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# **Huygens' principle for the non-self-adjoint scalar wave equation on Petrov type III space-times**

by

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**ABSTRACT.** – We prove that if the non-self-adjoint scalar wave equation satisfies Huygens' principle on Petrov type III space-times, then it is equivalent to the conformally invariant scalar wave equation. © Elsevier, Paris

*Key words:* Huygens' principle, scalar wave equation, Petrov type III, space-times.

**RÉSUMÉ.** – On démontre que la validité du principe de Huygens pour l'équation des ondes scalaires non-auto-adjointe sur un espace-temps général de type III de Petrov implique que l'équation est équivalente à l'équation invariante conforme des ondes scalaires. © Elsevier, Paris

## 1. INTRODUCTION

In this paper we consider the general linear second-order hyperbolic equation on a four-dimensional curved space-time  $V_4$ , with  $C^\infty$  coefficients:

$$Pu := g^{ab}\nabla_a\nabla_b u + A^a(x)\partial_a u + C(x)u = 0, \quad (1)$$

where  $g^{ab}$  is a contravariant pseudo-Riemannian metric with signature  $(+ - - -)$  on  $V_4$ ,  $u$  is the unknown scalar function,  $\nabla_a$  denotes the covariant derivative with respect to the Levi-Civita connection,  $A^a$  is a vector field and  $C$  is a scalar field in  $V_4$ . All considerations are restricted to a geodesically convex domain. The equation (1) is also called the *non-self-adjoint scalar wave equation*.

*Huygens' principle* is said to be valid for an equation of the form (1) if and only if for every Cauchy initial problem, and for each point  $x_0 \in V_4$ , the solution depends only on the Cauchy data in an arbitrarily small neighbourhood of  $S \cap C^-(x_0)$ , where  $S$  denotes the initial surface and  $C^-(x_0)$  the past null conoid of  $x_0$ . Such equations are called *Huygens' equations*.

Necessary conditions for the validity of Huygens' principle for (1) are given by

$$(I) \quad C := C - \frac{1}{2}A^i{}_{;i} - \frac{1}{4}A_i A^i - \frac{1}{6}R = 0, \quad (2)$$

$$(II) \quad H^k{}_{a;k} = 0, \quad (3)$$

$$(III) \quad S_{abk}{}^k - \frac{1}{2}C^k{}_{ab}{}^l L_{kl} = -5 \left( H_{ak} H_b{}^k - \frac{1}{4}g_{ab} H_{kl} H^{kl} \right), \quad (4)$$

$$(IV) \quad TS(3S_{abk} H^k{}_c + C^k{}_{ab}{}^l H_{ck;l}) = 0, \quad (5)$$

$$(V) \quad TS(3C^k{}_{ab}{}^l{}^m C_{kcd;l} + 8C^k{}_{ab}{}^l{}_{;c} S_{kl} + 40S_{ab}{}^k S_{cdk} - 8C^k{}_{ab}{}^l S_{kl;c} \\ - 24C^k{}_{ab}{}^l S_{cdk;l} + 4C^k{}_{ab}{}^l C_l{}^m{}_{ck} L_{dm} + 12C^k{}_{ab}{}^l C^m{}_{cd} L_{km} \\ + 12H_{ka;b} H^k{}_d - 16H_{ka;b} H^k{}_{c;d} - 84H^k{}_a C_{kbcl} H^l{}_d \\ - 18H_{ka} H^k{}_b L_{cd}) = 0, \quad (6)$$

$$(VI) \quad TS(36C^k{}_{ab}{}^l C_{lcdm;k} H^m{}_e - 6C^k{}_{ab}{}^l{}_{;c} C_{lde}{}^m H_{km} - 138S_{ab}{}^k C_{kcdl} H^l{}_e \\ + 6S_{abk} H^k{}_{c;de} + 6C^k{}_{ab}{}^l{}_{;c} H_{kd;le} - 24S_{abk;c} H^k{}_{d;e} \\ + 12C^k{}_{ab}{}^l L_{kc} H_{ld;e} - 9C^k{}_{ab}{}^l{}_{;c} L_{kd} H_{le} - 9S_{abk} L_{cd} H^k{}_e) = 0, \quad (7)$$

where

$$H_{ij} := A_{[i,j]}, \tag{8}$$

$$C_{ijkl} := R_{ijkl} - 2g_{[i[l}L_{j]k]}, \tag{9}$$

$$S_{ijk} := L_{i[j;k]}, \tag{10}$$

$$L_{ij} := -R_{ij} + \frac{R}{6}g_{ij}. \tag{11}$$

In the expressions above  $R_{abcd}$  denotes the Riemann tensor,  $C_{abcd}$  the Weyl tensor,  $R_{ab} := g^{cd}R_{cabd}$ , the Ricci tensor, and  $R := g^{ab}R_{ab}$  the Ricci scalar associated to the metric  $g_{ab}$ , and  $A_a := g_{ab}A^b$ . Conditions I-IV were obtained by Günther [7], condition V was derived by Wunsch [18] for the self-adjoint case and McLenaghan [8] for the non-self-adjoint case. Condition VI was obtained by Anderson and McLenaghan [8]. We shall use here the two-component spinor formalism of Penrose [14] and the spin-coefficient formalism of Newman and Penrose [13], [15], whose conventions we follow. The spinor equivalents of the tensors (8), (9), (10), (11) are given by:

$$C_{abcd} \leftrightarrow \Psi_{ABCD}\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}} + \bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}\epsilon_{AB}\epsilon_{DC}, \tag{12}$$

$$H_{ab} \leftrightarrow 2(\phi_{AB}\epsilon_{\dot{A}\dot{B}} + \bar{\phi}_{\dot{A}\dot{B}}), \tag{13}$$

$$L_{ab} \leftrightarrow 2(\Phi_{AB\dot{A}\dot{B}} - \Lambda\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}), \tag{14}$$

$$S_{abc} \leftrightarrow \Psi^D{}_{ABC;D\dot{A}}\epsilon_{\dot{C}\dot{B}} + \bar{\Psi}^D{}_{\dot{A}\dot{B}\dot{C};D\dot{A}}\epsilon_{CB}, \tag{15}$$

where  $\Psi_{ABCD} = \Psi_{(ABCD)}$  is the Weyl spinor,  $\Lambda = (1/24)R$ ,  $\phi_{AB} = \phi_{(AB)}$  is the Maxwell spinor, and  $\Phi_{AB\dot{A}\dot{B}} = \Phi_{(AB)(\dot{A}\dot{B})} = \bar{\Phi}_{\dot{A}\dot{B}AB}$  is the trace-free Ricci spinor.

We can set up, at each point of space-time, a *dyad basis*  $\{o^A, \iota^A\}$  satisfying the relation  $o_A\iota^A = 1$ . The NP-components of any spinor are defined by projecting the spinor into the the local basis  $\{o^A, \iota^A\}$ . For the curvature spinors and Maxwell spinor we have

$$\begin{aligned} \Psi_{ABCD} &= \Psi_0\iota_{ABCD} - 4\Psi_1o_{(A}\iota_{BCD)} + 6\Psi_2o_{(AB}\iota_{CD)} \\ &\quad - 4\Psi_3o_{(ABC}\iota_{D)} + \Psi_4o_{ABCD}, \end{aligned} \tag{16}$$

$$\begin{aligned} \Phi_{AB\dot{A}\dot{B}} &= \Phi_{22}o_{AB}\bar{o}_{\dot{A}\dot{B}} - 2\Phi_{21}o_{AB}\bar{o}_{(\dot{A}}\iota_{\dot{B})} - 2\Phi_{12}o_{(A}\iota_{B)}\bar{o}_{\dot{A}\dot{B}} \\ &\quad + \Phi_{20}o_{AB}\bar{\iota}_{\dot{A}\dot{B}} + \Phi_{02}\iota_{AB}\bar{\iota}_{\dot{A}\dot{B}} + 4\Phi_{11}o_{(A}\iota_{B)}\bar{o}_{(\dot{A}}\iota_{\dot{B})} \\ &\quad - 2\Phi_{10}o_{(A}\iota_{B)}\bar{o}_{(\dot{A}}\iota_{\dot{B})} - 2\Phi_{01}\iota_{AB}\bar{\iota}_{\dot{A}\dot{B}}, \end{aligned} \tag{17}$$

$$\phi_{AB} = \phi_0\iota_{AB} - 2\phi_1o_{(A}\iota_{B)} + \phi_2o_{AB}, \tag{18}$$

where  $\iota_{ABCD} = \iota_A\iota_B\iota_C\iota_D$ , etc.

The covariant derivatives of the dyad basis spinors are given in terms of the NP-spin coefficients by

$$o_{A;B\dot{B}} = o_A I_{B\dot{B}} + \iota_A II_{B\dot{B}}, \quad \iota_{A;B\dot{B}} = o_A III_{B\dot{B}} - \iota_A I_{B\dot{B}}, \quad (19)$$

where

$$\left. \begin{aligned} I_{B\dot{B}} &:= \gamma o_B \bar{o}_{\dot{B}} - \alpha o_B \bar{\iota}_{\dot{B}} - \beta \iota_B \bar{o}_{\dot{B}} + \varepsilon \iota_B \bar{\iota}_{\dot{B}}, \\ II_{B\dot{B}} &:= -\tau o_B \bar{o}_{\dot{B}} + \rho o_B \bar{\iota}_{\dot{B}} + \sigma \iota_B \bar{o}_{\dot{B}} - \kappa \iota_B \bar{\iota}_{\dot{B}}, \\ III_{B\dot{B}} &:= \nu o_B \bar{o}_{\dot{B}} - \lambda o_B \bar{\iota}_{\dot{B}} - \mu \iota_B \bar{o}_{\dot{B}} + \pi \iota_B \bar{\iota}_{\dot{B}}. \end{aligned} \right\} \quad (20)$$

The necessary conditions I-VI can be expressed in terms of dyad components, containing only the Newman-Penrose scalars. This conversion procedure consists in two steps. Firstly, we have to convert the tensorial expressions into spinor form, using (12)-(15). Secondly, the spinor equations must be expressed in terms of dyad components, using (16), (17), (18), and (19). We perform these lengthy calculations automatically with the NPspinor package [4], [6], available in the Maple computer algebra system.

Using the fact that  $H_{[ij;k]} = 0$ , it can be shown that the spinor form of condition II is given by:

$$(II_s) \quad \phi_{AK};^K_{\dot{A}} = 0. \quad (21)$$

For condition III a direct application of the correspondence relations yields

$$\begin{aligned} \Psi_{ABKL};^K_{\dot{A}}{}^L_{\dot{B}} + \bar{\Psi}_{\dot{A}\dot{B}\dot{K}\dot{L}};^{\dot{K}}_{\dot{A}}{}^{\dot{L}}_{\dot{B}} + \Psi_{AB}{}^{KL}\Phi_{K\dot{L}\dot{A}\dot{B}} \\ + \bar{\Psi}_{\dot{A}\dot{B}}{}^{\dot{K}\dot{L}}\Phi_{\dot{K}\dot{L}AB} + 10\phi_{AB}\bar{\phi}_{\dot{A}\dot{B}} = 0. \end{aligned} \quad (22)$$

Instead of (22) we shall use a stronger form of this condition, obtained by Wunsch [19] and McLenaghan and Williams [11]:

$$(III_s) \quad \nabla^K_{\dot{A}}\nabla^L_{\dot{B}}\Psi_{ABLK} + \Phi^{KL}_{\dot{A}\dot{B}}\Psi_{ABKL} + 5\phi_{AB}\bar{\phi}_{\dot{A}\dot{B}} = 0. \quad (23)$$

While the original necessary condition (22) is Hermitian, (23) is complex.

The conversion of the remaining conditions to the respective spinor form and the determination of the dyad components is done automatically by defining templates in the NPspinor package. The spinor form of the trace-free symmetric part of a tensor is obtained by taking the correspondent spinor equivalent of that tensor and symmetrizing with respect to all dotted and undotted indices [17].

McLenaghan and Walton [10] have shown that any non-self-adjoint equation (1) on any Petrov type N space-time satisfies Huygens' principle

if and only if it is equivalent to a scalar equation with  $A_a = 0$  and  $C = 0$ , on an space-time corresponding to the exact plane-wave metric.

In this paper we prove the correspondent result for Petrov type III space-times. In this case the Weyl spinor has the form:

$$\Psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\beta_{D)}, \tag{24}$$

where  $\alpha_A$  and  $\beta_A$  are the principal spinors. If we choose the spin basis such that  $\alpha_A$  is proportional to the dyad basis spinor  $o_A$  and  $\beta$  proportional to  $\iota_A$ , we obtain from (16):

$$\Psi_{ABCD} = -4\Psi_3 o_{(A}o_B\iota_C\iota_{D)}. \tag{25}$$

The dyad transformation

$$o' = e^{w/2}o, \quad \iota' = e^{-w/2}(\iota + qo), \tag{26}$$

where  $w$  and  $q$  are complex functions, induces the following transformation on  $\Psi_3$ :

$$\Psi'_3 = e^{-w}(\Psi_3 + 3q\Psi_2 + 3q^2\Psi_1 + q^3\Psi_0). \tag{27}$$

Thus, we can choose the tetrad such that  $\Psi_3 = -1$ , so that

$$\Psi_{ABCD} = 4o_{(A}o_B\iota_C\iota_{D)}. \tag{28}$$

In a recent paper, Anderson, McLenaghan and Walton [3] have proved the following theorems:

**THEOREM 1.** – *The validity of Huygens' principle for any non-self-adjoint scalar wave equation (1) in any Petrov type III space-time implies that the space-time is conformally related to one in which every repeated null vector field of the Weyl tensor  $l_a$ , is recurrent, i.e.,*

$$l_{[a}l_{b;c]} = 0. \tag{29}$$

**THEOREM 2.** – *There exist no non-self-adjoint Huygens' equations (1) on any Petrov type III space-time for which the following conditions hold*

$$\left. \begin{aligned} \Psi_{ABCD;E\dot{E}}\iota^A\iota^B\iota^C o^D\iota^E\bar{o}^{\dot{E}} &= 0, \\ \Psi_{ABCD;E\dot{E}}\iota^A\iota^B\iota^C o^D o^E\bar{o}^{\dot{E}} &= 0, \\ \Psi_{ABCD;E\dot{E}}\iota^A\iota^B\iota^C\iota^D o^E\bar{o}^{\dot{E}} &= 0. \end{aligned} \right\} \tag{30}$$

In the next section we show that the restrictions imposed in Theorem 2 may be removed.

## 2. MAIN THEOREM

The main result of this paper is expressed by the theorem:

**THEOREM 3.** – *If a non-self-adjoint scalar wave equation of the form (1) satisfies Huygens' principle on any Petrov type III space-time, then it must be equivalent to a conformally invariant scalar wave equation*

$$g^{ab}\nabla_a\nabla_bu + \frac{1}{6}Ru = 0. \quad (31)$$

*Proof.* – We shall first prove, using the necessary conditions II to VI given by (3)-(7), that the assumption  $A_{[a,b]} := H_{ab} \neq 0$  leads to a contradiction. In terms of the dyad components of the Maxwell tensor  $H_{ab}$ , this is the same as proving that the necessary conditions imply  $\phi_0 = \phi_1 = \phi_2 = 0$ . Finally we invoke a lemma by Günther [7] that states that every equation of the form (1) for which  $A_{[a,b]} := 0$  is related by a trivial transformation to one for which  $A_a = 0$ . It then follows from the necessary condition I (eq. (2)) that  $C = R/6$ . We use a notation for the dyad components of the necessary conditions in the form  $X_{ab}$ , where X is the Roman numeral corresponding to the necessary condition,  $a$  denotes the number of indices corresponding to the dyad spinor  $\iota$  and  $b$  the number of dotted indices corresponding to the dyad spinor  $\bar{\iota}$ . We shall refer to the Newman-Penrose field equations using the notation NP1, NP2, etc. as listed in the Appendix. The methods employed in this proof are similar to those used in [12].

We start with the result obtained by Anderson and McLenaghan [3], expressed in following lemma:

**LEMMA 1.** – *For the non-self-adjoint scalar equation of the form (1), in Petrov type III space-times, the necessary conditions II, III, IV, V and VI together with the assumption that the Maxwell spinor  $\phi_{AB}$  is nonzero, imply that there exists a spinor dyad  $\{o_A, \iota_A\}$  and a conformal transformation such that*

$$\left. \begin{aligned} \kappa = \sigma = \rho = \tau = \epsilon = 0, \\ \Psi_0 = \Psi_1 = \Psi_4 = 0, \Psi_3 = -1, \\ \Phi_{00} = \Phi_{01} = \Phi_{02} = \Lambda = 0, \\ D\alpha = D\beta = D\Phi_{11} = 0. \end{aligned} \right\} \quad (32)$$

In what follows we shall use the relations (32) where necessary. It was shown in [3] that  $\phi_0 = \phi_1 = 0$ . Thus, what remains to be proved is that the assumption  $\phi_2 \neq 0$  leads to a contradiction.

Let us assume initially that  $\alpha\beta\pi \neq 0$ .

From  $\Pi_{01}$ ,  $\Pi_{00}$ ,  $\text{III}_{10}$ , (NP6) and (NP25) we have, respectively,

$$D\phi_2 = 0, \tag{33}$$

$$\delta\phi_2 = -2\phi_2\beta, \tag{34}$$

$$\delta\beta = -\beta(\bar{\alpha} + \beta), \tag{35}$$

$$D\gamma = \alpha\bar{\pi} + \beta\pi + \Phi_{11}, \tag{36}$$

$$\delta\Phi_{21} = 2(\alpha + 2\pi + \lambda\Phi_{11} - \alpha\Phi_{21}). \tag{37}$$

By adding (NP22) to the complex conjugate of (NP23), and solving for  $D\bar{\Phi}_{12}$  we get

$$D\bar{\Phi}_{12} = 2\bar{\pi}\Phi_{11}, \tag{38}$$

and

$$\delta\Phi_{11} = 0. \tag{39}$$

Subtracting (NP24) from (NP29) and solving for  $\bar{\delta}\bar{\Phi}_{12}$  we obtain

$$\bar{\delta}\bar{\Phi}_{12} = 2(-\beta + \Phi_{11}\bar{\mu} - \bar{\beta}\Phi_{12}), \tag{40}$$

Adding (NP24) to two times (NP29) and solving for  $D\bar{\Phi}_{22}$  we get

$$D\bar{\Phi}_{22} = 2(-\beta - \bar{\beta} + \Phi_{21}\bar{\pi} + \Phi_{12}\pi). \tag{41}$$

By substituting (34) into  $\text{IV}_{10}$  we find

$$\bar{\delta}\phi_2 = 2(3\bar{\beta}\phi_2 - \alpha\phi_2 + 2\pi\bar{\phi}_2 + \alpha\bar{\phi}_2). \tag{42}$$

Using (41) and (42),  $\text{V}_{20}$  and  $\text{VI}_{03}$  can be written respectively as

$$-24\bar{\phi}_2\phi_2\beta^2 + \phi_2^2(-6\bar{\alpha}\bar{\pi} + 18\beta\bar{\pi} + 9\beta\bar{\alpha} - \epsilon\bar{\alpha}^2 - 2\delta\bar{\pi} - \delta\bar{\alpha}) + 12\bar{\alpha}^2 + 24\bar{\pi}\bar{\alpha} + 80\bar{\pi}\beta + 44\delta\bar{\alpha} + 8\delta\bar{\pi} + 44\beta\bar{\alpha} = 0, \tag{43}$$

$$\beta(\phi_2(\delta\bar{\alpha} + 2\delta\bar{\pi} + 3\bar{\alpha}^2 - 18\bar{\pi}\beta + 6\bar{\alpha}\bar{\pi} - 9\bar{\alpha}\beta) + 24\bar{\phi}_2\beta^2) = 0. \tag{44}$$

Eliminating  $\delta(\bar{\alpha} + 2\pi)$  from (43) and (44) we find

$$\frac{\beta^2}{\phi_2^2 - 4}(19\bar{\pi}\phi_2 + 10\bar{\alpha}\phi_2 - 12\beta\bar{\phi}_2) = 0. \tag{45}$$

The denominator  $-\phi_2^2 + 4$  in the expression above must be nonzero, since  $\phi_2$  cannot be constant. Otherwise, from (34), we would have  $\beta = 0$ .



In order to determine further side relations we still need to find the Pfaffians  $\delta\alpha$ ,  $\delta\bar{\beta}$ , and  $\delta\pi$ , in terms of  $\bar{\delta}\alpha$ . From (NP6), (NP7), (NP8), (NP9) and (NP12) we have

$$\bar{\delta}\pi = D\lambda - \pi^2 - \pi\alpha + \pi\bar{\beta}, \quad (46)$$

$$\delta\pi = D\mu - \pi\bar{\pi} + \pi\bar{\alpha} - \beta\pi, \quad (47)$$

$$D\nu = \Delta\pi + \pi\mu + \pi\bar{\mu} + \bar{\pi}\lambda + \pi\gamma - \pi\bar{\gamma} - 1 + \Phi_{21}, \quad (48)$$

$$\delta\alpha = \bar{\delta}\beta + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\bar{\beta}\bar{\alpha} + \Phi_{11}. \quad (49)$$

Using (38) and (41), we now evaluate the NP commutator  $[\delta, D]\Phi_{22} - [\Delta, D]\Phi_{12}$  to obtain

$$\begin{aligned} & -2\delta\bar{\beta} + D\Delta\Phi_{12} + 2\pi\Phi_{11}\bar{\gamma} + 2\beta\bar{\beta} + 4\beta^2 + 2\pi\bar{\lambda}\Phi_{11} - 6\beta\Phi_{21}\bar{\pi} + 4\pi\bar{\pi} \\ & + 4\beta\bar{\alpha} - D\delta\Phi_{22} - 4\bar{\alpha}\Phi_{12}\pi - 2\beta\Phi_{12}\pi - 2\bar{\alpha}\Phi_{21}\bar{\pi} + 2\pi\Phi_{11}\gamma - 2\pi\Phi_{11}\bar{\mu} \\ & + 4\pi\Phi_{11}\mu + 2\pi\Phi_{12}\pi + 2\Phi_{21}\delta\bar{\pi} + 2\Phi_{12}\delta\pi - 2\Phi_{11}\Delta\bar{\pi} + 2\bar{\alpha}\bar{\beta} - 6\pi\bar{\beta} \\ & + 2\Phi_{21}\bar{\pi}^2 + 2\pi\Phi_{12}\bar{\beta} + 2\pi\bar{\alpha} = 0. \end{aligned} \quad (50)$$

From  $[\Delta, D]\Phi_{12}$  we get

$$\begin{aligned} D\Delta\Phi_{12} &= 2\Phi_{11}\Delta\bar{\pi} - 2\pi\Phi_{11}\gamma - 2\pi\Phi_{11}\bar{\gamma} + 2\pi\bar{\alpha} + 4\pi\bar{\pi} + 2\pi\bar{\lambda}\Phi_{11} \\ & - 2\bar{\alpha}\Phi_{12}\pi - 2\pi\beta + 2\pi\Phi_{11}\bar{\mu} - 2\pi\Phi_{12}\bar{\beta}. \end{aligned} \quad (51)$$

From (NP26),

$$\delta\Phi_{22} = \Delta\Phi_{12} + 2\bar{\gamma} + 4\bar{\mu} - 2\nu\Phi_{11} + 2\bar{\lambda}\Phi_{21} + 2\Phi_{12}\bar{\gamma} + 2\Phi_{12}\mu - 2\Phi_{22}\beta - 2\Phi_{22}\bar{\alpha}. \quad (52)$$

Substituting (52), (51), (36), (46), (47) and (48) into (50) we have

$$\delta\bar{\beta} = -\bar{\alpha}\bar{\beta} - 4\pi\bar{\beta} - 2D\bar{\mu} - \beta\bar{\beta} + 2\pi\bar{\pi} - 2\Phi_{11}. \quad (53)$$

Now, using (33), (34), (42), (47), (49) and (53) in the commutator  $[\bar{\delta}, \delta]\phi_2$ , and solving for  $D\mu$  we obtain

$$\begin{aligned} D\mu &= -\frac{1}{12\bar{\phi}_2}(8\bar{\phi}_2\alpha\bar{\pi} + 24\bar{\phi}_2\beta\pi - 8\phi_2\alpha\bar{\pi} \\ & + 10\bar{\phi}_2\Phi_{11} + 12\beta\bar{\phi}_2\alpha - 12\bar{\alpha}\phi_2\bar{\beta} + 4\bar{\phi}_2\pi\bar{\pi} \\ & + 4\bar{\alpha}\bar{\phi}_2\alpha - 4\bar{\alpha}\phi_2\alpha - 2\phi_2\Phi_{11} - 24\phi_2\pi\bar{\beta} + 8\bar{\phi}_2\pi\bar{\alpha}). \end{aligned} \quad (54)$$

The explicit form of  $D\lambda$  can be determined from (43), (44) and (46), and is given by:

$$D\lambda = \frac{1}{2(-\bar{\phi}_2^2 + 4)} (\bar{\phi}_2^2(3\alpha^2 - 2\pi^2 + 4\alpha\pi - 16\bar{\beta}\pi - 9\bar{\beta}\alpha + \bar{\delta}\alpha) + 8\pi^2 - 16\pi\alpha - 12\alpha^2 + 24\bar{\phi}_2\phi_2\bar{\beta}^2 - 88\pi\bar{\beta} - 44\bar{\beta}\alpha - 4\bar{\delta}\alpha). \quad (55)$$

Finally, from  $V_{12}$  we get

$$D\pi = 0. \quad (56)$$

We have now determined all the Pfaffians needed for finding new side relations using integrability conditions. Before proceeding we go back to (45) and introduce a further simplification by expressing  $\phi_2$  in terms of  $\bar{\phi}_2$ ,

$$\phi_2 = \frac{\bar{\phi}_2}{12\bar{\beta}}(19\pi + 10\alpha). \quad (57)$$

Substituting (57) into the numerator of the complex conjugate of (45), we find

$$S_1 := 361\pi\bar{\pi} + 190\alpha\bar{\pi} + 190\pi\bar{\alpha} + 100\alpha\bar{\alpha} - 144\beta\bar{\beta} = 0. \quad (58)$$

Another side relation can be determined from the NP commutator  $[\bar{\delta}, \delta](\alpha + 2\pi)$ :

$$\begin{aligned} &(-820\alpha^2\bar{\Phi}_{11} + 95\pi^2\bar{\Phi}_{11} + 2680\alpha^3\bar{\alpha} - 5448\alpha\bar{\beta}\bar{\Phi}_{11} - 10236\pi\bar{\beta}\bar{\Phi}_{11} \\ &+ 5360\alpha^3\bar{\pi} - 9648\bar{\beta}\alpha^2\bar{\beta} - 38016\pi^2\bar{\beta}\bar{\beta} - 13926\pi^2\bar{\beta}\bar{\alpha} - 2508\pi^2\bar{\pi}\bar{\beta} \\ &- 3816\bar{\beta}\alpha^2\bar{\alpha} - 720\alpha^2\bar{\pi}\bar{\beta} + 61628\alpha\pi^2\bar{\pi} + 40128\pi^3\bar{\pi} + 20064\pi^3\bar{\alpha} \\ &+ 30814\pi^2\alpha\bar{\alpha} + 15752\pi\alpha^2\bar{\alpha} - 38376\pi\bar{\beta}\alpha\bar{\beta} + 31504\pi\alpha^2\bar{\pi} \\ &- 14592\alpha\pi\bar{\beta}\bar{\alpha} - 2688\pi\alpha\bar{\pi}\bar{\beta} - 1508\pi\alpha\bar{\Phi}_{11})/(19\pi + 10\alpha) = 0. \quad (59) \end{aligned}$$

It follows from (57) that the numerator  $19\pi + 10\alpha$  in the preceding equation must be non-zero. Solving this equation for  $\bar{\Phi}_{11}$  we obtain

$$\begin{aligned} \bar{\Phi}_{11} = &(2680\alpha^3\bar{\alpha} + 5360\alpha^3\bar{\pi} - 9648\bar{\beta}\alpha^2\bar{\beta} - 38016\pi^2\bar{\beta}\bar{\beta} - 13926\pi^2\bar{\beta}\bar{\alpha} \\ &- 2508\pi^2\bar{\pi}\bar{\beta} - 3816\bar{\beta}\alpha^2\bar{\alpha} - 720\alpha^2\bar{\pi}\bar{\beta} + 61628\alpha\pi^2\bar{\pi} + 40128\pi^3\bar{\pi} \\ &+ 20064\pi^3\bar{\alpha} + 30814\pi^2\alpha\bar{\alpha} + 15752\pi\alpha^2\bar{\alpha} - 38376\pi\bar{\beta}\alpha\bar{\beta} \\ &+ 31504\pi\alpha^2\bar{\pi} - 14592\alpha\pi\bar{\beta}\bar{\alpha} - 2688\pi\alpha\bar{\pi}\bar{\beta})/(820\alpha^2 - 95\pi^2 \\ &+ 5448\bar{\beta}\alpha + 10236\pi\bar{\beta} + 1508\pi\alpha), \quad (60) \end{aligned}$$

where, for now, we assume that the denominator in the expression above,

$$d_1 := 820\alpha^2 - 95\pi^2 + 5448\bar{\beta}\alpha + 10236\pi\bar{\beta} - 1508\pi\alpha, \quad (61)$$

is non-zero. Evaluating  $\delta\phi_2 + 2\phi_2\beta = 0$  (cf. Eq. (38)), using (57), and solving for  $\Phi_{11}$  we find

$$\begin{aligned} \Phi_{11} = & (700\alpha^2\bar{\alpha} + 5300\alpha\pi\bar{\pi} + 2650\pi\alpha\bar{\alpha} + 2508\pi^2\bar{\alpha} + 1400\alpha^2\bar{\pi} \\ & - 2520\bar{\beta}\alpha\beta - 4752\pi\beta\bar{\beta} - 1650\pi\bar{\beta}\bar{\alpha} - 132\pi\bar{\pi}\bar{\beta} + 5016\pi^2\bar{\pi} \\ & - 900\bar{\beta}\alpha\bar{\alpha} - 72\alpha\pi\bar{\beta}) / (-437\pi + 372\bar{\beta} - 230\alpha), \end{aligned} \quad (62)$$

where we assume, for now, that the denominator of (62), given by

$$d_2 := 437\pi - 372\bar{\beta} + 230\alpha \neq 0, \quad (63)$$

is nonzero. Evaluating the commutators  $[\bar{\delta}, \delta]\bar{\beta}$  and  $[\bar{\delta}, \delta]\bar{\phi}_2$ , and solving each one for  $\bar{\delta}\alpha$  we get, respectively,

$$\begin{aligned} \bar{\delta}\alpha = & (-308\pi\alpha\Phi_{11} + 8016\pi\alpha^2\bar{\pi} + 234\bar{\beta}\alpha^2\beta + 3972\pi\alpha^2\bar{\alpha} + 78\bar{\beta}\alpha^2\bar{\alpha} \\ & + 759\alpha\bar{\beta}\Phi_{11} - 29\alpha^2\Phi_{11} + 1440\alpha^3\bar{\pi} + 702\alpha^3\bar{\alpha} + 1386\pi\bar{\beta}\Phi_{11} \\ & - 513\pi^2\Phi_{11} + 1296\pi\bar{\beta}\alpha\beta + 528\pi^2\bar{\beta}\bar{\alpha} + 14872\alpha\pi^2\bar{\pi} + 7436\pi^2\alpha\bar{\alpha} \\ & + 1584\pi^2\beta\bar{\beta} + 4598\pi^3\bar{\alpha} + 9196\pi^3\bar{\pi} + 432\alpha\pi\bar{\beta}\bar{\alpha} - 108\pi\alpha^2\beta \\ & - 54\alpha^3\beta) / (12\pi\bar{\alpha} + 6\alpha\bar{\alpha} + 36\beta\pi + 18\beta\alpha + 3\Phi_{11}), \end{aligned} \quad (64)$$

$$\bar{\delta}\alpha = -\frac{6\alpha^2\bar{\pi} - 572\pi\bar{\pi}\bar{\beta} - 286\pi\bar{\beta}\bar{\alpha} - 157\bar{\beta}\alpha\bar{\alpha} + 3\alpha^2\bar{\alpha} - 314\alpha\bar{\pi}\bar{\beta}}{\bar{\alpha} + 2\bar{\pi}}, \quad (65)$$

where we assume, for the moment, that the denominators of (64) and (65), given by

$$d_3 := 4\pi\bar{\alpha} + 2\alpha\bar{\alpha} + 12\beta\pi + 6\beta\alpha + \Phi_{11}, \quad (66)$$

$$d_4 := \alpha + 2\pi, \quad (67)$$

are nonzero. By subtracting (64) from (65) and solving for  $\Phi_{11}$  we have

$$\begin{aligned} \Phi_{11} = & (-864\bar{\beta}\alpha^2\bar{\alpha} + 7436\pi^2\alpha\bar{\alpha} - 3168\alpha\pi\bar{\beta}\bar{\alpha} + 4008\pi\alpha^2\bar{\alpha} \\ & - 2904\pi^2\bar{\beta}\bar{\alpha} + 720\alpha^3\bar{\alpha} + 9196\pi^3\bar{\pi} - 8712\pi^2\beta\bar{\beta} + 14872\alpha\pi^2\bar{\pi} \\ & + 4598\pi^3\bar{\alpha} - 2592\bar{\beta}\alpha^2\beta + 8016\pi\alpha^2\bar{\pi} + 1440\alpha^3\bar{\pi} - 9504\pi\bar{\beta}\alpha\beta) \\ & / (-288\bar{\beta}\alpha + 20\alpha^2 - 528\pi\bar{\beta} + 513\pi^2 + 308\pi\alpha), \end{aligned} \quad (68)$$

where the denominator of (68),

$$d_5 := -288 \bar{\beta} \alpha + 20 \alpha^2 - 528 \pi \bar{\beta} + 513 \pi^2 + 308 \pi \alpha, \tag{69}$$

is assumed to be nonzero for the moment.

Subtracting (60) from (62) and taking the numerator we find

$$\begin{aligned} S_2 := & 39914208\pi^2 \alpha^2 \bar{\pi} + 19957104\pi^2 \alpha^2 \bar{\alpha} \\ & + 35332704\pi^3 \bar{\pi} \bar{\beta} + 12279168\pi^3 \bar{\beta} \bar{\alpha} \\ & + 1190400\bar{\alpha} \alpha^4 - 12773376\bar{\beta}^2 \bar{\alpha} \pi \alpha - 3483648\bar{\beta}^2 \bar{\alpha} \alpha^2 \\ & + 1200960\bar{\beta} \bar{\alpha} \alpha^3 - 11708928\bar{\beta}^2 \bar{\alpha} \pi^2 - 20739456\pi \bar{\beta} \alpha^2 \beta \\ & + 8008704\alpha^2 \pi \bar{\beta} \bar{\alpha} + 30335616\pi \alpha^2 \bar{\pi} \bar{\beta} + 56708640\pi^2 \alpha \bar{\pi} \bar{\beta} \\ & - 32440608\pi^2 \bar{\beta} \alpha \beta - 16161552\pi^3 \bar{\beta} \bar{\beta} + 8022720\pi \alpha^3 \bar{\alpha} \\ & + 5408640\alpha^3 \bar{\pi} \bar{\beta} \\ & + 16045440\pi \alpha^3 \bar{\pi} + 8529708\pi^4 \bar{\alpha} + 43221504\alpha \pi^3 \bar{\pi} + 21610752\pi^3 \alpha \bar{\alpha} \\ & + 17059416\pi^4 \bar{\pi} + 2380800\alpha^4 \bar{\pi} + 17343792\alpha \pi^2 \bar{\beta} \bar{\alpha} - 10139904\bar{\beta}^2 \alpha^2 \beta \\ & - 124416\alpha^2 \bar{\pi} \bar{\beta}^2 - 418176\pi^2 \bar{\pi} \bar{\beta}^2 - 456192\pi \alpha \bar{\pi} \bar{\beta}^2 \\ & - 4285440\bar{\beta} \alpha^3 \beta - 37407744\pi \bar{\beta}^2 \alpha \beta - 34499520\pi^2 \beta \bar{\beta}^2 = 0. \tag{70} \end{aligned}$$

By subtracting (62) from (68) and taking the numerator, we obtain a third side relation:

$$\begin{aligned} S_3 := & -9374472\pi^2 \alpha^2 \bar{\pi} - 4687236\pi^2 \alpha^2 \bar{\alpha} + 6137076\pi^3 \bar{\pi} \bar{\beta} + 5150178\pi^3 \bar{\beta} \bar{\alpha} \\ & - 179600\bar{\alpha} \alpha^4 - 2128896\bar{\beta}^2 \bar{\alpha} \pi \alpha - 580608\bar{\beta}^2 \bar{\alpha} \alpha^2 + 686160\bar{\beta} \bar{\alpha} \alpha^3 \\ & - 1951488\bar{\beta}^2 \bar{\alpha} \pi^2 + 4189824\pi \bar{\beta} \alpha^2 \beta + 4040184\alpha^2 \pi \bar{\beta} \bar{\alpha} \\ & + 5272368\pi \alpha^2 \bar{\pi} \bar{\beta} + 9852984\pi^2 \alpha \bar{\pi} \bar{\beta} + 8913384\pi^2 \bar{\beta} \alpha \beta \\ & + 6244920\pi^3 \bar{\beta} \bar{\beta} - 1505080\pi \alpha^3 \bar{\alpha} + 940320\alpha^3 \bar{\pi} \bar{\beta} - 3010160\pi \alpha^3 \bar{\pi} \\ & - 3295930\pi^4 \bar{\alpha} - 12877972\alpha \pi^3 \bar{\pi} - 6438986\pi^3 \alpha \bar{\alpha} - 6591860\pi^4 \bar{\pi} \\ & - 359200\alpha^4 \bar{\pi} + 7909932\alpha \pi^2 \bar{\beta} \bar{\alpha} - 1689984\bar{\beta}^2 \alpha^2 \beta - 20736\alpha^2 \bar{\pi} \bar{\beta}^2 \\ & - 69696\pi^2 \bar{\pi} \bar{\beta}^2 - 76032\pi \alpha \bar{\pi} \bar{\beta}^2 + 646560\bar{\beta} \alpha^3 \beta - 6234624\pi \bar{\beta}^2 \alpha \beta \\ & - 5749920\pi^2 \beta \bar{\beta}^2 = 0. \tag{71} \end{aligned}$$

We can eliminate  $\pi$  in the above equations by defining new variables  $x_1$  and  $x_2$  by

$$x_1 := \frac{\alpha}{\pi}, \tag{72}$$

$$x_2 := \frac{\beta}{\bar{\pi}}. \tag{73}$$

The side relations now assume the form (modulo non-zero factors)

$$S_1 := 361 + 190x_1 + 190\bar{x}_1 + 100x_1\bar{x}_1 - 144x_2\bar{x}_2 = 0, \quad (74)$$

$$\begin{aligned} S_2 := & -1663092x_1^2\bar{x}_1 - 99200\bar{x}_1x_1^4 - 2527968x_1^2\bar{x}_2 + 975744\bar{x}_2^2\bar{x}_1 \\ & - 1023264\bar{x}_2\bar{x}_1 + 2874960x_2\bar{x}_2^2 - 4725720x_1\bar{x}_2 - 198400x_1^4 \\ & - 3326184x_1^2 - 1337120x_1^3 + 34848\bar{x}_2^2 - 3601792x_1 - 710809\bar{x}_1 \\ & - 2944392\bar{x}_2 + 844992\bar{x}_2^2x_1^2x_2 + 1064448\bar{x}_2^2\bar{x}_1x_1 \\ & + 290304\bar{x}_2^2\bar{x}_1x_1^2 - 100080\bar{x}_2\bar{x}_1x_1^3 + 1728288\bar{x}_2x_1^2x_2 \\ & - 667392x_1^2\bar{x}_2\bar{x}_1 + 2703384\bar{x}_2x_1x_2 + 1346796x_2\bar{x}_2 - 668560x_1^3\bar{x}_1 \\ & - 450720x_1^3\bar{x}_2 - 1800896x_1\bar{x}_1 + 10368x_1^2\bar{x}_2^2 + 38016x_1\bar{x}_2^2 \\ & - 1445316x_1\bar{x}_2\bar{x}_1 + 357120\bar{x}_2x_1^3x_2 \\ & + 3117312\bar{x}_2^2x_1x_2 - 1421618 = 0, \end{aligned} \quad (75)$$

$$\begin{aligned} S_3 := & 2343618x_1^2\bar{x}_1 + 89800\bar{x}_1x_1^4 - 2636184x_1^2\bar{x}_2 + 975744\bar{x}_2^2\bar{x}_1 \\ & - 2575089\bar{x}_2\bar{x}_1 + 2874960x_2\bar{x}_2^2 + 3295930 - 4926492x_1\bar{x}_2 \\ & + 179600x_1^4 + 4687236x_1^2 + 1505080x_1^3 + 34848\bar{x}_2^2 \\ & + 6438986x_1 + 1647965\bar{x}_1 \\ & - 3068538\bar{x}_2 + 844992\bar{x}_2^2x_1^2x_2 + 1064448\bar{x}_2^2\bar{x}_1x_1 \\ & + 290304\bar{x}_2^2\bar{x}_1x_1^2 - 343080\bar{x}_2\bar{x}_1x_1^3 - 2094912\bar{x}_2x_1^2x_2 \\ & - 2020092x_1^2\bar{x}_2\bar{x}_1 - 4456692\bar{x}_2x_1x_2 - 3122460x_2\bar{x}_2 + 752540x_1^3\bar{x}_1 \\ & - 470160x_1^3\bar{x}_2 + 3219493x_1\bar{x}_1 + 10368x_1^2\bar{x}_2^2 + 38016x_1\bar{x}_2^2 \\ & - 3954966x_1\bar{x}_2\bar{x}_1 - 323280\bar{x}_2x_1^3x_2 + 3117312\bar{x}_2^2x_1x_2 = 0. \end{aligned} \quad (76)$$

In order to study the possible solutions of the polynomial systems that appear from the side relations, we apply the procedure `gsolve` which is part of the Maple package `grobner` [5]. This procedure computes a collection of reduced lexicographic Gröbner bases corresponding to a set of polynomials. The system corresponding to the set is first subdivided by factorization. Then a variant of Buchberger's algorithm which factors all intermediate results is applied to each subsystem. The result is a list of reduced subsystems whose roots are those of the original system, but whose variables have been successively eliminated and separated as far as possible. This means that instead of trying to find a Gröbner basis, the package attempts to factor the polynomials that form the system after each step of the reduction algorithm. In the algorithm, the variables  $x_1$ ,  $x_2$  and their complex conjugates,  $\bar{x}_1$  and  $\bar{x}_2$ , are treated as independent variables. In the subsequent analysis we use the fact that they are complex conjugates of each other.

Applying `gsolve` to the set of equations formed by  $S_2 = 0$ ,  $S_3 = 0$  (cf. (75) and (76)), their complex conjugates, and  $S_1 = 0$  (cf. (74)), we find the only possible solution for which  $x_1 \neq 0$  and  $x_2 \neq 0$  is given by

$$324x_2\bar{x}_2 - 1 = 0, \quad 6x_1 + 11 = 0, \quad 6\bar{x}_1 + 11 = 0. \quad (77)$$

By substituting (77) into any of the previous expressions for  $\Phi_{11}$  we find that  $\Phi_{11} = 0$ . Using this and  $\pi = -\alpha 6/11$  in  $V_{11}$  one gets

$$-1761\bar{\beta}\alpha\bar{\alpha} + 5\alpha^2\bar{\alpha} + 3267\beta\bar{\beta}^2 - 5445\beta\alpha\bar{\beta} = 0. \quad (78)$$

It is easy to verify that (78) and the first equation in (77), now given in the form  $1089\beta\bar{\beta} - \alpha\bar{\alpha} = 0$ , imply that  $\alpha = \beta = 0$ .

Let us consider first the cases in which each one of the denominators  $d_1, d_2, d_3, d_4$  and  $d_5$ , given respectively by (61), (63), (66), (67) and (69), is zero.

**(i)  $d_1 = 0$**

From (61) we have, in terms of the variables  $x_1$  and  $x_2$ :

$$820x_1^2 + 1508x_1 + 5448\bar{x}_2x_1 - 95 + 10236\bar{x}_2 = 0. \quad (79)$$

Since the numerator of (60) must also vanish, we have

$$\begin{aligned} &2680x_1^3 + 1340x_1^3\bar{x}_1 - 1908\bar{x}_2\bar{x}_1x_1^2 + 7876x_1^2\bar{x}_1 \\ &- 360\bar{x}_2x_1^2 + 15752x_1^2 - 4824\bar{x}_2x_2x_1^2 + 30814x_1 - 1344\bar{x}_2x_1 \\ &+ 15407x_1\bar{x}_1 - 19188\bar{x}_2x_2x_1 - 7296\bar{x}_2\bar{x}_1x_1 + 20064 + 10032\bar{x}_1 \\ &- 19008\bar{x}_2x_2 - 6963\bar{x}_2\bar{x}_1 - 1254\bar{x}_2 = 0. \end{aligned} \quad (80)$$

Applying `gsolve` to the set of equations consisting of (79), (80), their complex conjugates, and  $S_1 = 0$  (cf. eq. (74)) we find that all possible solutions require that  $x_2 = 0$ .

**(ii)  $d_2 = 0$**

In this case, from (63), we have

$$230x_1 + 437 - 372\bar{x}_2 = 0. \quad (81)$$

This implies that the numerator of (62) must vanish, so we have

$$\begin{aligned} &350x_1^2\bar{x}_1 + 700x_1^2 + 1325x_1\bar{x}_1 - 36\bar{x}_2x_1 + 2650x_1 + 2508 \\ &- 450\bar{x}_2\bar{x}_1x_1 - 1260\bar{x}_2x_2x_1 - 825\bar{x}_2\bar{x}_1 - 66\bar{x}_2 - 2376\bar{x}_2x_2 \\ &+ 1254\bar{x}_1 = 0. \end{aligned} \quad (82)$$

Applying `gsolve` to the set of equations consisting of (81), (82), their complex conjugates, and  $S_1 = 0$  we find again that all solutions require that  $x_2 = 0$ .

(iii)  $d_3 = 0$

From (66) we now have

$$\Phi_{11} = -2\pi\bar{\pi}(x_1 + 2)(3x_2 + \bar{x}_1). \quad (83)$$

By subtracting (83) from its complex conjugate we obtain

$$E_1 := -6x_2 - 3x_1x_2 - 2\bar{x}_1 + 6\bar{x}_2 + 3\bar{x}_2x_1 + 2x_1 = 0. \quad (84)$$

Subtracting (83) from (60) and taking the numerator, we get

$$\begin{aligned} E_2 := & 246x_2x_1^2 + 268x_1^2 + 216x_1^2\bar{x}_1^{-2} + 477x_1x_2 + 1152\bar{x}_2x_1x_2 \\ & - 36\bar{x}_2x_1 + 1066x_1 + 354\bar{x}_2x_1\bar{x}_1 + 2232x_2\bar{x}_2 + 518\bar{x}_1 - 66\bar{x}_2 \\ & + 711\bar{x}_2\bar{x}_1 - 30x_2 + 692x_1\bar{x}_1 + 1056 = 0. \end{aligned} \quad (85)$$

An additional side relation is obtained by subtracting the complex conjugate of (83) from (62) and taking the numerator of the resulting expression:

$$\begin{aligned} E_3 := & 144x_2\bar{x}_2 + 2691x_1x_2 + 2622x_2 + 144\bar{x}_2x_1x_2 + 690x_2x_1^2 \\ & - 120x_1^2\bar{x}_1 - 428x_1\bar{x}_1 + 81x_2x_1 - 380\bar{x}_1 - 2508 + 66\bar{x}_2 - 2650x_1 \\ & + 36\bar{x}_2x_1 - 700x_1^2 + 78x_2x_1\bar{x}_1 = 0. \end{aligned} \quad (86)$$

Applying `grobner` to the set consisting of  $E_2$ , its complex conjugate,  $E_1$ ,  $E_3$  and  $S_1$  we find that this system admits no solution.

(iv)  $d_4 = 0$

When the denominator of (65), given by (67), is zero, its numerator must be zero, implying that  $\alpha = \bar{\beta}(443/3)$ . This, on the other hand implies immediately, from  $S_1 = 0$  (cf. (58)), that  $\beta = 0$ .

(v)  $d_5 = 0$

When the denominator of (68) is zero, its numerator must be zero too. Thus, we get

$$(11 + 6x_1)^2(38 + 19\bar{x}_1^{-1}12\bar{x}_2\bar{x}_1 + 10x_1\bar{x}_1 + 20x_1 - 36x_2\bar{x}_2) = 0, \quad (87)$$

$$308x_1 + 513 + 20x_1^2 - 528\bar{x}_2 - 288\bar{x}_2x_1 = 0. \quad (88)$$

Applying `gsolve` to the polynomial system defined by the system of polynomials defined by (87), (88), their complex conjugates, and  $S_1 = 0$ , we find that there are no possible solutions.

The case  $\alpha = \beta = \pi = 0$ , which leads to Theorem 2, was considered in [3] and results in a contradiction to the hypothesis  $\phi_2 \neq 0$ . The more general case,  $\alpha\pi\beta = 0$ , also leads to a contradiction. The proof, found in [16], is tedious but straightforward, and will not be presented here. Thus, the necessary conditions I-VI for the validity of Huygens' principle imply that we must have  $H_{ab} := A_{[a,b]} = 0$ . This completes the proof of Theorem 3.

### APPENDIX

In this Appendix we give the Newman-Penrose field equations and commutation relations referred in the paper. We note that many lists in the literature contain typographic mistakes, or use different conventions

#### Bianchi identities

- (NP1)  $D\rho - \bar{\delta}\kappa = \rho^2 + \sigma\bar{\sigma} + (\epsilon + \bar{\epsilon})\rho - \bar{\kappa}\tau - (3\alpha + \bar{\beta} - \pi)\kappa + \Phi_{00},$
- (NP2)  $D\sigma - \delta\kappa = (\rho + \bar{\rho})\sigma + (3\epsilon - \bar{\epsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0,$
- (NP3)  $D\tau - \Delta\kappa = (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + (\epsilon - \bar{\epsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 + \Phi_{01},$
- (NP4)  $D\alpha - \bar{\delta}\epsilon = (\rho + \bar{\epsilon} - 2\epsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + (\epsilon + \rho)\pi + \Phi_{10},$
- (NP5)  $D\beta - \delta\epsilon = (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\epsilon})\beta - (\mu + \gamma)\kappa + (\bar{\pi} - \bar{\alpha})\epsilon + \Psi_1,$
- (NP6)  $D\gamma - \Delta\epsilon = (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta - (\epsilon + \bar{\epsilon})\gamma - (\gamma + \bar{\gamma})\epsilon + \tau\pi - \nu\kappa + \Psi_2 - \Lambda + \Phi_{11},$
- (NP7)  $D\lambda - \bar{\delta}\pi = \rho\lambda + \bar{\sigma}\mu + \pi^2 + (\alpha - \bar{\beta})\pi - \nu\bar{\kappa} + (\bar{\epsilon} - 3\epsilon)\lambda + \Phi_{20},$
- (NP8)  $D\mu - \delta\pi = \bar{\rho}\mu + \sigma\lambda + \pi\bar{\pi} - (\epsilon + \bar{\epsilon})\mu - (\bar{\alpha} - \beta)\pi - \nu\kappa + \Psi_2 + 2\Lambda,$
- (NP9)  $D\nu - \Delta\pi = (\bar{\tau} + \pi)\mu + (\tau + \bar{\pi})\lambda + (\gamma - \bar{\gamma})\pi - (3\epsilon + \bar{\epsilon})\nu + \Psi_3 + \Phi_{21},$
- (NP10)  $\Delta\lambda - \bar{\delta}\nu = (\bar{\gamma} - 3\gamma - \mu - \bar{\mu})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \Psi_4,$
- (NP11)  $\delta\rho - \bar{\delta}\sigma = (\bar{\alpha} + \beta)\rho - (3\alpha - \bar{\beta})\sigma + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 + \Phi_{01},$
- (NP12)  $\delta\alpha - \bar{\delta}\beta = \mu\rho - \sigma\lambda + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + (\rho - \bar{\rho})\gamma + (\mu - \bar{\mu})\epsilon - \Psi_2 + \Lambda + \Phi_{11},$
- (NP13)  $\delta\lambda - \bar{\delta}\mu = (\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + (\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3 + \Phi_{21},$
- (NP14)  $\delta\nu - \Delta\mu = \mu^2 + \lambda\bar{\lambda} + (\gamma + \bar{\gamma})\mu - \bar{\nu}\pi + (\tau - \bar{\alpha} - 3\beta)\nu + \Phi_{22},$
- (NP15)  $\delta\gamma - \Delta\beta = (\tau - \bar{\alpha} - \beta)\gamma + \mu\tau - \sigma\nu - \epsilon\bar{\nu} - (\gamma - \bar{\gamma} - \mu)\beta + \alpha\bar{\lambda} + \Phi_{12},$
- (NP16)  $\delta\tau - \Delta\sigma = \mu\sigma + \bar{\lambda}\rho + (\tau - \bar{\alpha} + \beta)\tau - (3\gamma - \bar{\gamma})\sigma - \kappa\bar{\nu} + \Phi_{02},$
- (NP17)  $\Delta\rho - \bar{\delta}\tau = -\rho\bar{\mu} - \sigma\lambda + (\gamma + \bar{\gamma})\rho - (\bar{\tau} + \alpha - \bar{\beta})\tau + \nu\kappa - \Psi_2 - 2\Lambda,$
- (NP18)  $\Delta\alpha - \bar{\delta}\gamma = (\epsilon + \rho)\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma - \Psi_3.$



**Ricci identities**

- (NP19)  $\bar{\delta}\Psi_0 - D\Psi_1 + D\Phi_{01} - \delta\Phi_{00} = (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1$   
 $+ 3\kappa\Psi_2 + (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} + 2(\epsilon + \bar{\rho})\Phi_{01}$   
 $+ 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02},$
- (NP20)  $\Delta\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1$   
 $+ 3\sigma\Psi_2 - \bar{\lambda}\Phi_{00} + 2(\bar{\pi} - \beta)\Phi_{01}$   
 $+ 2\sigma\Phi_{11} + (2\epsilon - 2\bar{\epsilon} + \bar{\rho})\Phi_{02} - 2\kappa\Phi_{12},$
- (NP21)  $3\bar{\delta}\Psi_1 - 3D\Psi_2 + 2D\Phi_{11} - 2\delta\Phi_{10} + \bar{\delta}\Phi_{01} - \Delta\Phi_{00}$   
 $= 3\lambda\Psi_0 - 9\rho\Psi_2 + 6(\alpha - \pi)\Psi_1 + 6\kappa\Psi_3$   
 $+ (\bar{\mu} - 2\mu - 2\gamma - 2\bar{\gamma})\Phi_{00} + (2\alpha + 2\pi + 2\bar{\tau})\Phi_{01}$   
 $+ 2(\tau - 2\bar{\alpha} + \bar{\pi})\Phi_{10} + 2(2\bar{\rho} - \rho)\Phi_{11}$   
 $+ 2\sigma\Phi_{20} - \bar{\sigma}\Phi_{02} - 2\bar{\kappa}\Phi_{12} - 2\kappa\Phi_{21},$
- (NP22)  $3\Delta\Psi_1 - 3\delta\Psi_2 + 2D\Phi_{12} - 2\delta\Phi_{11} + \bar{\delta}\Phi_{02} - \Delta\Phi_{01}$   
 $= 3\nu\Psi_0 + 6(\gamma - \mu)\Psi_1 - 9\tau\Psi_2 + 6\sigma\Psi_3 - \bar{\nu}\Phi_{00}$   
 $+ 2(\bar{\mu} - \mu - \gamma)\Phi_{01} - 2\bar{\lambda}\Phi_{10} + 2(\tau + 2\bar{\pi})\Phi_{11}$   
 $+ (2\alpha + 2\pi + \bar{\tau} - 2\bar{\beta})\Phi_{02}$   
 $+ (2\bar{\rho} - 2\rho - 4\bar{\epsilon})\Phi_{12} + 2\sigma\Phi_{21} - 2\kappa\Phi_{22},$
- (NP23)  $3D\Psi_3 + D\Phi_{21} - \delta\Phi_{20} + 2\bar{\delta}\Phi_{11} - 2\Delta\Phi_{10}$   
 $= 6\lambda\Psi_1 - 9\pi\Psi_2 + 6(\epsilon - \rho)\Psi_3 + 3\kappa\Psi_4 - 2\nu\Phi_{00}$   
 $+ 2(\bar{\mu} - \mu - 2\bar{\gamma})\Phi_{10} + (2\pi + 4\bar{\tau})\Phi_{11}$   
 $+ (2\beta + 2\tau + \bar{\pi} - 2\bar{\alpha})\Phi_{20} - 2\bar{\sigma}\Phi_{12}$   
 $+ 2(\bar{\rho} - \rho - \epsilon)\Phi_{21} - \bar{\kappa}\Phi_{22} + 2\lambda\Phi_{01},$
- (NP24)  $3\Delta\Psi_2 - 3\delta\Psi_3 + D\Phi_{22} - \delta\Phi_{21} + 2\bar{\delta}\Phi_{12} - 2\Delta\Phi_{11}$   
 $= 6\nu\Psi_1 - 9\mu\Psi_2 + 6(\beta - \tau)\Psi_3 + 3\sigma\Psi_4 - 2\nu\Phi_{01} - 2\bar{\nu}\Phi_{10}$   
 $+ 2(2\bar{\mu} - \mu)\Phi_{11} + 2\lambda\Phi_{02} - \bar{\lambda}\Phi_{20} + 2(\pi + \bar{\tau} - 2\bar{\beta})\Phi_{12}$   
 $+ 2(\beta + \tau + \bar{\pi})\Phi_{21} + (\bar{\rho} - 2\epsilon - 2\bar{\epsilon} - 2\rho)\Phi_{22}$
- (NP25)  $\bar{\delta}\Psi_3 - D\Psi_4 + \bar{\delta}\Phi_{21} - \Delta\Phi_{20} = 3\lambda\Psi_2 - 2(\alpha + 2\pi)\Psi_3$   
 $+ (4\epsilon - \rho)\Psi_4 - 2\nu\Phi_{10} + 2\lambda\Phi_{11} + (2\gamma - 2\bar{\gamma} + \bar{\mu})\Phi_{20}$   
 $+ 2(\bar{\tau} - \alpha)\Phi_{21} - \bar{\sigma}\Phi_{22}$
- (NP26)  $\Delta\Psi_3 - \delta\Psi_4 + \bar{\delta}\Phi_{22} - \Delta\Phi_{21} = 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3$   
 $+ (4\beta - \tau)\Psi_4 - 2\nu\Phi_{11} - \bar{\nu}\Phi_{20}$   
 $+ 2\lambda\Phi_{12} + 2(\gamma + \bar{\mu})\Phi_{21} + (\bar{\tau} - 2\bar{\beta} - 2\alpha)\Phi_{22}$

- (NP27)  $D\Phi_{11} - \delta\Phi_{10} - \bar{\delta}\Phi_{01} + \Delta\Phi_{00} + 3D\Lambda = (2\gamma - \mu + 2\bar{\gamma} - \bar{\mu})\Phi_{00}$   
 $+ (\pi - 2\alpha - 2\bar{\tau})\Phi_{01} + (\bar{\pi} - 2\bar{\alpha} - 2\bar{\tau})\Phi_{10} + 2(\rho + \bar{\rho})\Phi_{11}$   
 $+ \bar{\sigma}\Phi_{02} + \sigma\Phi_{20} - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21}$
- (NP28)  $D\Phi_{12} - \delta\Phi_{11} - \bar{\delta}\Phi_{02} + \Delta\Phi_{01} + 3\delta\Lambda = (2\gamma - \mu - 2\bar{\mu})\Phi_{01}$   
 $+ \bar{\nu}\Phi_{00} - \bar{\lambda}\Phi_{10} + 2(\bar{\pi} - \tau)\Phi_{11} + (\pi + 2\bar{\beta} - 2\alpha - \bar{\tau})\Phi_{02}$   
 $+ (2\rho + \bar{\rho} - 2\bar{\epsilon})\Phi_{12} + \sigma\Phi_{21} - \kappa\Phi_{22},$
- (NP29)  $D\Phi_{22} - \delta\Phi_{21} - \bar{\delta}\Phi_{12} + \Delta\Phi_{11} + 3\Delta\Lambda = \nu\Phi_{01}$   
 $+ \bar{\nu}\Phi_{10} - 2(\mu + \bar{\mu})\Phi_{11} - \lambda\Phi_{02} - \bar{\lambda}\Phi_{20} + (2\pi - \bar{\tau} + 2\bar{\beta})\Phi_{12}$   
 $+ (2\beta - \tau + 2\bar{\pi})\Phi_{21} + (\rho + \bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22}.$

**NP commutation relations**

$$\bar{\delta}\delta - \delta\bar{\delta} = (-\mu + \bar{\mu})D + (-\rho + \bar{\rho})\Delta + (\alpha - \bar{\beta})\delta + (-\bar{\alpha} + \beta)\bar{\delta},$$

$$\bar{\delta}\Delta - \Delta\bar{\delta} = -\nu D + (\bar{\tau} - \alpha - \bar{\beta})\Delta + \lambda\delta + (\bar{\mu} + \gamma - \bar{\gamma})\bar{\delta},$$

$$\bar{\delta}D - D\bar{\delta} = (\alpha + \bar{\beta} - \pi)D + \bar{\kappa}\Delta - \bar{\sigma}\delta - (\rho - \epsilon + \bar{\epsilon})\bar{\delta},$$

$$\Delta D - D\Delta = (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\bar{\tau} + \pi)\delta - (\tau + \bar{\pi})\bar{\delta}.$$

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