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Mean Green's function of the Anderson model at weak disorder with an infra-red cut-off

by

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ABSTRACT. – In this paper we develop a polymer expansion with large/small field conditions for the mean resolvent of a weakly disordered system. Then we show that we can apply our result to a two-dimensional model, for energies outside the unperturbed spectrum or in the free spectrum provided the potential has an infrared cut-off. This leads to an asymptotic expansion for the density of states. This is an important first step towards a rigorous analysis of the density of states in the free spectrum of a random Schrödinger operator at weak disorder. © Elsevier, Paris

RÉSUMÉ. – Dans cet article, je construis un développement de polymères, avec conditions de petit champ-grand champ, pour la résolvante moyenne d'un modèle d'Anderson faiblement désordonné. Je montre ensuite que ce développement s'applique à un modèle bidimensionnel, pour des énergies hors du spectre non-perturbé ou dans le spectre libre pourvu que le potentiel ait une coupure infrarouge. On peut ainsi obtenir un développement asymptotique de la densité d'états. C'est un premier pas important vers l'analyse rigoureuse de la densité d'états d'un opérateur de Schrödinger aléatoire à faible désordre et à l'intérieur du spectre libre. © Elsevier, Paris

1. INTRODUCTION

In the one-body approximation, the study of disordered systems amounts to the study of random Schrödinger operators of the form

$$H = H_0 + \lambda V \tag{1}$$

where H_0 is a kinetic term (*i.e.* a self-adjoint or essentially self-adjoint operator corresponding to some dispersion relation, typically a regularized version of $-\Delta$) and V is a real random potential (in the simplest case, V is a white noise). We work on a ultra-violet regular subspace of $L^2(\mathbb{R}^d)$ and we restrict ourselves to λ small so as to see λV as a kind of perturbation of the free Hamiltonian.

The properties of H are usually established through the behavior of the kernel of the resolvent operator or Green's function [1, 2, 3]

$$G_E(x,y) = \langle x|\frac{1}{H-E}|y\rangle \tag{2}$$

For instance, the density of states is given by

$$\rho(E) = \frac{1}{\pi}\lim_{\varepsilon\to 0} \text{Im } G_{E+i\varepsilon}(x,x) \tag{3}$$

The important point is that, in the thermodynamic limit, the system is self-averaging, *i.e.* mean properties are often almost sure ones. Thus the problem can be seen as a statistical field theory with respect to the random field V. In Statistical Mechanics, functional integrals in the weakly coupled regime are controlled through a cluster expansion (or polymer expansion) with small field versus large field conditions, the problem being then to control a Boltzmann weight [4, 5]).

In the first part of this paper, we derive a resolvent cluster expansion with large field versus small field conditions assuming that V satisfies some large deviations estimates. This would allow to prove the existence and the regularity of the mean Green's function (theorem 1) and to get an asymptotic expansion for the density of states.

In the second part, we show that the hypothesis of theorem 1 are satisfied in the case of a 2 dimensional model with a rotation invariant dispersion relation and an infra-red cut-off on the potential. From the point of view of *Renormalization Group* analysis, our result allows to control the model away from the singularity, *i.e.* to perform the first renormalization group steps and therefore to generate a fraction of the expected "mass".

Our large deviations estimates are obtained through an analogy with a random matrix problem. In dimension two, the potential in momentum space looks very much like an element of the *Gaussian Unitary Ensemble*, when restricted to the close neighborhood of the singularity. In higher dimension, we have to deal with new kind of random matrices which have constraints on their coefficients. Therefore, the problem is much more difficult [6].

2. MODEL AND RESULTS

2.1. The model

In $\mathbb{R}^d$ we consider

$$H = H_0 + \lambda V \tag{4}$$

where V is a Gaussian random field with covariance ξ whose smooth translation invariant kernel is rapidly decaying (we will note the associated measure $d\mu_\xi$). Because ξ is smooth, $d\mu_\xi$ as a measure on tempered distributions is in fact supported on $\mathcal{C}^\infty$ functions. We suppose also that $\hat{H}_0^{-1}$ has compact support in momentum space so that we do not have to deal with ultra-violet problems. We construct the finite volume model in $\mathbb{R}^d/_{\Lambda\mathbb{Z}^d}$ by replacing ξ and H_0 by their "Λ-periodization"

$$\xi_\Lambda(x,y) = \frac{1}{\Lambda^d} \sum_{p \in \frac{2\pi}{\Lambda}\mathbb{Z}^d} e^{ip(x-y)} \hat{\xi}(p) = \sum_{z \in \Lambda\mathbb{Z}^d} \xi(x-y+z) \tag{5}$$

$$H_0^{(\Lambda)}(x,y) = \ldots = \sum_{z \in \Lambda\mathbb{Z}^d} H_0(x-y+z) \tag{6}$$

Then we define

$$G_{\Lambda,\varepsilon}(E,\lambda,V) = \frac{1}{H_0^{(\Lambda)} + \lambda V - (E + i\varepsilon)} \tag{7}$$

$$\bar{G}_{\Lambda,\varepsilon}(E,\lambda) = \int d\mu_{\xi_\Lambda}(V)\, G_{\Lambda,\varepsilon}(E,\lambda,V) \tag{8}$$

$d\mu_{\xi_\Lambda}$ can be considered either as a measure on $\mathcal{C}^\infty\left(\mathbb{R}^d/_{\Lambda\mathbb{Z}^d}\right)$ or as a measure on $\mathcal{C}^\infty(\mathbb{R}^d)$ which is supported by the space of Λ-periodic functions. In the same way, $G_{\Lambda,\varepsilon}$ will be considered as an operator either on $L^2\left(\mathbb{R}^d/_{\Lambda\mathbb{Z}^d}\right)$ or on $L^2(\mathbb{R}^d)$. One can note that in momentum space, because of the cut-off, the problem reduces to a finite dimensional one.

Because V is a multiplicative operator, its operator norm is equal to its L^∞ norm which is measurable and almost surely finite. Therefore V is bounded and self-adjoint. Then $G_{\Lambda,\varepsilon}(E,\lambda,V)$ is almost surely an analytic operator-valued function of λ in a small domain (depending on V) around the origin. This domain can be extended to a V-dependent neighborhood of the real axis thanks to the identity (for $|\lambda-\mu|$ small enough)

$$G_{\Lambda,\varepsilon}(E,\mu,V) = G_{\Lambda,\varepsilon}(E,\lambda,V)\left\{ I + \sum_{n=1}^{\infty}(\lambda-\mu)^n [V G_{\Lambda,\varepsilon}(E,\lambda,V)]^n \right\} \tag{9}$$

In the same way, $G_{\Lambda,\varepsilon}(E,\lambda,V)$ is analytic in E. One can also check that $G_{\Lambda,\varepsilon}(E,\lambda,V)$ has a smooth kernel and is integrable with respect to $d\mu_{\xi_\Lambda}$. Furthermore, $\bar{G}_{\Lambda,\varepsilon}(E,\lambda)$ will have a translation invariant kernel because $d\mu_{\xi_\Lambda}$ is translation invariant.

2.2. Main result

We introduce a function θ which satisfies

- θ is an odd $\mathcal{C}^\infty$ function, increasing and bounded
- for any x, $|\theta(x)| \leqslant |x|$
- for any $|x| \leqslant 1$, $\theta(x) = x$
- the $\mathcal{L}^\infty$ norm of its derivatives does not grow too fast

Then for $\mu > 0$, we define the operators $C_{\Lambda,\mu}$, $D_{\Lambda,\mu}$ and $U_{\Lambda,\mu}$ through the Fourier transform of their kernel

$$\hat{C}_{\Lambda,\mu}^{-1}(p) = \hat{H}_0^{(\Lambda)}(p) - E - i\mu \tag{10}$$

$$\hat{D}_{\Lambda,\mu}(p) = \frac{1}{\left|\theta[\hat{H}_0^{(\Lambda)}(p) - E] - i\mu\right|^{1/2}} \tag{11}$$

$$\hat{U}_{\Lambda,\mu}^{-1}(p) = \hat{D}_{\Lambda,\mu}^2(p)\hat{C}_{\Lambda,\mu}^{-1}(p) \tag{12}$$

Given any characteristic length L we can divide the space into cubes Δ of side L and construct an associated $\mathcal{C}_0^\infty$ partition of unity

$$1 = \sum_{\Delta} \chi_\Delta \tag{13}$$

where χ_Δ has support in a close neighborhood of the cube Δ (*e.g.* on Δ and its nearest neighbors). This decomposition induces an orthogonal decomposition of V into a sum of fields V_Δ with covariance

$$\xi_\Lambda^\Delta(x,y) = \int dz\, \xi_\Lambda^{1/2}(x-z)\chi_\Delta(z)\xi_\Lambda^{1/2}(z-y) \tag{14}$$

For simplicity we will pretend that ξ and $\xi^{1/2}$ have compact support, so that V_Δ is almost surely supported on a close neighborhood of Δ, moreover we will take that it is restricted to Δ and its nearest neighbors. The generalization to a fast decaying ξ can be easily obtained by decomposing each V_Δ over the various cubes and write more complicated small/large field conditions that test the size of V_Δ in the various cubes. This leads to lengthy expressions that we want to avoid.

Finally, we note d_Λ the distance in $\mathbb{R}^d/_{\Lambda\mathbb{Z}^d}$

$$d_\Lambda(x,y) = \min_{z\in\Lambda\mathbb{Z}^d} |x-y+z| \tag{15}$$

In the following, C or $O(1)$ will stand as generic names for constants in order to avoid keeping track of the numerous constants that will appear. Furthermore we will not always make the distinction between a function and its Fourier transform but we will use x, y and z as space variables and p and q as momentum variables.

THEOREM 1. – *Suppose that*

- ξ *has a smooth, translation invariant with compact support kernel*
- $C_\xi = \sup_\Lambda \frac{1}{2\Lambda^d} \int_{[0,\Lambda]^d} \xi_\Lambda^{-1}(x,y)\, dx\, dy$ *exists*
- *for all* $E \in [E_1, E_2]$ *and all* μ, C_μ, D_μ *and* U_μ *have smooth kernels with fast decay over a length scale* L.
- *for all* n_1, *we have* C_{n_1} *such that for all* Λ *and all triplets* $(\Delta_1, \Delta_2, \Delta_3)$

$$\|\chi_{\Delta_1} D_{\Lambda,\mu} V_{\Delta_2} D_{\Lambda,\mu} \chi_{\Delta_3}\| \leqslant \frac{C_{n_1} \|D_{\Lambda,\mu} V_{\Delta_2} D_{\Lambda,\mu}\|}{[1 + L^{-1} d_\Lambda(\Delta_1, \Delta_2) + L^{-1} d_\Lambda(\Delta_2, \Delta_3)]^{n_1}} \tag{16}$$

- *there are constants* C_0, C_1, $\kappa > 0$ *and* $\alpha > 0$ *such that*

$$\begin{gathered}\forall \Lambda \leqslant \infty,\ \forall a > 1,\ \forall \Delta, \\ \mathbb{P}_\Lambda(\|D_{\Lambda,\mu} V_\Delta D_{\Lambda,\mu}\| \geqslant a C_0) \leqslant C_1 e^{-\kappa a^2 L^\alpha}\end{gathered} \tag{17}$$

where $\mathbb{P}_\Lambda(.)$ *denote the probability with respect to the measure* $d\mu_{\xi_\Lambda} \equiv \otimes d\mu_{\xi_\Lambda^\Delta}$ $(\xi_\infty \equiv \xi)$

$$\mathbb{P}_\Lambda(X) = \int d\mu_{\xi_\Lambda}(V)\, \mathbb{1}_X(V) = \mu_{\xi_\Lambda}(X) \tag{18}$$

Then let $\mu_0 = L^{-d/2} C_\xi^{1/2}$, $\mu = \lambda \mu_0$ *and*

$$\bar{T}_{\Lambda,\varepsilon} = D^{-1}_{\Lambda,\mu} \bar{G}_{\Lambda,\varepsilon} D^{-1}_{\Lambda,\mu} \tag{19}$$

For all $\lambda \leqslant \lambda_0 = O(1)$ *and for all* ε *small enough (in a* λ*-dependent way),* $\bar{T}_{\Lambda,\varepsilon}(E, \lambda)$ *is uniformly bounded in* Λ *and admits the following development (in the operator norm sense)*

$$1_{\Omega_\Lambda} \bar{T}_{\Lambda,\varepsilon}(E, \lambda) 1_{\Omega_\Lambda} = 1_{\Omega_\Lambda} \bar{T}(E + i\varepsilon, \lambda) 1_{\Omega_\Lambda} + \mathrm{O}\left(\frac{1}{\Lambda}\right) \tag{20}$$

where $\Omega_\Lambda = [-\Lambda^{1/2}; \Lambda^{1/2}]^d$, *and* 1_{Ω_Λ} *is the characteristic function of* Ω_Λ.

Furthermore we have the following properties for $\bar{G} = D_\mu \bar{T} D_\mu$*:*

- $\bar{G}$ *has a smooth, translation invariant kernel*
- $\bar{G}_{\Lambda,\varepsilon}$ *and* $\bar{G}$ *have high power decay*

$$\exists n_0 \text{ large, } \exists C_G(n_0) \text{ such that } \forall (\Delta, \Delta'),$$
$$\|1_\Delta \bar{G}_{\Lambda,\varepsilon} 1_{\Delta'}\| \leqslant \frac{C_G(n_0)}{[1 + L^{-1} d_\Lambda(\Delta, \Delta')]^{n_0}} \tag{21}$$

and a similar relation for $\bar{G}$ *with* d_Λ *being replaced by* d.

- $\bar{G}(E, \lambda)$ *is an analytic operator valued function of* E *for all* E *in* $]E_1, E_2[$ *with a small* λ*-dependent radius of analyticity.*
- $\bar{G}(E, \lambda)$ *is a* $\mathcal{C}^\infty$ *operator-valued function of* λ *and admits an asymptotic expansion to all orders in* λ*. which is the formal perturbative expansion of*

$$\int d\mu_\xi(V)\, e^{\frac{\mu_0^2}{2}\langle 1, \xi^{-1} 1\rangle}\, e^{i\mu_0 \langle V, \xi^{-1} 1\rangle}\, \frac{1}{H_0 - E + \lambda V - i(\mu + 0^+)} \tag{22}$$

(<> denotes the scalar product, i.e. $\langle f, Af\rangle = \int \bar{f}(x) A(x,y) f(y)\, dx\, dy$*) Asymptoticity in the sense that there exist some constants* C *and* α *such that the difference between* $\bar{G}$ *and its* n^{th} *perturbative expansion has high power decay and is bounded in norm by*

$$\|R_n\| \leqslant C^n \lambda^n (n!)^\alpha \tag{23}$$

This theorem is formulated in a rather general way so as to apply with minimum transformation to various situations (lattice or continuous models) and in any dimension. Then we construct a concrete example with a two-dimensional model. One can also refer to [6] for a $d = 3$ case.

2.3. Anderson model with an infra-red cut-off in dimension d=2

We consider

$$H = -\Delta_\eta + \lambda \eta_E V \eta_E \tag{24}$$

where

- Δ_η^{-1} is a ultra-violet regularized inverse Laplacian, *i.e.* there is a $\mathcal{C}_0^\infty$ function η_{UV} equal to 1 on "low" momenta such that

$$\Delta_\eta^{-1}(p) = \frac{\eta_{UV}(p)}{p^2} \tag{25}$$

We will note p^2 instead of $-\Delta_\eta$, the UV-cutoff being then implicit.
- we are interested in the mean Green's function for an energy $E = O(1)$
- η_E is an infra-red cut-off which enforces

$$|p^2 - E| \geqslant A\lambda^2 |\log \lambda|^2 \tag{26}$$

for some large constant A
- V has covariance ξ which is a $\mathcal{C}_0^\infty$ approximation of a δ-function

This corresponds to the model away from the singularity $p^2 = E$ in a multi-scale renormalization group analysis, we will show that it generates a small fraction of the expected imaginary part which is $O(\lambda^2)$.

Let $M^{1/2}$ be an even integer greater than 2, we define $j_0 \in \mathbb{N}$ such that

$$M^{-j_0} \leqslant \inf_{\mathrm{Supp}(\eta_E)} |p^2 - E| \leqslant M^{-(j_0 - 1)} \tag{27}$$

Next, we construct a smooth partition of unity into cubes of side M^{j_0} (they form a lattice $\mathbb{D}_{j_0}$) and we construct the fields V_Δ's accordingly.

THEOREM 2. – *There exist constants C_0 and C_1 such that for any Λ, $a \geqslant 1$ and $\Delta \in \mathbb{D}_{j_0}$ we have*

$$\mathbb{P}_\Lambda\Big(\|D_{\Lambda,\mu}\eta_E V_\Delta \eta_E D_{\Lambda,\mu}\| \geqslant a C_0 j_0 M^{j_0/2}\Big) \leqslant C_1 e^{-\frac{1}{2}a^2 M^{j_0/6}} \tag{28}$$

Furthermore theorem 1 *applies and $\bar{G}_E = \eta_E \bar{G} \eta_E$ is asymptotic to its perturbative expansion*

$$\bar{G}_E \sim \eta_E \frac{1}{p^2 - E - i\eta_E O(\lambda^2 |\log \lambda|^{-2})\eta_E} \eta_E \tag{29}$$

It is easy to extend this result to the case of a rotation invariant dispersion relation and for energies outside the free spectrum not too close to the band edge. In this case, the cut-off is no longer needed so that the result apply to the full model.

3. RESOLVENT POLYMER EXPANSION WITH LARGE FIELD VERSUS SMALL FIELD CONDITIONS

3.1. Sketch of proof for theorem 1

We give here the global strategy for proving theorem 1, the main ingredient being the polymer expansion that we will detail in the following.

First we recall (without proving them) some quite standard properties of Gaussian measures.

LEMMA 1. – *Complex translation*

Let X be a Gaussian random field with covariance C and let $d\mu_C$ be the associated measure. For any regular functional $\mathcal{F}(X)$ and any function $f \in$ Ran C, we have the following identity

$$\int d\mu_C(X)\,\mathcal{F}(X) = e^{\frac{1}{2}\langle f, C^{-1} f\rangle} \int d\mu_C(X)\,\mathcal{F}(X - if)\, e^{i\langle X, C^{-1} f\rangle} \tag{30}$$

LEMMA 2. – *Integration by part*

With the same notations than above we have

$$\int d\mu_C(X)\, X(x)\mathcal{F}(X) = \int dy\, C(x,y) \int d\mu_C(X) \frac{\delta}{\delta X(y)} \mathcal{F}(X) \tag{31}$$

Those lemmas could for instance be easily proved for polynomial functionals and extended through a density argument to a wide class of functionals. ■

Our starting point is obtained by applying lemma 1 with $f = \mu_0 1$.

$$\bar{G}_{\Lambda,\varepsilon}(E+z,\lambda) = \int d\mu_{\xi_\Lambda}(V)\, e^{\frac{\mu_0^2}{2}\langle 1, \xi_\Lambda^{-1} 1\rangle + i\mu_0\langle V, \xi_\Lambda^{-1} 1\rangle} \frac{1}{H_0^{(\Lambda)} - (E + i\mu) + \lambda V - i\varepsilon - z} \tag{32}$$

$$\bar{T}_{\Lambda,\varepsilon}(E+z,\lambda) = \int d\mu_{\xi_\Lambda}(V)\, e^{\frac{\mu_0^2}{2}\langle 1, \xi_\Lambda^{-1} 1\rangle + i\mu_0\langle V, \xi_\Lambda^{-1} 1\rangle} \frac{1}{U_{\Lambda,\mu}^{-1} + \lambda D_{\Lambda,\mu} V D_{\Lambda,\mu} - (z + i\varepsilon) D_{\Lambda,\mu}^2} \tag{33}$$

On one hand we earned something because now the resolvent operator in the integral is bounded in norm independently of ε (in the following we will note z instead of $z + i\varepsilon$ and show convergence for any z such that $|z| \ll \mu$, this would allow to prove analyticity in z). But on the other hand we have a huge normalization factor to pay. However, we can remark that this normalization factor is in fact equivalent to a factor e per L-cube.

Most of the demonstration amounts to a polymer expansion of $\bar{T}_{\Lambda,\varepsilon}$, *i.e.* we write it as a sum over polymers of polymer activities

$$T_{out,in} = \chi_{\Delta_{out}} \bar{T}_{\Lambda,\varepsilon} \chi_{\Delta_{in}} \tag{34}$$

$$T_{out,in} = \chi_{\Delta_{out}} \left[U_{\Lambda,\mu} + \frac{\lambda^{c_1} \sum_{Y \in \mathcal{A}} \lambda^{c_2 |Y|} \Gamma_Y T(Y)}{[1 + L^{-1} d_\Lambda(\Delta_{in}, \Delta_{out})]^{n_0}} \right] \chi_{\Delta_{in}} \tag{35}$$

where c_1 and c_2 are small constants, Γ_Y has decay in the spatial extension of Y and $\| \sum_{Y \in \mathcal{A}} T(Y) \|$ is bounded. Furthermore, $G(Y)$ is given by a functional integration over fields V_Δ's corresponding to cubes in the support of the polymer Y. This show that $T_{\Lambda,\varepsilon}$ is bounded and has a high power decay uniformly in Λ.

Next, when we consider $1_{\Omega_\Lambda} \bar{T}_{\Lambda,\varepsilon} 1_{\Omega_\Lambda}$ we can divide the sum over polymers into a sum over polymers with a large spatial extension (say $\Lambda^{2/3}$) and sum over "small" polymers. The large polymers will have a total contribution small as Λ^{-1} to some large power. For the small polymers, since we are far away from the boundaries, their contribution calculated with $d\mu_{\xi_\Lambda}$ will be equal to their contribution calculated with $d\mu_\xi$ up to a factor Λ^{-n}. In this way we can prove the development (20). Smoothness of the kernel will be obtained because we will show that we can write

$$T(Y) = U_{\Lambda,\mu} \tilde{T}(Y) U_{\Lambda,\mu} \tag{36}$$

The convergence for any $z \ll \mu$ allows to show analyticity (we write z-derivatives as Cauchy integrals so that we can show that they all exist and do not grow too fast). Then an asymptotic expansion can be generated through the repeated use of resolvent identity. After n steps we have

$$\tilde{R}_n(V) = (-\lambda C V)^n G \tag{37}$$

The factor $(-\lambda CV)^n$ yields a factor $\lambda^{\varepsilon n}$ in the small field region while in the large field region, the control of the products of fields with the Gaussian measure yields an extra $(n!)^\nu$. Therefore we can define

$$R_n = \left\langle \tilde{R}_{n/\varepsilon} \right\rangle_{d\mu} + \textit{Perturbative terms from } n \textit{ to } n/\varepsilon \tag{38}$$

Finally, for the density of states, we just need to remark that

$$G(0,0) = \int dp\, dq\, G(p,q) = \left\langle \tilde{\delta}, G\, \tilde{\delta} \right\rangle \tag{39}$$

where $\tilde{\delta}$ is a regularized δ-function because of the presence of the ultra-violet cut-off. Thus an asymptotic expansion for G with respect to the operator norm will yield an asymptotic expansion for the density of states.

3.2. Improved polymer expansions

Cluster expansions in constructive field theory lay heavily on a clever application of the Taylor formula with integral remainder. Writing the full Taylor series would amount to completely expand the perturbation series, which most often diverges, and therefore should be avoided. A rather instructive example of minimal convergent expansion is the Brydges-Kennedy forest formula: you have a function defined on a set of links between pair of cubes and you expand it not on all possible graphs but only on forests, *cf.* [4].

For more complex objects a way to generalize such a formula can be found in [7], and we refer the reader to it for a more careful treatment and for various proofs. Let us assume that we have a set of objects that we call monomers. A sequence of monomers will be called a polymer, then we will expand a function defined on a set of monomers into a sum over allowed polymers.

To be more precise, let $\mathcal{X}$ be a set of monomers, we define the set $\mathcal{Y}$ of polymers on $\mathcal{X}$ as the set of all finite sequences (possibly empty) of elements of $\mathcal{X}$. Then a monomer can be identified to a polymer of length 1. The empty sequence or empty polymer will be noted $\emptyset$. We define on $\mathcal{Y}$

- a concatenation operator:
  for $Y = (X_1, \ldots, X_n)$ and $Y' = (X'_1, \ldots, X'_{n'})$, we define

$$Y \cup Y' = (X_1, \ldots, X_n, X'_1, \ldots, X'_{n'}) \tag{40}$$

- the notion of starting sequence:
  we say that Y_1 is a starting sequence of Y (equivalently that Y is a continuation of Y_1) and we note $Y_1 \subset Y$ iff there exists Y_2 such that $Y = Y_1 \cup Y_2$

Then we call allowed set (of polymers) any finite subset $\mathcal{A} \subset \mathcal{Y}$ such that

- $\forall Y, Y' \quad Y' \subset Y$ and $Y \in \mathcal{A} \Rightarrow Y' \in \mathcal{A}$
- $\forall X, Y, Y' \quad Y \subset Y'$ and $Y \cup X \notin \mathcal{A} \Rightarrow Y' \cup X \notin \mathcal{A}$

the first condition implies that $\emptyset \in \mathcal{A}$ whenever $\mathcal{A}$ is non-empty. Finally, for Y belonging to some allowed set $\mathcal{A}$, a monomer X is said to be admissible for Y (according to $\mathcal{A}$) iff $Y \cup X \in \mathcal{A}$.

LEMMA 3. – *Let $\mathcal{X} = \{X\}$ be a set of N monomers and $\mathcal{Y}$ the set of polymers on $\mathcal{X}$. We assume that we have an indexation of $\mathbb{R}^N$ by $\mathcal{X}$, i.e. a bijection from $\mathcal{X}$ to $\{1, \ldots, N\}$ so that an element of $\mathbb{R}^N$ can be noted $\vec{z} = (z_X)_{X \in \mathcal{X}}$.*

For $\mathcal{F}$ a regular function from $\mathbb{R}^N$ to some Banach space $\mathcal{B}$ and an allowed set $\mathcal{A} \subset \mathcal{Y}$, the polymer expansion of $\mathcal{F}$ according to $\mathcal{A}$ is given through the following identity

$$\begin{aligned} \mathcal{F}(\vec{1}) &\equiv \mathcal{F}(1, \ldots, 1) \\ &= \sum_{n \geqslant 0} \sum_{Y = (X_1, \ldots, X_n) \in \mathcal{A}} \int_{1 > h_1 > \ldots h_n > 0} \\ &\quad dh_1 \ldots dh_n \left(\prod_{X \in Y} \frac{\partial}{\partial z_X} \right) \mathcal{F}[\vec{z}(Y, \{h_i\})] \end{aligned} \tag{41}$$

where $\vec{z}(Y, \{h_i\})$ is given by

$$z_X(Y, \{h_i\}) = \begin{cases} 0 & \textit{if } X \textit{ is admissible for } Y \\ 1 & \textit{if } X \textit{ is not admissible for } \emptyset \\ h_i & \textit{if } X \textit{ not admissible for } Y \textit{ and } X = X_j \textit{ for some } j, \\ & \textit{in which case } i = \max\{j / X = X_j\} \\ h_i & \textit{with } i = \min\{j / X \textit{ not admissible for } (X_1, \ldots X_j)\}, \\ & \textit{otherwise} \end{cases} \tag{42}$$

Proof. – The proof is made through an inductive iteration of a first order Taylor formula. We start with $\mathcal{F}(\vec{1})$ and put a common interpolating parameter h_1 on all admissible monomers for the empty set, i.e. we make a first order Taylor expansion with integral remainder of $\mathcal{F}\left[h_1 \vec{z}_1 + (\vec{1} - \vec{z}_1)\right]$ between 0 and 1, with $\vec{z}_1$ being the vector with entries 1 or 0 according to whether the corresponding monomer is admissible or not. Then each partial derivative acting on $\mathcal{F}$ can be seen as taking down the corresponding monomer so that terms can be seen as growing polymers. The iteration goes as follow: for a term of order n corresponding to a given polymer Y and having n interpolating parameters $1 > h_0 > \ldots > h_n > 0$ we put a common parameter h_{n+1} interpolating between 0 and h_n on all monomers admissible for Y. It is easy to check that the process is finite since $\mathcal{A}$ is finite and that one obtains the desired formula. ■

In the following our monomers are sets of cubes (that we call the support of the monomer) and links between those cubes. When we take down a polymer, we connect all the cubes in its support and maybe some more cubes. Thus a polymer is made of several connected regions, we will say that it is connected if it has a single connected component. The rules of admissibility will be to never take down a monomer whose support is totally contained in a connected region.

In this case, one can show that the interpolating parameters depend only of the connected component to which the corresponding monomer belongs so that one can think to "factorize" the connected components. We define generalized polymers as sets of connected polymers. Then a generalized polymer $Y = \{Y_1, \dots, Y_p\}$ is allowed if the polymer $Y_1 \cup \dots \cup Y_p$ is allowed (this does not depend of the order of the $Y_i' s$). Equation (41) becomes

$$\mathcal{F}(\vec{1}) = \sum_{\substack{Y=\{Y_1,\dots,Y_p\} \\ Y_i=(X_i^1,\dots X_i^{n_i})}} \left(\prod_{i=1}^{p} \int_{1>h_i^1>\dots h_i^{n_i}>0} dh_i^1 \dots dh_i^{n_i} \right) \left(\prod_{X\in Y} \frac{\partial}{\partial z_X} \right) \mathcal{F}\left[\vec{z}(Y, \{h_i^j\})\right] \tag{43}$$

where the sum extends on all allowed generalized polymers, and $\vec{z}(Y, \{h_i^j\})$ is given by

$$z_X(Y, \{h_i^j\}) = \begin{cases} 0 & \text{if } X \text{ is admissible for } Y, \textit{ i.e.} \text{ for } Y_1 \cup \dots \cup Y_p \\ 1 & \text{if } X \text{ is not admissible for } \emptyset \\ h_i^j & \text{if } X = X_i^j \text{ for some } i \text{ and } j \\ h_i^j & \text{where } X \text{ is not admissible for } Y_i \text{ and} \\ & j = \min\{k/X \text{ not admissible for } (X_i^1, \dots, X_i^k)\}, \\ & \text{otherwise} \end{cases}$$

3.3. Large/small field decomposition

Semi-perturbative expansion (like cluster expansions) are convergent only when the "perturbation" is small (in our case the operators V_Δ's). Thus it is very important to distinguish between the so called *small field regions* where perturbations will work and the *large field regions* where we must find other estimates (they will come mostly from the exponentially small probabilistic factor attached to those regions).

We take a C_0^∞ function ε such that

- $0 \leqslant \varepsilon \leqslant 1$
- $\text{Supp}(\varepsilon) \subset [0,2]$
- $\varepsilon_{|[0,1]} = 1$

Then for each Δ we define

$$\varepsilon_\Delta(V_\Delta) = \varepsilon\left(\frac{\|D_{\Lambda,\mu} V_{\vec{\Delta}} D_{\Lambda,\mu}\|}{a\lambda^{-1/4} C_0}\right) \quad \text{and} \quad \eta_\Delta = 1 - \varepsilon_\Delta \tag{44}$$

where $a = O(1)$. We can expand

$$1 = \prod_\Delta (\varepsilon_\Delta + \eta_\Delta) = \sum_{N \geqslant 0} \sum_{\Omega = \{\Delta_1, \ldots, \Delta_N\}} \left(\prod_{\Delta \in \Omega} \eta_\Delta\right)\left(\prod_{\Delta \notin \Omega} \varepsilon_\Delta\right) \tag{45}$$

where Ω is the large field region whose contribution will be isolated through the following lemma.

LEMMA 4. – *Let Ω be a large field region made of N cubes $\Delta_1, \ldots, \Delta_N$ and A any operator such that*

$$\forall D \subset \{1, \ldots N\}, \quad A + \sum_{i \in D} B_i \text{ is invertible} \tag{46}$$

(B_i stands for $B_{\Delta_i} \equiv \lambda D_{\Lambda,\mu} V_{\Delta_i} D_{\Lambda,\mu}$).

We have the following identity

$$\frac{1}{A + \sum B_i} = \sum_{n=0}^{N} (-1)^n \sum_{i_1 \in \{1 \ldots N\}} \sum_{\substack{i_2 \in \{1 \ldots N\} \\ i_2 \notin \{i_1\}}} \cdots \sum_{\substack{i_n \in \{1 \ldots N\} \\ i_n \notin \{i_1 \ldots i_{n-1}\}}} \frac{1}{A} O_n \frac{1}{A} \ldots O_1 \frac{1}{A} \tag{47}$$

where

$$O_p = B_{i_p} - \left(\sum_{j \in \{i_1 \ldots i_p\}} B_j\right) \frac{1}{A + \sum_{j \in \{i_1 \ldots i_p\}} B_j} B_{i_p} \tag{48}$$

Proof. – The proof relies on resolvent expansion identities

$$\frac{1}{A+B} = \frac{1}{A}\left(I - B\frac{1}{A+B}\right) = \left(I - \frac{1}{A+B} B\right)\frac{1}{A} \tag{49}$$

We show by induction that for all $m \in \{1, \ldots, N\}$ we have

$$\frac{1}{A+\sum B_i} = \sum_{n=0}^{m-1} (-1)^n \sum_{\substack{(i_1,\ldots i_n)\\ i_k \notin \{i_1 \ldots i_{k-1}\}}} \frac{1}{A} O_n \frac{1}{A} \ldots O_1 \frac{1}{A} + (-1)^m R_m \quad (50)$$

$$R_m = \sum_{\substack{(i_1,\ldots i_m)\\ i_k \notin \{i_1 \ldots i_{k-1}\}}} \frac{1}{A+\sum B_i} B_{i_m} \frac{1}{A} O_{m-1} \frac{1}{A} \ldots O_1 \frac{1}{A} \quad (51)$$

The case $m = 1$ is obtained by a resolvent expansion

$$\frac{1}{A+\sum B_i} = \frac{1}{A} - \sum_{i_1} \frac{1}{A+\sum B_i} B_{i_1} \frac{1}{A} \quad (52)$$

Then we go from m to $m+1$ with 2 steps of resolvent expansion. We write

$$\frac{1}{A+\sum B_i} = \left(I - \sum_{i_{m+1} \notin \{i_1 \ldots i_m\}} \frac{1}{A+\sum B_i} B_{i_{m+1}} \right) \frac{1}{A + \sum_{k=1}^{m} B_{i_k}} \quad (53)$$

$$= \left(I - \sum_{i_{m+1} \notin \{i_1 \ldots i_m\}} \frac{1}{A+\sum B_i} B_{i_{m+1}} \right) \frac{1}{A} \left(I - \sum_{k=1}^{m} B_{i_k} \frac{1}{A + \sum_{l=1}^{m} B_{i_l}} \right) \quad (54)$$

Finally, for $m = N$ we make a last resolvent expansion on the rest term R_N by writing

$$\frac{1}{A+\sum B_i} = \frac{1}{A} \left(I - \sum B_i \frac{1}{A+\sum B_i} \right) \quad (55)$$

■

If we look at

$$\chi_{\Delta_{out}} \frac{1}{A+\sum B_i} \chi_{\Delta_{in}} \quad (56)$$

and fix $\{\Delta_{i_1},\dots\Delta_{i_n}\}$, we can see that summing over the sequences $(i_1,\dots i_n)$ and choosing a particular term for each O_p amounts to construct a tree on $\{\Delta_{in},\Delta_{out},\Delta_{i_1},\dots\Delta_{i_n}\}$.

We define an oriented link l as a couple of cubes that we note $(l.y, l.x)$, then $\vec{\mathcal{L}}$ is the set of oriented links. Given two cubes Δ_{in} and Δ_{out} and a set of cubes $\Omega = \{\Delta_1,\dots\Delta_n\}$ we construct the set $\mathcal{T}_R(\Delta_{in},\Delta_{out},\Omega)$ of oriented trees going form Δ_{in} to Δ_{out} through Ω as the sequences $(l_1,\dots l_{n+1}) \in \vec{\mathcal{L}}^{n+1}$ which satisfy

- $l_1.x = \Delta_{in}$
- $l_{n+1}.y = \Delta_{out}$
- $\forall k \in \{1,\dots n\},\ l_k.y \in \Omega$
- $\forall k \in \{2,\dots n+1\},\ l_k.x \in \{l_1.y,\dots l_{k-1}.y\}$
- $\forall k \in \{2,\dots n\},\ l_k.y \notin \{l_1.y,\dots l_{k-1}.y\}$

Then we have the following equivalent formulation of lemma 4.

LEMMA 5. – *Let Ω be a large field region made of N cubes $\Delta_1,\dots,\Delta_N$ and A any operator such that*

$$\forall D \subset \{1,\dots N\},\quad A + \sum_{i\in D} B_i \text{ is invertible} \tag{57}$$

We have the following identity

$$\chi_{\Delta'}\frac{1}{A+\sum B_i}\chi_\Delta = \sum_{n=0}^{N}(-1)^n \sum_{\substack{\Omega'\subset\Omega\\ \Omega'=\{\Delta'_1,\dots\Delta'_n\}}}\ \sum_{\substack{\mathcal{T}\in\mathcal{T}_R(\Delta,\Delta',\Omega')\\ \mathcal{T}=(l_1,\dots l_{n+1})}} \chi_{\Delta'}\frac{1}{A}O_n(l_{n+1}.x, l_n.y)\frac{1}{A}\dots O_1(l_{2.x}, l_1.y)\frac{1}{A}\chi_\Delta \tag{58}$$

where

$$O_p(\Delta_j,\Delta_i) = B_{\Delta_i}\delta_{\Delta_i\Delta_j} - B_{\Delta_j}\frac{1}{A + \sum_{i\in\{1\dots p\}} B_{l_i.y}} B_{\Delta_i} \tag{59}$$

The proof being just a rewriting of lemma 4 is quite immediate. ■

Thanks to this lemma we can factorize out the contribution of the large field region, then we need to extract spatial decay for the resolvent in the small field region. However a kind of Combes-Thomas estimate [8] would not be enough because of the normalization factor that we must pay. For this reason, we will make a polymer expansion to determine which region really contributes to the resolvent.

3.4. Polymer expansion for the resolvent in the small field region

For some large field region Ω, we want to prove the decay of

$$\frac{1}{U_{\Lambda,\mu}^{-1} + \lambda \sum_{\Delta \notin \Omega} D_{\Lambda,\mu} V_{\Delta} D_{\Lambda,\mu} - z D_{\Lambda,\mu}^2} \equiv RU_{\Lambda,\mu} \tag{60}$$

and get something to pay for the normalization factor.

We will pretend that $z = 0$ for simplicity but it is easy to adapt our expansion to treat the general case. We define a 2-link as a pair of cubes and a 3-link as a 2-link with an intermediate cube. We note $\mathcal{L}_2$ the set of 2-links and $\mathcal{L}_3$ for the 3-links.

$$\mathcal{L}_2 = \{\{\Delta, \Delta'\}\} \tag{61}$$

$$\mathcal{L}_3 = \{(\{\Delta_x, \Delta_y\}, \Delta_z)\} \tag{62}$$

For $l = (\Delta, \Delta') \in \mathcal{L}_2$ and $L = (\{\Delta_x, \Delta_y\}, \Delta_z) \in \mathcal{L}_3$, we set

$$\begin{aligned} Q_L = \lambda(&\chi_{\Delta_x} U_{\Lambda,\mu} D_{\Lambda,\mu} V_{\Delta_z} D_{\Lambda,\mu} \chi_{\Delta_y} \\ &+ \chi_{\Delta_y} U_{\Lambda,\mu} D_{\Lambda,\mu} V_{\Delta_z} D_{\Lambda,\mu} \chi_{\Delta_x}) \end{aligned} \tag{63}$$

$$R_l = \chi_{\Delta} R \chi_{\Delta'} + \chi_{\Delta'} R \chi_{\Delta} \tag{64}$$

$$U_l = \chi_{\Delta} U_{\Lambda,\mu} \chi_{\Delta'} + \chi_{\Delta'} U_{\Lambda,\mu} \chi_{\Delta} \tag{65}$$

Afterwards we define

$$\mathcal{L}_3(\Omega) = \{(\{\Delta_x, \Delta_y\}, \Delta_z) \in \mathcal{L}_3 | \Delta_z \notin \Omega\} \tag{66}$$

so that we can write

$$R(\Omega) = \frac{1}{I + \sum_{L \in \mathcal{L}_3(\Omega)} Q_L} = \left(\frac{1}{I + \sum_{L \in \mathcal{L}_3(\Omega)} x_L Q_L} \right) (1, \ldots, 1) \tag{67}$$

Now, for a given 2-link $l_0 = \{\Delta_0, \Delta_0'\}$, we expand $R_{l_0}(\Omega)$ on $\mathcal{L}_3(\Omega)$ (*cf.* lemma 3). We consider that a 3-link connects the cubes of its support with the additional rules that for a "growing" polymer:

- if we have two adjacent connected components Y_1 and Y_2 (such that $d_\Lambda(Y_1, Y_2) = 0$) we connect the two components
- we connect Δ_0 (resp. Δ_0') to any adjacent polymer component

This allows to take into account that operators localized on a pair of cubes have their support extending to the neighboring cubes.

Let us notice that if A and B have disjoint support, we have

$$\frac{1}{1+A+B} = \frac{1}{1+A} \times \frac{1}{1+B} \tag{68}$$

Then it is easy to see that the expansion of R_{l_0} involves only totally connected polymers which connect Δ_0 to Δ_0', because the other terms necessarily contain a product of two operators with disjoint supports which gives zero. We note $\mathcal{A}(\Omega, l_0)$ the corresponding set of polymers which is a decreasing function of Ω, *i.e.*

$$\Omega' \subset \Omega \Rightarrow \mathcal{A}(\Omega, l_0) \subset \mathcal{A}(\Omega', l_0) \tag{69}$$

According to (43), our expansion looks like

$$R_{l_0}(\Omega) = \sum_{n \geqslant 0} \sum_{\substack{Y \in \mathcal{A}(\Omega, l_0) \\ Y = (L_1, \ldots, L_n)}} \int_{1 > h_1 > \ldots h_n > 0} dh_1 \ldots dh_n$$
$$\left(\prod_{L \in Y} \frac{\partial}{\partial z_L}\right) \frac{1}{1 + \sum_{L \in \mathcal{L}_3(\Omega)} z_L Q_L} [\vec{z}(Y, \{h_i\})] \tag{70}$$

$$= \sum_{Y \in \mathcal{A}(\Omega, l_0)} \int \prod_i dh_i \left(\prod_{L \in Y} \frac{\partial}{\partial u_L}\right)$$
$$\frac{1}{1 + \sum_{L \in \mathcal{L}_3(\Omega)} z_L Q_L + \sum_{L \in Y} u_L Q_L} \left[\vec{z}(Y, \{h_i\}), \vec{0}\right] \tag{71}$$

In the second expression, we rewrite the derivatives as Cauchy integrals so that

$$R_{l_0}(\Omega) = \sum_{Y \in \mathcal{A}(\Omega, l_0)} \int \prod_i dh_i \left(\prod_{L \in Y} \oint \frac{du_L}{2i\pi u_L^2}\right)$$
$$\frac{1}{1 + \sum_{L \in \mathcal{L}(\Omega)} z_L(Y, \{h_i\}) Q_L + \sum_{L \in Y} u_L Q_L} \tag{72}$$

$$\equiv \sum_{Y \in \mathcal{A}(\Omega, l_0)} R(Y) = 1 + \sum_{Y \in \mathcal{A}^*(\Omega, l_0)} R(Y) \tag{73}$$

where $\mathcal{A}^*(\Omega, l_0) = \mathcal{A}(\Omega, l_0)/\{\emptyset\}$.

We suppose that we fixed n_1 the power rate of decay of $\|\chi_{\Delta_1} D_{\Lambda,\mu} V_{\Delta_2} D_{\Lambda,\mu} \chi_{\Delta_3}\|$ in $d_\Lambda(\Delta_1, \Delta_2)$ and $d_\Lambda(\Delta_2, \Delta_3)$. We will note

$$\gamma_{\{\Delta,\Delta'\}} = \left[1 + L^{-1} d_\Lambda(\Delta, \Delta')\right]^{-1} \tag{74}$$

Then we have the following lemma

LEMMA 6. – *For $n_2 = n_1 - 3(d+1)$ and λ small enough, we have*

$$\begin{aligned} &\forall l_0 = \{\Delta_0, \Delta_0'\},\ \forall Y \in \mathcal{A}^*(\emptyset, l_0) \equiv \mathcal{A}^*(l_0), \\ &\qquad \|R(Y)\| \leqslant \lambda^{|Y|/4} \gamma^{n_2}_{\{\Delta_0,\Delta_0'\}} \Gamma(Y), \ \textit{with} \sum_{Y \in \mathcal{A}^*(l_0)} \Gamma(Y) \leqslant 1 \end{aligned} \tag{75}$$

where $|Y|$ is the number of monomers in Y.

Proof. – To any 3-link we can associate a pair of 2-links in the following way

$$L = (\{\Delta_x, \Delta_y\}, \Delta_z) \mapsto \{L_x, L_y\} = \{\{\Delta_x, \Delta_z\}, \{\Delta_y, \Delta_z\}\} \tag{76}$$

Since we are in the small field region

$$\forall L \in \mathcal{L}_3(\Omega), \quad \|Q_L\| \leqslant O(1)\, \lambda^{3/4} \gamma_{L_x}^{n_1-(d+1)} \gamma_{L_y}^{n_1-(d+1)} \tag{77}$$

Then in (72) we can integrate each u_L on a circle of radius

$$\rho_L = \lambda^{-1/2} \gamma_{L_x}^{-n_1+2(d+1)} \gamma_{L_y}^{-n_1+2(d+1)} \tag{78}$$

while staying in the domain of analyticity for u_L and have a resolvent bounded in norm by say 2 (if λ is small enough). Thus

$$\|R(Y)\| \leqslant C \int_{1>h_1>\ldots h_{|Y|}>0} dh_1 \ldots dh_{|Y|} \left(\prod_{L \in Y} \lambda^{1/2} \gamma_{L_x}^{n_2+(d+1)} \gamma_{L_y}^{n_2+(d+1)} \right) \tag{79}$$

$$\leqslant C \lambda^{|Y|/4} \gamma_{l_0}^{n_2} \times \frac{\lambda^{|Y|/4}}{|Y|!} \left(\prod_{L \in Y} \gamma_{L_x}^{d+1} \gamma_{L_y}^{d+1} \right) \tag{80}$$

this demonstrates the first part of the lemma with

$$\Gamma(Y) = \frac{\lambda^{|Y|/4}}{|Y|!}\left(\prod_{L\in Y}\gamma_{L_x}^{d+1}\gamma_{L_y}^{d+1}\right) \tag{81}$$

The links of Y are ordered but we can take them to be unordered by eating up the factor $1/|Y|!$. This amounts to sum on a larger set of graphs.

$$\sum_{Y\in\mathcal{A}^*(l_0)}\Gamma(Y) \leqslant \sum_{Y\in\bar{\mathcal{A}}^*(l_0)}\bar{\Gamma}(Y) \tag{82}$$

with

$$\bar{\mathcal{A}}^*(l_0) = \{\{L_1,\ldots,L_n\}\,|\,\exists\,\sigma\in\mathcal{S}_n \text{ s.t. } (L_{\sigma_1},\ldots,L_{\sigma_n})\in\mathcal{A}^*(l_0)\} \tag{83}$$

$$\bar{\Gamma}(Y) = |Y|!\,\Gamma(Y) = \lambda^{|Y|/4}\left(\prod_{L\in Y}\gamma_{L_x}^{d+1}\gamma_{L_y}^{d+1}\right) \tag{84}$$

For a link $l = \{\Delta,\Delta'\}$ in Y, we have several possibilities:

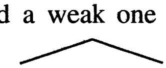
- both cubes Δ and Δ' collapse and we have a *tadpole* ;
- $\Delta \neq \Delta'$ and the cubes were not previously connected, we have a *strong link* ——;
- $\Delta \neq \Delta'$ but both cubes are already connected, it is a *weak link* ----.

We can note that a tadpole is a particular weak link.

We can forget about the *proximity links* (the fact that we connect adjacents components) because it is an irrelevant complication which only add a factor $O(1)^{|Y|}$ in the end.

Our expansion rules insure that each monomer connects a new cube except for the 3-links making a double tadpole but there is at most one of them per cube.

Thus, the only allowed 3-links are:

- a double tadpole ;
- a strong link and a weak one ;
- two strong links .

Now we remark that:

- the "contribution" of a 3-link factorizes on the corresponding pair of 2-links

$$\lambda^{1/4}\gamma_{L_x}^{d+1}\gamma_{L_y}^{d+1} = \left(\lambda^{1/8}\gamma_{L_x}^{d+1}\right) \times \left(\lambda^{1/8}\gamma_{L_y}^{d+1}\right) \tag{85}$$

- and that the set $\bar{\mathcal{A}}^*(l_0)$ is a subset of $\mathcal{T}(l_0)$ which is the set of trees connecting Δ_0 and Δ_0', built with the links {——, ——↘, ↙——}, with at most one double tadpole per vertex and with the convention that a dashed line must contract back to some existing vertex.

Then we can write

$$\sum_{Y \in \bar{\mathcal{A}}^*(l_0)} \bar{\Gamma}(Y) \leqslant \sum_{Y \in \mathcal{T}(l_0)} \Gamma'(Y) \tag{86}$$

$$\Gamma'(Y) = \lambda^{|Y|/8}\left(\prod_{l \in Y} \gamma_l\right) \tag{87}$$

The sum over $\mathcal{T}(l_0)$ can be decomposed as:

- choose $m \geqslant 1$, the number of strong links;
- choose $0 \leqslant n \leqslant m$, the number of weak links;
- choose $m-1$ cubes $\{\Delta_1, \ldots, \Delta_{m-1}\}$;
- choose a tree T on $\{\Delta_0, \Delta_0', \Delta_1, \ldots, \Delta_{m-1}\}$;
- choose which links of T carry the weak links and whether they are on the *right* or on the *left*;
- choose the cubes $(\Delta_1', \ldots, \Delta_n')$ where the weak links contract back;
- choose $0 \leqslant p \leqslant m+1$
- and finally place p tadpoles on $\{\Delta_0, \Delta_0', \Delta_1, \ldots, \Delta_{m-1}\}$.

We can perform the sum on tadpole configurations because for p tadpoles, we have a factor $\lambda^{p/4}$ coming from the tadpoles and at most $\binom{m+1}{p}$ configurations.

$$\sum_{Y \in \mathcal{T}(l_0)} \Gamma'(Y) \leqslant \sum_{\substack{m \geqslant 1 \\ \{\Delta_1, \ldots, \Delta_{m-1}\}}} \sum_{T} \sum_{\substack{0 \leqslant n \leqslant m \\ (\Delta_1', \ldots, \Delta_n')}} \binom{m}{n} 2^n \lambda^{\frac{m+n}{8}} \left(\prod_l \gamma_l\right)$$
$$\times \sum_{p=0}^{m+1} \binom{m+1}{p} \lambda^{p/4} \tag{88}$$

$$\leqslant \sum_{\substack{m \geqslant 1 \\ \{\Delta_1, \ldots, \Delta_{m-1}\}}} \sum_{T} \sum_{\substack{0 \leqslant n \leqslant m \\ (\Delta_1', \ldots, \Delta_n')}} \binom{m}{n} 2^n \lambda^{\frac{m+n}{8}} \left(\prod_l \gamma_l\right)$$
$$\times (1+\lambda^{1/4})^{m+1} \tag{89}$$

Then we can sum on the "free ends" of the weak links since we have

$$\sum_{\Delta_i' \in \{\Delta_0, \Delta_0', \Delta_1, \ldots, \Delta_{m-1}\}} \gamma_{\{\Delta_{k(i)}, \Delta_i'\}} \leqslant \sum_{\Delta_i'} \gamma_{\{\Delta_{k(i)}, \Delta_i'\}} \leqslant C, \tag{90}$$

$\Delta_{k(i)}$ being the cube from which starts the weak link number i.

$$\sum_{Y \in \mathcal{T}(l_0)} \Gamma'(Y) \leqslant \sum_{\substack{m \geqslant 1 \\ \{\Delta_1, \ldots, \Delta_{m-1}\}}} (1 + \lambda^{1/4})^{m+1} \lambda^{m/8} \sum_T \left(\prod_{l \in T} \gamma_l \right)$$

$$\times \sum_{0 \leqslant n \leqslant m} \binom{m}{n} \left(2C\lambda^{1/8} \right)^n \tag{91}$$

$$\leqslant C \sum_{\substack{m \geqslant 1 \\ \{\Delta_1, \ldots, \Delta_{m-1}\}}} \lambda_1^m \sum_T \left(\prod_{l \in T} \gamma_l \right) \tag{92}$$

$$\lambda_1 = (1 + \lambda^{1/4})(1 + 2C\lambda^{1/8})\lambda^{1/8} \tag{93}$$

Then we fix the form of T and we sum over the positions of $\Delta_1, \ldots, \Delta_{m-1}$. Since the cubes are now labeled we get $(m-1)!$ the desired sum.

$$\sum_{Y \in \mathcal{T}(l_0)} \Gamma'(Y) \leqslant C \sum_{m \geqslant 1} \frac{\lambda_1^m}{(m-1)!} \sum_T \sum_{(\Delta_1, \ldots, \Delta_{m-1})} \left(\prod_{l \in T} \gamma_l \right) \tag{94}$$

We choose Δ_0 as the root of our tree and suppose that the position of Δ_0' is not fixed. Then the sum over the position of the cubes is made starting from the leaves thanks to the decaying factors γ_l (cf. [9]), this costs a factor $O(1)^m$.

Finally, the sum over T, which is a sum over unordered trees, is performed using Cayley's theorem which states that there are $(m+1)^{m-1}$ such trees.

$$\sum_{Y \in \mathcal{T}(l_0)} \Gamma'(Y) \leqslant C \sum_{m \geqslant 1} (C\lambda_1)^m \frac{(m+1)^{m-1}}{(m-1)!} \leqslant C'\lambda_1 \leqslant 1 \tag{95}$$

for λ small enough. ■

We note that we can perform the same expansion on

$$R' = U_{\Lambda,\mu}^{-1} \frac{1}{U_{\Lambda,\mu}^{-1} + \lambda \sum_{\Delta \notin \Omega} D_{\Lambda,\mu} V_\Delta D_{\Lambda,\mu} - z D_{\Lambda,\mu}^2} \tag{96}$$

3.5. Summation and bonds on T

We define

$$T_{out,in} = \chi_{\Delta_{out}} T_{\Lambda,\varepsilon} \chi_{\Delta_{in}} \tag{97}$$

We can combine equations (32), (45), (58) and (73) to write

$$\begin{aligned} T_{out,in} = & \int \otimes d\mu_{\xi^{\Delta}_{\Lambda}}(V_{\Delta})\, e^{\frac{\mu_0^2}{2}<1,\xi_{\Lambda}^{-1}1>+i\mu_0<V,\xi_{\Lambda}^{-1}1>} \sum_{N\geqslant 0} \sum_{\Omega=\{\Delta_1,\ldots\Delta_N\}} \\ & \left(\prod_{\Delta\in\Omega}\eta_{\Delta}\right)\left(\prod_{\Delta\notin\Omega}\varepsilon_{\Delta}\right)\sum_{n=0}^{N}(-1)^n \\ & \sum_{\substack{\Omega'\subset\Omega \\ |\Omega'|=n}} \sum_{\substack{\mathcal{T}\in\mathcal{T}_R(\Delta_{in},\Delta_{out},\Omega') \\ \mathcal{T}=(l_1,\ldots l_{n+1})}} \sum_{\substack{(\Delta_2^x,\ldots\Delta_{n+1}^x) \\ (\Delta_1^y,\ldots\Delta_n^y)}} \\ & \sum_{(\Delta_1^z,\ldots\Delta_{n+1}^z)} \sum_{\substack{(Y_1,\ldots Y_{n+1}) \\ Y_i\in\mathcal{A}(\Omega,\{\Delta_i^y,\Delta_i^z\})}} \\ & U_{\Delta_{out},\Delta_{n+1}^z} R'_{\Delta_{n+1}^z,\Delta_{n+1}^x} T_n(l_{n+1}.x, l_n.y) R_{\Delta_n^y,\Delta_n^z} \\ & U_{\Delta_n^z,\Delta_n^x}\ldots T_1(l_{2.x}, l_1.y) R_{\Delta_1^y,\Delta_1^z} U_{\Delta_1^z,\Delta_{in}} \end{aligned} \tag{98}$$

where we pretend that the χ_{Δ}'s are sharp otherwise we would have to deal with adjacent cubes but it's an irrelevant complication. Furthermore, for the leftmost term we made a polymer expansion of $U_{\Lambda,\mu}R'$ instead of $RU_{\Lambda,\mu}$ so that we can write $T_{out,in}$ as

$$T_{out,in} = \chi_{\Delta_{out}}\Big(U_{\Lambda,\mu} + U_{\Lambda,\mu}\tilde{T}U_{\Lambda,\mu}\Big)\chi_{\Delta_{in}} \tag{99}$$

The various small field V_{Δ}'s which have been suppressed from all the small field resolvents still appear in the large field insertions O_j. We note $B_{\Delta} = \lambda D_{\Lambda,\mu} V_{\Delta} D_{\Lambda,\mu}$, then we must deal with factor of the following kind (*cf.* lemma 4)

$$\frac{1}{A + \sum_{j\in\{i_1,\ldots,i_p\}} B_j + B_{\Delta}} B_{i_p} = GB_{i_p}, \tag{100}$$

The B_j's are the large field regions that have been visited at this point.

We use resolvent identity to write

$$GB_{i_p} = \tilde{G}B_{i_p} - GB_\Delta \frac{1}{A}(I - \sum B_j \tilde{G})B_{i_p} \tag{101}$$

$$\tilde{G} = \frac{1}{A + \sum B_j} \tag{102}$$

In this way, either V_Δ has disappeared or we added it as a leaf on the tree connecting the large field regions.

The crucial point is to notice that for any cube Δ, each term where Δ appears in Ω but not in Ω' pairs with a corresponding term where $\Delta \notin \Omega$ and $\Delta \notin \bigcup \mathrm{Supp}(Y_i)$ (*i.e.* Δ has been killed in every polymer expansion). Then the corresponding ε_Δ and η_Δ add up back to 1 so that

$$\begin{aligned} T_{out,in} = \sum_{n\geqslant 0}(-1)^n & \sum_{\Omega=\{\Delta_1,\ldots\Delta_n\}} \sum_{\substack{\mathcal{T}\in\mathcal{T}_R(\Delta_{in},\Delta_{out},\Omega)\\ \mathcal{T}=(l_1,\ldots l_{n+1})}} \sum_{\substack{(\Delta_2^x,\ldots\Delta_{n+1}^x)\\ (\Delta_1^y,\ldots\Delta_n^y)}} \sum_{(\Delta_1^z,\ldots\Delta_{n+1}^z)} \\ & \sum_{\substack{(Y_1,\ldots Y_{n+1})\\ Y_i\in\mathcal{A}(\Omega,\{\Delta_i^y,\Delta_i^z\})}} \int \otimes d\mu_{\xi_\Lambda^\Delta}(V_\Delta)\, e^{\frac{\mu_0^2}{2}\langle 1,\xi_\Lambda^{-1}1\rangle + i\mu_0\langle V,\xi_\Lambda^{-1}1\rangle} \left(\prod_{\Delta\in\Omega}\eta_\Delta\right) \\ & \left(\prod_{\Delta\in\cup\mathrm{Supp}(Y_i)}\varepsilon_\Delta\right) U_{\Delta_{out},\Delta_{n+1}^z} R'_{\Delta_{n+1}^z,\Delta_{n+1}^x} T_n(l_{n+1}.x, l_n.y) R_{\Delta_n^y,\Delta_n^z} \\ & U_{\Delta_n^z,\Delta_n^x} \ldots T_1(l_{2.x}, l_1.y) R_{\Delta_1^y,\Delta_1^z} U_{\Delta_1^z,\Delta_{in}} \end{aligned} \tag{103}$$

The factor $e^{\frac{\mu_0^2}{2}<1,\xi_\Lambda^{-1}1>+i\mu_0<V,\xi_\Lambda^{-1}1>}$ corresponds to the translation of V by $-i\mu_0$, this is equivalent to have translated all the V_Δ's by $-i\mu_0\chi_\Delta$ therefore we can write it as

$$\prod_\Delta e^{\frac{\mu_0^2}{2}<\chi_\Delta,(\xi_\Lambda^\Delta)^{-1}\chi_\Delta>+i\mu_0<V_\Delta,(\xi_\Lambda^\Delta)^{-1}\chi_\Delta>} \tag{104}$$

then we can perform the integration on all $V_\Delta \notin \Omega \cup (\bigcup \mathrm{Supp}(Y_i))$ so that the normalization factor reduces to

$$\prod_{\Delta\in\Omega\cup(\cup\mathrm{Supp}(Y_i))} e^{\frac{\mu_0^2}{2}<\chi_\Delta,(\xi_\Lambda^\Delta)^{-1}\chi_\Delta>+i\mu_0<V_\Delta,(\xi_\Lambda^\Delta)^{-1}\chi_\Delta>} \tag{105}$$

This amounts to pay a constant per cube of $\Omega \cup (\bigcup \mathrm{Supp}(Y_i))$, this is done in Ω with a fraction of the probabilistic factor coming from the large field condition and in $\bigcup \mathrm{Supp}(Y_i)$ with a fraction of the factor $\lambda^{\sum |Y_i|/4}$ coming from the $R's$.

The sums over the various Y_i's are controlled by lemma 6 and we are left with a sum over a tree that we perform much in the same way we did in lemma 6. Indeed one can check that spatial decay appears through factors of the form

$$\sum_{\Delta_i^x, \Delta_i^y, \Delta_i^z} V_{\Delta_{l_i \cdot y}} D_{\Lambda,\mu} \chi_{\Delta_i^y} R \chi_{\Delta_i^z} U_{\Lambda,\mu} \chi_{\Delta_i^x} D_{\Lambda,\mu} V_{\Delta_{l_i \cdot x}} \tag{106}$$

thus we can extract decay in $d_\Lambda(l_i.y, l_i.x)$ time a bound in $\prod \|D_{\Lambda,\mu} V_{\Delta_{l_i \cdot y}} D_{\Lambda,\mu}\|$ when we combine all these factors.

Yet we need some extra features to deal with the product of O_i's, each of them being bounded in norm by $O(1)\,\mu^{-1} \|D_{\Lambda,\mu} V_i D_{\Lambda,\mu}\| \|D_{\Lambda,\mu} V_{k_i} D_{\Lambda,\mu}\|$ for some k_i.

The factor μ^{-1} can be controlled with a small fraction of the probabilistic factor attached to the cube Δ_i

If a given $D_{\Lambda,\mu} V_i D_{\Lambda,\mu}$ appears at a large power it has necessarily a large number of links attached to it. Because of the tree structure, the links must go further and further so that the decay of the links together with the Gaussian measure allow to control the factorial coming from the accumulation of fields. This is quite standard and the reader can refer to [9] for instance.

Finally we can write $T_{out,in}$ as a sum over polymers of the form

$$T_{out,in} = \chi_{\Delta_{out}} (U_{\Lambda,\mu} + \delta T) \chi_{\Delta_{in}} \tag{107}$$

$$\delta T = \frac{\lambda^{c_1}}{[1 + L^{-1} d_\Lambda(\Delta_{in}, \Delta_{out})]^{n_3}} \sum_{Y \in \mathcal{A}^*(\Delta_{in}, \Delta_{out})} \lambda^{c_2 |Y|} \Gamma_Y T(Y) \tag{108}$$

where c_1 and c_2 are small constant, Γ_Y has decay in the spatial extension of Y and $\|\sum_{Y \in \mathcal{A}^*(\Delta_{in}, \Delta_{out})} T(Y)\|$ is bounded. ■

4. ANDERSON MODEL WITH AN INFRA-RED CUT-OFF IN DIMENSION $d = 2$

We are interested now in the particular case

$$H = -\Delta_\eta + \lambda \eta_E V \eta_E \tag{109}$$

where

- Δ_η^{-1} is a ultra-violet regularized inverse Laplacian, that we will note $-p^2$
- η_E is an infra-red cut-off that forces $|p^2 - E| \geqslant A\lambda^2 |\log \lambda|^2$
- V has covariance ξ which is a $\mathcal{C}_0^\infty$ approximation of a δ-function
- $M^{1/2}$ is an even integer greater than 2, and $j_0 \in \mathbb{N}$ is such that

$$M^{-j_0} \leqslant \inf_{\text{Supp}(\eta_E)} |p^2 - E| \leqslant M^{-(j_0-1)} \tag{110}$$

For each $0 \leqslant j \leqslant j_0$, we construct a smooth partition of unity into cubes of side M^j which form a lattice $\mathbb{D}_j$. It follows a decomposition of V in fields V_{Δ_j} and we will assume for simplicity that for $j < k$ and $\Delta_k \in \mathbb{D}_k$

$$V_{\Delta_k} = \sum_{\substack{\Delta_j \in \mathbb{D}_j \\ \Delta_j \subset \Delta_k}} V_{\Delta_j} \tag{111}$$

even if it is not totally true because of irrelevant border effects.

4.1. The matrix model

We make a partition of unity according to the size of $p^2 - E$ thanks to a function $\hat{\eta}$ which satisfies

- $\hat{\eta}$ is in $\mathcal{C}_0^\infty(\mathbb{R}_+)$ with value in $[0, 1]$
- $\hat{\eta}$ has its support inside $[0, 2]$ and is equal to 1 on $[0, 1]$
- the L^∞ norm of the derivatives of $\hat{\eta}$ does not grow too fast

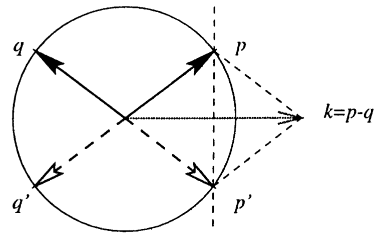
Then we construct

$$\begin{cases} \hat{\eta}_0(p) = 1 - \hat{\eta}\big[M^2(p^2 - E)^2\big] \\ \hat{\eta}_j(p) = \hat{\eta}\big[M^{2j}(p^2 - E)^2\big] - \hat{\eta}\Big[M^{2(j+1)}(p^2 - E)^2\Big] \quad \text{for } j > 0 \end{cases} \tag{112}$$

In order to shorten expressions, we assume that

$$\eta_E = \sum_{j=0}^{j_0} \eta_j \tag{113}$$

We expect that most of the physics will come from the neighborhood of the singularity $p^2 = E$ of the free propagator. As an operator in momentum space, V has a kernel $\hat{V}(p, q) \equiv \hat{V}(p - q)$. But since p and q have more or less the same norm, there are only two configurations which give the sum $p - q$ [10, 11].



We can see this in another way. If we make perturbations and integrate on V we will get Feynman graphs with four-legged vertices where the incoming momenta have a fixed norm and must add to zero (or almost zero) because of (approximate) translation invariance. Then the four momenta approximately form a rhombus which happens to be a parallelogram. It implies that they must be more or less opposite 2 by 2. Thus the problem looks like a vectorial model because the angular direction of the momentum is preserved.

In order to have this feature more explicit, we decompose the slice $\Sigma^j \equiv \text{Supp}(\hat{\eta}_j)$ into $M^{j/2}$ angular sectors. We introduce $\hat{\eta}_S$ with

- $\hat{\eta}_S$ is an even function in $\mathcal{C}_0^\infty(\mathbb{R})$ with value in $[0,1]$
- $\hat{\eta}_S$ has its support inside $[-1-\frac{1}{M}, 1+\frac{1}{M}]$ and is equal to 1 on $[-1,1]$
- $\hat{\eta}_S(1+x) = 1 - \hat{\eta}\big(1+\frac{1}{M}-x\big)$ for $|x| \leqslant \frac{1}{M}$
- the L^∞ norm of the derivatives of $\hat{\eta}_S$ does not grow too fast

Then we define $\theta_j = \pi M^{-j/2}$ and construct sectors S^j_α of angular width $\theta_j(1+\frac{1}{M})$ centered around $k_\alpha \equiv e^{i\alpha}$ (identifying $\mathbb{R}^2$ and $\mathbb{C}$), with $\alpha \in \theta_j\, \mathbb{Z}/_{M^{j/2}\mathbb{Z}}$.

$$\hat{\eta}_j = \sum_\alpha (\hat{\eta}^j_\alpha)^2 \quad \text{where} \quad \big(\hat{\eta}^j_\alpha\big)^2\big(|p|e^{i\theta}\big) \equiv \hat{\eta}_j(|p|)\, \hat{\eta}_S\Big(\frac{\theta-\alpha}{2\theta_j}\Big) \tag{114}$$

Afterwards, we define the operators η^j_α's by their kernel

$$\eta^j_\alpha(x,y) = \int dp\, e^{ip(x-y)}\hat{\eta}^j_\alpha(p) \tag{115}$$

They form a positive, self-adjoint partition of identity.

$$I = \sum_{j,\alpha} \big(\eta^j_\alpha\big)^2 \tag{116}$$

We will map our problem to an operator-valued matrix problem with the following lemma whose proof is quite obvious.

LEMMA 7. – *Let $\mathcal{H}$ be an Hilbert space and suppose that we have a set of indices $\mathcal{I}$ and a partition of unity*

$$I = \sum_{i \in \mathcal{I}} \eta_i^2 \tag{117}$$

where I is the identity in $\mathcal{L}(\mathcal{H})$ and the η_i's are self-adjoint positive operators.

For all $i \in \mathcal{I}$, we define

$$\mathcal{H}_i = \eta_i(\mathcal{H}) \tag{118}$$

then $\mathcal{H}$ and $\mathcal{L}(\mathcal{H})$ are naturally isomorphic to $\bigotimes_{i \in \mathcal{I}} \mathcal{H}_i$ and $\mathcal{L}\left(\bigotimes_{i \in \mathcal{I}} \mathcal{H}_i\right)$ thanks to

$$x \in \mathcal{H} \mapsto (x_i)_{i \in \mathcal{I}} \quad \textit{where } x_i = \eta_i x \tag{119}$$

$$A \in \mathcal{L}(\mathcal{H}) \mapsto (A_{ij})_{i,j \in \mathcal{I}} \quad \textit{where } A_{ij} = \eta_i A \eta_j \tag{120}$$

In our case, we define $\mathcal{I}_j$ as the set of sectors in the slice j and $\mathcal{I} = \cup \mathcal{I}_j$ so that we can construct the operator-valued matrices $\mathbf{V}_\Delta$'s as

$$(\mathbf{V}_\Delta)_{\alpha\beta}^{jk} = \eta_\alpha^j V_\Delta \eta_\beta^k \tag{121}$$

For a slice Σ^l, we define the enlarged slice

$$\bar{\Sigma}^l = \bigcup_{m \geqslant l} \Sigma^m \tag{122}$$

Then an angular sector S_α^l of Σ^l has a natural extension into an angular sector $\bar{S}_\alpha^l$ of $\bar{\Sigma}^l$ and we have the corresponding operator $\bar{\eta}_\alpha^l$.

4.2. Size of the $\mathbf{V}_\Delta^{jk}$'s

Let $\mathbf{V}_\Delta^{jk}$ be defined by

$$\left(\mathbf{V}_\Delta^{jk}\right)_{\alpha\beta}^{lm} = \delta_{jl}\delta_{km}\eta_\alpha^l V_\Delta \eta_\beta^m \tag{123}$$

where the δ are Kronecker's ones. We can remark that

$$\left(V_\Delta^{jk}\right)^\dagger = V_\Delta^{kj} \quad \Rightarrow \quad \|V_\Delta^{jk}\| = \|V_\Delta^{kj}\| \tag{124}$$

Then we have the following large deviations result

THEOREM 3. – *There exist constants C and $C_\varepsilon(\varepsilon)$ such that for all Λ, $j \leqslant k$, $a \geqslant 1$ and $\Delta \in \mathbb{D}_k$*

$$\mathbb{P}_\Lambda\Big(\|\mathbf{V}_\Delta^{jk}\| \geqslant aCM^{-j/2}\Big) \leqslant C_\varepsilon e^{-(1-\varepsilon)a^2 M^{(\frac{k}{2}-\frac{j}{3})}} \tag{125}$$

where C_ε behaves like $1/\varepsilon$.

Proof. – We use the bound

$$\|\mathbf{V}_\Delta^{jk}\|^{2m_0} \leqslant \mathrm{Tr}\left[\left(\mathbf{V}_\Delta^{jk}\right)\left(\mathbf{V}_\Delta^{jk}\right)^\dagger\right]^{m_0} \tag{126}$$

where

$$\left(\mathbf{A}^\dagger\right)_{\alpha\beta}^{lm} = \left(\mathbf{A}_{\beta\alpha}^{ml}\right)^\dagger \tag{127}$$

$$\mathrm{Tr}\ \mathbf{A} = \sum_{l,\alpha} \mathrm{tr}\ \mathbf{A}_{\alpha\alpha}^{ll} = \sum_{l,\alpha} \int \mathbf{A}_{\alpha\alpha}^{ll}(x,x)\,dx \tag{128}$$

Thus for any m_0

$$\mathbb{P}\Big(\|\mathbf{V}_\Delta^{jk}\| \geqslant aCM^{-j/2}\Big)$$

$$= \int \mathbb{1}_{\left(\|\mathbf{V}_\Delta^{jk}\| \geqslant aCM^{-j/2}\right)} d\mu_{\xi_\Lambda}(V) \tag{129}$$

$$\leqslant \frac{1}{(aCM^{j/2})^{2m_0}} \int \|\mathbf{V}_\Delta^{jk}\|^{2m_0} d\mu_{\xi_\Lambda}(V) \tag{130}$$

$$\leqslant \frac{1}{(aCM^{-j/2})^{2m_0}} \int \mathrm{Tr}\left[\left(\mathbf{V}_\Delta^{jk}\right)\left(\mathbf{V}_\Delta^{jk}\right)^\dagger\right]^{m_0} d\mu_{\xi_\Lambda} \tag{131}$$

Let us note

$$\mathcal{I}_{m_0} \equiv \mathrm{Tr}\left[\left(\mathbf{V}_\Delta^{jk}\right)\left(\mathbf{V}_\Delta^{jk}\right)^\dagger\right]^{m_0} = \mathrm{tr}\left[\sum_\alpha (\eta_\alpha^j)^2\, V_\Delta \sum_\beta (\eta_\beta^k)^2\, V_\Delta\right]^{m_0} \tag{132}$$

We have the following lemma

LEMMA 8. – *There exists a constant C such that for all m_0 we have the following bound*

$$< \mathcal{I}_{m_0} > \leqslant C^{2m_0} M^{-jm_0}\Big[1 + M^{-m_0(\frac{k}{2}-\frac{j}{3})} m_0!\Big] \tag{133}$$

This lemma is the core of the demonstration but its proof is quite long so that we postpone it until the end of this part. It leads to

$$\mathbb{P}\Big(\|\mathbf{V}_{\Delta}^{jk}\| \geqslant aCM^{-j/2}\Big) \leqslant a^{-2m_0}\Big[1 + M^{-m_0(\frac{k}{2}-\frac{j}{3})}m_0!\Big] \tag{134}$$

We take $m_0 = a^2M^{\frac{k}{2}-\frac{j}{3}}$ and use the rough bound

$$n! \leqslant n^{(n+1)}e^{-(n-1)} \tag{135}$$

to get the desired estimate. ■

In fact, in the proof of lemma 8 it is easy to see that η_j and η_k can be replaced by $\bar{\eta}_j$ and $\bar{\eta}_k$ with the same result. Furthermore, thanks to the locality of V and to the decay of the η's , the sum of several V_{Δ}^{jk}'s is more or less an orthogonal sum. More precisely, for any cube Δ_0 we define $D_m(\Delta_0)$ as the set of cubes of $\mathbb{D}_m$ which are contained in Δ_0. Then given two sets Ω_1 and Ω_2 and their smoothed characteristic functions χ_{Ω_1} and χ_{Ω_2} we have

LEMMA 9. – *For any n and C there is a constant C_n such that for any $j \leqslant k$ and $\Delta_0 \in \mathbb{D}_k$*

$$\|\chi_{\Omega_1}\bar{\eta}_j V_{\Delta_0}\bar{\eta}_k\chi_{\Omega_2}\| \leqslant \frac{C_n}{[1+M^{-nj}d(\Omega_1,\Delta_0)^n][1+M^{-nk}d(\Omega_2,\Delta_0)^n]} \max\left[\|\bar{\eta}_j V_{\Delta_0}\bar{\eta}_k\|, \sup_{\substack{m<j\\ n<k}} \sup_{\Delta\in D_{m\wedge n}(\Delta_0)} M^{-C(j-m+k-n)}\|\eta_m V_{\Delta}\eta_n\|\right] \tag{136}$$

where $m \wedge n = \min(m,n)$.

Proof. – We introduce $\chi_{\bar{\Delta}_0}$ a C_0^∞ function equal to 1 on the support of V_{Δ_0} then we write

$$\chi_{\Omega_1}\bar{\eta}_j V_{\Delta_0}\bar{\eta}_k\chi_{\Omega_2} = \chi_{\Omega_1}\bar{\eta}_j\chi_{\bar{\Delta}_0}\bar{\eta}_j V_{\Delta_0}\bar{\eta}_k\chi_{\bar{\Delta}_0}\bar{\eta}_k\chi_{\Omega_2} + \sum_{\substack{m<j\\ n<k}} \chi_{\Omega_1}\bar{\eta}_j\chi_{\bar{\Delta}_0}\eta_m V_{\Delta_0}\eta_n\chi_{\bar{\Delta}_0}\bar{\eta}_k\chi_{\Omega_2} \tag{137}$$

Afterwards we introduce the sectors and the matrix formulation and we notice that when we want to compute for instance the norm of the function $\eta_\gamma^n\chi_{\bar{\Delta}_0}\bar{\chi}_\alpha^k\chi_{\Omega_2}$, momentum conservation tells us that we can convolve $\chi_{\bar{\Delta}_0}$ by a function which is restricted in momentum space to the neighborhood of $S^n_\gamma - \bar{S}^k_\alpha$. In this way it is quite easy to see that we can extract at the same time spatial decay and momentum conservation decay. ■

4.3. Proof of theorem 2

Let $\Delta_0 \in \mathbb{D}_{j_0}$, we call $X_{C_x,a}$ and $Y_{C_y,a}$ the events

$$X_{C_x,a} = \Big[\exists j \leqslant k, \exists \Delta \in D_k(\Delta_0) \text{ s. t. } \|\bar{\eta}_j V_\Delta \bar{\eta}_k\| \geqslant a C_x M^{-j/2} M^{\frac{j_0-k}{4}}\Big] \tag{138}$$

$$Y_{C_y,a} = \Big[\|D_{\Lambda,\mu}\eta_E V_{\Delta_0}\eta_E D_{\Lambda,\mu}\| \geqslant a C_y j_0 M^{j_0/2}\Big] \tag{139}$$

We will note $\bar{Z}$ the contrary event of Z.

Theorem 3 tells us that

$$\mathbb{P}(X_{C,a}) \leqslant \sum_k \sum_{j\leqslant k} \sum_{\Delta\in D_k(\Delta_0)} C' e^{-\frac{3}{4}a^2 M^{\frac{j_0-k}{2}} M^{(\frac{k}{2}-\frac{j}{3})}} \tag{140}$$

$$\leqslant \sum_k O(1) M^{2(j_0-k)} e^{-\frac{3}{4}a^2 M^{\frac{j_0}{6}} M^{\frac{j_0-k}{3}}} \tag{141}$$

$$\leqslant C_1 e^{-\frac{1}{2}a^2 M^{\frac{j_0}{6}}} \tag{142}$$

One can see that thanks to lemma 9, $\bar{X}_{C,a}$ implies $\bar{Y}_{O(1)C,a}$. Thus if we call $C_0 = O(1)C$

$$\mathbb{P}(Y_{C_0,a}) \leqslant \mathbb{P}(X_{C,a}) \leqslant C_1 e^{-\frac{1}{2}a^2 M^{\frac{j_0}{6}}} \tag{143}$$

Furthermore, if we work with respect to $\bar{X}_{C,a}$ which is stronger than $\bar{Y}_{C_0,a}$ everything goes as if one had

$$\|\chi_{\Delta_1} D_{\Lambda,\mu} V_{\Delta_2} D_{\Lambda,\mu}\chi_{\Delta_3}\| \leqslant \frac{C_{n_1}\|D_{\Lambda,\mu}V_{\Delta_2}D_{\Lambda,\mu}\|}{[1+L^{-1}d_\Lambda(\Delta_1,\Delta_2)]^{n_1}[1+L^{-1}d_\Lambda(\Delta_2,\Delta_3)]^{n_1}} \tag{144}$$

Thus we will be able to apply theorem 1 with an effective coupling constant

$$\lambda_{\text{eff}} = \lambda j_0 M^{j_0/2} \tag{145}$$

and a length scale $L = M^{j_0}$.

If we want to make perturbations it is clever to perturb around the expected Green's function without cut-off, *i.e.* we write

$$\frac{1}{p^2 - E - i\mu + \lambda V} = \frac{1}{p^2 - E - i\mu_0 + \lambda V + i\delta\mu} \tag{146}$$

where μ_0 is the expected contribution of the tadpole given by the self-consistent condition

$$\mu_0 = \lambda^2 \mathrm{Im} \int \frac{1}{p^2 - E - i\mu_0} dp \tag{147}$$

Afterwards, when we compute the perturbative expansion, the tadpole with cut-off will eat up a fraction $\lambda^2 M^j \sim O(|\log\lambda|^{-2})$ of the counter-term so that

$$G \sim \frac{1}{p^2 - E - i\eta_E O(\lambda^2 |\log\lambda|^{-2})\eta_E} \tag{148}$$

■

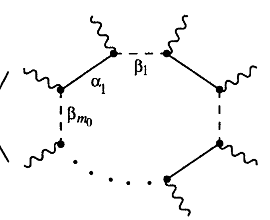
In fact since the tadpole has a real part, it implies that we should also renormalize the energy by a shift

$$\delta E = O\big(\lambda^2 \log[\text{UV cut-off scale}]\big) \tag{149}$$

4.4. Proof of lemma 8

We will note $J_\alpha \equiv (\eta^j_\alpha)^2$, $K_\beta \equiv (\eta^k_\beta)^2$ and X as either J or K.

We can perform the integration on V_Δ so that $\langle \mathcal{I}_{m_0} \rangle$ appears as a sum of Feynman graphs.

$$\langle \mathcal{I}_{m_0} \rangle = \sum_{\alpha_1 \ldots \alpha_{m_0} \beta_1 \ldots \beta_{m_0}} \Big\langle$$  $$\Big\rangle = \sum_{\mathcal{G}} \mathcal{A}(\mathcal{G}) \tag{150}$$

where a solid line stands for a J_{α_i}, a dashed line stands for a K_{β_i} and a wavy line represents the insertion of a V_Δ. In the following, we will prove the theorem in infinite volume with V having a covariance δ in order to have shorter expressions. The proof can then easily be extended to short range covariances and finite volume except for the first few slices where one must pay attention to the ultra-violet cut-off but this is irrelevant because it will cost only a factor $O(1)$.

The integration on V_Δ consists in contracting the wavy lines together, then both ends are identified and bear an extra χ_Δ which restricts their position.

The X's will stand as propagators and the contraction of the V_Δ's will give birth to 4-legged vertices.

4.4.1. Momentum conservation at vertices

First, we notice that if we note $\bar{\alpha}$ the opposite sector of α

$$X_\alpha(x,y) = X_{\bar{\alpha}}(y,x) \tag{151}$$

Then we put an orientation on each propagator, so that if a X_α goes from a vertex at z to a vertex at z' it gives a $X_\alpha(z,z') = X_{\bar{\alpha}}(z',z)$, *i.e.* it is equivalent to have an incoming $X_{\bar{\alpha}}$ at z and an incoming X_α at z'. Now, for a given vertex with incoming propagators X_{α_1}, X_{α_2}, X_{α_3}, X_{α_4}, the spatial integration over its position gives a term of the form

$$\Gamma_{\alpha_1\ldots\alpha_4}(x_1,x_2,x_3,x_4) = \int X_{\alpha_1}(x_1,z)X_{\alpha_2}(x_2,z)X_{\alpha_3}(x_3,z) \\ X_{\alpha_4}(x_4,z)\chi_\Delta(z)\,dz \tag{152}$$

In momentum space, it becomes

$$\Gamma_{\alpha_1\ldots\alpha_4}(p_1,\ldots,p_4) = X_{\alpha_1}(p_1)\ldots X_{\alpha_4}(p_4)\int \chi_\Delta(k)\,\delta(p_1+\ldots+p_4-k)\,dk \tag{153}$$

where we use the same notation for a function and its Fourier transform.

In x-space, χ_Δ is a $\mathcal{C}_0^\infty$ function with support inside a box of side $O(1)M^{j_0}$, it means that in momentum space, it is a $\mathcal{C}^\infty$ function with fast decay over a scale M^{-j_0}. Thus for all n there exists $C_\chi(n)$ such that

$$|\chi_\Delta(k)| \leqslant \frac{C_\chi(n)M^{2j_0}}{(1+M^{2j_0}|k|^2)^{n+1}} \tag{154}$$

We make a decomposition of χ_Δ

$$\chi_\Delta(k) = \sum_{s=0}^{j_0/2} \chi_s(k) \tag{155}$$

where χ_0 has its support inside the ball of radius $2MM^{-j_0}$, $\chi_{j_0/2}(\equiv \chi_\infty)$ has its support outside the ball of radius $M^{j_0/2}M^{-j_0}$ and χ_s forces $|k|$ to be in the interval $[M^s M^{-j_0}; 2MM^s M^{-j_0}]$. In this way, we can decompose each vertex v into a sum of vertices v_s, where a vertex v_s forces momentum conservation up to $O(1)M^s M^{-j_0}$ and has a factor coming from

$$|\chi_s(x)| \leqslant C'_\chi(n) M^{-sn} \times M^{-sn} \tag{156}$$

We split the factor in order to have a small factor per vertex and yet retain some decay to perform the sum on s.

4.4.2. Tadpole elimination

A graph will present tadpoles when two neighboring V_Δ's contract together thus yielding a $X(z,z)$. Suppose that we have a j-tadpole then at the corresponding vertex we will have something of the form

$$\int dz\, \chi_u(z)\, K_\beta(x,z)\, J_\alpha(z,z)\, K_{\beta'}(z,y) \tag{157}$$

Between the two K's, momentum will be preserved up to $2M\, M^u M^{-j_A}$ which in most case is much smaller than $M^{-j_0/2}$ so that β' is very close to β. Then we would like to forget about J_α by summing over α and see the whole thing as a kind of new K_β. Now if per chance the new K_β makes a tadpole we will erase it, and so on recursively.

First, we define the propagators as propagators (or links) of order 0

$$^0J^{(0,0)}_{\alpha\alpha'} = \delta_{\alpha\alpha'} J_\alpha \tag{158}$$

$$^0K^{(0,0)}_{\beta\beta'} = \delta_{\beta\beta'} K_\beta \tag{159}$$

Then we define links of order 1

$$^1K^{(p,0)}_{\beta\beta'}(x,y) = \sum_{\beta_1\ldots\beta_{p-1}} \sum_{\substack{\alpha_1\ldots\alpha_p \\ \alpha'_1\ldots\alpha'_p}} \int dz_1 \ldots dz_p\, K_\beta(x,z_1)\ {}^0J^{(0,0)}_{\alpha_1\alpha'_1}(z_1)\chi_{u_1}(z_1)$$
$$K_{\beta_1}(z_1,z_2)\ldots\ {}^0J^{(0,0)}_{\alpha_p\alpha'_p}(z_p)\chi u_p(z_p) K_{\beta'}(z_p,y) \tag{160}$$

where we don't write the momentum conservation indices for shortness. We have a similar definition for ${}^{(1)}J^{(0,q)}_{\alpha\alpha'}$ (obtained by erasing q K-tadpoles of order 0).

We will note $X^{(),t}$ to indicate that momentum is preserved up to tM^{-j_0} between the leftmost and the rightmost X's or $X^{(),\infty}$ if momentum conservation is worse than $M^{-j_0/2}$.

Now, we can iterate the process in an obvious way. Yet, we must add an important restriction: we will erase a $X^{(),\infty}$ tadpole only if it is attached to a v_∞ vertex.

LEMMA 10. – *There exist constants C_1 and C_2 (independent of j and k) such that for any tadpole obtained by erasing a total of p J-tadpoles and q K-tadpoles we have the following bond*

$$\left|X^{(p,q)}_{\gamma\gamma'}(z,z)\right| \leqslant C_1 C_2^{p+q} M^{-pj-qk} M^{-3x/2}\mathcal{F}(X), x = \begin{cases} j \text{ if } X = J \\ k \text{ if } X = K \end{cases} \tag{161}$$

where $\mathcal{F}(X)$ is a small factor coming from the various χ_s that appear in the expression of X. Thus, $\mathcal{F}$ gets smaller as momentum conservation gets worse.

Proof. – First we will prove this result when momentum is well preserved, *i.e.* up to $M^{-j_0/2}$ at worst, then we will see what has to be adapted when there is a bad momentum conservation.

The proof is by induction on the order of the tadpole. We define C_1, C_2 and C_3 such that

$$|X_\gamma(x,y)| \leqslant C_1 M^{-3x/2} \tag{162}$$

$$\sup_x \int dy\, |X_\gamma(x,y)| \leqslant C_3 \tag{163}$$

$$C_2 = 9C_1C_3 \tag{164}$$

It is easy to see that for level 0 tadpoles

$$\left|{}^0X_{\gamma\gamma'}(z,z)\right| \leqslant C_1 M^{-3x/2}\mathcal{F}(X) \tag{165}$$

Now, consider ${}^mJ^{(p,q)}_{\alpha\alpha'}$ a J-tadpole of order m and weight (p,q) obtained by erasing n K-tadpoles of order $m-1$ and weights $(p_1,q_1),\ldots,(p_n,q_n)$. We have

$$p = p_1 + \ldots + p_n \qquad q = q_1 + \ldots + q_n + n \tag{166}$$

The expression of ${}^mJ$ will be of the form

$$\begin{aligned}&{}^mJ^{(p,q)}_{\alpha\alpha'}(z,z)\\ &= \sum_{\alpha_1\ldots\alpha_{n-1}} \sum_{\substack{\beta_1\ldots\beta_n\\ \beta'_1\ldots\beta'_n}} \int dz_1\ldots dz_n\, J_\alpha(z,z_1)\ {}^{m-1}K^{(p_1,q_1)}_{\beta_1\beta'_1}(z_1,z_1)\\ &\chi_{u_1}(z_1)\, J_{\alpha_1}(z_1,z_2)\ldots K^{(p_n,q_n)}_{\beta_n\beta'_n}(z_n,z_n)\,\chi_{u_n}(z_n) J_{\alpha'}(z_n,z)\end{aligned} \tag{167}$$

Since we supposed that we have momentum conservation up to $M^{-j_0/2}$, the α_i's will be either α_{i-1} or one of its neighbors and β'_i will be either β_i or one of its neighbors. Thus the sum on sector attribution will give a factor $3^{2n-1}M^{nk/2} \leqslant 9^n M^{nk/2}$.

We have $n+1$ J's but only n spatial integrations because we have a tadpole. This gives a factor $C_3^n C_1 M^{-3j/2}$ (we forget about the momentum conservation factor for the moment).

Finally the ${}^{m-1}K$'s bring their factor so that

$$\begin{aligned} \left|{}^m J_{\alpha\alpha'}^{(p,q)}(z,z)\right| &\leqslant 9^n M^{nk/2} C_3^n C_1 M^{-3j/2} (9C_1C_3)^{\sum p_i + \sum q_i} \\ &\quad M^{-j\sum p_i - k\sum q_i}\left(C_1 M^{-3k/2}\right)^n \mathcal{F}(X) \\ &\leqslant C_1 (9C_1C_3)^{\sum p_i + \sum q_i + n} M^{-j\sum p_i - k(\sum q_i + n)} \\ &\quad M^{-3j/2}\mathcal{F}(X) \end{aligned} \tag{168}$$

which is precisely what we want. Then we can do the same for the ${}^m K$'s.

Now we must consider the cases with bad momentum conservation. First, let us suppose that momentum conservation is bad overall for ${}^m J$ but was good for the ${}^{m-1}K$'s, then the previous argument will work except if there are some v_∞ vertices. In this case we will have to pay a factor $M^{j/2}$ to find the following α_i instead of a factor 3. But from the corresponding χ_∞ we have a small factor

$$\frac{1}{1+M^{j_0 N'/2}} \tag{169}$$

from which we can take a fraction to pay the $M^{j/2}$ and retain a small factor for $\mathcal{F}(X)$.

Finally, if a ${}^{m-1}K$ has a bad momentum conservation it is necessarily attached to a v_∞ vertex (otherwise we would not erase it). In this case we must pay a factor $M^{j/2}M^{k/2}$ (to find β'_i and α_i) but again we can take a fraction of the factor of χ_∞ to do so.

When tadpole elimination has been completed, we have erased t_j J-tadpoles and t_k K-tadpoles and we are left with $m'_0 = m_0 - t_j - t_k$ vertices linked together by m'_0 J's and m'_0 K's (a tadpole which has not been erased being seen as a propagator).

For a $X_{\alpha\alpha'}^{(p,q)}(x,y)$, it is quite easy to see that to integrate on y with fixed x amounts more or less to the same problem for $O(1)^{p+q}X_\alpha$ and that to find α' knowing α costs a factor $O(1)^{p+q}$.

4.4.3. Sector conservation at the vertex

LEMMA 11. – *Let $(\bar{S}^l_{\alpha_1}, \ldots, \bar{S}^l_{\alpha_4})$ be a quadruplet of sectors of the enlarged slice $\bar{\Sigma}^l$ and $0 \leqslant r \leqslant O(1)M^{l/2}$ such that there are $p_1 \in \bar{S}^l_{\alpha_1}, \ldots, p_4 \in \bar{S}^l_{\alpha_4}$ verifying*

$$|p_1 + \ldots + p_4| \leqslant rM^{-l} \tag{170}$$

Then we can find $\{\alpha, \alpha', \beta, \beta'\} = \{\alpha_1, \ldots, \alpha_4\}$ satisfying

$$\begin{cases} |\alpha' - \bar{\alpha}| \leqslant (a\sqrt{r} + b)\, M^{-l/2} \\ |\beta' - \bar{\beta}| \leqslant (a\sqrt{r} + b)\, M^{-l/2} \end{cases} \tag{171}$$

where a and b are some constants independent of l and r.

Proof. – If we can prove the result for $l \geqslant O(1)$ then we will be able to enlarge the result to any l provided maybe we take some slightly bigger a and b. Therefore we assume that this is the case in the following.

We define $(\alpha, \alpha', \beta, \beta')$ by

- $\{\alpha, \alpha', \beta, \beta'\} = \{\alpha_1, \ldots, \alpha_4\}$
- $\alpha = \alpha_1$
- $|\alpha - \beta| = \min_{i \in \{2,3,4\}} |\alpha - \alpha_i|$
- $|\bar{\alpha} - \alpha'| \leqslant |\bar{\alpha} - \beta'|$

Then, if $|\alpha - \beta| \leqslant |\alpha' - \beta'|$ we exchange (α, β) and (α', β').

A sector $\bar{S}^l_\gamma$ is included in a tube, of center $k_\gamma \equiv e^{i\gamma}$ and whose direction is orthogonal to the direction γ, of size

$$\begin{cases} \text{length} : L = \pi M^{-l/2}(1 + \frac{2}{M}) \\ \text{width} \ : 2M^{-l} \end{cases} \tag{172}$$

We define

$$k_{\alpha\beta} = k_\alpha + k_\beta = 2\cos\left|\frac{\alpha - \beta}{2}\right| e^{i\frac{\alpha+\beta}{2}} \equiv 2\cos x\, e^{i\theta} \equiv re^{i\theta} \tag{173}$$

$$k_{\bar{\alpha}'\bar{\beta}'} = -k_{\alpha'\beta'} \equiv 2\cos \bar{x}'\, e^{i\bar{\theta}'} = -2\cos x' e^{i\theta'} \tag{174}$$

If we can prove that

$$\begin{cases} |x' - x| \leqslant (a'\sqrt{r} + b')\, M^{-l/2} \\ |\theta' - \theta| \leqslant (a'\sqrt{r} + b')\, M^{-l/2} \end{cases} \tag{175}$$

then we will be able to conclude, with $a = 2a'$ and $b = 2b'$.

It is easy to check that by construction, we have

- $0 \leqslant x' \leqslant x$
- $|\alpha - \beta| \leqslant \frac{2\pi}{3} \Rightarrow \cos x \geqslant \frac{1}{2}$

We have a trivial bound

$$2\tan\left|\frac{\theta'-\theta}{2}\right| \leqslant |k_{\alpha\beta} - k_{\alpha'\beta'}| \leqslant rM^{-l} + 2L + 4M^{-l} \equiv R \tag{176}$$

Therefore

$$|\theta'-\theta| \leqslant 2\tan\left|\frac{\theta'-\theta}{2}\right| \leqslant R \leqslant O(1)M^{-l/2} \tag{177}$$

We can see that θ is very well conserved.
If $\sin x \leqslant (a_1\sqrt{r} + b_1)M^{-l/2}$ then $|x - x'| \leqslant x \leqslant (a_2\sqrt{r} + b_2)M^{-l/2}$.
Otherwise, let us remark that $\bar{S}^l_\alpha + \bar{S}^l_\beta$ is at a distance at most $2M^{-l}$ from a rhombus $R_{\alpha\beta}$ of center $k_{\alpha\beta}$ and of diagonals

$$\begin{cases} 2L\sin x & \text{in the direction } \mathbf{u}_r \equiv \dfrac{\alpha+\beta}{2} \\ 2L\cos x & \text{in the direction } \mathbf{u}_\theta \equiv \dfrac{\alpha+\beta}{2} + \dfrac{\pi}{2} \end{cases} \tag{178}$$

Then, $R_{\alpha\beta} - R_{\alpha'\beta'}$ is at a distance at most $4M^{-l}$ from a rectangle $\mathcal{R}$ of center $k_{\alpha\beta} - k_{\alpha'\beta'}$ and of sides

$$\begin{cases} L_r = 2L(\sin x + \sin x' \cos|\theta'-\theta| + \cos x' \sin|\theta'-\theta|) \text{ in } \mathbf{u}_r \\ L_\theta = 2L(\cos x + \cos x' \cos|\theta'-\theta| + \sin x' \sin|\theta'-\theta|) \text{ in } \mathbf{u}_\theta \end{cases} \tag{179}$$

Since $|\theta'-\theta| \leqslant O(1)M^{-l/2}$, we have $|\cos(\theta'-\theta) - 1| \leqslant O(1)M^{-l}$. We define a z axis in the direction $\mathbf{u}_r$. $(k_{\bar{\alpha}'\bar{\beta}'} - k_{\alpha\beta})$ has a z coordinate $2(\cos x' - \cos x) + O(1)M^{-l}$. This leads to the condition

$$2|\cos x' - \cos x| \leqslant rM^{-l} + 2L(\sin x + \sin x') + b_3M^{-l} \tag{180}$$
$$\leqslant (r+b_3)M^{-l} + 4L\sin x \tag{181}$$

Let us note that $|\cos(x-u) - \cos x|$ is an increasing function of u and that we have

$$\cos(x-u) - \cos x = \sin x\, u - \frac{1}{2}\cos x\, u^2 + u^3\varepsilon(u) \quad \text{with } |\varepsilon(u)| \leqslant \frac{1}{6} \tag{182}$$

We take $u = (\sqrt{r} + b_4)M^{-l/2} + 2L \equiv (\sqrt{r} + b'_4)M^{-l/2} \leqslant O(M^{-l/4})$.

$$|\cos(x - u) - \cos x| \geqslant \sin x\, u - \left(\frac{1}{2} + \frac{u}{6}\right)u^2 \tag{183}$$

$$\geqslant 2L \sin x + \sin x(\sqrt{r} + b_4)M^{-l/2} - (\sqrt{r} + b'_4)^2 M^{-l} \tag{184}$$

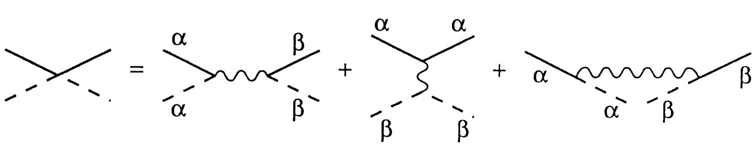
Since we are in the case $\sin x \geqslant (a_1\sqrt{r} + b_1)M^{-l/2}$, for a_1 and b_1 large enough we will have

$$2|\cos(x - u) - \cos x| \geqslant (r + b_3)M^{-l} + 4L \sin x \tag{185}$$

Therefore we must have $|x' - x| \leqslant u \leqslant (\sqrt{r} + b'_4)M^{-l/2}$ which allows us to conclude. ∎

4.4.4. Size of a graph

The previous section shows that, at each vertex, momenta come approximately by pairs of opposite sectors. Thus, for all the vertices which haven't been erased by the tadpole elimination process, we can choose by a factor 3 how to pair the sectors. Then we split the vertices in two half-vertices according to this pairing. We represent graphically this as



This gives $3^{m'_0}$ (split) graphs that we will consider as our basic graphs in the following.

A graph is decomposed into a number of *momentum cycles* connected together by wavy lines. We will follow those cycles to fix momentum sectors. Finding the enlarged sectors (of level j) will cost a factor $M^{j/2}$ per cycle times a constant per vertex. Then we will pay an extra $M^{(k-j)/2}$ for each K propagator to find its sector.

We define c the total number of momentum cycles that we decompose into t tadpoles, b bubbles (with 2 vertices) and l large cycles (with 3 or more vertices). We have

$$t + b + l = c \tag{186}$$

$$t + 2b + 3l \leqslant 2m'_0 \tag{187}$$

and the sector attribution costs

$$\mathcal{A}_1 = C^{m_0} M^{cj/2} M^{m'_0(k-j)/2} \tag{188}$$

Notice that the constant has an exponent m_0 because of the tadpole elimination process.

The spatial integration of the vertices will be made with the short J links whenever possible. We can decompose each graph into J-cycles linked together by K links (because there are 2 incoming J's at each vertex), this allows to integrate all the vertices but one per cycle with a J link. The total cost is (noticing that the last vertex is integrated in the whole cube Δ)

$$\mathcal{A}_2 = C^{m_0} M^{3(m'_0-c')j/2} M^{3(c'-1)k/2} M^{2k} \tag{189}$$

where c' is the total number of short cycles that we decompose into t' short tadpoles, b' short bubbles and l' large short cycles.

$$t' + b' + l' = c' \tag{190}$$
$$t' + 2b' + 3l' \leqslant m'_0 \tag{191}$$

The scaling of the tadpoles and the propagators give a factor

$$\mathcal{A}_3 = M^{-jt_j - kt_k} M^{-3m'_0(j+k)/2} \tag{192}$$

Tadpoles that have been obtained by erasing a few vertices (say $O(M^{j_0/4})$ for instance) will have an extra small factor because they strongly violate momentum conservation, we can take it to be a power of $M^{-j_0/4}$. Tadpoles with higher weights will not have this good factor but we will see that they bring a better combinatoric. The t momentum tadpoles will consist in t_1 low weight ones and t_2 others while the t' short tadpoles split into t'_1 low weight ones and t'_2 others. We can manage to have a factor

$$\mathcal{A}_4 = M^{-2t_1 j_0} M^{-2t'_1 j_0} \tag{193}$$

If we have a short bubble we will have four incoming long propagators whose momenta must add up to zero up to xM^{-j_0}. If we apply lemma 11, we can see that knowing 3 of these momenta it cost only a factor $O(1)\sqrt{x}$ to find the fourth momentum sparing us a factor $M^{(k-j)/2}$ obtained by naïvely fixing first the enlarged sector at slice j. If the bubble has a weight p, *i.e.* the two short propagators have been obtained after erasing p vertices, and a momentum conservation worse than $O(1)pM^{-j_0}$ then the small factor of bad momentum conservation will pay for the $O(1)\sqrt{p}$. We will have b'_1 such good bubbles, each of them bringing a factor $M^{-(k-j)/2}$. In addition, we will have b'_3 bad bubble of weight greater than $M^{(k-j)}$ for which we earn nothing and b'_2 bad bubbles of weight p_i bringing a factor $C\sqrt{p_i}M^{-(k-j)/2}$. This gives a factor

$$\mathcal{A}_5 = M^{-(b'_1+b'_3)(k-j)/2}C^{b'_3}\prod_i \sqrt{p_i} \tag{194}$$

Finally we have the following bound for the contribution of a graph

$$\begin{aligned}|\mathcal{A}(\mathcal{G})| &\leqslant \mathcal{A}_1 \ldots \mathcal{A}_5 \\ &\leqslant C^{m_0} M^{cj/2} M^{m'_0(k-j)/2} M^{3(m'_0-c')j/2} M^{3(c'-1)k/2} M^{2k} M^{-jt_j} \\ &\quad M^{-kt_k} M^{-3m'_0(j+k)/2} M^{-2j_0(t_1+t'_1)} M^{-(b'_1+b'_2)(k-j)/2} \prod_i \sqrt{p_i} &(195) \\ &\leqslant C^{m_0} M^{k/2} M^{cj/2} M^{3c'(k-j)/2} M^{-m'_0(k+j/2)} M^{-jt_j-kt_k)} \\ &\quad M^{-2j_0(t_1+t'_1)} M^{-(b'_1+b'_2)(k-j)/2} \prod_i \sqrt{p_i} &(196)\end{aligned}$$

If we use equations (187) and (191) we obtain

$$\begin{aligned}|\mathcal{A}(\mathcal{G})| \leqslant\ & C^{m_0} M^{k/2} M^{-m'_0(k+j)/2} M^{-jt_j-kt_k} M^{-m'_0 j/6} M^{bj/6} M^{t_2 j/3} \\ & M^{t'_2(k-j)} M^{b'_3(k-j)/2} M^{-t_1(2j_0-j/3)} M^{-t'_1(2j_0+j-k)} \prod_i \sqrt{p_i} &(197)\end{aligned}$$

we will take $m_0 \geqslant M^{k/6}$ so that $M^{k/2} \leqslant C^{m_0}$. Furthermore, t and t' are at most equal to $M^{-j_0/4}m_0$ thus $M^{kt} \leqslant C^{m_0}$. It allows us to rewrite the bound

$$|\mathcal{A}(\mathcal{G})| \leqslant C^{m_0} M^{-m'_0(k+j)/2} M^{-jt_j-kt_k} M^{-(m'_0-b)j/6} M^{b'_3(k-j)/2} \prod_i \sqrt{p_i} \tag{198}$$

4.5. Graph counting

LEMMA 12. – *Let $T(p)$ be the number of ways to contract $2p$ adjacent V's so as to make only generalized tadpoles. We have*

$$T(p) = \frac{(2p)!}{p!(p+1)!} \tag{199}$$

Proof. – It is easy to see that a good contraction scheme, *i.e* one that gives only generalized tadpoles, corresponds to have no crossing contractions. It means that if we label the fields $V_0 \dots V_{2p-1}$ according to their order and if V_i and V_j contract respectively to V_k and V_l then

$$i < j \Rightarrow k < j \text{ or } k > l \tag{200}$$

We have $T(1) = 1$. For $p > 1$, we contract first V_0 to some V_i. $V_1, \dots, V_{i-1}$ will necessarily contract among themselves making only generalized tadpoles and so will do $V_{i+1}, \dots, V_{2p-1}$. Thus i is necessarily odd and we have

$$T(p) = \sum_{k=0}^{p-1} T(k)T(p-1-k) \tag{201}$$

where by convention $T(0) = 1$.

We introduce the generating function

$$t(z) = \sum_{p=0}^{\infty} T(p)z^p \tag{202}$$

The recursion formula (201) can be translated into an equation for t which is

$$t(z) = zt^2(z) + 1 \tag{203}$$

whose resolution yields

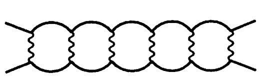
$$t(z) = \frac{1+\sqrt{1-4z}}{2z} \text{ or } \frac{1-\sqrt{1-4z}}{2z} \tag{204}$$

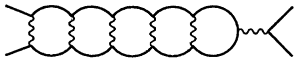
Since the second solution is analytic around $z = 0$, we can take it as $t(z)$ and the coefficients of its power expansion will give us $T(p)$. An easy computation leads to the desired formula. ■

LEMMA 13. – *The number $\mathcal{N}_M(B)$ of graphs with B possible momentum bubbles obtained in the contraction of a cycle of $2M$ V's has the following bound*

$$\mathcal{N}_M(B) \leqslant C^M (M-B)! \tag{205}$$

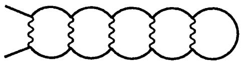
Proof. – First lets us remark that bubbles come in chains (possibly with a tadpole at one end) of two possible types

• type 1: 

- type 2: 

where a solid line stands here either for a J or a K.

We have two special cases

-  which can be seen as a type 1 chain
-  which can generate only one momentum bubble so that we can see it as a type 2 chain of length 1.

Having chosen the V's there are only two contraction schemes that yield a type 1 chain and a unique contraction scheme for a type 2 chain. If we fix explicitly the subgraphs corresponding to B bubbles and contract the remaining V's in any way we will get all the desired graphs plus some extra ones so that we can bound $\mathcal{N}_M(B)$.

We construct r_1 type 1 chains of lengths $\beta_1, \ldots, \beta_{r_1}$ and r_2 type 2 chains of lengths $\gamma_1, \ldots, \gamma_{r_2}$. We set

$$B_1 = \sum_i \beta_i \tag{206}$$

$$B_2 = \sum_i \gamma_i \tag{207}$$

$$B = B_1 + B_2 \tag{208}$$

To count the contraction schemes, first we cut the cycle of $2M$ V's into a sequence of V's (there are $2M$ ways to do so). Then it is easy to check that in order to build a type 1 chain we must choose two sets $\mathcal{B}_i$ and $\bar{\mathcal{B}}_i$ of $\beta_i + 1$ adjacent V's while for a type 2 we need a set $\mathcal{D}_i$ of $2\gamma_i + 1$ adjacent V's. We distribute those $2r_1 + r_2$ objects in $(2M - 2B_1 - 2r_1 - 2B_2 - r_2) + 2r_1 + r_2$ boxes in an ordered way, and for the ith type 1 chain the respective order of $\mathcal{B}_i$ and $\bar{\mathcal{B}}_i$ will fix the contraction scheme. Then, there remain $2M - 2B - 2r_1$ V's to contract so that we have the following number of configurations

$$\mathcal{N}_M(B) \leqslant 2M \sum_{B_1+B_2=B} \sum_{\substack{r_1 \leqslant B_1 \\ r_2 \leqslant B_2}} \sum_{\substack{\beta_1+\ldots\beta_{r_1}=B_1, \beta_i \geqslant 1 \\ \gamma_1+\ldots\gamma_{r_2}=B_2, \gamma_i \geqslant 1}} \frac{1}{r_1!} \frac{1}{r_2!} \frac{(2M-2B)!}{(2M-2B-2r_1-r_2)!} (2M-2B-2r_1-1)!! \tag{209}$$

We can compute this

$$\mathcal{N}_M(B) \leqslant 2M \sum_{B_1+B_2=B} \sum_{\substack{r_1 \leqslant B_1 \\ r_2 \leqslant B_2}} \binom{B_1-1}{r_1-1}\binom{B_2-1}{r_2-1}\frac{(2r_1)!}{r_1!}$$

$$\frac{[2(M-B-r_1)]!}{2^{M-B-r_1}(M-B-r_1)!}\frac{[2(M-B)]!}{[2(M-B)-2r_1-r_2]!(2r_1)!r_2!} \tag{210}$$

$$\leqslant 2M \sum_{B_1+B_2=B} \sum_{\substack{r_1 \leqslant B_1 \\ r_2 \leqslant B_2}} \binom{B_1-1}{r_1-1}\binom{B_2-1}{r_2-1} 3^{2(M-B)}2^{2r_1}r_1!$$

$$2^{M-B-r_1}(M-B-r_1)! \tag{211}$$

$$\leqslant 2M \sum_{B_1+B_2=B} 2^{B_1-1}2^{B_2-1}9^{M-B}2^M(M-B)! \tag{212}$$

$$\leqslant 2M(B+1)18^M(M-B)! \tag{213}$$

■

4.6. Bounds

Now we can achieve the proof of lemma 8 in bounding

$$\langle \mathcal{I}_{m_0} \rangle \leqslant \sum_{\mathcal{G}} |\mathcal{A}(\mathcal{G})| \tag{214}$$

In order to compute this sum, we fix first t_j, t_k and $\bar{b}$, where $\bar{b}$ is the number of possible momentum bubbles and therefore is greater than b.

Then we define the set $\Omega(t_j, t_k, \bar{b}, n, q_1, \ldots, q_n)$ has the set of graphs with the corresponding t_j, t_k and $\bar{b}$ and for which the erased tadpoles form n sets of $2q_i$ adjacent V's. We can write

$$\langle \mathcal{I}_{m_0} \rangle \leqslant \sum_{t_j, t_k, \bar{b}} \sum_{n=1}^{t_j+t_k} \frac{1}{n!} \sum_{\substack{q_1,\ldots q_n \\ q_i \geqslant 1}} \sum_{\mathcal{G} \in \Omega(t_j, t_k, \bar{b}, n, q_1, \ldots q_n)}$$

$$\prod (q_i+1) \prod \frac{1}{q_i+1} |\mathcal{A}(\mathcal{G})| \tag{215}$$

To bound $|\mathcal{A}(\mathcal{G})|/\prod(q_i+1)$ we notice that when a graph has a bad bubble of weight p_i it means that we have erased two set of generalized tadpoles q_{1_i} and q_{2_i} on the two propagators of the bubble with $q_{1_i} + q_{2_i} = p_i$. Thus we have a corresponding factor $(q_{1_i}+1)^{-1}(q_{2_i}+1)^{-1}$ which control the bad factor $\sqrt{p_i}$ of the bad bubble so that

$$\langle \mathcal{I}_{m_0} \rangle \leqslant \sum_{t_j,t_k,\bar{b}} \sum_{n=1}^{t_j+t_k} \frac{1}{n!} \sum_{\substack{q_1,\dots q_n \\ q_i \geqslant 1}} \sum_{\mathcal{G}\in\Omega(\dots)} \prod (q_i+1) C^{m_0} M^{-m_0'(k+j)/2} M^{-jt_j-kt_k} M^{-(m_0'-\bar{b})j/6} \tag{216}$$

The number of graphs in $\Omega(\dots)$ has the following bound

$$\mathcal{N}[\Omega(\dots)] \leqslant 2^{m_0} \prod T(q_i) 3^{m_0'} C^{m_0'} (m_0'-\bar{b})! \leqslant C^{m_0} \prod \frac{2^{2q_i}}{q_i+1} (m_0'-\bar{b})! \tag{217}$$

This leads to

$$\langle \mathcal{I}_{m_0} \rangle \leqslant \sum_{t_j,t_k,\bar{b}} C^{m_0} M^{-m_0'(k+j)/2} M^{-jt_j-kt_k} M^{-(m_0'-\bar{b})j/6} (m_0'-\bar{b})! \sum_{n=1}^{t_j+t_k} \frac{1}{n!} \sum_{\substack{q_1+\dots q_n = t_j+t_k \\ q_i \geqslant 1}} 1 \tag{218}$$

$$\leqslant \sum_{t_j,t_k,\bar{b}} C^{m_0} M^{-m_0'(k+j)/2} M^{-jt_j-kt_k} M^{-(m_0'-\bar{b})j/6} (m_0'-\bar{b})! \sum_{n=1}^{t_j+t_k} \binom{t_j+t_k-1}{n-1} \tag{219}$$

$$\leqslant \sum_{t_j,t_k,\bar{b}} C^{m_0} M^{-m_0'\frac{(k+j)}{2}} M^{-jt_j-kt_k} M^{-(m_0'-\bar{b})\frac{j}{6}} (m_0'-\bar{b})! \tag{220}$$

Summing on t_j and t_k is equivalent to sum over m_0' and t_k with $t_j = m_0 - t_k - m_0'$. The sum over $\bar{b}$ is roughly evaluated by taking the supremum over $\bar{b}$, the result depends whether m_0' is greater than $M^{j/6}$ or not.

$$\langle \mathcal{I}_{m_0} \rangle \leqslant \sum_{m_0' \leqslant M^{j/6}} \sum_{t_k} C^{m_0} M^{-m_0'(k+j)/2} M^{-jt_j-kt_k} + \sum_{m_0' > M^{j/6}} \sum_{t_k} C^{m_0} M^{-m_0'(k+j)/2} M^{-jt_j-kt_k} \max\left[1, M^{-m_0' j/6} (m_0'-\bar{b})!\right] \tag{221}$$

$$\leqslant C^{m_0} M^{-m_0 j} \sum_{m_0',t_k} M^{-m_0'(k-j)/2} M^{-t_k(k-j)} \left[1 + 1_{(m_0' > M^{j/6})} M^{-m_0' j/6} (m_0'-\bar{b})!\right] \tag{222}$$

Finally, the sum over t_k is easy and we bound the sum over m_0' by finding the supremum. One can check that it gives the announced result. ■

5. CONCLUSION

Understanding the effect of perturbations on the free spectrum of Hamiltonian operators is an outstanding challenge, yet we claim that in two dimension the control of the mean Green's function at weak disorder is within reach. In the present paper we derived fine probabilistic estimates and a polymer expansion which allows to control the model up to the neighborhood of the singularity. Therefore, we must deal with the "last slice" in order to control the full model. This will be partially done in [13] where we can investigate the mean Green's function for imaginary part much smaller that the final self-energy but still finite.

In the last slice, we contract some V's and split the resulting vertices in the low momentum channel (thanks to sector conservation). In this way, either we generate graphs that are still a resolvent with low momenta insertions and therefore are small, or we detach bubbles with low ingoing momenta. We show that this is quite analogous to the control of infra-red divergences in Quantum Electrodynamics so that we can develop approximate "Ward identities" which bring small factors for these bubbles.

One can note that in dimension $d = 3$, the situation is quite different because momenta are no longer planar so that sector conservation is up to a twist. Nevertheless, the imaginary part is expected to come mostly from the planar graphs while the twisted vertices are small. Therefore, one might think to develop a similar treatment in that case.

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