lambda(I_\eta) u_{i_q}^{\frac{1}{2}} A_\Lambda^{-\frac{1}{2}}) \psi_k^{\{i\}} \rangle \right\} \leq C_1 \, \eta. \tag{2.28}$$

where C_1 is finite according to (H6). An analogous calculation leads to a similar estimate for the second term involving $K_i^{(2)}$. From (2.26)–(2.28), we obtain an upper bound for the first term on the right hand side of (2.18),

$$C(q, E_0, g)\, \eta \left(\sum_{i_1,\ldots,i_q \in \widetilde{\Lambda^q}} \|K_{\{i\}}^{(1)}\|_1 + \sum_{i_1,\ldots,i_{q+1} \in \widetilde{\Lambda^{q+1}}} \|K_{\{i\}}^{(2)}\|_1 \right). \quad (2.29)$$

In the Appendix 2, we prove in Proposition 8.2 that $\exists$ finite constant $C_{E_0} > 0$, depending only on $[\mathrm{dist}(\sigma(H_0), E_0)]^{-1}$ and the dimension $d \geq 1$, such that (2.29) is bounded above by

$$C(q, E_0, g)\, C_{E_0}\, \eta |\Lambda|. \quad (2.30)$$

Results (2.18) and (2.30) prove the theorem. □

3. GREEN'S FUNCTION ESTIMATES FOR MULTIPLICATIVELY PERTURBED OPERATORS

The proof of Theorem 1.1 for random families of multiplicatively perturbed operators of the form (1.1), where the random coefficient $A_{\omega,\omega'}$ is given in (1.12)–(1.15), requires an initial length-scale estimate for the local Hamiltonians $H_\Lambda = H_{\Lambda,\omega} = H_{\Lambda,\omega,\omega'}$. We prove this estimate, called $[\mathrm{H1}](\gamma_0, \ell_0)$ in [4, 8], in this section. At the end of this section, we show how to use the results of sections 2 and 3 to prove the almost sure exponential decay for the Green's function. The Schrödinger operator version of these results are given in [4]. We must prove that if the local Hamiltonians create eigenvalues in the unperturbed spectral gap G of H_0, then we can control the distance from these eigenvalues to the band edges of Σ with a good probability. (At the end of this section, we consider the question of proving the existence of such eigenvalues.) This will allow us to apply the Combes-Thomas estimate of [4] (see Appendix 3) and prove the assumption $[\mathrm{H1}](\ell_0, \gamma_0)$. We will often use the elementary fact that $\mu \in \sigma_d(H_{\Lambda,\omega})$ if and only if $0 \in \sigma_d(H_0 - \mu A_{\Lambda,\omega})$.

We continue to use the notation of section 2. The finite volume Hamiltonians $H_\Lambda = H_{\Lambda,\omega,\omega'}$ are defined as

$$H_{\Lambda,\omega,\omega'} \equiv (1 + gV_{\Lambda,\omega,\omega'})^{-1/2} H_0 (1 + gV_{\Lambda,\omega,\omega'})^{-1/2}. \quad (3.1)$$

We will suppress the index ω' in this section as it will play no role in the calculations. Since $V_{\Lambda,\omega}$ has compact support, it is a relatively compact perturbation of H_0 (or, $(1 - P_0)H_0$, in the Maxwell case) and hence $\sigma_{ess}(H_0) = \sigma_{ess}(H_{\Lambda,\omega})$. One of our first tasks is to locate precisely the eigenvalues of $H_{\Lambda,\omega}$ in the gap $G = (B_-, B_+)$ of H_0, with good probability. We will often write H_Λ for $H_{\Lambda,\omega}$ and $A_{\Lambda,\omega}$ for $(1 + gV_{\Lambda,\omega})$. Let us recall that g is fixed so that A_Λ^{-1} exists. When no confusion will result, we will drop the subscript Λ.

The condition $[\text{H1}](\gamma_0, \ell_0)$ on the resolvent of $H_{\Lambda,\omega}$, written as $R_\Lambda(z) = (H_{\Lambda,\omega} - z)^{-1}$, when it exists, is the following. For any $\chi \in C^2$, we define the first-order, local differential operator $W(\chi)$ by

$$\begin{aligned} W(\chi) &\equiv A^{-1/2}[H_0, \chi]A^{-1/2} \\ &= -iA^{-1/2}\{h_0^T \cdot \nabla\chi + \nabla\chi \cdot h_0\}A^{-1/2}, \end{aligned} \tag{3.2}$$

where h_0 is the first-order differential operator defined in the acoustic case by

$$h_0 \equiv -(C_0\rho^{-1/2})p \cdot (C_0\rho^{1/2}), \tag{3.3}$$

and in the Maxwell case by

$$h_0 \equiv -(1 + \epsilon_0)^{-1/2}(\epsilon \cdot p)\epsilon(1 + \epsilon_0)^{-1/2}, \tag{3.4}$$

where $p \equiv -i\nabla$. This operator is localized on the support of $\nabla\chi$. For any fixed $\delta > 0$ small, we let $\Lambda_{\ell,\delta} \equiv \{x \in \Lambda_\ell \mid \text{dist}(\partial\Lambda_\ell, x) > \delta\}$. We will use χ_ℓ to denote a function satisfying $\chi_\ell|\Lambda_{\ell,\delta} = 1$, $\text{supp}\,\chi_\ell \subset \Lambda_\ell$, and $\chi_\ell \geq 0$. It follows that $\text{supp}\,\nabla\chi_\ell \subset \Lambda_\ell\backslash\Lambda_{\ell,\delta}$ and $W(\chi_\ell)$ is also localized in this region. The condition $[\text{H1}](\gamma_0, \ell_0)$ that we must verify is the following

[H1](γ_0, ℓ_0): $\exists\gamma_0 > 0$ *and* $\ell_0 >> 1$ *such that* $\gamma_0\ell_0 >> 1$ *and*

$$I\!P\{\|W(\chi_\ell)R_{\Lambda_\ell}(E + i\varepsilon)\chi_{\ell/3}\| < e^{-\gamma_0\ell_0}\|\} \geq 1 - \ell_0^{-\xi}, \tag{3.5}$$

for E near the band edges $\tilde{B}_\pm$ of Σ and for some $\xi > 2d$.

We prove this estimate in two steps.

1. We first prove that for $\delta > 0$ small,

$$\text{dist}\,(\sigma(H_{\Lambda,\omega}), \tilde{B}_\pm) > \delta,$$

with good probability.

2. We then apply the improved Combes-Thomas result of [4] to conclude exponential decay at energies $E \in (\tilde{B}_- - \delta/2, \tilde{B}) \cup (\tilde{B}_+, \tilde{B}_+ + \delta/2)$

with a good probability. This allows us to verify $[\mathrm{H1}](\gamma_0, \ell_0)$ for energies in this interval with an appropriate choice of γ_0 and ℓ_0.

We now discuss the location of the spectrum of the finite volume Hamiltonians $H_{\Lambda,\omega}$ in the unperturbed spectral gap G. Recall that we assumed that the family $\{H_\omega\}$ has an almost sure spectrum Σ and that this set has an open spectral gap $(\tilde{B}_-, \tilde{B}_+)$. The probability space is $\Omega = (\operatorname{supp} h)^{\boldsymbol{Z}^d}$.

LEMMA 3.1. – *If $\mu \in \sigma_d(H_{\Lambda,\omega_0}) \cap G$, for some $\omega_0 \in \Omega$, then $\mu \in \Sigma$.*

Proof. – Let ψ_{ω_0} be an eigenfunction of H_{Λ,ω_0} with eigenvalue $\mu_{\Lambda,\omega_0} \equiv \mu : H_{\Lambda,\omega_0}\psi_{\omega_0} = \mu\psi_{\omega_0}$, $\|\psi_{\omega_0}\| = 1$. We set $\phi_{\omega_0} \equiv A_{\Lambda,\omega_0}^{-1/2}\psi_{\omega_0}$, and note that there exist finite, nonzero constants C_1, C_2, depending only on g, u, and supp h such that $C_1 \leq \|\phi_{\omega_0}\| \leq C_2$. For any R, such that $\Lambda \subset\subset \Lambda_R$, and for any $\nu > 0$, consider the following events

$$I_\nu \equiv \left\{\omega \in \Omega \mid |\lambda_i(\omega_0) - \lambda_i(\omega)| \leq \nu(6C_2\mu\,|\Lambda|\,\|u\|_\infty)^{-1},\ \forall i \in \widetilde{\Lambda}\right\},$$

and

$$E_{R,\nu} \equiv \left\{\omega \in \Omega \mid |\lambda_i(\omega)| < \nu(6C_2\mu\,|\Lambda_R\backslash\Lambda|\,\|u\|_\infty)^{-1},\ \forall i \in \widetilde{\Lambda}_R\backslash\widetilde{\Lambda}\right\}.$$

We set $B_{R,\nu} \equiv I_\nu \cap E_{R,\nu}$. Let $\chi \in C^2$ be a smoothed characteristic function with supp $\chi \subset \Lambda_2$, $0 \leq \chi \leq 1$, and $\chi|\Lambda_1 = 1$. For $R > 1$, set $\chi_R(x) \equiv \chi(R^{-1}x)$ so that $\|\partial^\alpha\chi_R\| = \mathcal{O}(R^{-|\alpha|})$, for $|\alpha| = 0, 1, 2$. Choose R_1 sufficiently large so $\|\chi_{R_1}\phi_{\omega_0}\| > 1/(2C_1)$, and for $R > R_1$, we define $\phi_R \equiv \|\chi_R\phi_{\omega_0}\|^{-1}\chi_R\phi_{\omega_0}$, so that $\|\phi_R\| = 1$. Then, by the definition of ϕ_R and the local Hamiltonians,

$$\begin{aligned}(H_0 - \mu A_\omega)\phi_R = (H_0 - \mu A_{\Lambda,\omega_0})\phi_R &- \mu\sum_{i\in\widetilde{\Lambda}}(\lambda_i(\omega) - \lambda_i(\omega_0))u_i\phi_R \\ &- \mu\sum_{i\in\widetilde{\Lambda}_R\backslash\widetilde{\Lambda}}\lambda_i(\omega)u_i\phi_R,\end{aligned}$$

and it follows that for all $\omega \in B_{R,\nu}$,

$$\|(H_0 - \mu A_\omega)\phi_R\| \leq 2C_1\|\,[H_0, \chi_R]\phi_{\omega_0}\| + \frac{1}{3}\nu \tag{3.6}$$

The commutator is estimated as follows. For the acoustic model, the unperturbed Hamiltonian (1.6) is $H_0 = -C_0\rho^{1/2}\nabla\cdot\rho^{-1}\nabla\rho^{1/2}C_0$, so we have,

$$[H_0, \chi_R]\phi_{\omega_0} = -\,2C_0\rho^{1/2}(\nabla\chi_R)\cdot\rho^{-1}\nabla C_0\rho^{1/2}\phi_{\omega_0} \tag{3.7}$$

$$-\,C_0^2\rho\nabla\cdot(\rho^{-1}\nabla\chi_R)\phi_{\omega_0}. \tag{3.8}$$

For the Maxwell case $H_0 = -(1+\epsilon_0)^{-1/2}(\epsilon\cdot p)^2(1+\epsilon_0)^{-1/2}$, so we obtain,

$$(1+\epsilon_0)^{-1/2}[(\epsilon\cdot p)^2, \chi_R](1+\epsilon_0)^{-1/2}\phi_{\omega_0} = \\ -2i(1+\epsilon_0)^{-1/2}(\epsilon\cdot\nabla\chi_R)(\epsilon\cdot p)(1+\epsilon_0)^{-1/2}\phi_{\omega_0} + (1+\epsilon_0)^{-1}I(\partial)\chi_R\phi_{\omega_0}, \quad (3.9)$$

where

$$I(\partial)_{ij}\chi_R = \delta_{ij}\Delta\chi_R - \partial_i\partial_j\chi_R. \quad (3.10)$$

Now ϕ_{ω_0} is an eigenfunction of $H_0 - \mu A_{\Lambda,\omega_0}$ for eigenvalue zero, so we estimate the first term on the right in (3.7) by

$$\begin{aligned} \|2C_0\rho^{1/2}(\partial_l\chi_R)\rho^{-1}\partial_l C_0\rho^{1/2}\phi_{\omega_0}\|^2 &\le R^{-1}C_3\langle\phi_{\omega_0}, H_0\phi_{\omega_0}\rangle \\ &= R^{-1}C_3\langle\phi_{\omega_0}, \mu A_{\Lambda,\omega_0}\phi_{\omega_0}\rangle \\ &= R^{-1}C_3\langle\psi_{\omega_0}, \mu\psi_{\omega_0}\rangle \\ &\le R^{-1}\mu C_3, \end{aligned} \quad (3.11)$$

and the constant C_3 is independent of R. A similar estimate holds for the first term on the right in (3.9). Hence, by taking R sufficiently large, it follows from (3.4) that

$$\|(H_0 - \mu A_\omega)\psi_R\| \le \frac{2}{3}\nu.$$

This shows that for any $\nu > 0$, $\sigma(H_0 - \mu A_\omega) \cap [-\nu, \nu] \neq \emptyset$ with probability $I\!P(B_{R,\nu}) = I\!P(E_{R,\nu})I\!P(I_\nu) > 0$. Since the spectrum of the family $\{H_\omega\}$ is deterministic, this implies $\mu \in \Sigma$. □

LEMMA 3.2. – *Suppose that the local potential $V_{\Lambda,\omega}$ is positive. Let ψ_ω be a normalized eigenfunction of $H_{\Lambda,\omega}$ for eigenvalue μ. Define $\phi_\omega \equiv A_\omega^{-1/2}\psi_\omega$, so that ϕ_ω satisfies*

$$(H_0 - \mu A_{\Lambda,\omega})\phi_\omega = 0. \quad (3.12)$$

Then, we have the lower bound

$$\langle\phi_\omega, g\mu V_{\Lambda,\omega}\phi_\omega\rangle \ge \frac{[dist(\sigma(H_0),\mu)]^2}{g\mu M_\infty}. \quad (3.13)$$

Proof. – Since $M_\infty V_{\Lambda,\omega} \ge (V_{\Lambda,\omega})^2$ under the hypothesis that $V_{\Lambda,\omega} \ge 0$, we have

$$\begin{aligned} \langle\phi_\omega, g\mu V_{\Lambda,\omega}\phi_\omega\rangle &= (g\mu M_\infty)^{-1}\langle\phi_\omega, (g\mu)^2 M_\infty V_{\Lambda,\omega}\phi_\omega\rangle \\ &\ge (g\mu M_\infty)^{-1}\|g\mu V_{\Lambda,\omega}\phi_\omega\|^2. \end{aligned}$$

The eigenvalue equation gives $g\mu V_{\Lambda,\omega}\phi_\omega = -(H_0 - \mu)\phi_\omega$, so that

$$\begin{aligned}\langle \phi_\omega, g\mu V_{\Lambda,\omega}\phi_\omega\rangle &\geq (g\mu M_\infty)^{-1}\|(H_0-\mu)\phi_\omega\|^2 \\ &\geq (g\mu M_\infty)^{-1}[\text{dist}(\sigma(H_0),\mu)]^2,\end{aligned}$$

which is (3.13). □

LEMMA 3.3. – *Let* $\delta_\pm \equiv \frac{1}{2}|\tilde{B}_\pm - B_\pm|$, *and set* $0 < \delta < (gB_\pm M_\infty)^{-1} [min(\delta_+, \delta_-)]^2$. *Suppose that the random coefficients satisfy either of the conditions,* $\{\lambda_i(\omega) | \; i \in \tilde{\Lambda}\}$ *satisfy*

$$0 < \lambda_i(\omega) < (1 - \delta g B_\pm M_\infty [min(\delta_-, \delta_+)]^{-2})X, \tag{3.14}$$

where we take $X = m$ *for* B_- *and* $X = M$ *for* B_+. *Then, under the* B_- *condition, we have*

$$sup\, \{\sigma(H_{\Lambda,\omega}) \cap (-\infty, \tilde{B}_-)\} < \tilde{B}_- - \delta, \tag{3.15}$$

and, under the B_+ *condition, we have*

$$inf\, \{\sigma(H_{\Lambda,\omega}) \cap (\tilde{B}_+, \infty)\} > \tilde{B}_+ + \delta. \tag{3.16}$$

Proof. – Without loss of generality, we assume $H_{\Lambda,\omega}$ has an eigenvalue $\mu_{\Lambda,\omega} \equiv \mu \in [\tilde{B}_+, \tilde{B}_+ + \delta]$. Furthermore, we can assume that $V_{\Lambda,\omega} \geq 0$, since by Lemma 3.1, we always have $\mu \geq \tilde{B}_+$ and the eigenvalues of $H_{\Lambda,\omega}$ are decreasing functions of the coupling constants $\left\{\lambda_i(\omega) | i \in \tilde{\Lambda}\right\}$. This fact follows, for example, from a variant of the Feynman-Hellman formula and the positivity of u. Indeed, an eigenfunction ψ_ω of $H_{\Lambda,\omega}$ for eigenvalue μ satisfies the identity

$$0 = \langle \psi_\omega, A_\omega^{-1/2}(H_0 - \mu A_\omega)A_\omega^{-1/2}\psi_\omega\rangle. \tag{3.17}$$

Upon differentiating this equation with respect to λ_i, we obtain,

$$\frac{\partial \mu_{\Lambda,\omega}}{\partial \lambda_i} = -g\mu_{\Lambda,\omega}\langle \psi_\omega, u_i A_{\Lambda,\omega}^{-1}\psi_\omega\rangle \leq 0. \tag{3.18}$$

For $\theta \in I\!R$, define $A(\theta) \equiv 1 + g\theta V_{\Lambda,\omega}$. The family $T(\theta) \equiv A(\theta)^{-1/2}H_0A(\theta)^{-1/2}$, for θ in a small neighborhood of $\theta_0 = 1$, is an analytic type A family which is self-adjoint for θ real. If μ has multiplicity m, there are at most m functions $\mu^{(k)}(\theta)$, analytic in θ for θ near $\theta_0 = 1$, and which satisfy $\lim_{\theta\to\theta_0=1}\mu^{(k)}(\theta) = \mu$. Let $\psi^{(k)}(\theta)$ be an eigenfunction corresponding to the eigenvalue $\mu^{(k)}(\theta)$, with $\|\psi^{(k)}(\theta)\| = 1$, for θ real and $|\theta - 1|$ small. Applying the modified Feynman-Hellman formula again, we find

$$\frac{d\mu^{(k)}(\theta)}{d\theta} = -\theta^{-1}\left\langle \psi^{(k)}(\theta), \left(\frac{g\mu^{(k)}(\theta)\theta V_{\Lambda,\omega}}{1+g\theta V_{\Lambda,\omega}}\right)\psi^{(k)}(\theta)\right\rangle. \tag{3.19}$$

We now assume condition (3.14) that $\lambda_i(\omega) < (1 - \delta g B_+ M_\infty [\min(\delta_-, \delta_+)]^{-2})M$, $\forall i \in \widetilde{\Lambda}$, and fix

$$\theta_1 = \min_{i\in\widetilde{\Lambda}}\left(\frac{M}{\lambda_i(\omega)}\right) \geq \left(1 - \delta g B_+ M_\infty [\min(\delta_-, \delta_+)]^{-2}\right)^{-1} > 1.$$

Applying Lemma 3.2 to $\theta V_{\Lambda,\omega}$ in (3.19) under these conditions, and using the fact that $\mu^{(k)}(\theta) \leq \mu \leq B_+$, we obtain

$$\begin{aligned}\frac{d\mu^{(k)}(\theta)}{d\theta} &\leq -\theta^{-1}(g\mu^{(k)}(\theta)M_\infty)^{-1}\left[\operatorname{dist}(\mu^{(k)}(\theta), \sigma(H_0))\right]^2 \\ &\leq -\theta^{-1}(gB_+M_\infty)^{-1}\left[\operatorname{dist}(\mu^{(k)}(\theta), \sigma(H_0))\right]^2.\end{aligned} \tag{3.20}$$

Upon integrating over $[1, \theta_1]$, we get, by monotonicity of $\mu^{(k)}(\theta)$ for θ real,

$$\begin{aligned}\mu^{(k)}(\theta_1) &\leq \mu - (log\theta_1)(gB_+M_\infty)^{-1}\min \\ &\quad \left\{\left[\operatorname{dist}(\mu^{(k)}(\theta_1), \sigma(H_0))\right]^2,\ [\operatorname{dist}(\mu, \sigma(H_0))]^2\right\} \\ &\leq \mu - \delta < \tilde{B}_+.\end{aligned}$$

This shows that the local Hamiltonian

$$(1 + g\sum_{i\in\widetilde{\Lambda}} Mu_i)^{-1/2} H_0 (1 + g\sum_{i\in\widetilde{\Lambda}} Mu_i)^{-1/2}$$

has an eigenvalue outside of Σ which contradicts Lemma 3.1. □

Remark. – Equation (3.18) shows that the eigenvalues $\mu_{\Lambda,\omega}$ are decreasing functions of the coupling constants. This is in contrast to the Schrödinger case for which the eigenvalues are increasing functions of the coupling constants (see [4], equation (5.5)).

The main consequence of these three lemmas is the following proposition on the location of the spectrum of the local Hamiltonians.

PROPOSITION 3.1. – *For* $0 < \delta < (gB_+M_\infty)^{-1}[\min(\delta_+, \delta_-)]^2$, *we have*

$$\sup\left\{\sigma(H_{\Lambda,\omega}) \cap (-\infty, \tilde{B}_-)\right\} < \tilde{B}_- - \delta,$$

and

$$\inf\left\{\sigma(H_{\Lambda,\omega}) \cap (\tilde{B}_+, \infty)\right\} > \tilde{B}_+ + \delta,$$

with a probability larger than

$$1 - |\Lambda| \max_{X=m,M} \left| \int_{(1-\delta g B_+ M_\infty [\min(\delta_+,\delta_-)]^{-2})X}^{X} h(s)ds \right|.$$

Proof. – The probability that $\lambda_i(\omega) < (1-\delta g B_+ M_\infty [\min(\delta_+, \delta_-)]^{-2})X$, $\forall i \in \tilde{\Lambda}$, is given by

$$\left[1 - \left| \int_{(1-\delta g B_+ M_\infty \Delta^{-2})X}^{X} h(s)ds \right| \right]^{|\Lambda|}.$$

We choose $X = m$ for B_- and $X = M$ for B_+, and write $\Delta \equiv \min(\delta_-, \delta_+)$. The proposition now follows by expanding this probability and from Lemma 3.3. □

We note that hypothesis (H6) on the decay of the tail of the density h near the endpoints of its support m and M is essential in order to control the probability in Proposition 3.1.

It remains to verify the initial step of the multiscale analysis, namely $[\mathrm{H1}](\gamma_0, l_0)$ of [8]. Since there is no classically forbidden region for $H^X_{\omega,\omega'}$, we use the Combes-Thomas technique [13] rather than tunneling estimates. We verify $[\mathrm{H1}](\gamma_0, \ell_0)$ by combining Proposition 3.1 on the location of the spectrum of $H_{\Lambda_\ell,\omega}$ and the Combes-Thomas exponential decay estimate. The improved Combes-Thomas estimate of [4] is discussed in Appendix 3. We first give the decay estimate for the localized resolvent and then comment on the gradient term.

PROPOSITION 3.2. – *Let* $\chi_i, i = 1, 2$, *be two functions with* $\|\chi_i\|_\infty \leq 1$, *supp* $\chi_1 \subset \Lambda_{\ell/3}$ *and supp* χ_2 *localized near* $\partial\Lambda_\ell$ *and* $\delta_\pm \equiv \frac{1}{2}|\tilde{B}_\pm - B_\pm|$. *For* $\beta > 0$ *as in (H6), consider any* $\nu > 0$ *such that* $0 < \nu < 4\beta(2\beta + 3d)^{-1}$. *Then* $\exists \ell_0^\star \equiv \ell_0^\star(M_\infty, \delta_+, \delta_-, M)$ *such that* $\forall \ell_0 > \ell_0^\star$ *and* $\forall E \in (\tilde{B}_- - \ell_0^{\nu-2}, \tilde{B}_-] \cup [\tilde{B}_+, \tilde{B}_+ + \ell_0^{\nu-2})$,

$$\sup_{\varepsilon>0} \|\chi_2 R_{\Lambda_{\ell_0}}(E + i\varepsilon)\chi_1\| \leq e^{-\ell_0^{\nu/3}},$$

with probability $\geq 1 - \ell_0^{-\xi}$, *for some* $\xi > 2d$.

Proof. – From Proposition 3.1 and (H6), we compute the probability that $\sigma\big(H_{\Lambda_{\ell_0},\omega}\big)$ is at a distance $\delta = 2\ell_0^{\nu-2}$ from $\tilde{B}_\pm$,

$$\mathbb{P}\Big\{\text{dist}\Big(\sigma\big(H_{\Lambda_{\ell_0},\omega}\big), \tilde{B}_\pm\Big) > 2\delta\Big\} \geq 1 - \ell_0^d\Big(2\ell_0^{\nu-2} g B_+ M_\infty [\min(\delta_+, \delta_-)]^{-2} X\Big)^{3d/2+\beta}, \quad (3.21)$$

where $X = m$ for $\tilde{B}_-$ and $X = M$ for $\tilde{B}_+$. A simple computation shows that the right side of (3.21) is bounded below by $1 - \ell_0^{-\xi}$ for some $\xi > 2d$ provided ν satisfies $0 < \nu < 4\beta(2\beta + 3d)^{-1}$. We now apply Theorem 9.1 to $H_{\Lambda_{\ell_0},\omega}$. Let $E \in [\tilde{B}_- - \ell_0^{\nu-2}, \tilde{B}_-)$ and following the notation of Theorem 9.1, let $\Delta_- \equiv \text{dist}(\tilde{B}_- - \delta, E) > \delta/2 = \ell_0^{\nu-2}$ and $\Delta_+ \geq |\tilde{B}_+ - \tilde{B}_-|$. Since $\text{dist}(\text{supp}\,\chi_2, \text{supp}\,\chi_1) \geq \ell_0/3$ (in dimension $d > 9$, this is no longer true; one has to replace $\ell_0/3$ by $\ell_0/(3\sqrt{d})$, for the diameter of the inner cube), we obtain from Theorem 9.1,

$$\|\chi_2 R_{\Lambda_{\ell_0}}(E+i\varepsilon)\chi_1\| \leq C_2 \sup\Big(|\tilde{B}_+ - \tilde{B}_-|^{-1}, \ell_0^{2-\nu}\Big) \times e^{-\inf(\alpha_0, C_1\ell_0^{\nu/2-1}|\tilde{B}_+ - \tilde{B}_-|^{1/2}, (A_0/2)^{1/2}|\tilde{B}_+ - \tilde{B}_-|^{1/2})\ell_0/6}$$

The result follows by taking ℓ_0 large. □

PROPOSITION 3.3. – *$\exists \ell_0^\star$ such that $\forall \ell_0 > \ell_0^\star$, hypothesis* [H1] (γ_0, ℓ_0) *holds $\forall E \in (\tilde{B}_- - \ell_0^{\nu-2}, \tilde{B}_-] \cup [\tilde{B}_+, \tilde{B}_+ + \ell_0^{\nu-2})$ and any ν satisfying $0 < \nu < 4\beta(2\beta + 3d)^{-1}$, with β as in (H6).*

Proof. – We write the details for the acoustic case. The Maxwell case can be treated the same way with $p \equiv -i\nabla$ replaced by $p \cdot \epsilon$. As in Lemma 3.1, of [9], we write

$$\|W(\chi_{\ell,v}) R_{\Lambda_\ell} \chi_{\ell/3}\| \leq \|(\Delta\chi_{\ell,v}) R_{\Lambda_\ell} \chi_{\ell/3}\| + 2\sum_{j=1}^{d} \|(\partial_j \chi_{\ell,v})\, p_j A^{-1/2} R_{\Lambda_\ell} \chi_{\ell/3}\| \quad (3.22)$$

for a function $\chi_{\ell,v}$ localized within distance v of $\partial\Lambda_\ell$. Let χ_i, $i = 1, 2$, be smooth functions such that $\chi_i \chi_{\ell,v} = \chi_{\ell,v}$, $\chi_2\chi_1 = \chi_1$, and $\text{supp}\,\chi_i$ is localized within a distance $2v$ for $i = 1$ and $3v$ for $i = 2$, of $\partial\Lambda_\ell$. Then, we write for each j and any $u \in L^2(\mathbb{R}^d)$,

$$\begin{aligned}\|(\partial_j \chi_{\ell,v})\, p_j A^{-1/2} R_{\Lambda_\ell} u\|^2 &\leq C_0 \langle p_j A^{-1/2} R_{\Lambda_\ell} u, \chi_1 p_j A^{-1/2} \chi_2 R_{\Lambda_\ell} u\rangle \\ &\leq C_0 \|\chi_2 R_{\Lambda_\ell} u\|\; \|(p - A)_j \chi_1 (p - A)_j R_{\Lambda_\ell} u\|\end{aligned} \quad (3.23)$$

Taking $u = \chi_{\ell/3} f$, we see that (3.22) is bounded above as in Proposition 3.1 provided $\ell_0^\star$ is large enough and for any $\ell > \ell_0^\star$, with a probability $\geq 1 - \ell^{-\xi}$ since we have that $\|p^2 A^{-1/2} R_{\Lambda_\ell} u\|$ is bounded. □

Finally, we consider the question as to whether local, multiplicative perturbations can actually create eigenvalues in the spectral gap G of H_0. Recall that for Schrödinger operators, the proof of the existence of eigenvalues in a spectral gap G of a Schrödinger operator H_0 is rather easy under certain conditions (cf. [14] for a complete discussion). Consider a local, additive perturbation of H_0 of the form $H(\lambda) = H_0 - \lambda W$, where supp W is compact and the potential W has fixed sign. In this case, the Birman-Schwinger principle states that $E \in G$ is an eigenvalue of $H(\lambda)$ provided $1/\lambda$ is an eigenvalue of the nonzero, compact, self-adjoint operator $W^{1/2}(H_0 - E)^{-1} W^{1/2}$. By adjusting λ, we see that any $E \in G$ is an eigenvalue for some $H(\lambda)$.

The Birman-Schwinger principle for multiplicatively perturbed operators is similar. Suppose that H_0 has a spectral gap G as in (H2) and $E \in G$. Let V be a positive potential of compact support and define $A(\lambda) = 1 + \lambda V > 0$ for $\lambda \geq 0$. It is easy to show that E is an eigenvalue of $H = A(\lambda)^{-1/2} H_0 A(\lambda)^{-1/2}$ if and only if the compact operator

$$K(E) \equiv V^{1/2}(H_0 - E)^{-1} V^{1/2}, \tag{3.24}$$

has an eigenvalue $1/\lambda E$. As $K(E)$ is a nonzero, compact, self-adjoint operator, it follows that each $E \in G$ is an eigenvalue for some $H(\lambda)$. This proves that a local, multiplicative perturbation $A = 1 + V$ of $\tilde{H}_0$ can create an eigenvalue E in the unperturbed spectral gap G. Consequently, by Lemma 3.1, the almost sure spectrum $\Sigma \cap G \neq \emptyset$ at least in the large disorder regime. This verifies hypothesis (H8) for these examples.

4. SPECTRAL AVERAGING AND KOTANI'S TRICK

We apply the method of differential inequalities as used in [10] to obtain the following key technical lemma. We consider a one-parameter family H_λ of self-adjoint operators on a Hilbert space $\mathcal{H}$, which are multiplicative perturbations of a self-adjoint operator H_0 (note that this H_0 is not necessarily the operator appearing in (1.1)). We show at the end of section 5 how to associate a one-parameter family of multiplicatively perturbed operators to our acoustic and Maxwell models (1.1). Let $C_\lambda \equiv 1 + \lambda\eta$, with $\eta \geq 0$, $\sup \eta \equiv \eta_0 < \infty$. For $0 < \kappa < 1$, and any M_0 satisfying

$0 < M_0 \leq \infty$, let $J_\kappa \equiv [(\kappa - 1)\eta_0^{-1}, M_0)$. Then for $\lambda \in J_\kappa$, the function C_λ is invertible and $(1+\eta_0 M_0)^{-1} < C_\lambda^{-1} < 1/\kappa$. We define H_λ, $\lambda \in J_\kappa$, by $H_\lambda \equiv C_\lambda^{-1/2} H_0 C_\lambda^{-1/2}$. It is convenient to consider the family of operators $\tilde{H}_\lambda(z) \equiv H_0 - zC_\lambda$. For $\delta \equiv \text{Im} z \neq 0$, and $\lambda \in J_\kappa$, the operator $\tilde{H}_\lambda(z)$ is invertible. The injectivity of this family is easily seen from the estimate

$$|\delta|\kappa \parallel \phi \parallel^2 \leq |\text{Im}\langle\phi, \tilde{H}_\lambda(z)\phi\rangle| \leq \parallel \phi \parallel \parallel \tilde{H}_\lambda(z)\phi \parallel, \tag{4.1}$$

which holds for all $\phi \in D(H_0)$. An analogous estimate holds for $\tilde{H}_\lambda(z)^*$. We set $\tilde{R}_\lambda(z) \equiv \tilde{H}_\lambda(z)^{-1}$, $\text{Im} z \neq 0$. The boundedness of the inverse follows from the estimate,

$$|\delta|\kappa \parallel \tilde{R}_\lambda(z)\phi \parallel^2 \leq |\text{Im}\langle\phi, \tilde{R}_\lambda(z)\phi\rangle| \leq \parallel \phi \parallel \parallel \tilde{R}_\lambda(z)\phi \parallel . \tag{4.2}$$

Note that we have

$$(H_\lambda - z)^{-1} = C_\lambda^{-1/2} \tilde{R}_\lambda(z) C_\lambda^{-1/2}. \tag{4.3}$$

Below, a dot will denote differentiation with respect to λ.

LEMMA 4.1. – *For $E > 0$ and $\delta > 0$, define*

$$K(\lambda, \delta, \epsilon) \equiv \eta^{1/2}(\tilde{H}_\lambda(E + i\delta) + i\epsilon\dot{\tilde{H}}_\lambda(E + i\delta))^{-1}\eta^{1/2}. \tag{4.4}$$

The $\exists\, c_0 > 0$, independent of δ, such that for any real positive $g \in C_0^2(J_\kappa)$ and $\forall\, \phi \in \mathcal{H}$,

$$\sup_{\delta > 0} \left| \int_{J_\kappa} g(\lambda)\langle\phi, \eta^{1/2}\tilde{R}_\lambda(E + i\delta)\eta^{1/2}\phi\rangle \, d\lambda \right| \leq c_0 E^{-1} \parallel \phi \parallel^2 . \tag{4.5}$$

The constant c_0 depends only on η and $\parallel g^{(p)} \parallel_1$, $p = 0, 1,$ and 2.

Proof.

1. Let $z = E + i\delta$. We compute the first two λ derivatives of $\tilde{H}_\lambda(z)$. These are

$$\frac{d\tilde{H}_\lambda}{d\lambda}(z) = -z\eta, \tag{4.6}$$

and

$$\frac{d^2\tilde{H}_\lambda}{d\lambda^2}(z) = 0. \tag{4.7}$$

From (4.4) and (4.6), we obtain,

$$K(\lambda, \delta, \epsilon) = \eta^{1/2}(H_0 - (E + i\delta)C_\lambda + \delta\epsilon\eta - i\epsilon E\eta)^{-1}\eta^{1/2}. \tag{4.8}$$

We compute a lower bound

$$-\mathrm{Im}\,\langle \phi, K\phi\rangle = \langle \eta^{1/2}\phi, \tilde{K}^*(\delta C_\lambda + \epsilon E\eta)\tilde{K}\eta^{1/2}\phi\rangle \geq \epsilon E \parallel K\phi \parallel^2, \quad (4.9)$$

where

$$\tilde{K}(\lambda, \delta, \epsilon) \equiv (\tilde{H}_\lambda(E + i\delta) - i\epsilon\dot{\tilde{H}}_\lambda(E + i\delta))^{-1}. \quad (4.10)$$

By the Schwarz inequality and (4.9), we obtain

$$\parallel K(\lambda, \delta, \epsilon)\phi \parallel \leq (E\epsilon)^{-1} \parallel \phi \parallel . \quad (4.11)$$

Similarly, one establishes the inequalities,

$$(E\epsilon) \parallel K(\lambda, \delta, \epsilon)^*\phi \parallel^2 \leq \mathrm{Im}\,\langle \phi, K(\lambda, \delta, \epsilon)^*\phi\rangle \leq \parallel K(\lambda, \delta, \epsilon)^*\phi \parallel \parallel \phi \parallel, \quad (4.12)$$

so an estimate similar to (4.11) holds for K^*.

2. We define for $g \geq 0$ and $g \in C_0^2(J_\kappa)$, and for all $\phi \in \mathcal{H}$, with $\parallel \phi \parallel = 1$,

$$F(\delta, \epsilon) \equiv \int_{J_\kappa} g(\lambda)\langle \phi, K(\lambda, \delta, \epsilon)\phi\rangle \, d\lambda. \quad (4.13)$$

From (4.11), we have the *a priori* estimate

$$|F(\delta, \epsilon)| \leq \parallel g \parallel_1 (E\kappa_0\epsilon)^{-1}. \quad (4.14)$$

Our goal is to obtain an a priori estimate on $dF(\delta, \epsilon)/d\epsilon$. To this end, we calculate two derivatives

$$\frac{dK}{d\epsilon}(\lambda, \delta, \epsilon) = \eta^{1/2}\tilde{K}(-i\dot{\tilde{H}}_\lambda)\tilde{K}\eta^{1/2}, \quad (4.15)$$

and

$$\frac{dK}{d\lambda}(\lambda, \delta, \epsilon) = \eta^{1/2}\tilde{K}(-\dot{\tilde{H}}_\lambda)\tilde{K}\eta^{1/2}, \quad (4.16)$$

since, by (4.7), the second derivative of $\tilde{H}_\lambda$ with respect to λ vanishes.

This shows that $dK/d\epsilon$ can be replaced by $dK/d\lambda$. Using this result and differentiating $F(\delta, \epsilon)$, we obtain

$$\frac{dF}{d\epsilon}(\delta, \epsilon) = i \int_{J_\kappa} g(\lambda)\langle \phi, \dot{K}\phi\rangle. \quad (4.17)$$

After integrating by parts in (4.17), we obtain

$$\left|\frac{dF}{d\epsilon}(\delta, \epsilon)\right| \leq \left|\int_{J_\kappa} g'(\lambda)\langle \phi, K\phi\rangle\right|. \quad (4.18)$$

Using the estimate (4.11), this gives

$$\left|\frac{dF}{d\epsilon}(\delta,\epsilon)\right| \leq (E\epsilon)^{-1} \parallel g^{'} \parallel_1 . \tag{4.19}$$

We integrate this inequality to obtain

$$|F(\delta,\epsilon)| \leq |F(1,\delta)| + c_1 E^{-1} |\log \epsilon|. \tag{4.20}$$

Recalling that $|F(1,\delta)| \leq c_2 E^{-1}$, we get

$$|F(\delta,\epsilon)| \leq E^{-1}(c_2 + c_1 |\log \epsilon|), \tag{4.21}$$

where c_i are independent of ϵ, E, and δ. Note that this improves (4.14).
3. To iterate this procedure once more, we need an estimate similar to (4.21) for $\tilde{F}(\delta,\epsilon)$ defined by

$$\tilde{F}(\delta,\epsilon) \equiv \int_{J_\kappa} g^{'}(\lambda)\langle\phi, K(\lambda,\delta,\epsilon)\phi\rangle \, d\lambda.$$

Repeating the derivations above, we arrive at (4.17) with g replaced by g'. Consequently, we get as above

$$\left|\frac{d\tilde{F}}{d\epsilon}(\delta,\epsilon)\right| \leq (E\epsilon)^{-1} \parallel g'' \parallel_1 . \tag{4.22}$$

Integrating this inequality results in the bound

$$|\tilde{F}(\delta,\epsilon)| \leq E^{-1}(c_3 + c_4 |\log \epsilon|). \tag{4.23}$$

4. Finally, we use estimate (4.23) in the right side of (4.18) to obtain

$$\left|\frac{dF}{d\epsilon}(\delta,\epsilon)\right| \leq E^{-1}(c_5 + c_6 |\log \epsilon|).$$

Integrating this inequality, we find that $\exists\, c_0 > 0$ depending only on $\|g^{(i)}\|_1, i = 0, 1, 2$, so that

$$|F(\delta,\epsilon)| \leq E^{-1} c_0.$$

Taking $\epsilon \rightarrow 0$ and applying the Lebesgue Dominated Convergence Theorem yields the result. □

Remark. – We note the differential inequality method used in the proof applies to more general families of multiplicatively perturbed operators of the form $H_\lambda = C_\lambda^\sigma H_0 C_\lambda^\sigma$, $\sigma \in \mathbb{R}$, than treated here. For the family of operators $\tilde{H}_\lambda(z)$ described above, the dependence on λ is linear. In this case, the method of Simon and Wolff [36] can also be used (see also [8]).

Next, we obtain a version of Kotani's trick from Lemma 4.1. Let $E_\lambda(\cdot)$ be the spectral family for H_λ.

COROLLARY 4.1. – *In addition to the assumptions of Lemma 4.1, assume that* Ran η *is cyclic for* H_λ *in the sense that* $\{f(H_\lambda)\eta\phi | f \in L^\infty(\mathbb{R})$ *and* $\phi \in \mathcal{H}\}$ *is dense in* $\mathcal{H}$ *for each* $\lambda \in J_\kappa$. *Then for any Borel subset* $L \subset R^+$ *with* $|L| = 0$, *one has* $E_\lambda(L) = 0$ *Lebesgue a.e.* $\lambda \in J_\kappa$.

Proof. – It follows from Stone's formula that for any Borel set $K \subset \boldsymbol{R}^+$ and for all $\phi \in \mathcal{H}$,

$$\begin{aligned}&\langle \eta\phi, C_\lambda^{-1/2} E_\lambda(K) C_\lambda^{-1/2} \eta\phi\rangle \\ &\leq \frac{1}{\pi} \overline{\lim_{\delta\downarrow 0}} \operatorname{Im} \int_K dE\, \langle \eta\phi, C_\lambda^{-1/2}(H_0 - (E + i\delta))^{-1} C_\lambda^{-1/2} \eta\phi\rangle. \qquad (4.24)\end{aligned}$$

Then for any $g \geq 0$, $g \in C_0^2(J_\kappa)$, Fubini's Theorem and (4.24) imply that

$$\begin{aligned}&\int_{J_\kappa} g(\lambda) \langle \eta\phi, C_\lambda^{-1/2} E_\lambda(K) C_\lambda^{-1/2} \eta\phi\rangle \, d\lambda \\ &\leq \frac{1}{\pi} \overline{\lim_{\delta\downarrow 0}} \operatorname{Im} \int_K dE \int_{J_\kappa} d\lambda\, g(\lambda) \langle \eta\phi, C_\lambda^{-1/2}(H_0 - (E+i\delta))^{-1} C_\lambda)^{-1/2} \eta\phi\rangle.\end{aligned}$$

By Lemma 4.1, the integral over J_κ is uniformly bounded in δ so

$$\int_{J_\kappa} g(\lambda) \langle \eta\phi, C_\lambda^{-1/2} E_\lambda(K) C_\lambda^{-1/2} \eta\phi\rangle \, d\lambda \leq c(K) \parallel \phi \parallel^2 \ |K|, \qquad (4.25)$$

where $c(K)$ depends on $||\eta||_\infty$, $\inf K$, and c_0 as in Lemma 4.1. Finally using the positivity of g, if $L \subset \boldsymbol{R}^+$ s.t. $|L| = 0$, then from (4.25) and the cyclicity assumption,

$$C_\lambda^{-1/2} E_\lambda(L) C_\lambda^{-1/2} = 0,$$

for Lebesgue a.e. $\lambda \in J_\kappa$. Since C_λ is invertible, the result follows. □

5. PERTURBATION OF SINGULAR SPECTRA

We continue our study of multiplicatively perturbed operators of the form H_λ, $\lambda \in J_\kappa$, on the Hilbert space $\mathcal{H}$, as in section 4. We keep all notation of that section, so

$$H_\lambda = C_\lambda^{-1/2} H_0 C_\lambda^{-1/2}, \tag{5.1}$$

and

$$C_\lambda = 1 + \lambda\eta, \tag{5.2}$$

for some non-negative, bounded function η and for $\lambda \in J_\kappa$. We introduce a bounded operator $B_\lambda \equiv 1 - C_\lambda^{-1}$, and note that by (5.2)

$$B_\lambda = \lambda\eta C_\lambda^{-1}. \tag{5.3}$$

As in section 4, we define $\tilde{R}_\lambda(z) \equiv (H_0 - zC_\lambda)^{-1}$, which exists for $\operatorname{Im} z \neq 0$, and we write $R_\lambda(z) = (H_\lambda - z)^{-1}$, whenever it exists. As in [8], we consider two *a priori* assumptions on $R_0(z)$ relative to an interval $I \subset \mathbb{R}^+$ disjoint from zero.

(A1) $R_0(z)\eta$ is compact for all $z \in \mathbb{C}$, $\operatorname{Im} z > 0$.

(A2) $\exists I_0 \subset I$, $|I_0| = |I|$, such that for all $E \in I_0$,

$$\sup_{\epsilon \neq 0} \| R_0(E + i\epsilon)\eta \| \leq C(E) < \infty.$$

In the Maxwell case, the operator H_0 in condition (A1) is replaced by $H_0 P_0^\perp$. Let $\tilde{\mathcal{H}}_\lambda \equiv [f(H_\lambda)\eta\phi,\ f \in L^\infty(\mathbb{R}),\ \phi \in \mathcal{H}]^{cl}$ and denote by $\tilde{H}_\lambda$ the restriction of H_λ to this cyclic subspace.

THEOREM 5.1. – *Assume (A1)-(A2). Then* $\sigma_{ac}(\tilde{H}_\lambda) \cap I = \phi\ \ \forall \lambda \in J_\kappa$, *and* $\sigma(\tilde{H}_\lambda) \cap I \subset \bar{I}_0$ *and is pure point for a.e.* $\lambda \in J_\kappa$.

Proof. – We first derive the analog of the Aronszajn-Donoghue formula. For $z \in \mathbb{C}$, $\operatorname{Im} z > 0$,

$$\begin{aligned} R_\lambda(z) &= C_\lambda^{1/2}(H_0 - C_\lambda z)^{-1} C_\lambda^{1/2} \\ &= -C_\lambda^{1/2}\{(H_0 - z)^{-1} - (H_0 - C_\lambda z)^{-1}\} C_\lambda^{1/2} + C_\lambda^{1/2} R_0(z) C_\lambda^{1/2} \\ &= zC_\lambda^{1/2} R_0(z) C_\lambda^{1/2} B_\lambda R_\lambda(z) + C_\lambda^{1/2} R_0(z) C_\lambda^{1/2}. \end{aligned}$$

Multiplying this formula on the right by η and re-arranging it, we find

$$C_\lambda^{1/2} R_0(z)\eta C_\lambda^{1/2} = (1 - zC_\lambda^{1/2} R_0(z) C_\lambda^{1/2} B_\lambda) R_\lambda(z)\eta. \tag{5.4}$$

In the Maxwell case, the resolvent $R_0(z)$ in (5.4) is written as $R_0(z) = R_0(z)P_0^\perp + (-z^{-1})P_0$. For $|\lambda| < 1$, the term $(1 + C_\lambda^{1/2} P_0 C_\lambda^{1/2} B_\lambda)$, on the right side of (5.4), has norm less than one. (This can be assumed, without loss of generality, by adjusting the parameter g, if necessary.) Hence, this term, which is independent of z, is invertible. After inverting this term, the term containing $P_0^\perp$ is compact and invertible for almost all values of E, as described below. Condition (A1) implies that $C_\lambda^{1/2} R_0 B_\lambda C_\lambda^{1/2}$ is compact for $Imz \neq 0$. It follows that $1 - zC_\lambda^{1/2} R_0(z) C_\lambda^{1/2} B_\lambda$ is invertible by the Fredholm Theorem. For, if it is not invertible at some z with $Imz \neq 0$, it follows that $1 - zC_\lambda^{1/2} R_0(z) C_\lambda^{1/2} B_\lambda$ has an eigenvalue 0. It is easy to see that this implies that H_λ would have an eigenvalue z. Furthermore, for $E \in I_0$, condition (A2) implies that

$$\begin{aligned} n - \lim_{\epsilon \to 0} (1 - (E + i\epsilon) C_\lambda^{1/2} R_0(E + i\epsilon) C_\lambda^{1/2} B_\lambda) \\ = 1 - E C_\lambda^{1/2} R_0(E) C_\lambda^{1/2} B_\lambda, \end{aligned}$$

exists and that this non-trivial operator is compact. Applying Fredholm theory and the Bounded Inverse Theorem, we see that this limit is boundedly invertible if and only if there is no non-trivial, $\mathcal{H}$-valued solution of

$$(1 - E C_\lambda^{1/2} R_0(E) C_\lambda^{1/2} B_\lambda)\xi = 0, \tag{5.5}$$

for $E \in I_0$. We show that when (5.5) holds, E is an eigenvalue of H_λ. We sketch the proof as it follows the same lines as [8]. Let us define a vector $\psi_{\epsilon,\lambda} \equiv R_0(E + i\epsilon) B_\lambda C_\lambda^{1/2} \xi$. It follows from the above that $w - \lim_{\epsilon \to 0} \psi_{\epsilon,\lambda} \equiv \psi_\lambda$ exists in $\mathcal{H}$. For any $\phi \in \mathcal{H}$,

$$\begin{aligned} \langle E\psi_\lambda, C_\lambda^{1/2}\phi\rangle &= \lim_{\epsilon \to 0} \langle z\psi_{\epsilon,\lambda}, C_\lambda^{1/2}\phi\rangle \\ &= \langle \xi, \phi\rangle, \end{aligned} \tag{5.6}$$

so $\psi_\lambda \neq 0$ since $\xi \neq 0$ and $\phi \in \mathcal{H}$ is arbitrary. It follows that

$$\langle \psi_\lambda, (H_0 - E)\phi\rangle = \langle B_\lambda C_\lambda^{1/2} \xi, \phi\rangle. \tag{5.7}$$

Replacing ϕ in both equations (5.6)–(5.7) by $C_\lambda^{1/2}\phi$, we obtain

$$\langle \psi_\lambda, (H_0 - E)\phi\rangle = E\langle \psi_\lambda, B_\lambda C_\lambda \phi\rangle. \tag{5.8}$$

By the identity $(1 + B_\lambda C_\lambda) = C_\lambda$, it now follows from (5.8) that

$$\langle \psi_\lambda, (H_0 - C_\lambda E)\phi\rangle = 0,$$

so $C_\lambda^{1/2}\psi_\lambda$ is an eigenfunction of H_λ with eigenvalue E. Since the set of eigenvalues of H_λ is countable, there exists a set $I_\lambda \subset I$ of full measure for which

$$n - \lim_{\epsilon \to 0} (1 - (E + i\epsilon) C_\lambda^{1/2} R_0(E + i\epsilon) C_\lambda^{1/2} B_\lambda)^{-1},$$

exists $\forall E \in I_\lambda$. The resolvent formula (5.4), condition (A.2), and this result imply that $\forall E \in I_\lambda$,

$$\limsup_{\epsilon \to 0} \|R_\lambda(E + i\epsilon)\eta\| \leq C(E),$$

for some finite constant $C(E)$. This now implies that $\sigma_{ac}(\tilde{H}_\lambda) \cap I = \phi$, $\forall \lambda \in J_\kappa$. By the theorem of de la Vallée-Poussin, these results also imply that $\sigma_{sc}(\tilde{H}_\lambda) \cap I \subset I \setminus I_0$, which has measure zero. As $I \subset \boldsymbol{R}^+$ is disjoint from zero, it now follows from Corollary 4.1 that $\sigma_{sc}(\tilde{H}_\lambda) \cap I = \phi$ a.e. $\lambda \in J_\kappa$. This proves the theorem. □

We apply Theorem 5.1 to families $H^X_{\omega,\omega'}$ of multiplicatively perturbed operators (1.13)–(1.16) as follows. Fix the parameter $\omega' \in \Omega'$ and consider variations of ω of the form $\omega \to \tilde{\omega}$ with

$$\lambda_i(\omega) = \lambda_i(\tilde{\omega}), \quad i \neq 0,$$

and

$$\lambda_0(\omega) \neq \lambda_0(\tilde{\omega}).$$

Dropping the frozen subscript ω', we have

$$\begin{aligned} H_{\tilde{\omega}} &= (1 + gV_{\tilde{\omega}})^{-1/2} H_0 (1 + gV_{\tilde{\omega}})^{-1/2} \\ &= \left(\frac{1 + gV_\omega}{1 + gV_{\tilde{\omega}}}\right)^{1/2} H_\omega \left(\frac{1 + gV_\omega}{1 + gV_{\tilde{\omega}}}\right)^{1/2}, \end{aligned}$$

where $V_\omega(x) = \sum_{i \in \boldsymbol{Z}^d} \lambda_i(\omega) u(x - i - \xi_i(\omega'))$. Let $C_\lambda \equiv (1 + gV_{\tilde{\omega}})(1 + gV_\omega)^{-1}$, in analogy with the multiplier C_λ introduced above. To see the connection, let us define a parameter $\lambda \equiv \lambda_0(\tilde{\omega}) - \lambda_0(\omega)$. Because of the supp$h$, we have $\lambda \in [-m - \lambda_0(\omega), M - \lambda_0(\omega)]$. By definition, we have

$$\begin{aligned} 1 + gV_{\tilde{\omega}} &= 1 + g(\lambda_0(\tilde{\omega}) - \lambda_0(\omega))u + gV_{\tilde{\omega}} \\ &= 1 + g\lambda u + gV_\omega, \end{aligned}$$

so that

$$C_\lambda = 1 + g\lambda u(1 + gV_\omega)^{-1}.$$

Let $\eta \equiv gu(1 + gV_\omega)^{-1}$, and note that $\eta > 0$. We observe that C_λ so defined is invertible for $\lambda \in [-m - \lambda_0(\omega), M - \lambda_0(\omega)]$. Consequently, the results of Lemma 4.1, Corollary 4.1, and Theorem 5.1 hold for

$$H_\lambda = C_\lambda^{-1/2} H_\omega C_\lambda^{-1/2},$$

with ω frozen and for λ in the interval specified above.

We must also clarify the cyclicity assumption of Theorem 5.1 when applied to our models. For the acoustic model, the Hilbert space $\tilde{\mathcal{H}}_\lambda$ is actually equal to the entire Hilbert space $\mathcal{H}=L^2(\mathbb{R}^d)$. Because of the reduction described above, it suffices to show that $\{f(H_\omega)\eta\phi \mid \phi \in L^2(\mathbb{R}^d)\}$ is dense in $L^2(\mathbb{R}^d)$ for any ω. Due to the invertibility of $A_\omega^{-1/2}$, this result is implied by the density of $\{f(H_0 - E_0 A_\omega)\eta\phi \mid \phi \in L^2(\mathbb{R}^d)\}$, for some real E_0 in the resolvent set of H_ω. In the case that $H_0 = -\Delta + V_0$, it is shown in Appendix 2 of [8] that for $V_0 \in L^\infty(\mathbb{R}^d)$ and for any bounded, open, non-empty subset $\Lambda \subset \mathbb{R}^d$, the span of the set $\{f(H_0)\phi | \phi \in C_0^\infty(\Lambda)$ and $f \in L^\infty(\mathbb{R}^d)\}$ is dense in $L^2(\mathbb{R}^d)$. The demonstration of this is based on a theorem of F. H. Lin [33] concerning unique continuation for the parabolic equation associated with H_0.

In the Maxwell case, we consider vectors of the form $f(H_\omega)\eta\phi$, where $\phi \in L^2(\mathbb{R}^3, \mathbb{C}^3)$. The Maxwell operator has the form

$$H_\omega = -(1 + \epsilon_\omega)^{-1/2} \Delta \Pi (1 + \epsilon_\omega)^{-1/2} = Q^* Q, \tag{5.9}$$

as described in section 1. We need to prove density in the orthogonal complement of ker Q. It is easy to check that

$$\ker Q = \{(1 + \epsilon_\omega)^{-1/2}\xi \mid \xi \in \ker \Pi\}. \tag{5.10}$$

Since the coefficient $(1 + \epsilon_\omega)$ is boundedly invertible, it suffices to examine the states of the form $\Pi \Delta I_3 u \psi$, for ψ in some dense set. The matrix ΔI_3 is diagonal. It follows as above that then set of vectors of the form $\Delta I_3 u \psi$ is dense in $L^2(\mathbb{R}^3, \mathbb{C}^3)$. Since the projector Π is onto the subspace $\Pi L^2(\mathbb{R}^3, \mathbb{C}^3)$, the image of the dense set is dense. Finally, it follows from the coments on the kernel of Q that the set of vectors of the form $(1 + \epsilon_\omega)^{-1/2} \Pi \Delta u \psi$ is dense on the orthogonal complement of ker Q, and the result follows.

6. PROOF OF LOCALIZATION

The proof of localization near the unperturbed spectral gaps follows as in [4, 8], given the Wegner estimate, Theorem 2.1, and the initial decay estimate [H1](γ_0, l_0) as established in Proposition 3.3. A multiscale analysis is used to prove the uniform boundedness estimate, assumption (A1) of section 5, needed in the proof of Theorem 5.1. With regard to the multiscale analysis as presented in Appendix 1 of [8], we mention the modifications necessary in the present case. If $\Lambda \subset I\!R^d$ is a bounded region, we define the local operator $H_\Lambda = A_\Lambda^{-1/2} H_0 A_\Lambda^{-1/2}$, as above, on $L^2(I\!R^d)$ in the acoustic case, and on $L^2(I\!R^3, \mathbb{C}^3)$ in the Maxwell case. The geometric resolvent equation (GRE) takes the following form. Suppose $\Lambda \subset \Lambda'$ are two bounded regions and χ_Λ is a smoothed characteristic function on Λ such that $|\nabla \chi_\Lambda|$ is localized near the boundary of Λ. In the Maxwell case, we write χ_Λ for the diagonal matrix $\chi_\Lambda I_3$. The GRE equation is

$$\chi_\Lambda R_{\Lambda'}(z) = R_\Lambda(z)\chi_\Lambda + R_\Lambda(z) A_\omega^{-1/2} [H_0, \chi_\Lambda] A_\omega^{-1/2} R_{\Lambda'}(z). \tag{6.1}$$

The commutator is a first-order differential operator (a matrix operator in the Maxwell case) with support on $\text{supp}\,|\nabla \chi_\Lambda|$. We let

$$W_{\Lambda,\omega} \equiv A_\omega^{-1/2} [H_0, \chi_\Lambda] A_\omega^{-1/2}, \tag{6.2}$$

where H_0 is given by

$$H_0 = \begin{cases} (1+\epsilon_0)^{-1/2} (\epsilon \cdot p)^2 (1+\epsilon_0)^{-1/2} & \textit{Maxwell} \\ -\, C_0 \rho^{1/2} \nabla \cdot \rho^{-1} \nabla C_0 \rho^{1/2} & \textit{Acoustic.} \end{cases} \tag{6.3}$$

With this definition, the GRE assumes its usual form

$$\chi_\Lambda R_{\Lambda'}(z) = R_\Lambda(z)\chi_\Lambda + R_\Lambda(z) W_{\Lambda,\omega} R_{\Lambda'}(z). \tag{6.4}$$

Although $W_{\Lambda,\omega}$ now depends on ω, the dependence is not important due to the uniform boundedness of $A_\omega^{-1/2}$.

Next, we recall an equality of Agmon ([1], pg. 20, eqn. 1.16), which is useful for bounding the product of a gradient and the resolvent:

$$\|\nabla(\psi u)\|^2 = \text{Re}\,\langle \nabla u, \nabla(\psi^2 u)\rangle + \|\,|\nabla\psi| u\|^2, \tag{6.5}$$

which is valid for all suitable $u \in H^1(I\!R^d)$ and smooth ψ. (We mention that this identity was incorrectly stated on page 175 of [8], although the inequality (A.11) is correct.) In the acoustic case, the operator $W_{\Lambda,\omega}$ has the form,

$$W_{\Lambda,\omega} = -A_{\Lambda}^{-1}C_0^2\rho\nabla\cdot(\rho^{-1}\nabla\chi_{\Lambda}) \\ - 2(A\rho)^{-1/2}C_0(\nabla\chi_{\Lambda}\cdot\nabla)\rho^{1/2}C_0A_{\Lambda}^{-1/2}. \quad (6.6)$$

We apply the Agmon identity (6.5) to the second term on the right side of (6.6). Because of the boundedness of the coefficient $(A\rho)^{-1/2}C_0$, we take ψu on the left side of (6.5) to be of the form $\psi_j u$, with $\psi_j = \partial_j\chi_{\Lambda_{ql'}}$ and $u = \rho^{1/2}C_0A_{\Lambda_{ql'}}^{-1/2}R_{ql'}(E+i\epsilon)\chi_{l'/3}\phi$, for any normalized ϕ. In computing the right side of (6.5), we use the fact that $(\partial_i\chi_{l'})\ \chi_{l'/3} = 0$. To obtain an estimate on $W_{l'}R_{ql'}$, we note that

$$\|\partial_k(\psi_j u)\|^2 \leq \|\nabla(\psi_j u)\|^2,$$

Setting $k = j$ and summing over this index, we find that there exist constants $C_1 > 0$ and $C_2 > 0$, depending on d, C_0, ρ, and $\|\partial^\alpha\chi_{l'}\|_\infty$, for a multi-index $|\alpha| = 0, 1,$ and 2, such that

$$\|W_{l'}R_{ql'}(E+i\epsilon)\chi_{l'/3}\| \leq (C_1 + C_2\sqrt{E})\|\tilde{\chi}_{l'}R_{ql'}\chi_{l'/3}\|, \quad (6.7)$$

as in [8], (A.11). A similar calculation in the Maxwell case leads to the analog of (6.7).

With these modifications, the proof of (A1) using the multiscale analysis proceeds as in [8]. The probabilistic part of the argument is the same as in [8].

These results suffice to prove Theorem 1.1 in the manner as given in [8].

APPENDIX 1

Properties of multiplicatively perturbed, periodic families

We summarize the situation when the background operator H_0 is invariant with respect to the group of translations on $\mathbb{R}^d$, with $d = 3$ in the Maxwell case. For example, suppose the unperturbed dielectric function $(1 + \epsilon_0)$ in (1.10), or the unperturbed sound speed C_0 and density ρ in (1.6), are Γ-periodic functions, where Γ is a d-dimensional subgroup of $\mathbb{Z}^d$. Then, due to the nature of the Anderson-type perturbation (1.12), the perturbed families (1.16) will be Γ-periodic. We would like to verify the conditions necessary so that the family of random operators (1.16) has a deterministic spectrum, as discussed after condition (H7) in section 1.

As in section 1, the families of random operators have the form

$$H^X_{\omega,\omega'} = (A^X_{\omega,\omega'})^{-1/2} H_0 (A^X_{\omega,\omega'})^{-1/2}, \tag{7.1}$$

for $X = A$ or $X = M$, the acoustic or Maxwell case, respectively. The random coefficients $A^X_{\omega,\omega'}$ are given in (1.12)–(1.16). It is clear that under these conditions these operators are self-adjoint on $H^2(\mathbb{R}^d)$ for $X = A$, and on $H^2(\mathbb{R}^3, \mathbb{C}^3)$ for $X = M$.

LEMMA 7.1. – *The family of random operators defined in (7.1) is measurable.*

Proof. – We establish the measurability of the function $(\omega, \omega') \in \Omega \times \Omega' \rightarrow H^X_{\omega,\omega'}$. It suffices (cf., [12]) to prove that the resolvent $R_{\omega,\omega'}(z) \equiv (H^X_{\omega,\omega'} - z)^{-1}$ is weakly measurable. Under assumptions on the random variables, the potential u, and the coupling constant g, it is clear that

$$0 \leq A^{-1}_{\omega,\omega'} \leq C_0, \tag{7.2}$$

for some constant $0 < C_0 < \infty$, and for all $(\omega, \omega') \in \Omega \times \Omega'$. The bounded multiplication operator $A^{-1/2}_{\omega,\omega'}$ is measurable. Let $\rho : \mathbb{R}^+ \rightarrow [0, 1]$ be compactly supported and such that $\rho \geq 0$, $\rho|[0,1] = 1$. Define $\rho_\lambda(x) \equiv \rho(x/\lambda)$ for $\lambda > 0$. For any non-negative, self-adjoint operator A, $(1 - \rho_\lambda(A)) \rightarrow 0$ strongly as $\lambda \rightarrow \infty$. We define the resolvent of a cut-off H_0 by

$$R_{\omega,\omega'}(z; \lambda) \equiv (A^{-1/2}_{\omega,\omega'}(H_0\ \rho_\lambda(H_0)) A^{-1/2}_{\omega,\omega'} - z)^{-1}. \tag{7.3}$$

By the resolvent formula, we obtain,

$$\begin{aligned} &R_{\omega,\omega'}(z) - R_{\omega,\omega'}(z;\lambda) \\ &= -R_{\omega,\omega'}(z;\lambda) A^{-1/2}_{\omega,\omega'}(1 - \rho_\lambda(H_0)) H_0 A^{-1/2}_{\omega,\omega'} R_{\omega,\omega'}(z). \end{aligned} \tag{7.4}$$

The right side converges strongly to zero since $H_0 A^{-1/2}_{\omega,\omega'} R_{\omega,\omega'}(z)$ is bounded (recall that $A_{\omega,\omega'}$ is bounded). Now the operator $A^{-1/2}_{\omega,\omega'}(H_0\ \ \rho_\lambda(H_0)) A^{-1/2}_{\omega,\omega'}$ is measurable. To see this, recall that if $\{u_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis, and $\{E_i \equiv |u_i\rangle\langle u_i|\}$ is the set of corresponding rank-one projections, then $\sum_{i=1}^{n} E_i \rightarrow 1$ strongly. Consequently, we have

$$\begin{aligned}&\langle f, A_{\omega,\omega'}^{-1/2}(H_0\ \rho_\lambda(H_0))A_{\omega,\omega'}^{-1/2} g\rangle \\ &= \sum_{i=1}^{\infty} \langle A_{\omega,\omega'}^{-1/2} f, (H_0\ \rho_\lambda(H_0))u_i\rangle\langle u_i, A_{\omega,\omega'}^{-1/2} g\rangle,\end{aligned} \tag{7.5}$$

and each term on the right side is measurable. Consequently, $R_{\omega,\omega'}(z;\lambda)$ is measurable and by the strong convergence of (7.4), we get the measurability of $H^X_{\omega,\omega'}$. □

PROPOSITION 7.1. – *The family of random operators defined in (7.1) with the coefficients satisfying hypotheses (H5) and (H7) has a deterministic spectrum. That is, there exists a closed subset* $\Sigma \subset I\!R$ *such that* $\sigma(H_{\omega,\omega'}) = \Sigma$ *for almost every* (ω,ω').

Proof. – This follows now by standard results ([12], Chapter 5) given the measurability of the family, the ergodicity of the translation subgroup Γ, and the invariance of the operators under the action of Γ. Note that in the acoustic case, the two coefficients ϵ_0 and ρ of H_0 must both be invariant with respect to the subgroup Γ. □

We now turn to some examples of periodic background operators H_0 for which there exists a gap in the spectrum as demanded in hypothesis (H2). Materials with periodic dielectric properties, for which the periodicity matches optical wavelengths, are called *Photonic Crystals*. The classical wave spectrum of such materials will consist of band structure, which can be established using standard Floquet theory (cf. [32]). The main question is whether there are materials with open gaps in the spectrum. The presence of an open gap implies that electromagnetic waves at certain frequencies will not propagate in the material. The existence of open gaps in the spectrum of periodic acoustic and Maxwell operators has been studied for certain models by Figotin and Kuchment [22, 23]. They prove the existence of open gaps for two-component media with high dielectric contrast. These materials consist of host medium with a dielectric constant $\epsilon_h > 1$ into which are embedded "atoms" modeled by cubic regions of dielectric constant $\epsilon_a = 1$. The "atomic cubes" are nonoverlapping and centered on the lattice. The host material fills the corridor regions between the cubes and it is assumed that the diameter $\delta << 1$ of these corridors is small. Kuchment and Figotin prove the existence of gaps in the spectrum of the periodic acoustic operator in $d = 2, 3$ dimensions, and the periodic Maxwell equations in $d = 2$ dimensions when, roughly speaking, ϵ_h is sufficiently large and δ is sufficiently small.

A refinement of this modeling of the atoms is given by replacing the regions where $\epsilon_a = 1$ by a more realistic approximation allowing radiative transitions. This type of periodic media is called an *optical atomic lattice*. Physically, these lattices are composed of periodic arrays of atoms held together by the interference patterns created by crossed laser beams. The band structure of a dipolar lattice was computed in [39]. This band structure is characterized by two dimensionless parameters. The authors found conditions on these parameters for which there always exists an open gap in the optical spectrum. They also confirmed the presence of open gaps for certain values of these parameters in the scalar wave approximation (acoustic waves). These materials are of interest because radiation from atoms in the material, whose frequency lies in the optical band gap, will be suppressed.

APPENDIX 2

A summary of certain trace ideal estimates

In this appendix, we summarize the trace ideal estimates needed in the proof of the Wegner estimate, Theorem 2.1. Recall that the pair of multi-indexed operators $K_{\{i\}}^{(j)}$, $j = 1, 2$, are defined in (2.22) and (2.23), respectively. The following lemma is a direct consequence of the trace ideal assumption (H3) and the Hölder inequality for trace ideals:

$$||AB||_r \leq ||A||_p \; ||B||_q, \tag{8.1}$$

for nonnegative numbers $r, s,$ q satisfying $1/r = (1/p) + (1/q)$ (cf. [35]).

LEMMA 8.1. – *Assume (H1)-(H3) and (H7). Then* $K_{\{i\}}^{(j)}$ *is a trace class operator where, for* $j = 1$, *the set* $\{i\}$ *is a q-tuple, and for* $j = 2$, *the set* $\{i\}$ *is a* $q+1$*-tuple. There exists a finite constant* $\widetilde{C}_{E_0} > 0$, *depending only on dist*$(\sigma(H_0), E_0)^{-1}$ *and* $d \geq 1$, *such that* $||K_{\{i\}}^{(j)}||_1 \leq \widetilde{C}_{E_0}$.

The main result of this appendix is the following proposition which establishes (2.30).

PROPOSITION 8.1. – *Under the assumptions of Lemma 8.1, for any* $E_0 \in (B_-, B_+)$, $\exists$ *finite constant* $C_{E_0} > 0$, *depending only on dist*$(\sigma(H_0), E_0)^{-1}$ *and* $d \geq 1$, *such that*

$$\sum_{i_1, \ldots, i_{q'} \in \widetilde{\Lambda}} ||K_{\{i\}}^{(j)}||_1 \leq C_{E_0} |\Lambda|, \tag{8.2}$$

where $q' = q$, *for* $j = 1$, *and* $q' = q + 1$, *for* $j = 2$.

For the sake of completeness, we will sketch the outline of the proof of Proposition 8.1 in this appendix, and refer the reader to [4] for the details. The following calculations concern only the unperturbed Hamiltonians H_0, as given in (1.6) and (1.10). We assume the existence of a spectral gap for H_0 and fix $E_0 \in (B_-, B_+) \subset \rho(H_0)$. To simplify the notation, we write $R_0 \equiv (H_0 - E_0)^{-1}$.

The first step in the proof of Proposition 8.1 is a variant of the Combes-Thomas result (see Appendix 3) on the decay of the localized resolvent. The proof follows that of Lemma 7.3 in [4].

LEMMA 8.2. – *Assume (H1)-(H3) and (H7). Suppose that* $\chi_1, \chi_2 \in C^\infty(\mathbb{R}^d)$, *with supp* χ_1 *compact and such that supp* χ_2 *lies in a half-space disjoint from supp* χ_1, $\|\chi_i\|_\infty = 1$, *dist(supp* χ_1, *supp* $\chi_2) \geq a > 0$, *for some* $a > 0$. *Then, the operator* $\chi_1 R_0 \chi_2 \in \mathcal{J}_1$. *Furthermore, there exist finite constants* $D > 0$, $\alpha > 0$ *such that*

$$\|\chi_1 R_0 \chi_2\|_1 \leq D\, e^{-\alpha a}, \tag{8.3}$$

where D *and* α *depend only on dist*$(\sigma(H_0), E_0)^{-1}$.

The main idea in the proof of Proposition 8.1 is to use the exponential decay, as described in Lemma 8.2, to control the summations whenever two sites i_m and i_n are sufficiently far apart. This is done as follows. We divide the set of indices $\{i\} \subset \tilde{\Lambda}^{q'}$ into two sets, according to whether there is a pair of successive indices far apart relative to the size of the support of u. Let $\eta \equiv 2\text{diam supp}u$ and choose any $a > 0$ so that $\eta < a < 2\eta$. Then, if $\|i_m - i_n\| > a$, we have $\text{dist}(\text{supp}u_{i_n}, \text{supp}u_{i_m}) > a/2$. By Lemma 8.2, this implies that $\|u_{i_n} R_0 u_{i_m}\|_{1,2}$ decays exponentially. A q'-tuple $\{i\}$ is in $I_1 \subset \tilde{\Lambda}^{q'}$ if $\|i_{k-1} - i_k\| > a \forall k = 2, 3, \ldots q'$. Let I_2 be the complementary set of indices. The sum over I_1 is easily seen to be bounded above by $C(a, d, q)\ |\Lambda|$, for a constant depending on supp$u$, the dimension d, and the integer q. As for the sum over I_2, we first sum over all i_1 such that there exists some $q' - 1$-tuple $(i_2, \ldots, i'_q)$ for which $(i_1, \ldots, i'_q) \in I_2$. Whenever $\|i_1 - i_2\| > a$, we use Lemma 8.2 to evaluate the trace norm. Whenever $\|i_1 - i_2\| < a$, we simply bound the operator norm of $u_1 R_0 u_2$ by the constant $C(a, d, 1)$, which is independent of $|\Lambda|$. We continue in this way until getting to the last pair of summation indices. Then, we use the exponential decay when the points are separate or the fact that the sum is independent of $|\Lambda|$ when the points are close. We remark that $K^{(1)}_{\{i\}}$ can be evaluated exactly as in [4] if we first take the adjoint, or, we can commute u_{i_1} through the first resolvent.

The last facts that we need in section 2 are given in the following proposition. The proof is given in [4] and follows the same lines as sketched above.

PROPOSITION 8.2. – *Let* $K_0 \equiv A^{1/2} R_0 V_\Lambda A^{-1/2}$, *then there exists a finite constant* C_0, *depending only on* $(dist(\sigma(H_0), E_0))^{-1}$, *the dimension d, and q, so that*

$$\mathbb{E}(\|K_0\|_q{}^q) \leq C_0\,|\Lambda|. \tag{8.4}$$

APPENDIX 3

Combes-Thomas estimates on the resolvent

We summarize the major parts of the improved Combes-Thomas estimate [13] given in [4].

LEMMA 9.1. – *Let* A *and* B *be two self-adjoint operators such that* $d_\pm \equiv \mathrm{dist}(\sigma(A) \cap \mathbb{R}^\pm, 0) > 0$, *and* $\|B\| < 1$. *Then,*

(i) *For* $\beta \in \mathbb{R}$ *s.t.* $|\beta| < \frac{1}{2}\sqrt{d_+ d_-}$, *one has* $0 \in \rho(A + i\beta B)$,

(ii) *For* $\beta \in \mathbb{R}$ *as in (i),*

$$\left\|(A + i\beta B)^{-1}\right\| \leq 2\sup(d_+^{-1}, d_-^{-1}).$$

Proof. – Let $P_\pm$ be the spectral projectors for A corresponding to the sets $\sigma(A) \cap \mathbb{R}^\pm$, respectively and define $u_\pm \equiv P_\pm u$. By the Schwarz inequality one has

$$\begin{aligned} \|u\|\;\|(A + i\beta B)u\| &\geq Re\langle (u_+ - u_-), (A + i\beta B)(u_+ + u_-)\rangle \\ &\geq d_+\|u_+\|^2 + d_-\|u_-\|^2 - 2\beta Im\langle u_+, Bu_-\rangle \quad (9.1) \\ &\geq \frac{1}{2}(d_+\|u_+\|^2 + d_-\|u_-\|^2), \end{aligned}$$

where we again used the Schwarz inequality to estimate the inner product. It follows that

$$\|(A + i\beta B)u\| \geq \frac{1}{2}\min(d_+, d_-)\|u\|,$$

and since this is independent of the sign of β, the lemma follows. □

Remark. – We have assumed that both $d_\pm$ are finite so that 0 belongs to a spectral gap of A of finite width. If 0 is below the bottom of the spectrum of A, the distance d_- is not defined. In this case, the usual Combes-Thomas argument gives an exponential factor $\sqrt{d_+}$.

PROPOSITION 9.1. – *Let $\widetilde{H}$ be a semibounded self-adjoint operator with a spectral gap $G \equiv (E_-, E_+) \subset \rho(\widetilde{H})$. Let W be a symmetric operator such that $D(W) \supset D((\widetilde{H}+C_0)^{\frac{1}{2}})$ and $\|(\widetilde{H}+C_0)^{-\frac{1}{2}}W(\widetilde{H}+C_0)^{-\frac{1}{2}}\| < 1$, for some C_0 such that $\widetilde{H}+C_0 > 1$. For any $E \in G$, let $\Delta_\pm \equiv \mathrm{dist}(E_\pm, E)$. Then, we have*

(i) *The energy $E \in \rho(\widetilde{H}+i\beta W)$ for all real β satisfying*

$$|\beta| < \frac{1}{2}\left\{\frac{\Delta_+\Delta_-}{(E_+ + C_0)(E_- + C_0)}\right\}^{\frac{1}{2}};$$

(ii) *for any real β and energy E as in (i),*

$$\|(\widetilde{H}+i\beta W - E)^{-1}\| \leq 2\sup\left(\frac{E_+ + C_0}{\Delta_+}, \frac{E_- + C_0}{\Delta_-}\right).$$

Proof. – Let $E \in G$ and C_0 be as above. Define a self-adjoint operator $A \equiv (\widetilde{H}+C_0)^{-1}(\widetilde{H}-E)$ and $B \equiv (\widetilde{H}+C_0)^{-\frac{1}{2}}W(\widetilde{H}+C_0)^{-\frac{1}{2}}$. By hypothesis, the operator B is self-adjoint and satisfies $\|B\| < 1$. Note that $0 \in \rho(A)$ and

$$d_\pm \equiv \mathrm{dist}(\sigma(A) \cap I\!R^\pm, 0) = \Delta_\pm(E_\pm + C_0)^{-1} > 0 \tag{9.2}$$

Applying Lemma 3.1 to these operators A and B, we see that for β as in (i), $0 \in \rho(A + i\beta B)$ and that

$$\|(A+i\beta B)^{-1}\| \leq 2\sup\left(\frac{E_+ + C_0}{\Delta_+}, \frac{E_- + C_0}{\Delta_-}\right).$$

Let $P_\pm$ be as in the proof of Lemma 3.1. For any $w \in D(\widetilde{H})$,

$$\begin{aligned}\|(\widetilde{H}+i\beta W - E)w\| &= \|(\widetilde{H}+C_0)^{\frac{1}{2}}(A+i\beta B)(\widetilde{H}+C_0)^{\frac{1}{2}}w\| \\ &\geq \|(A+i\beta B)(\widetilde{H}+C_0)^{\frac{1}{2}}w\|,\end{aligned}$$

since $(\widetilde{H}+C_0) \geq 1$. We now repeat estimate (3.1) taking $u \equiv (\widetilde{H}+C_0)^{\frac{1}{2}}w$. This gives

$$\begin{aligned}\|(\widetilde{H}+i\beta W - E)w\| &\geq \frac{1}{2}\|(\widetilde{H}+C_0)^{\frac{1}{2}}u\|^{-1}\Big(d_+\|P_+(\widetilde{H}+C_0)^{\frac{1}{2}}w\|^2 \\ &\quad +(d_-\|P_-(\widetilde{H}+C_0)^{\frac{1}{2}}w\|^2\Big) \qquad (9.3)\\ &\geq \frac{1}{2}\min(d_+, d_-)\|(\widetilde{H}+C_0)^{\frac{1}{2}}w\|.\end{aligned}$$

Since $\|(\widetilde{H}+C_0)^{\frac{1}{2}}w\| \geq \|w\|$ and $d_\pm$ are defined in (3.2), result (ii) follows from (3.3) and Lemma 3.1. □

We now sketch the application of these results to multiplicatively perturbed operators of the form

$$H = A^{-1/2} H_0 A^{-1/2}, \tag{9.4}$$

where H_0 has the form

$$H_0 = \begin{cases} -(1+\epsilon_0)^{-1/2}(\epsilon \cdot p)^2(1+\epsilon_0)^{-1/2} & Maxwell \\ C_0\rho^{1/2}p \cdot \rho^{-1}pC_0\rho^{1/2} & Acoustic. \end{cases} \tag{9.5}$$

Since the coefficients play no role in this discussion, let us define $\tilde{A}$ as follows:

$$\tilde{A} \equiv \begin{cases} (1+\epsilon_0)^{-1}A & Maxwell \\ C_0^2\rho A & Acoustic. \end{cases} \tag{9.6}$$

We define a smoothed distance function $d(x) \equiv (1+\|x\|^2)^{1/2}$. Let $\alpha \in \mathbb{R}$. We compute the unitary conjugation of H_0 by $e^{i\alpha d}$. In either case, the operator can be written in the form

$$\begin{aligned} H(\alpha) &\equiv e^{i\alpha d} H_0 e^{-i\alpha d} \\ &= H_0 - \alpha \tilde{A}^{-1/2} Q_1 \tilde{A}^{-1/2} + \alpha^2 \tilde{A}^{-1} Q_2, \end{aligned} \tag{9.7}$$

where

$$Q_1 = \begin{cases} (\epsilon \cdot \nabla d)(\epsilon \cdot p) + (\epsilon \cdot p)(\epsilon \cdot \nabla d) & Maxwell \\ \rho^{-1}\nabla d \cdot p + p \cdot \rho^{-1}\nabla & Acoustic \end{cases} \tag{9.8}$$

and

$$Q_2 = \begin{cases} (\epsilon \cdot \nabla d)^2 & Maxwell \\ \rho^{-1}|\nabla d|^2 & Acoustic \end{cases} \tag{9.9}$$

Under the hypotheses of section 1, we show that $H(\alpha)$ admits an extension to an analytic type-A family of operators in a strip $S_{\alpha_0} = \{z \in \mathbb{C} \mid |\mathrm{Im} z| < \alpha_0\}$, for any $\alpha_0 > 0$. Let $Q(\alpha) = -\alpha \tilde{A}^{-1/2} Q_1 \tilde{A}^{-1/2} + \alpha^2 \tilde{A}^{-1} Q_2$. Then it suffices to show that for some $z \in \mathbb{C} \backslash \mathbb{R}$,

$$\|Q(\alpha)(H-z)^{-1}\| < 1. \tag{9.10}$$

It is easy to check that for any $\alpha_0 > 0$, we have the bound

$$\|Q(\alpha)(H-z)^{-1}\| \leq [\mathrm{dist}(\sigma(H), z)]^{-1} C_0, \tag{9.11}$$

for a constant depending on α_0, $\|\tilde{A}\|$, and $\|\partial^\beta \rho\|$, for $|\beta| = 1, 2$.

THEOREM 9.1. – *Let H_0 be given as in (1.17)–(1.18) satisfying (H1), and let A be a bounded, invertible multiplication operator satisfying $0 < A_0 \leq \|A\| \leq A_1$, for two finite constants A_0 and A_1. Then the dilated operator $H(\alpha) \equiv e^{i\alpha d} H e^{-i\alpha d}$, $\alpha \in I\!R$, admits an analytic continuation to a type-A family on the strip $S(\alpha_0)$, for any $\alpha_0 > 0$. Suppose H_0 satisfies (H2) and that H has a spectral gap $G \equiv (E_-, E_+) \subset (B_-, B_+)$ $(E_- \neq E_+)$. For $E \in G$, define $\Delta_\pm \equiv \mathrm{dist}(E_\pm, E)$. Then there exist finite constants C_1, $C_2 > 0$, depending only on H_0 and A, such that*

(i) *for any real β satisfying $|\beta| < \min(\alpha_0, C_1\sqrt{\Delta_+\Delta_-}, \sqrt{A_0\Delta_+/2})$, the energy $E \in \rho(H(i\beta))$;*

(ii) *for any real β as in (i),*

$$\|(H(i\beta) - E)^{-1}\| \leq C_2 \max(\Delta_+^{-1}, \Delta_-^{-1}). \tag{9.12}$$

Proof. – From the calculations above, we have,

$$H(\alpha) = H + \alpha^2 A^{-1} Q_2 + \alpha W,$$

where $\alpha \in I\!R$ and $W = -A^{-1/2} Q_1 A^{-1/2}$ is symmetric. The existence of the analytic extension in α to $S(\alpha_0)$ is proved above. Taking $\alpha = i\beta, \beta$ real and $|\beta| < \alpha_0$, we have

$$H(i\beta) = H - \beta^2 A^{-1} Q_2 + i\beta W$$

We apply Proposition 9.1 to this operator taking $\widetilde{H} \equiv H - \beta^2 |\nabla\rho|^2$. This operator has a spectral gap which contains $(\widetilde{E}_-, \widetilde{E}_+)$, where $\widetilde{E}_- = E_-$ and $\widetilde{E}_+ = E_+ - \beta^2 A_0^{-1}$. In order that $\widetilde{\Delta}_+ \equiv \mathrm{dist}(\widetilde{E}_+, E) > (\Delta_+/2)$, we require $|\beta| < \sqrt{A_0\Delta_+/2}$. (Note that $\widetilde{\Delta}_- = \Delta_-$). We can now apply Proposition 9.2 to conclude $E \in \rho(H(i\beta))$ for $|\beta| < \min\{\alpha_0, C_1\sqrt{\Delta_+\Delta_-}, \sqrt{A_0\Delta_+/2}\}$ and that (9.4) holds. □

APPENDIX 4

Maxwell's equations and the hamiltonian

We show how to reduce Maxwell's equations to an operator of the form (1.1). We set $\mu = 1$, so $B = H$, and write Maxwell's equations for a medium with dielectric function ϵ and no sources as

$$\partial_t B = -\nabla \times (\epsilon^{-1} D) \tag{10.1}$$

$$\partial_t D = \nabla \times H \tag{10.2}$$

$$\nabla \cdot D = 0 \tag{10.3}$$

$$\nabla \cdot B = 0, \tag{10.4}$$

where $D \equiv \epsilon E$. For this reduction and further details concerning Maxwell's equations, we refer to [15]. We introduce a six-component state vector F as

$$F = \begin{pmatrix} \epsilon^{1/2} E \\ B \end{pmatrix}, \tag{10.5}$$

and note that Maxwell's equations (10.1)–(10.4) can be written in the form

$$i\partial_t F = KF, \tag{10.6}$$

where

$$K = \begin{pmatrix} 0 & \epsilon^{-1/2} \boldsymbol{\epsilon} \cdot p \\ -\boldsymbol{\epsilon} \cdot p \epsilon^{-1/2} & 0 \end{pmatrix}, \tag{10.7}$$

is a 6×6 matrix, $p \equiv -i\nabla$, and $\boldsymbol{\epsilon}$ is the Levi-Civita tensor. With this definition, the energy of the electromagnetic field is

$$W = \frac{1}{2} \int |F(x)|^2 \, dx, \tag{10.8}$$

so that on $L^2(\boldsymbol{R}^3, \mathbb{C}^6)$, the time-evolution given by (10.6) is formally unitary. The divergence free conditions (10.3)–(10.4) imply that

$$\begin{pmatrix} p\epsilon^{1/2} & 0 \\ 0 & p \end{pmatrix} F = 0. \tag{10.9}$$

Because of (10.6), F satisfies the wave equation

$$\partial_t^2 F + K^2 F = 0, \tag{10.10}$$

where

$$K^2 = \begin{pmatrix} -\epsilon^{-1/2}(\boldsymbol{\epsilon} \cdot p)^2 \epsilon^{-1/2} & 0 \\ 0 & -(\boldsymbol{\epsilon} \cdot p)\epsilon^{-1}(\boldsymbol{\epsilon} \cdot p) \end{pmatrix}. \tag{10.11}$$

Let $Q \equiv (\boldsymbol{\epsilon} \cdot p)\epsilon^{-1/2}$, so $Q^* = -\epsilon^{-1/2}(\boldsymbol{\epsilon} \cdot p)$, and write $F = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. Then the wave equation (10.10) becomes the pair of equations

$$\partial_t^2 \phi_1 + Q^* Q \phi_1 = 0, \tag{10.12}$$

$$\partial_t^2 \phi_2 + QQ^* \phi_2 = 0. \tag{10.13}$$

The two operators Q^*Q and QQ^* can be written as

$$Q^*Q = -\epsilon^{-1/2}\Delta\Pi\epsilon^{-1/2}, \tag{10.14}$$

and

$$QQ^* = -\Pi\nabla\cdot\epsilon^{-1}\nabla\Pi, \tag{10.15}$$

where Π is the orthogonal projection on $L^2(\mathbb{R}^3, \mathbb{C}^3)$ defined as follows. For $g \in L^2(\mathbb{R}^3, \mathbb{C}^3)$, let $\hat{g}$ be the Fourier transform. Then

$$(\Pi g)^\wedge(p) = (1 - \hat{p}^T \cdot \hat{p})\hat{g}(p), \tag{10.16}$$

where

$$\begin{aligned} \hat{p}^T \cdot \hat{p} &= |p|^{-2}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \cdot (p_1\ p_2\ p_3) \\ &= |p|^{-2}\begin{pmatrix} p_1^2 & p_1p_2 & p_1p_3 \\ p_1p_2 & p_2^2 & p_2p_3 \\ p_1p_3 & p_2p_3 & p_3^2 \end{pmatrix}. \end{aligned} \tag{10.17}$$

The projection Π has the following property. If $g \in L^2(\mathbb{R}^3, \mathbb{C}^3)$ has a Fourier transform $\hat{g}$ of the form

$$\hat{g}(p) = \hat{p}\xi(p), \quad \hat{p} = p|p|^{-1}, \tag{10.18}$$

for some $\xi \in L^2(\mathbb{R}^3)$, then

$$\Pi g = 0. \tag{10.19}$$

That is, the operator Π projects onto states transverse to p. When $\epsilon = 1$, it follows from (10.12)–(10.15) that only transverse modes of the electromagnetic field in free space propagate. The orthogonal complement of Ran Π consists of longitudinal modes.

The situation for nonconstant ϵ is as follows. If the field vector $F = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ satisfies (10.12)–(10.13) and the divergence-free conditions (10.3)–(10.4), then the component $\phi_2 \in \Pi L^2(\mathbb{R}^3, \mathbb{C}^3)$, and is purely transverse ($\phi_2 = B$ and $\mu = 1$), whereas ϕ_1 must satisfy

$$\nabla \cdot \epsilon^{1/2}\phi_1 = 0. \tag{10.20}$$

Both of these divergence-free conditions (10.3)–(10.4) are preserved under the time-evolution generated by K^2, as follows from (10.1)–(10.2). Consequently, the generator QQ^* on $L^2(\mathbb{R}^3, \mathbb{C}^3)$ preserves the subspace $\Pi L^2(\mathbb{R}^3, \mathbb{C}^3)$, and the generator Q^*Q on $L^2(\mathbb{R}^3, \mathbb{C}^3)$ preserves the subspace determined by (10.20). Let P_ϵ be the projection onto this invariant subspace. When ϵ is a random variable, this projector selects a random subspace of $L^2(\mathbb{R}^3, \mathbb{C}^3)$. We are interested in computing $\sigma(P_\epsilon Q^*Q)$ almost surely. This can be obtained from a study of $\sigma(Q^*Q)$ a. s. , since the orthogonal complement of $P_\epsilon L^2(\mathbb{R}^3, \mathbb{C}^3)$ consists precisely of the null space of Q^*Q. The same is true for QQ^*. Hence, we can consider either equation (10.12) or (10.13) on $L^2(\mathbb{R}^3, \mathbb{C}^3)$. Furthermore, it follows from general results (cf. [25]) that $\sigma(QQ^*) \setminus \{0\} = \sigma(Q^*Q) \setminus \{0\}$. Recall that for multiplicative perturbations, localization only occurs at internal band edges. Hence, the nature of the spectrum at zero will not concern us. The operator H_0, and its perturbations, does have an infinite-dimensional subspace corresponding to the eigenvalue zero. The relative compactness condition (H3) requires that we project out this subspace and work with the reduced Hamiltonian. As this causes only minor changes in the argument, we simply point out in the text where these modifications are required.

We will concentrate on (10.12) and Q^*Q given in (10.14). As in section 1, we will write Q^*Q as

$$H_{\omega,\omega'} = -A_{\omega,\omega'}^{-1/2} \Pi \Delta A_{\omega,\omega'}^{-1/2}, \tag{10.21}$$

which is related to (10.14) by $A_{\omega,\omega'} = \epsilon_{\omega,\omega'}$. We take $\epsilon_{\omega,\omega'}^X$ as in section 1, (1.13)–(1.15).

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