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Global properties of vacuum states in de Sitter space

by

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ABSTRACT. – Starting from the assumption that vacuum states in de Sitter space look for any geodesic observer like equilibrium states with some *a priori* arbitrary temperature, an analysis of their global properties is carried out in the algebraic framework of local quantum physics. It is shown that these states have the Reeh–Schlieder property and that any primary vacuum state is also pure and weakly mixing. Moreover, the geodesic temperature of vacuum states has to be equal to the Gibbons–Hawking temperature and this fact is closely related to the existence of a discrete PCT–like symmetry. It is also shown that the global algebras of observables in vacuum sectors have the same structure as their counterparts in Minkowski space theories. © Elsevier, Paris

RÉSUMÉ. – Partant de l'hypothèse que les états de vide dans un espace de De Sitter sont perçus par tout observateur géodésique comme des états d'équilibre ayant *à priori* une température quelconque, nous analysons leurs propriétés globales dans le cadre algébrique de la physique quantique locale. Nous montrons que ces états possèdent la propriété de Reeh–Schlieder et que tout état de vide primitif est pur et faiblement mélangeant. De plus, la température géodésique d'un état de vide doit obligatoirement coïncider avec celle de Gibbons–Hawking, propriété reliée étroitement à l'extérieur d'une symétrie discrète du type PCT. Nous montrons enfin que les algèbres d'observables globales dans les secteurs de vide ont la même structure que leurs analogues dans les théories d'espace de Minkowski. © Elsevier, Paris

1. INTRODUCTION

It is a well established fact that the most elementary states in de Sitter space, corresponding to vacuum states in Minkowski space, look for any geodesic observer like thermal states with a certain specific temperature which depends on the radius of the space. This universal (model independent) feature can be traced back to the Unruh effect [1], the thermalizing effects of event horizons [2,3,4,5] or to stability properties of the elementary states which manifest themselves in the form of specific analyticity properties [6,7].

In the present article we take this characteristic feature of elementary states (called vacuum states in the following) as input in a general analysis of their global properties. These properties were recently also discussed in [7]. In the present analysis, which is carried out in the algebraic framework of local quantum physics [8], we reproduce the results in [7] under slightly less restrictive assumptions and exhibit further interesting properties of vacuum states in de Sitter space which closely resemble those of their counterparts in Minkowski space.

Following is a brief outline of our results: In Sec. 2 we collect some basic properties of de Sitter space, the de Sitter group and of the unitary representations of this group. After these preparations we state in Sec. 3 our assumptions and establish a Reeh–Schlieder theorem for vacuum states. In Sec. 4 we show that the global algebras of observables are, in any vacuum sector, of type I according to the classification of Murray and von Neumann and have an abelian commutant. In particular, any primary vacuum state is also pure and weakly mixing. Invariant means of local observables with respect to certain specific one–parameter subgroups of the de Sitter group and their relation to the center of the global algebras are discussed in Sec. 5. Finally, we establish in Sec. 6 a PCT theorem in de Sitter space and present an argument showing that the temperature of de Sitter space has to be equal to the Gibbons–Hawking temperature. The article concludes with a brief summary.

2. DE SITTER SPACE AND DE SITTER GROUP

For the convenience of the reader we compile here some relevant properties of the de Sitter space and the de Sitter group as well as some information on the continuous unitary representations of this group. (For an extensive list of references on this subject cf. [9].)

The n–dimensional de Sitter space $\mathcal{S}^n$ can conveniently be described in the $n+1$–dimensional ambient Minkowski space $\mathbb{R}^{n+1}$. Assuming that $n > 1$, it corresponds to a one–sheeted hyperboloid which, in proper coordinates, is given by

$$\mathcal{S}^n = \{x \in \mathbb{R}^{n+1} : x_0^2 - x_1^2 \ldots - x_n^2 = -1\}. \tag{2.1}$$

The metric and causal structure on $\mathcal{S}^n$ are induced by the Minkowskian metric on $\mathbb{R}^{n+1}$. Accordingly, the isometry group of $\mathcal{S}^n$ is the group $O(1,n)$, called the de Sitter group. Its action on $\mathcal{S}^n$ is given by the familiar action of the Lorentz group in the ambient space. We restrict attention here to the identity component of $O(1,n)$ which is usually denoted by $SO_0(1,n)$.

In the following we deal with certain distinguished subregions $\mathcal{W} \subset \mathcal{S}^n$, called wedges. These wedges are defined as the causal completions of timelike geodesics in $\mathcal{S}^n$. Thus they are those parts of de Sitter space which are both, visible and accessible for observers moving along the respective geodesics. Each wedge $\mathcal{W}$ can be represented as intersection of de Sitter space $\mathcal{S}^n$ with a wedge shaped region in the ambient space, such as

$$\mathcal{W}_j = \{x \in \mathbb{R}^{n+1} : x_j > |x_0|\} \cap \mathcal{S}^n, \quad j = 1, \ldots n. \tag{2.2}$$

We note that any wedge $\mathcal{W} \subset \mathcal{S}^n$ is obtained from a fixed one, say $\mathcal{W}_1$, by the action of some element of $SO_0(1,n)$. Moreover, the spacelike complement $\mathcal{W}'$ of a wedge $\mathcal{W}$ is again a wedge.

Given a wedge $\mathcal{W}$ there is a unique one–parameter subgroup of $SO_0(1,n)$ which leaves $\mathcal{W}$ invariant and induces a future directed Killing vector field in that region. We denote this group by $\Lambda_{\mathcal{W}}(t)$, $t \in \mathbb{R}$, and call it the group of boosts associated with $\mathcal{W}$. It describes the time evolution for the geodesic observer in $\mathcal{W}$. The causal complement $\mathcal{W}'$ of $\mathcal{W}$ is also invariant under the action of $\Lambda_{\mathcal{W}}(t)$, $t \in \mathbb{R}$, but the corresponding Killing vector field is past directed in that region. Hence there holds $\Lambda_{\mathcal{W}'}(t) = \Lambda_{\mathcal{W}}(-t)$, $t \in \mathbb{R}$.

Let us now turn to a discussion of the continuous unitary representations of $SO_0(1,n)$. Given any such representation U on some Hilbert space $\mathcal{H}$ we denote the corresponding selfadjoint generators with respect to the chosen coordinate system by $M_{\mu\nu}$, $\mu, \nu = 0, 1, \ldots n$. They satisfy on a canonical domain of analytic vectors [10] the Lie–algebra relations

$$[M_{\mu\nu}, M_{\rho\sigma}] = -\,i g_{\mu\rho} M_{\nu\sigma} + i g_{\mu\sigma} M_{\nu\rho} + i g_{\nu\rho} M_{\mu\sigma} - i g_{\nu\sigma} M_{\mu\rho}, \tag{2.3}$$

where $g_{\mu\nu}$ is the metric tensor of the ambient Minkowski space. The operators M_{0j} generate the action of the boosts associated with the wedges $\mathcal{W}_j$,

$$U(\Lambda_{\mathcal{W}_j}(t)) = \mathrm{e}^{itM_{0j}}, \quad t \in \mathbb{R}, \ j = 1, \ldots n, \tag{2.4}$$

and the M_{jk}, $j, k = 1, \ldots n$, are the generators of spatial rotations.

For fixed $j \neq k$ the operators M_{0j}, M_{0k} and M_{jk} form a Lie sub–algebra and there holds for $s, t \in \mathbb{R}$

$$\mathrm{e}^{isM_{0j}}\,\mathrm{e}^{itM_{0k}}\,\mathrm{e}^{-\,isM_{0j}} = \mathrm{e}^{it(\mathrm{ch}\,(s)M_{0k}+\mathrm{sh}\,(s)M_{jk})} \tag{2.5}$$

$$\mathrm{e}^{isM_{jk}}\,\mathrm{e}^{itM_{0k}}\,\mathrm{e}^{-\,isM_{jk}} = \mathrm{e}^{it(\cos(s)M_{0k}+\sin(s)M_{0j})}. \tag{2.6}$$

These relations are repeatedly used in the proofs of the following results.

LEMMA 2.1. – *Let $\mathcal{N} \subset SO_0(1,n)$ be any open neighborhood of the unit element in $SO_0(1,n)$ and let $\Lambda_{\mathcal{W}}(t)$, $t \in \mathbb{R}$, be the boosts associated with a given wedge $\mathcal{W}$. Then the strong closure of the group generated by the unitary operators $U(\Lambda\Lambda_{\mathcal{W}}(t)\Lambda^{-1})$ with $t \in \mathbb{R}$, $\Lambda \in \mathcal{N}$, coincides with $U(SO_0(1,n))$.*

Proof. – Because of the de Sitter invariance of the problem we can assume without loss of generality that $\mathcal{W}$ is the wedge $\mathcal{W}_1$. The statement can then be established by the following computation. Let $U_{\mathcal{N}}$ be the closed unitary group generated by $U(\Lambda\Lambda_{\mathcal{W}_1}(t)\Lambda^{-1})$ with $t \in \mathbb{R}$ and $\Lambda \in \mathcal{N}$. It follows from (2.5) that for sufficiently small $|s|$ and any $t \in \mathbb{R}$ there holds $\mathrm{e}^{it(\mathrm{ch}\,(s)M_{01}+\mathrm{sh}\,(s)M_{j1})} \in U_{\mathcal{N}}$ for $j = 2, \dots n$. Keeping $s \neq 0$ fixed one sees by an application of the Trotter product formula [11] to the product of the one–parameter groups $\mathrm{e}^{it(\mathrm{ch}\,(s)M_{01}+\mathrm{sh}\,(s)M_{j1})}$ and $\mathrm{e}^{-it\mathrm{ch}\,(s)M_{01}}$ that the rotations $\mathrm{e}^{it\mathrm{sh}\,(s)M_{j1}}$, $t \in \mathbb{R}$, $j = 2, \dots n$, belong to $U_{\mathcal{N}}$. Relation (2.6) then implies that also $\mathrm{e}^{itM_{0j}} \in U_{\mathcal{N}}$ for $t \in \mathbb{R}$, $j = 1, \dots n$. Since these operators generate $U(SO_0(1,n))$ there holds $U_{\mathcal{N}} = U(SO_0(1,n))$, proving the statement. ∎

LEMMA 2.2. – *Let $\Psi \in \mathcal{H}$ be invariant under the action of $U(\Lambda_{\mathcal{W}}(t))$, $t \in \mathbb{R}$, where $\Lambda_{\mathcal{W}}(t)$, $t \in \mathbb{R}$, is the group of boosts associated with a given wedge $\mathcal{W}$. Then Ψ is invariant under the action of $U(SO_0(1,n))$.*

Proof. – As in the proof of the preceding lemma we may assume without restriction of generality that $\mathcal{W}$ is the wedge $\mathcal{W}_1$. Putting $t = 2r\mathrm{e}^{-|s|}$ in relation (2.5) it follows from the continuity of the representation U that in the sense of strong operator convergence on $\mathcal{H}$

$$\lim_{s\to\pm\infty} \mathrm{e}^{isM_{01}}\mathrm{e}^{2ir\mathrm{e}^{-|s|}M_{0j}}\mathrm{e}^{-isM_{01}} = \mathrm{e}^{ir(M_{0j}\pm M_{1j})} \tag{2.7}$$

for $j = 2, \dots n$. On the other hand, since Ψ is invariant under the action of the unitary operators $\mathrm{e}^{isM_{01}}$, $s \in \mathbb{R}$, and since $\mathrm{e}^{2ir\mathrm{e}^{-|s|}M_{0j}}$ converges to 1 in the strong operator topology for $s \to \pm\infty$ and fixed r, we get

$$\lim_{s\to\pm\infty} ||\mathrm{e}^{isM_{01}}\mathrm{e}^{2ir\mathrm{e}^{-|s|}M_{0j}}\mathrm{e}^{-isM_{01}}\Psi - \Psi|| = \lim_{s\to\pm\infty} ||\mathrm{e}^{2ir\mathrm{e}^{-|s|}M_{0j}}\Psi - \Psi|| = 0. \tag{2.8}$$

Combining these relations we obtain

$$e^{ir(M_{0j} \pm M_{1j})}\Psi = \Psi \quad \text{for} \quad r \in \mathbb{R},\, j = 2, \ldots n. \tag{2.9}$$

By a similar argument as in the proof of the preceding lemma it then follows that $U(\Lambda)\Psi = \Psi$ for any $\Lambda \in SO_0(1,n)$. ■

We conclude this section by recalling a result of Nelson [10] on the existence of analytic vectors for generators of unitary representations of Lie–groups. We state this result in a form which is convenient for the subsequent applications.

LEMMA 2.3. – *Let $\mathcal{C}$ be a sufficiently small neighborhood of the origin in $\mathbb{C}$. There exists a dense set of vectors $\Phi \in \mathcal{H}$ such that*

$$\sum_{n=0}^{\infty} \frac{||(uM_{0k} + vM_{jk})^n\, \Phi||}{n!} < \infty \tag{2.10}$$

for $u, v \in \mathcal{C}$ and $j, k = 1, \ldots n$. Phrased differently, the vectors Φ are analytic for the respective generators with a uniform radius of convergence.

Proof. – The statement follows from Theorem 3 in [10] by taking also into account the quantitative estimates in Corollary 3.1 and Lemma 6.2 of that reference. ■

3. REEH–SCHLIEDER PROPERTY OF VACUUM STATES

Before we turn now to the analysis of vacuum states in de Sitter space we briefly list our assumptions, establish our notation and add a few comments.

1. *(Locality)* There is an inclusion preserving mapping

$$\mathcal{O} \to \mathcal{A}(\mathcal{O}) \tag{3.1}$$

from the set of open, bounded, contractible regions $\mathcal{O} \subset \mathcal{S}^n$ to von Neumann algebras $\mathcal{A}(\mathcal{O})$ on some Hilbert space $\mathcal{H}$. We interpret each $\mathcal{A}(\mathcal{O})$ as the algebra generated by all observables which can be measured in $\mathcal{O}$. For any wedge $\mathcal{W} \subset \mathcal{S}^n$ the corresponding algebra $\mathcal{A}(\mathcal{W})$ is defined as the von Neumann algebra generated by the local algebras $\mathcal{A}(\mathcal{O})$ with $\mathcal{O} \subset \mathcal{W}$, and $\mathcal{A}$ denotes the von Neumann algebra generated by all local algebras $\mathcal{A}(\mathcal{O})$. The local algebras are supposed to satisfy the condition of locality, i.e.

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)' \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2', \tag{3.2}$$

where $\mathcal{O}'$ denotes the spacelike complement of $\mathcal{O}$ in $\mathcal{S}^n$ and $\mathcal{A}(\mathcal{O})'$ the commutant of $\mathcal{A}(\mathcal{O})$ in $\mathcal{B}(\mathcal{H})$.

2. *(Covariance)* On $\mathcal{H}$ there is a continuous unitary representation U of the de Sitter group $SO_0(1,n)$ which induces automorphisms α of $\mathcal{B}(\mathcal{H})$ acting covariantly on the local algebras. More concretely, putting $\alpha_\Lambda(\,\cdot\,) \doteq U(\Lambda)\,\cdot\,U(\Lambda)^{-1}$, $\Lambda \in SO_0(1,n)$, there holds for each region $\mathcal{O} \subset \mathcal{S}^n$

$$\alpha_\Lambda(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\Lambda\mathcal{O}). \tag{3.3}$$

3. *(de Sitter Vacuum)* There is a unit vector $\Omega \in \mathcal{H}$, describing the vacuum, which is invariant under the action of $U(SO_0(1,n))$ and cyclic for the global algebra $\mathcal{A}$. The corresponding vector state ω on $\mathcal{A}$, given by

$$\omega(A) = (\Omega,\, A\,\Omega), \quad A \in \mathcal{A}, \tag{3.4}$$

has the following *geodesic KMS–property* suggested by the results of Gibbons and Hawking [2]: For every wedge $\mathcal{W}$ the restriction (partial state) $\omega \upharpoonright \mathcal{A}(\mathcal{W})$ satisfies the KMS–condition at some inverse temperature $\beta > 0$ with respect to the time evolution (boosts) $\Lambda_\mathcal{W}(t)$, $t \in \mathbb{R}$, associated with $\mathcal{W}$. In other words, for any pair of operators $A, B \in \mathcal{A}(\mathcal{W})$ there exists an analytic function F in the strip $\{z \in \mathbb{C} : 0 < \mathrm{Im} z < \beta\}$ with continuous boundary values at $\mathrm{Im} z = 0$ and $\mathrm{Im} z = \beta$, which are given respectively (for $t \in \mathbb{R}$) by

$$F(t) = \omega(A\alpha_{\Lambda_\mathcal{W}(t)}(B)), \quad F(t+i\beta) = \omega(\alpha_{\Lambda_\mathcal{W}(t)}(B)A). \tag{3.5}$$

In order to cover also the case of degenerate vacuum states we do *not* assume here that the vacuum vector Ω is (up to a phase) unique.

The inverse temperature β in the preceding condition has to be the same for all wedges $\mathcal{W}$ because of the invariance of Ω under the action of the de Sitter group. Its actual value has been determined by several authors in a general setting by starting from various assumptions, such as the condition of local stability [12,5], the weak spectral condition [7] or the condition of modular covariance on lightlike hyper-surfaces [13]. As we shall see, the present assumptions already fix the value of β.

Our last condition expresses the idea that all observables are built from strictly local ones. It is a standard assumption in the case of Minkowski space theories.

4. *(Weak additivity)* For each open region $\mathcal{O} \subset \mathcal{S}^n$ there holds

$$\bigvee_{\Lambda \in SO_0(1,n)} \mathcal{A}(\Lambda\mathcal{O}) = \mathcal{A}. \tag{3.6}$$

Note that, for $n > 1$, $\{\Lambda\mathcal{O} : \Lambda \in SO_0(1,n)\}$ defines a covering of $\mathcal{S}^n$ since $SO_0(1,n)$ acts transitively on that space. So the condition is clearly satisfied if the local algebras are generated by Wightman fields.

We turn now to the analysis of the cyclicity properties of Ω with respect to the local algebras $\mathcal{A}(\mathcal{O})$.

DEFINITION 3.1. – *Let $\mathcal{O} \subset \mathcal{S}^n$ be any open region. The $*$–algebra $\mathcal{B}(\mathcal{O})$ is defined as the set of operators $B \in \mathcal{A}(\mathcal{O})$ for which there exists some neighborhood $\mathcal{N} \subset SO_0(1,n)$ of the unit element in $SO_0(1,n)$ (depending on B) such that*

$$\alpha_\Lambda(B) \in \mathcal{A}(\mathcal{O}) \quad \textit{for} \quad \Lambda \in \mathcal{N}. \tag{3.7}$$

It is apparent that $\mathcal{B}(\mathcal{O})$ is indeed a $*$–algebra and that $\mathcal{A}(\mathcal{O}_0) \subset \mathcal{B}(\mathcal{O})$ for any region $\mathcal{O}_0$ whose closure satisfies $\overline{\mathcal{O}_0} \subset \mathcal{O}$.

In the subsequent lemmas we establish some technical properties of the orthogonal complements of the spaces $\mathcal{B}(\mathcal{O})\Omega$ in $\mathcal{H}$. (Cf. [14] for a similar discussion in case of Minkowski space theories.) It suffices for our purposes to consider regions $\mathcal{O} \subset \mathcal{S}^n$ which are so small that there exists a wedge $\mathcal{W}$ and an open neighborhood $\mathcal{N} \subset SO_0(1,n)$ of the unit element in $SO_0(1,n)$ such that $\Lambda^{-1}\mathcal{O} \subset \mathcal{W}$ for all $\Lambda \in \mathcal{N}$. Then, if $\Lambda_\mathcal{W}(t)$, $t \in \mathbb{R}$, is the one–parameter group of boosts associated with $\mathcal{W}$, there holds

$$\Lambda\Lambda_\mathcal{W}(t)\Lambda^{-1}\mathcal{O} \subset \Lambda\mathcal{W} \quad \text{for} \quad \Lambda \in \mathcal{N},\ t \in \mathbb{R}, \tag{3.8}$$

where $\Lambda\Lambda_\mathcal{W}(t)\Lambda^{-1}$, $t \in \mathbb{R}$, are the boosts associated with the wedge $\Lambda\mathcal{W}$.

LEMMA 3.2. – *Let $\mathcal{O} \subset \mathcal{S}^n$ be a sufficiently small region (in the sense described above) and let $\Psi \in \mathcal{H}$ be a vector with the property that*

$$(\Psi, B\,\Omega) = 0 \quad \textit{for} \quad B \in \mathcal{B}(\mathcal{O}). \tag{3.9}$$

Then the vectors $U(\Lambda)\Psi$, $\Lambda \in SO_0(1,n)$, have the same property.

Proof. – Let $B \in \mathcal{B}(\mathcal{O})$ and $\Lambda \in \mathcal{N}$ with $\mathcal{N}$ as in relation (3.8). It follows from the definition of $\mathcal{B}(\mathcal{O})$ and the continuity of the boosts that there is an $\varepsilon > 0$ such that $\alpha_{\Lambda\Lambda_\mathcal{W}(t)\Lambda^{-1}}(B) \in \mathcal{B}(\mathcal{O})$ for $|t| < \varepsilon$ and consequently

$$(\Psi, \alpha_{\Lambda\Lambda_\mathcal{W}(t)\Lambda^{-1}}(B)\,\Omega) = 0 \quad \text{for} \quad |t| < \varepsilon. \tag{3.10}$$

On the other hand, since $\Lambda\Lambda_\mathcal{W}(t)\Lambda^{-1}\mathcal{O} \subset \Lambda\mathcal{W}$, there holds $\alpha_{\Lambda\Lambda_\mathcal{W}(t)\Lambda^{-1}}(B) \in \mathcal{A}(\Lambda\mathcal{W})$ for $t \in \mathbb{R}$. So the geodesic KMS–property of Ω implies that

$$t \longrightarrow \alpha_{\Lambda\Lambda_\mathcal{W}(t)\Lambda^{-1}}(B)\,\Omega, \quad t \in \mathbb{R} \tag{3.11}$$

extends analytically to some vector–valued function in the strip $\{z \in \mathbb{C} : 0 < \mathrm{Im} z < \beta/2\}$ [15]. Combining these two informations it follows that

$$(U(\Lambda\Lambda_{\mathcal{W}}(t)\Lambda^{-1})\Psi, B\,\Omega) = (\Psi, \alpha_{\Lambda\Lambda_{\mathcal{W}}(-t)\Lambda^{-1}}(B)\,\Omega) = 0 \tag{3.12}$$

for all $t \in \mathbb{R}$ and $B \in \mathcal{B}(\mathcal{O})$. Since $\Lambda \in \mathcal{N}$ was arbitrary, we conclude by repetition of the preceding argument that for any $\Lambda_1, \ldots \Lambda_k \in \mathcal{N}$ and $t_1, \ldots t_k \in \mathbb{R}$

$$(U(\Lambda_1\Lambda_{\mathcal{W}}(t_1)\Lambda_1^{-1}) \ldots U(\Lambda_k\Lambda_{\mathcal{W}}(t_k)\Lambda_k^{-1})\Psi, B\,\Omega) = 0. \tag{3.13}$$

As $U(SO_0(1,n))$ is generated by products of the boost operators $U(\Lambda\Lambda_{\mathcal{W}}(t)\Lambda^{-1})$ with $\Lambda \in \mathcal{N}$, $t \in \mathbb{R}$, cf. Lemma 2.1, the assertion follows. ■

LEMMA 3.3. – *Let $\mathcal{O} \subset S^n$ and $\Psi \in \mathcal{H}$ be as in the preceding lemma. There holds for $\Lambda_1, \ldots \Lambda_k \in \mathcal{N}$ and $B_1, \ldots B_k \in \mathcal{B}(\mathcal{O})$*

$$(\Psi, \alpha_{\Lambda_1}(B_1) \cdots \alpha_{\Lambda_k}(B_k)\,\Omega) = 0. \tag{3.14}$$

Proof. – As $\mathcal{B}(\mathcal{O})$ is a $*$–algebra we see from the preceding lemma that $B^*U(\Lambda)\,\Psi$ is orthogonal to $\mathcal{B}(\mathcal{O})\,\Omega$ for any $B \in \mathcal{B}(\mathcal{O})$ and $\Lambda \in SO_0(1,n)$. So the statement follows by induction. ■

We are now in a position to establish the Reeh–Schlieder property of Ω, i.e. the fact that Ω is a cyclic vector for all local algebras.

THEOREM 3.4. – *For any open region $\mathcal{O} \subset S^n$ there holds*

$$\overline{\mathcal{A}(\mathcal{O})\,\Omega} = \mathcal{H}. \tag{3.15}$$

Proof. – We may assume that $\mathcal{O}$ is so small that the preceding lemma can be applied. Now if $\Psi \in \mathcal{H}$ is orthogonal to $\mathcal{A}(\mathcal{O})\,\Omega$ it is also orthogonal to $\mathcal{B}(\mathcal{O})\,\Omega$ and consequently to $\Big(\bigvee_{\Lambda \in SO_0(1,n)} \alpha_\Lambda(\mathcal{B}(\mathcal{O}))\Big)\Omega$. As $\mathcal{A}(\mathcal{O}_0) \subset \mathcal{B}(\mathcal{O})$ if $\overline{\mathcal{O}_0} \subset \mathcal{O}$ there holds

$$\begin{aligned} \bigvee_{\Lambda \in SO_0(1,n)} \alpha_\Lambda(\mathcal{B}(\mathcal{O})) &\supset \bigvee_{\Lambda \in SO_0(1,n)} \alpha_\Lambda(\mathcal{A}(\mathcal{O}_0)) \\ &= \bigvee_{\Lambda \in SO_0(1,n)} \mathcal{A}(\Lambda\mathcal{O}_0) = \mathcal{A}, \end{aligned} \tag{3.16}$$

where in the last equality we made use of weak additivity. Since Ω is cyclic for $\mathcal{A}$ it follows that $\Psi = 0$, completing the proof. ■

4. TYPE OF THE GLOBAL ALGEBRA $\mathcal{A}$

We turn now to the analysis of the global algebra $\mathcal{A}$, where we will make use of modular theory, cf. for example [15]. The geodesic KMS–property implies that $\alpha_{\Lambda_{\mathcal{W}}(-t)}$, $t \in \mathbb{R}$, is (apart from a rescaling of the parameter t by β) the group of modular automorphisms associated with the pair $\{\mathcal{A}(\mathcal{W}), \Omega\}$ for any wedge $\mathcal{W} \subset \mathcal{S}^n$. Hence, by the basic results of modular theory, $\alpha_{\Lambda_{\mathcal{W}}(t)}$, $t \in \mathbb{R}$, is the modular group of $\{\mathcal{A}(\mathcal{W})', \Omega\}$. This fact will be used at various points in the subsequent investigation.

We begin our discussion with two preparatory propositions which are of interest in their own right.

PROPOSITION 4.1. – *The commutant $\mathcal{A}'$ of $\mathcal{A}$ is pointwise invariant under the adjoint action of $U(\Lambda)$, $\Lambda \in SO_0(1,n)$, i.e. the representation U of the de Sitter group is contained in the global algebra of observables $\mathcal{A}$.*

Proof. – We fix a wedge $\mathcal{W}$ and consider the corresponding automorphisms $\alpha_{\Lambda_{\mathcal{W}}(t)}$, $t \in \mathbb{R}$. As $\mathcal{W}$ is invariant under the boosts $\Lambda_{\mathcal{W}}(t)$ the algebras $\mathcal{A}(\mathcal{W})'$ and $\mathcal{A}(\mathcal{W})' \cap \mathcal{A}$ are invariant under the action of these automorphisms. Both algebras contain $\mathcal{A}(\mathcal{W}')$ and thus have Ω as a cyclic vector.

If $X \in \mathcal{A}' \subset \mathcal{A}(\mathcal{W})'$ and $A \in \mathcal{A}(\mathcal{W})' \cap \mathcal{A}$ it follows from modular theory that the function $t \longrightarrow (\Omega, A\alpha_{\Lambda_{\mathcal{W}}(t)}(X)\,\Omega)$ extends to a bounded analytic function F in the strip $\{z \in \mathbb{C} : 0 > \mathrm{Im} z > -\beta\}$. Moreover, the boundary value of F at $\mathrm{Im} z = -\beta$ is given by $F(t - i\beta) = (\Omega, \alpha_{\Lambda_{\mathcal{W}}(t)}(X) A\,\Omega) = (\Omega, A\alpha_{\Lambda_{\mathcal{W}}(t)}(X)\,\Omega)$, where in the second equality we have used the commutativity of $\mathcal{A}'$ and $\mathcal{A}(\mathcal{W})' \cap \mathcal{A}$. Hence F can be extended by periodicity to a bounded analytic function on $\mathbb{C}$ and thus is constant. Since $A \in \mathcal{A}(\mathcal{W})' \cap \mathcal{A}$ is arbitrary and Ω is cyclic for $\mathcal{A}(\mathcal{W})' \cap \mathcal{A}$ and separating for $\mathcal{A}'$ we conclude that $X = \alpha_{\Lambda_{\mathcal{W}}(t)}(X)$, i.e. X commutes with the unitaries $U(\Lambda_{\mathcal{W}}(t))$ for $t \in \mathbb{R}$ and every wedge $\mathcal{W}$. As these unitaries generate the group $U(SO_0(1,n))$, the proof is complete. ∎

PROPOSITION 4.2. – *Let $\mathcal{O} \subset \mathcal{S}^n$ be any open region and let E_0 be the projection onto the space of $U(SO_0(1,n))$–invariant vectors in $\mathcal{H}$. Then the von Neumann algebra generated by E_0 and $\mathcal{A}(\mathcal{O})$ coincides with $\mathcal{A}$.*

Proof. – Given $\mathcal{O} \subset \mathcal{S}^n$ we pick another open region $\mathcal{O}_0$ such that for some neigbourhood $\mathcal{N}$ of the unit element of $SO_0(1,n)$ there holds $\Lambda\mathcal{O}_0 \subset \mathcal{O}$ for $\Lambda \in \mathcal{N}$. Now let $C \in \{\mathcal{A}(\mathcal{O}), E_0\}'$. Then there holds for any $A \in \mathcal{A}(\mathcal{O}_0)$ and $\Lambda \in \mathcal{N}$

$$\begin{aligned} U(\Lambda)^{-1}CU(\Lambda)A\,\Omega &= AU(\Lambda)^{-1}CE_0\,\Omega \\ &= AU(\Lambda)^{-1}E_0C\,\Omega = AC\,\Omega = CA\,\Omega. \end{aligned} \quad (4.1)$$

Since Ω is cyclic for $\mathcal{A}(\mathcal{O}_0)$ it follows that $U(\Lambda)^{-1}CU(\Lambda) = C$ for $\Lambda \in \mathcal{N}$ and therefore for $\Lambda \in SO_0(1,n)$. Hence C commutes also with $\bigvee_{\Lambda \in SO_0(1,n)} \mathcal{A}(\Lambda\mathcal{O}_0) = \mathcal{A}$, where we have used weak additivity, and consequently $\{E_0, \mathcal{A}(\mathcal{O})\}' \subset \{E_0, \mathcal{A}\}'$. But $E_0 \in U(SO_0(1,n))'' \subset \mathcal{A}$, where the inclusion follows from the preceding proposition. Hence $\mathcal{A} \subset \{E_0, \mathcal{A}(\mathcal{O})\}'' \subset \mathcal{A}$ as claimed. ■

With this information we can now establish the following theorem.

THEOREM 4.3. – *In the vacuum sector of any de Sitter theory there holds:*

(a) *The commutant $\mathcal{A}'$ of $\mathcal{A}$ is abelian (i.e. $\mathcal{A}$ is of type I and $\mathcal{A}'$ is the center of $\mathcal{A}$).*

(b) *The projection E_0 onto the space of all $U(SO_0(1,n))$–invariant vectors in $\mathcal{H}$ is an abelian projection in $\mathcal{A}$ with central support* 1.

Proof. – Let $\mathcal{W}$ be any wedge and let $X_1, X_2 \in \mathcal{A}' \subset \mathcal{A}(\mathcal{W})'$, $A \in \mathcal{A}(\mathcal{W})' \cap \mathcal{A}$. As in the proof or Proposition 4.1 we make use of the modular theory for $\{\mathcal{A}(\mathcal{W})', \Omega\}$ and consider the function $t \to (\Omega, AX_1\alpha_{\Lambda_\mathcal{W}(t)}(X_2)\,\Omega)$. It extends to an analytic function F in the strip $\{z \in \mathbb{C} : 0 > \text{Im}z > -\beta\}$ whose boundary value at $\text{Im}z = -\beta$ is given by $F(t - i\beta) = (\Omega, \alpha_{\Lambda_\mathcal{W}(t)}(X_2)AX_1\,\Omega)$. On the other hand, the pointwise invariance of $\mathcal{A}'$ under the action of $\alpha_{\Lambda_\mathcal{W}(t)}$, cf. Proposition 4.1, implies that F is constant. Combining these two facts we get

$$(\Omega, AX_1X_2\,\Omega) = (\Omega, X_2AX_1\,\Omega) = (\Omega, AX_2X_1\,\Omega), \quad (4.2)$$

where in the second equality we made use of the commutativity of A and X_2. The cyclicity of Ω for $\mathcal{A}(\mathcal{W})' \cap \mathcal{A}$ implies $[X_1, X_2]\,\Omega = 0$. Since Ω is separating for $\mathcal{A}'$ it follows that $[X_1, X_2] = 0$. So $\mathcal{A}'$ is abelian, proving the first part of the statement.

For the proof of the second part we pick a wedge $\mathcal{W}$ and choose $A \in \mathcal{A}(\mathcal{W})$ and $B \in \mathcal{A}(\mathcal{W}')$. By the mean ergodic theorem (see e.g. [16]) there holds in the sense of strong operator convergence

$$\lim_{T\to\infty} T^{-1} \int_0^T dt\, U(\Lambda_\mathcal{W}(\pm t)) = F_0, \quad (4.3)$$

where F_0 denotes the projection onto the subspace of vectors in $\mathcal{H}$ which are invariant under the action of the unitaries $U(\Lambda_\mathcal{W}(t))$, $t \in \mathbb{R}$. Hence, making use of locality and the invariance of E_0 under left and right multiplication with $U(\Lambda_{\mathcal{W}}(t))$, we get

$$E_0BF_0AE_0 = \lim_{T\to\infty} T^{-1}\int_0^T dt\, E_0B\alpha_{\Lambda_{\mathcal{W}}(t)}(A)E_0$$
$$= \lim_{T\to\infty} T^{-1}\int_0^T dt\, E_0\alpha_{\Lambda_{\mathcal{W}}(t)}(A)BE_0 = E_0AF_0BE_0. \quad (4.4)$$

According to Lemma 2.2 F_0 coincides with E_0 and hence

$$E_0BE_0AE_0 = E_0AE_0BE_0. \quad (4.5)$$

This shows that the algebras $\{E_0\mathcal{A}(\mathcal{W})E_0 \restriction E_0\mathcal{H}\}''$ and $\{E_0\mathcal{A}(\mathcal{W}')E_0 \restriction E_0\mathcal{H}\}''$ commute. By Proposition 4.2 both algebras coincide with $E_0\mathcal{A}E_0$, hence relation (4.5) holds for all $A, B \in \mathcal{A}$, proving that E_0 is an abelian projection.

Finally, let E be any projection in the center of $\mathcal{A}$ which dominates E_0, i.e. $EE_0 = E_0$. Then there holds in particular $E\,\Omega = \Omega$ and since Ω is separating for the center it follows that $E = 1$. So E_0 has central support 1. ■

COROLLARY 4.4. – *The following statements are equivalent for any vacuum state ω:*

(a) *ω is a primary state*

(b) *ω is a pure state*

(c) *ω is weakly mixing with respect to the action of boosts.*

Proof. – If ω is primary $\mathcal{A}$ has a trivial center. But according to part (a) of the preceding theorem the center of $\mathcal{A}$ is equal to $\mathcal{A}'$ and consequently $\mathcal{A}' = \mathbb{C}\,1$. Hence ω is a pure state.

In the latter case there holds $\mathcal{A} = \mathcal{B}(\mathcal{H})$ which implies $E_0\mathcal{A}E_0 = \mathcal{B}(E_0\mathcal{H})$. According to part (b) of the preceding theorem the algebra $E_0\mathcal{A}E_0$ is abelian, so E_0 must be a one-dimensional projection. Thus by the mean ergodic theorem, 2.2

$$\lim_{T\to\infty} T^{-1}\int_0^T dt\,\omega(B\alpha_{\Lambda_{\mathcal{W}}(t)}(A)) = (\Omega, BE_0A\,\Omega) = \omega(B)\omega(A) \quad (4.6)$$

for any $A, B \in \mathcal{A}$ which shows that ω is weakly mixing.

Conversely, relation (4.6) implies that the projection $E_0 \in \mathcal{A}$ is one–dimensional. Hence if $X \in \mathcal{A}'$ there holds $X\,\Omega = E_0X\,\Omega = \omega(X)\,\Omega$. Since Ω is separating for $\mathcal{A}'$ it follows that $\mathcal{A}' = \mathbb{C}\,1$. So the state ω is pure and *a fortiori* primary. ■

5. INVARIANT MEANS AND THE CENTER OF $\mathcal{A}$

We analyze now the properties of the invariant means on $\mathcal{B}(\mathcal{H})$ which are induced by the adjoint action of the boost operators $U(\Lambda_{\mathcal{W}}(t))$, $t \in \mathbb{R}$, associated with arbitrary wedges $\mathcal{W} \subset \mathcal{S}^n$. Since $\mathbb{R}$ is amenable such means exist in the space of linear mappings on $\mathcal{B}(\mathcal{H})$ as limit points of the nets

$$T^{-1} \int_0^T dt\, U(\Lambda_{\mathcal{W}}(t)) \cdot U(\Lambda_{\mathcal{W}}(t))^{-1}, \quad T \to \infty, \tag{5.1}$$

in the so-called point-weak-open topology. We denote the respective limits by $M_{\mathcal{W}}$ and note that they are, for given $\mathcal{W}$, in general neither unique nor normal. Therefore the following result is of some interest.

PROPOSITION 5.1. – *Let $\mathcal{W}$ be any wedge and let $M_{\mathcal{W}}$ be a corresponding mean on $\mathcal{B}(\mathcal{H})$. The restriction $M_{\mathcal{W}} \upharpoonright \mathcal{A}(\mathcal{W})$ is unique, normal, and its range lies in $\mathcal{A}(\mathcal{W})$ and coincides with the center of $\mathcal{A}$.*

Proof. – Let $A \in \mathcal{A}(\mathcal{W})$. From the invariance of $\mathcal{A}(\mathcal{W})$ under the adjoint action of $U(\Lambda_{\mathcal{W}}(t))$, $t \in \mathbb{R}$, it follows that $M_{\mathcal{W}}(A)$ belongs to $\mathcal{A}(\mathcal{W})$ and commutes with the unitary operators $U(\Lambda_{\mathcal{W}}(t))$, $t \in \mathbb{R}$. So Lemma 2.2 implies that $M_{\mathcal{W}}(A)\,\Omega \in E_0\mathcal{H}$. Now let $\mathcal{N} \subset SO_0(1,n)$ be a neighbourhood of the unit element of the de Sitter group such that $\Lambda\mathcal{W} \cup \mathcal{W}$ has an open spacelike complement for $\Lambda \in \mathcal{N}$. Then, because of locality and the Reeh–Schlieder property, Ω is separating for $\mathcal{A}(\Lambda\mathcal{W}) \bigvee \mathcal{A}(\mathcal{W})$. Moreover, by the mean ergodic theorem and Lemma 2.2, $U(\Lambda) M_{\mathcal{W}}(A) U(\Lambda)^{-1}\,\Omega = E_0 A\,\Omega = M_{\mathcal{W}}(A)\,\Omega$. So there holds $\alpha_\Lambda(M_{\mathcal{W}}(A)) = M_{\mathcal{W}}(A)$ for $\Lambda \in \mathcal{N}$ and consequently for all $\Lambda \in SO_0(1,n)$. As the operators $M_{\mathcal{W}}(A)$, $A \in \mathcal{A}(\mathcal{W})$, commute with $\mathcal{A}(\mathcal{W}')$ it follows that they also commute with $\bigvee_{\Lambda \in SO_0(1,n)} \mathcal{A}(\Lambda\mathcal{W}') = \mathcal{A}$ and thus belong to the center of $\mathcal{A}$ by Theorem 4.3.

Because of the fact that Ω is separating for the center and the relation $M_{\mathcal{W}}(A)\Omega = E_0 A\Omega$ it is then clear that $M_{\mathcal{W}} \upharpoonright \mathcal{A}(\mathcal{W})$ is unique and normal.

The preceding results imply that $M_{\mathcal{W}}(\mathcal{A}(\mathcal{W}))$ is contained in $\mathcal{A}(\mathcal{W})$ and a subset of the center of $\mathcal{A}$. As the elements of the center are pointwise invariant under the action of $M_{\mathcal{W}}$ it is also clear that $M_{\mathcal{W}}(\mathcal{A}(\mathcal{W}))$ is a von Neumann algebra. Its restriction to $E_0\mathcal{H}$ has Ω as a cyclic vector by the Reeh–Schlieder property. So it is maximally abelian in $E_0\mathcal{H}$ and therefore contains the restriction of the center of $\mathcal{A}$ to that space. But the center of $\mathcal{A}$ is faithfully represented on $E_0\mathcal{H}$, so the assertion follows. ∎

We mention as an aside that it follows from this proposition that every wedge algebra $\mathcal{A}(\mathcal{W})$ is of type III_1 according to the classification of

Connes. For it implies that the centralizer of ω in $\mathcal{A}(\mathcal{W})$ coincides with the center. By central decomposition one may therefore restrict attention to the case where the wedge algebras are factors and the centralizers are trivial. Making also use of the fact that the modular groups $U(\Lambda_{\mathcal{W}}(t))$, $t \in \mathbb{R}$, cannot be cyclic because of the group structure of $SO_0(1,n)$ (unless the representation U is trivial) the assertion then follows from the well–known results of Connes in [17].

It is neither clear what can be said about the action of $M_{\mathcal{W}}$ on algebras of arbitrary (even bounded) regions, nor how these means depend on the choice of the wedge $\mathcal{W}$. Nevertheless it is possible to define a universal invariant mean M of the operators in the set–theoretic union of algebras $\bigcup_{\mathcal{W}\subset\mathcal{S}^n} \mathcal{A}(\mathcal{W})$ with values in the center of $\mathcal{A}$. (Note that this union is neither an algebra nor a vector space.) We define M by setting for any wedge $\mathcal{W}$

$$M \restriction \mathcal{A}(\mathcal{W}) \doteq M_{\mathcal{W}} \restriction \mathcal{A}(\mathcal{W}). \tag{5.2}$$

For the proof that this definition is consistent, let $\mathcal{W}_1, \mathcal{W}_2$ be wedges and let $A \in \mathcal{A}(\mathcal{W}_1) \cap \mathcal{A}(\mathcal{W}_2)$. Then $M_{\mathcal{W}_1}(A), M_{\mathcal{W}_2}(A)$ are elements of the center of $\mathcal{A}$ and there holds $M_{\mathcal{W}_1}(A)\,\Omega = E_0 A\,\Omega = M_{\mathcal{W}_2}(A)\,\Omega$. As Ω is separating for the center, this implies $M_{\mathcal{W}_1}(A) = M_{\mathcal{W}_2}(A)$, proving the consistency.

Since for $A \in \mathcal{A}(\mathcal{W})$ and $\Lambda \in SO_0(1,n)$ there holds $\alpha_\Lambda(A) \in \mathcal{A}(\Lambda\mathcal{W})$ one obtains $M(\alpha_\Lambda(A))\,\Omega = E_0\alpha_\Lambda(A)\,\Omega = E_0 A\,\Omega = M(A)\,\Omega$. It follows that $M(\alpha_\Lambda(A)) = M(A)$ for $\Lambda \in SO_0(1,n)$, $A \in \mathcal{A}(\mathcal{W})$ and any wedge $\mathcal{W}$. So we have established the following proposition.

PROPOSITION 5.2. – *There exists a unique map*

$$M : \bigcup_{\mathcal{W}\subset\mathcal{S}^n} \mathcal{A}(\mathcal{W}) \longrightarrow \textit{center}\,(\mathcal{A}) \tag{5.3}$$

which is invariant under the right and left action of $\alpha_\Lambda, \Lambda \in SO_0(1,n)$, *and whose restriction to* $\mathcal{A}(\mathcal{W})$ *coincides with the corresponding mean* $M_{\mathcal{W}}$, $\mathcal{W} \subset \mathcal{S}^n$.

It is probably not meaningful to extend M to operators which are localized in regions larger than wedges.

6. PCT AND THE TEMPERATURE OF DE SITTER SPACE

We finally discuss the implications of the geodesic KMS–property of vacuum states for the modular conjugations $J_{\mathcal{W}}$ associated with the wedge algebras $\mathcal{A}(\mathcal{W})$ and the vacuum vector Ω. The following proposition is an easy consequence of standard results in modular theory.

PROPOSITION 6.1. – *There holds wedge duality for any wedge $\mathcal{W} \subset \mathcal{S}^n$,*

$$J_{\mathcal{W}} \mathcal{A}(\mathcal{W}) J_{\mathcal{W}} = \mathcal{A}(\mathcal{W})' = \mathcal{A}(\mathcal{W}'). \tag{6.1}$$

Proof. – The first equality in the statement is a basic result of modular theory. For the proof of the second equality it suffices to note that (i) $\mathcal{A}(\mathcal{W}') \subset \mathcal{A}(\mathcal{W})'$ because of locality, (ii) Ω is cyclic for $\mathcal{A}(\mathcal{W}')$ by the Reeh–Schlieder property and (iii) $\mathcal{A}(\mathcal{W}')$ is stable under the action of the modular group $\alpha_{\Lambda_{\mathcal{W}}(t)}$, $t \in \mathbb{R}$, associated with the pair $\{\mathcal{A}(\mathcal{W})', \Omega\}$. It then follows from a well known result in modular theory [15, Theorem 9.2.36] that $\mathcal{A}(\mathcal{W}') = \mathcal{A}(\mathcal{W})'$. ■

We will show next that the existence of the modular conjugations $J_{\mathcal{W}}$ fixes the inverse temperature β. Moreover, the specific form of the adjoint action of these conjugations on the unitary group $U(SO_0(1,n))$ can be computed.

For the proof we consider the wedge $\mathcal{W}_1$, cf. (2.2), and the corresponding modular group $\mathrm{e}^{-itM_{01}}$, $t \in \mathbb{R}$, and conjugation $J_{\mathcal{W}_1}$ associated with $\mathcal{A}(\mathcal{W}_1)$ and Ω. We also pick a region $\mathcal{O} \subset \mathcal{W}_1$ such that $\Lambda\mathcal{O} \subset \mathcal{W}_1$ for all Λ in some neighborhood of the unit element in $SO_0(1,n)$. Thus, for sufficiently small $s \in \mathbb{R}$, there holds

$$\mathrm{e}^{isM_{0j}} \mathcal{A}(\mathcal{O}) \mathrm{e}^{-isM_{0j}} \subset \mathcal{A}(\mathcal{W}_1), \tag{6.2}$$

where $\mathrm{e}^{isM_{0j}}$ are the boost operators associated with the wedges $\mathcal{W}_j$, $j = 1, \dots n$.

Making use of relation (2.5) we get for $A \in \mathcal{A}(\mathcal{O})$ and $j = 2, \dots n$

$$\mathrm{e}^{-itM_{01}} \left(\mathrm{e}^{isM_{0j}} A \mathrm{e}^{-isM_{0j}}\right) \Omega = \mathrm{e}^{is(\mathrm{ch}\,(t) M_{0j} - \mathrm{sh}\,(t) M_{1j})} \, \mathrm{e}^{-itM_{01}} A \, \Omega. \tag{6.3}$$

According to the geodesic KMS–property of the vacuum and modular theory [15], the vector–valued functions

$$t \longrightarrow \mathrm{e}^{-itM_{01}} B \, \Omega, \quad B \in \mathcal{A}(\mathcal{W}_1), \tag{6.4}$$

can be analytically continued into the strip $\{z \in \mathbb{C} : 0 > \mathrm{Im} z > -\beta/2\}$ and have continuous boundary values at $z = -i\beta/2$, given by

$$\mathrm{e}^{-(\beta/2) M_{01}} B \, \Omega = J_{\mathcal{W}_1} B^* \, \Omega. \tag{6.5}$$

Moreover, for given $\gamma > \beta/2$ and sufficiently small s, the vector–valued function

$$t \longrightarrow \mathrm{e}^{-is(\mathrm{ch}\,(t)M_{0j} - \mathrm{sh}\,(t)M_{1j})}\,\Phi, \quad |t| < \gamma, \tag{6.6}$$

where Φ is any element of the dense set of analytic vectors described in Lemma 2.3, can be analytically continued into the complex circle $\{z \in \mathbb{C} : |z| < \gamma\}$. The continuation is given by

$$\mathrm{e}^{-is(\mathrm{ch}\,(z)M_{0j} - \mathrm{sh}\,(z)M_{1j})}\,\Phi, \tag{6.7}$$

where the exponential function is defined in the sense of power series. Taking scalar products of the vectors in equation 6.3 with Φ we therefore obtain for sufficiently small s by analytic continuation in t the equality

$$(\Phi, \mathrm{e}^{-izM_{01}}\mathrm{e}^{isM_{0j}}A\,\Omega) = (\mathrm{e}^{-is(\mathrm{ch}\,(\overline{z})M_{0j} - \mathrm{sh}\,(\overline{z})M_{1j})}\,\Phi, \mathrm{e}^{-izM_{1j}}A\,\Omega) \tag{6.8}$$

for z in $\{z \in \mathbb{C} : |z| < \gamma\} \cap \{z \in \mathbb{C} : 0 > \mathrm{Im} z > -\beta/2\}$. Proceeding to the boundary point $z = -i\beta/2$ and making use of relations (6.2) and (6.5) we arrive at

$$(\Phi, J_{W_1}\mathrm{e}^{isM_{0j}}A^*\,\Omega) = (\mathrm{e}^{-is(\mathrm{ch}\,(i\beta/2)M_{0j} - \mathrm{sh}\,(i\beta/2)M_{1j})}\,\Phi, J_{W_1}A^*\,\Omega). \tag{6.9}$$

Since the operators J_{W_1} and $\mathrm{e}^{isM_{0j}}$ are (anti–)unitary and the vectors Φ and $A^*\,\Omega$ are arbitrary elements of two dense sets in $\mathcal{H}$ we conclude that $\mathrm{e}^{-is(\mathrm{ch}\,(i\beta/2)M_{0j} - \mathrm{sh}\,(i\beta/2)M_{1j})}$ has to be unitary. If $s \neq 0$ this is only possible if β is an integer multiple of 2π. As a matter of fact there holds $\beta = 2\pi$ as we will show next.

If $\beta \geq 4\pi$ then $2\pi \leq \beta/2$ and hence the vectors in $\mathcal{A}(\mathcal{W}_1)\,\Omega$ are in the domain of $\mathrm{e}^{-2\pi M_{01}}$. Now from (6.8) we see that for $A \in \mathcal{A}(\mathcal{O})$ and sufficiently small s, t such that $\mathrm{e}^{isM_{0j}}\mathrm{e}^{itM_{01}}A\mathrm{e}^{-itM_{01}}\mathrm{e}^{-isM_{0j}} \in \mathcal{A}(\mathcal{W}_1)$ there holds

$$\mathrm{e}^{-\pi M_{01}}\mathrm{e}^{isM_{0j}}\mathrm{e}^{itM_{01}}A\,\Omega = \mathrm{e}^{-isM_{0j}}\mathrm{e}^{itM_{01}}\mathrm{e}^{-\pi M_{01}}A\,\Omega. \tag{6.10}$$

By multiplication of this equation with the spectral projections $P(\Delta)$ of M_{01}, where $\Delta \subset \mathbb{R}$ is compact, we proceed to

$$\mathrm{e}^{-\pi M_{01}}P(\Delta)\,\mathrm{e}^{isM_{0j}}\mathrm{e}^{itM_{01}}A\,\Omega = P(\Delta)\,\mathrm{e}^{-isM_{0j}}\mathrm{e}^{itM_{01}}\mathrm{e}^{-\pi M_{01}}A\,\Omega. \tag{6.11}$$

Since $\mathrm{e}^{-\pi M_{01}}P(\Delta)$ is a bounded operator the vector–valued functions on both sides of this equality can be analytically continued in t into the strip

$\{z \in \mathbb{C} : 0 < \text{Im}z < \pi\}$. Therefore the equality holds for all $t \in \mathbb{R}$ and consequently

$$e^{-\pi M_{01}} P(\Delta)\, e^{isM_{0j}} P(\Delta)\, A\,\Omega = P(\Delta)\, e^{-isM_{0j}} P(\Delta)\, e^{-\pi M_{01}} A\,\Omega. \quad (6.12)$$

As $\mathcal{A}(\mathcal{O})\,\Omega$ is dense in $\mathcal{H}$ we get $e^{-\pi M_{01}} P(\Delta)\, e^{isM_{0j}} P(\Delta) = P(\Delta)\, e^{-isM_{0j}} P(\Delta)\, e^{-\pi M_{01}}$ on all spectral subspaces of M_{01}. So by left multiplication of this equation with $e^{-\pi M_{01}}$ we conclude that $P(\Delta)\, e^{isM_{0j}} P(\Delta)$ and $e^{-2\pi M_{01}}$ commute. Since Δ was arbitrary, it follows that $e^{itM_{01}}$ and $e^{isM_{0j}}$ commute for $j = 2, \ldots n$ which is only possible if U is the trivial representation. So we have proved:

THEOREM 6.2. – *The geodesic temperature has the Gibbons–Hawking value, $\beta = 2\pi$.*

With the help of relation (6.9) we will now compute the adjoint action of the modular conjugation $J_{\mathcal{W}_1}$ on $U(SO_0(1,n))$. As $\beta = 2\pi$ we obtain from (6.9) for small s

$$J_{W_1} e^{isM_{0j}} = e^{-isM_{0j}} J_{W_1}, \quad j = 2, \ldots n, \quad (6.13)$$

and it is then apparent that this relation holds for arbitrary $s \in \mathbb{R}$. The modular theory, on the other hand, implies that

$$J_{W_1} e^{isM_{01}} = e^{isM_{01}} J_{W_1}. \quad (6.14)$$

Since the boost operators $e^{isM_{0j}}$, $j = 1, \ldots n$, generate $U(SO_0(1,n))$, the adjoint action of J_{W_1} on this group can be read off from these relations. After a moments reflection one sees that J_{W_1} is an anti–unitary representer of the element $TP_1 \in O(1,n)$, where T denotes time reflection and P_1 the reflection along the spatial 1-direction in the chosen coordinate system. Moreover, from Proposition 6.1, applied to J_{W_1}, and the preceding two equalities one obtains

$$J_{W_1} \mathcal{A}(\mathcal{W}) J_{W_1} = \mathcal{A}(TP_1\mathcal{W}) \text{ for } \mathcal{W} \subset \mathcal{S}^n. \quad (6.15)$$

Summarizing these results, we have established the following version of a PCT–Theorem in de Sitter space.

THEOREM 6.3. – *The modular conjugation J_{W_1} associated with the wedge $\mathcal{W}_1$ is an anti–unitary representer of the reflection $TP_1 \in O(1,n)$ which induces the corresponding action on $U(SO_0(1,n))$ and on the wedge algebras.*

An analogous result for wedges other than $\mathcal{W}_1$ is obtained by de Sitter covariance.

7. CONCLUSIONS

Starting from the physically meaningful assumption that vacuum states in de Sitter space look like equilibrium states for all geodesic observers with an *a priori* arbitrary temperature, we have analysed in a general setting the global structure of these states. It turned out that they have essentially the same properties as vacuum states in Minkowski space except that they are not ground states.

For mixed vacuum states it follows from the results of Sec. 4 that the respective sub-ensembles belong to different superselection sectors (phases) which can be distinguished by elements of the center of the algebra of observables. By central decomposition one can always proceed to pure vacuum states which are weakly mixing. It of interest in this context that this central decomposition can be performed by any geodesic observer. As has been shown in Sec. 5, the relevant macroscopic order parameters can be constructed in every wedge by suitable "time averages" of local observables.

The geodesic temperature has the value predicted by Gibbons and Hawking also in the present general setting. This result could be established without any further "stability assumptions" by making use of the analytic structure of the de Sitter group, which was also essential in the proof of an analogue of the PCT–Theorem in de Sitter space.

Our results provide evidence to the effect that the vacuum states, as defined in the present investigation, indeed describe the envisaged physical situation. It would therefore be of interest to clarify the relation between our setting and the apparently more restrictive framework of maximal analyticity, proposed in [7] to characterize vacuum states in de Sitter space.

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