ANNALES DE L'I. H. P., SECTION A

FUMIHIKO NAKANO

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Annales de l'I. H. P., section A, tome 69, nº 4 (1998), p. 441-455

<http://www.numdam.org/item?id=AIHPA_1998__69_4_441_0>

Ann. Inst. Henri Poincaré,

Vol. 69, n° 4, 1998, p. 441-455. *Physique théorique*

Calculation of the Hall conductivity by Abel limit

by

Fumihiko NAKANO

Mathematical Institute, Tohoku University

ABSTRACT. – As the rigorous justification of Bellissard's theory [B2], we define and calculate the non-diagonal component of the conductivity tensor (called the Hall conductivity) of the 2-dimensional electron system in a uniform magnetic field. Our model is the 2-dimensional Schrödinger operator with uniform magnetic and electric fields. To calculate the Hall conductivity concretely, we consider Abel limit. © Elsevier, Paris

Key words: Integer quantum hall effect, hall conductivity, relaxation time approximation, Chern character, Abel limit.

RÉSUMÉ. – Comme justification rigoureuse de la théorie de Bellissard, nous définissons et calculons la composante non-diagonale du tenseur de conductivité (appelé la conductivité Hall) d'un système électronique bi-dimensionnel dans un champ magnétique uniforme. Notre modèle est l'opérateur de Schrödinger bi-dimensionnel en champs magnétique et électrique uniformes. Pour calculer la conductivité Hall de manière concrète, nous considérons la limite d'Abel. © Elsevier, Paris

1. INTRODUCTION

In many studies of integer quantum Hall effect, the model is no interacting spinless-fermions on $\mathbf{R}^2$ or bounded regions of it submitted to uniform magnetic field perpendicular to $\mathbf{R}^2$ (e.g., [AS, ASY, B1, 2, L, TKNN] and references therein). They calculate the Hall conductivity σ_H and prove that σ_H is equal to a suitable topological invariant up to a constant; therefore, σ_H is quantized. There are, however, several ways of this calculation and several kinds of topological invariants. Among this, what is concerned here is Bellissard's theory [B2]. In [B2], there used the relaxation time approximation to calculate the Hall conductivity σ_H and proved σ_H is equal to the Chern character which is introduced by Connes in the non-commutative geometry [C]. And the purpose of this paper is to justify the relaxation time approximation on the rigorous mathematical footing.

Our Hamiltonian is one-body Schrödinger operator with uniform electric and magnetic fields on $L^2(\mathbf{R}^2)$:

$$H_{\mathcal{E},\omega} := H_\omega - \mathcal{E} \cdot x_1, \quad x = (x_1, x_2) \in \mathbf{R}^2,\ \omega \in \Omega, \tag{1.1}$$

where H_ω is the Landau Hamiltonian plus potential:

$$H_\omega := (-i\nabla - A(x))^2 + V_\omega(x). \tag{1.2}$$

The vector potential $A(x) := (-Bx_2/2, Bx_1/2)$ corresponds to the uniform magnetic field of strength $B(> 0)$ perpendicular to $\mathbf{R}^2$. Ω, which can be considered to describe the disorder configuration, is a probability space and $V_\omega(x) \in L^\infty(\mathbf{R}^2), \omega \in \Omega$ is a random potential. We assume that, for arbitrary $a \in \mathbf{R}^2$, there exists a translation $T^a : \Omega \to \Omega$ such that

$$V_{T^a\omega}(x) = V_\omega(x + a), \quad \omega \in \Omega, \quad x \in \mathbf{R}^2. \tag{1.3}$$

In fact, Ω and $V_\omega(x)$ can be constructed from a potential $V(x) \in L^\infty(\mathbf{R}^2)$ by considering all of its translations and by identifying Ω with the weak-$*$ closure of the set $\{V(x - a) : a \in \mathbf{R}^2\}$. That means to consider the all translations of the system which is needed to study the homogeneous media [B3], and (1.3) makes H_ω satisfy the covariance relation (1.12). Let $\mathbf{P}$ be a T-invariant, normalized, and ergodic measure on Ω. $\mathcal{E} \in \mathbf{R}$ is the uniform electric field along the x_1-direction enforced to particles.

Let us first consider the charge transport in the time interval $[0, S](S > 0)$ raised by the electric field. It is defined as the integral from $t = 1$ to $t = S$ of the thermal average per area of the current operator $J_\omega(t) = e^{itH_{\mathcal{E},\omega}} i[H_\omega, x] e^{-itH_{\mathcal{E},\omega}}$ at temperature $T > 0$ in the grand canonical ensemble [B2]:

$$j^\omega(T, \mathcal{E}, S) := \int_0^S dt\, \mathcal{T}\big(e^{itH_{\mathcal{E},\omega}} i[H_\omega, x] e^{-itH_{\mathcal{E},\omega}} f_{T,\epsilon_F}(H_\omega)\big), \tag{1.4}$$

where $\mathcal{T}$ is the trace per area for an operator A on $L^2(\mathbf{R}^2)$:

$$\mathcal{T}(A) := \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \mathrm{Trace}(\chi_\Lambda A \chi_\Lambda), \tag{1.5}$$

whenever it exists. In (1.5), $\Lambda = [-L, L]\times[-L, L] \subset \mathbf{R}^2, L > 0, |\Lambda| = 4L^2$ is its area, and χ_Λ is the characteristic function of Λ. $f_{T,\epsilon_F}(\lambda)$ is the Fermi-Dirac distribution function:

$$f_{T,\epsilon_F}(\lambda) := (1 + \exp((\lambda - \epsilon_F)/kT))^{-1}, \tag{1.6}$$

where $\epsilon_F \in \mathbf{R}$ is the Fermi level, k is the Boltzmann constant, and $T(> 0)$ is the temperature. The operator $e^{-itH_{\mathcal{E},\omega}}$ is the time evolution of $H_{\mathcal{E},\omega}$. The well-definedness of (1.4) is confirmed in Lemma 3.1 in section 3.

Now we define the Hall conductivity σ_H^ω, that is the charge transport along the direction perpendicular to the electric field (namely, the x_2-direction), per unit electric field, and per unit time interval. The natural definition would be:

$$\sigma_H^\omega \sim \lim_{S\uparrow\infty} \lim_{\mathcal{E}\downarrow 0, T\downarrow 0} \sigma_H^\omega(T, \mathcal{E}, S), \tag{1.7}$$

$$\sigma_H^\omega(T, \mathcal{E}, S) := \frac{1}{S}\frac{1}{\mathcal{E}} \int_0^S dt\, \mathcal{T}(e^{itH_{\mathcal{E},\omega}} i[H_\omega, x_2] e^{-itH_{\mathcal{E},\omega}} f_{T,\epsilon_F}(H_\omega)). \tag{1.8}$$

$\sim$ means that the above $S \uparrow \infty$ limit in (1.7) does not necessarily exists, though (1.8) is proved to be well-defined for $S > 0$, $\mathcal{E} > 0$ in section 3. Instead of (1.7), we consider its Abel limit and define:

$$\sigma_H^\omega := \lim_{\delta\downarrow 0,} \lim_{\mathcal{E}\downarrow 0, T\downarrow 0} \sigma_H^{\omega,\delta}(T, \mathcal{E}), \tag{1.9}$$

$$\sigma_H^{\omega,\delta}(T, \mathcal{E}) := \frac{1}{\mathcal{E}} \int_0^\infty dt\, \delta e^{-\delta t}\, \mathcal{T}(e^{itH_{\mathcal{E},\omega}} i[H_\omega, x_2] e^{-itH_{\mathcal{E},\omega}} f_{T,\epsilon_F}(H_\omega)). \tag{1.10}$$

The purpose of this paper is to prove that the RHS of (1.9) exists and is equal to the Chern character introduced by Connes. The result is:

THEOREM. – *We assume that the Fermi level ϵ_F lies in the gap of the spectrum of H_ω. When $T \downarrow 0$, $\mathcal{E} \downarrow 0$, and $\delta \downarrow 0$, $\sigma_H^{\omega,\delta}(T, \mathcal{E})$ has the following limit for $\mathbf{P}$-a.e. ω:*

$$\lim_{\delta\downarrow 0} \lim_{T\downarrow 0, \mathcal{E}\downarrow 0} \sigma_H^{\omega,\delta}(T, \mathcal{E}) = i\mathcal{T}(P_\omega[\partial_1 P_\omega, \partial_2 P_\omega] P_\omega), \tag{1.11}$$

where $P_\omega := \chi_{(-\infty,\epsilon_F)}(H_\omega)$ is the Fermi projection of H_ω and $\partial_j P_\omega := i[P_\omega, x_j]$, $j = 1, 2$.

Remarks.

1. Since the width of gaps between Landau levels are proportional to B, and V_ω belongs to $L^\infty(\mathbf{R}^2)$, the spectral gap of H_ω always exists when B is sufficiently large. But the gap assumption is undesirable when we study the quantum Hall effect and this gap assumption is needed only for mathematical technicalities (Lemma 3.3). However, it is showed that, if the localization length is finite in the neighborhood of the Fermi level, the RHS of (1.11) is constant under the small variation of the Fermi level, at least in the lattice model [B2].
2. H_ω satisfies the following covariance relation:
$$U(a)H_\omega U(a)^* = H_{T^a\omega}, \quad \omega \in \Omega, \quad a \in \mathbf{R}^2, \tag{1.12}$$
where $U(a)$ is the magnetic translation:
$$\{U(a)\psi\}(x) := e^{iBx\wedge a/2}\psi(x-a), \quad \psi \in L^2(\mathbf{R}^2) \tag{1.13}$$
This makes the spectrum of H_ω be independent of $\omega \in \Omega$.
3. the RHS of (1.11) is called the Chern-character and is proved to be equal to the Fredholm index of an operator [ASS, B2]. Thus σ_H^ω is proved to be quantized.
4. We can exchange $\lim_{T\downarrow 0}$ for $\lim_{\mathcal{E}\downarrow 0}$. But we must let $\delta \downarrow 0$ at last.
5. We can exchange the limit of $\mathcal{E} \downarrow 0$, $T \downarrow 0$ for the integral $\int_0^\infty dt$ in (1.9) and (1.10). But since the $S \uparrow \infty$ limit in (1.7) does not necessarily exists, we could not justify the Abel limit argument (this is referred to at the end of section 3).
6. The Hamiltonian in Bellissard's theory has the kick-rotor term which is not included in (1.7). The reason why we do not need the kick-rotor term is that we assume the gap hypothesis. We will see this in the proof of Lemma 3.3 in section 3. On the other hand, we can expect that the Fermi projection P_ω include the states with diverging localization length and the Hall conductivity is nonzero and finite.

The following sections are devoted to prove Theorem by justifying Bellissard's theory. In section 2, we will recall some notations and results of the operator algebra [B1, 2, 3, NB]. And prove some basic results which are often used to prove Theorem in section 3, that is, we consider the space of the "Hilbert-Schmidt" class of operators, which is called $L^2(\mathcal{A}, \mathcal{T})$ where the usual trace is replaced by the trace per area, define an anti self-adjoint operator on $L^2(\mathcal{A}, \mathcal{T})$, and study the properties of its resolvent and unitary evolution group. In section 3, we prove the well-definedness of $j^\omega(T, \mathcal{E}, S)$, $\sigma_H^\omega(T, \mathcal{E}, S)$, and $\sigma_H^{\omega,\delta}(T, \mathcal{E})$, and we study the $T \downarrow 0$, $\mathcal{E} \downarrow 0$, and $\delta \downarrow 0$ limit of $\sigma_H^{\omega,\delta}(T, \mathcal{E})$.

2. PREPARATION

At first, we shall recall some notations and results of the operator algebra [B1, 2, 3, NB]. Let $\mathcal{A}_0 := C_0(\Omega \times \mathbf{R}^2)$ be the set of continuous functions with compact support on $\Omega \times \mathbf{R}^2$ (weak$*$-topology is introduced on Ω). We define the $*$-algebra structure on $\mathcal{A}_0$:

$$AB(\omega, x) = \int_{\mathbf{R}^2} dy\, A(\omega, y)B(T^{-y}\omega, x - y)e^{\frac{iB}{2}x\wedge y}, \tag{2.1}$$

$$A^*(\omega, x) = \bar{A}(T^{-x}\omega, -x), \tag{2.2}$$

for $A, B \in \mathcal{A}_0$, $\omega \in \Omega$, $x \in \mathbf{R}^2$. For $\omega \in \Omega$, this $*$-algebra has a representation π_ω on $L^2(\mathbf{R}^2)$:

$$(\pi_\omega(A)\psi)(x) = \int_{\mathbf{R}^2} dy\, A(T^{-x}\omega, y - x)e^{\frac{iB}{2}y\wedge x}\psi(y), \tag{2.3}$$

for $A \in \mathcal{A}_0, \psi \in L^2(\mathbf{R}^2)$. It is easy to see the following properties.

1. $\pi_\omega(AB) = \pi_\omega(A)\pi_\omega(B)$, $\pi_\omega(A)^* = \pi_\omega(A^*)$.
2. $\pi_\omega(A)$ is bounded on $L^2(\mathbf{R}^2)$.
3. $\pi_\omega(A)$ satisfies the covariance relation:

$$U(a)\pi_\omega(A)U(a)^* = \pi_{T^a\omega}(A). \tag{2.4}$$

On the other hand, for an arbitrary integral operator K on $L^2(\mathbf{R}^2)$ which satisfies the covariance relation (2.4), and whose integral kernel $K(\omega; x, y)$ is continuous w.r.t. $(\omega, x, y) \in \Omega \times \mathbf{R}^2 \times \mathbf{R}^2$ and vanishes for large $|x - y|$, there is an $A \in \mathcal{A}_0$ such that $\pi_\omega(A) = K$.

We define a C^*-norm:

$$\|A\| := \sup_{\omega\in\Omega} \|\pi_\omega(A)\|_{op},$$

on $\mathcal{A}_0$, where $\|\cdot\|_{op}$ is the operator norm on $L^2(\mathbf{R}^2)$. Let $\mathcal{A} = C^*(\Omega \times \mathbf{R}^2)$ be the completion of $\mathcal{A}_0$ with respect to $\|\cdot\|$. It is well known that $(z - H_\omega)^{-1}(z \in \rho(H_\omega)), P_\omega$, and $f_{T,\epsilon_F}(H_\omega)$ belong to $\pi_\omega(\mathcal{A})$.

A trace on $\mathcal{A}_0$ is given by

$$\mathcal{T}_{\mathbf{P}}(A) := \int_\Omega d\mathbf{P}(\omega)A(\omega, 0), \quad \text{for } A \in \mathcal{A}_0. \tag{2.5}$$

By Birkhoff's ergodic theorem, $\mathcal{T}_{\mathbf{P}}(A) = \mathcal{T}(\pi_\omega(A))$ for $\mathbf{P}$-a.e.ω. Hence, for $A \in \mathcal{A}_0$, $\mathcal{T}(\pi_\omega(A))$ is finite and constant almost surely.

For $p \geq 1$, we define $L^p(\mathcal{A}, \mathcal{T})$ as the completion of $\mathcal{A}_0$ under the norm

$$||A||_{L^p} := \left(\mathcal{T}_{\mathbf{P}}(\{A^*A\}^{p/2})\right)^{1/p}, \quad \text{for } A \in \mathcal{A}_0. \tag{2.6}$$

$L^2(\mathcal{A}, \mathcal{T})$ becomes the Hilbert space under the inner product $< A|B >:= \mathcal{T}_{\mathbf{P}}(A^*B)$.

The differential structure on $\mathcal{A}_0$ is given by

$$\partial_j A(\omega, x) := ix_j A(\omega, x), \quad j = 1, 2, \quad A \in \mathcal{A}_0. \tag{2.7}$$

It corresponds to the commutator with x_j on the representation $\pi_\omega(A)$: $\pi_\omega(\partial_j A) = i[\pi_\omega(A), x_j]$ $(A \in \mathcal{A}_0, j = 1, 2)$. And moreover, we also define

$$\partial_j A := i[A, x_j], \quad j = 1, 2, \tag{2.8}$$

where A is an arbitrary operator on $L^2(\mathbf{R}^2)$, and the commutator $[A, B]$ is first defined as a form on $D(A) \cap D(B)$ and then extended as an operator. Using (2.7), we define the "Sobolev space" $\mathcal{H}^1$ which is the completion of $\mathcal{A}_0$ under the inner product

$$< A|B >_{\mathcal{H}^1} := < A|B > + < \nabla A|\nabla B >, \tag{2.9}$$

where $\nabla := (\partial_1, \partial_2)$.

Let us quote some results about the integral kernel of functions of H_ω.

LEMMA 2.1.

(1) [ASS] *If the Fermi level* ϵ_F *lies in the gap of* $\sigma(H_\omega)$, *the integral kernel* $P(\omega; x, y)$ *of* P_ω *has the exponential decay property, i.e., there exist constants* $C > 0$, $d > 0$ *such that*

$$|P(\omega; x, y)| \leq C \exp(-d|x - y|), \quad x, y \in \mathbf{R}^2, \omega \in \Omega. \tag{2.10}$$

(2) [Y] *Let* $f(x) \in \mathcal{S}$ *(the space of rapidly decreasing functions on* $\mathbf{R}$*). Then the integral kernel* $F(\omega; x, y)$ *of* $f(H_\omega)$ *has the rapidly decreasing property, i.e., for arbitrary* $N \in \mathbf{Z}$, *there exists a constant* $C_N > 0$ *such that*

$$|F(\omega; x, y)| \leq C_N(1 + |x - y|)^{-N}, \quad x, y \in \mathbf{R}^2, \omega \in \Omega. \tag{2.11}$$

The definition of the differential (2.7) and Lemma 2.1(1) immediately imply

COROLLARY 2.2. – *If the Fermi level ϵ_F lies in the gap of $\sigma(H_\omega)$, then the Chern character $\mathcal{T}(P_\omega[\partial_1 P_\omega, \partial_2 P_\omega]P_\omega)$ exists for* $\mathbf{P} - a.e.\omega$.

We should remark that there are several ways of proving Corollary 2.2 (see, e.g., [B2, NB]). Next, we define the following operator on $L^2(\mathcal{A}, \mathcal{T})$.

$$\mathcal{L}_{H_{\mathcal{E},\omega}}(B) := i[H_{\mathcal{E},\omega}, B], \quad B \in L^2(\mathcal{A}, \mathcal{T}). \tag{2.12}$$

In the rest of this section, we study some properties of $\mathcal{L}_{H_{\mathcal{E},\omega}}$ using the above set up.

LEMMA 2.3. – $\mathcal{L}_{H_{\mathcal{E},\omega}}$ *is anti self-adjoint on* $L^2(\mathcal{A}, \mathcal{T})$.

Proof. – Let $\mathcal{A}_0^\infty := C_0(\Omega; C_0^\infty(\mathbf{R}^2))$. It is easy to see $\mathcal{L}_{H_{\mathcal{E},\omega}}$ is anti-symmetric on $\mathcal{A}_0^\infty$ so that it is sufficient to show that $e^{t\mathcal{L}_{H_{\mathcal{E},\omega}}}$ is unitary and strongly continuous by Stone's theorem. The unitarity is obvious. To see the strong continuity, we write

$$e^{t\mathcal{L}_{H_{\mathcal{E},\omega}}} A = \int_0^t ds\; e^{isH_{\mathcal{E},\omega}} i[H_{\mathcal{E},\omega}, A] e^{-isH_{\mathcal{E},\omega}}, \quad A \in L^2(\mathcal{A}, \mathcal{T}) \quad \square$$

To study the properties of $\mathcal{L}_{H_{\mathcal{E},\omega}}$ further, we introduce the following closed subspace of $L^2(\mathcal{A}, \mathcal{T})$:

$$X^m := \text{closure of } \mathcal{A}_0 \text{ w.r.t. the norm } \|A\|_m,$$
$$\|A\|_m := \sum_{k=0}^{m} \|(b + H_\omega)^k A (b + H_\omega)^{m-k}\|_{L^2}, \tag{2.13}$$

for $m \geq 0 (\in \mathbf{Z})$. $b > 0$ is taken sufficiently large if necessary such that $(b + H_\omega)$ is positive. For $m < 0 (\in \mathbf{Z})$, we define $X^m := (X^{-m})^*$. We will often use the following lemma in section 3.

LEMMA 2.4. – *The followings hold for* $\mathbf{P} - a.e.\omega$.

(1) $\nabla H_\omega \in X^{-2}$.

(2) *If* $|\mathcal{E}|$ *is sufficiently small,* $(\delta - \mathcal{L}_H - \mathcal{E} \cdot \nabla)^{-1}$ *is bounded on* X^m *for any* $m \in \mathbf{Z}$ *(in this section, the electric field* $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2) \in \mathbf{R}^2$ *is the 2-dimensional vector to consider the more general situation).*

(3) *Let* $Z^n := \{A \in X^{n+1} | \nabla A \in X^n\}$ *and* $Z^\infty := \cap_{n=2}^\infty Z^n$. *Then,* $f_{T,\epsilon_F}(H_\omega)$, $\nabla f_{T,\epsilon_F}(H_\omega)$, P_ω, *and* ∇P_ω *belong to* Z^∞.

(4) *If* $A \in X^m$ *and* $B \in X^{-m}$, *then*

$$< (\delta - \mathcal{L}_{H_\omega} - \mathcal{E} \cdot \nabla)^{-1} A | B > = - < A | (\delta - \mathcal{L}_{H_\omega} - \mathcal{E} \cdot \nabla)^{-1} B >, \tag{2.14}$$

for sufficiently small $|\mathcal{E}|$, *where* $< A|B >$ *is the dual coupling between* X^m *and* X^{-m}.

(5) $e^{t\mathcal{L}_{H_{\mathcal{E},\omega}}} A$ *is uniformly bounded in* X^m *w.r.t.* $t \in \mathbf{R}$ *and* $\mathcal{E} \in \mathbf{R}^2$*(for* $|\mathcal{E}|$ *small) for* $A \in X^m$ *(*$m \leq 0$*).*

(6) *If* $A \in Z^\infty$*, then it holds that*

$$(\delta - \mathcal{L}_H - \mathcal{E}\cdot\nabla)^{-1}A - (\delta - \mathcal{L}_H)^{-1}A = (\delta - \mathcal{L}_H - \mathcal{E}\cdot\nabla)^{-1}\mathcal{E}\cdot\nabla(\delta - \mathcal{L}_H)^{-1}A, \quad (2.15)$$

for sufficiently small $|\mathcal{E}|$.

Proof. – (1) We introduce the closed subspace of $L^2(\mathbf{R}^2)$: $Y^m := \{f \in L^2(\mathbf{R}^2) | (-i\nabla - A)^m f \in L^2(\mathbf{R}^2)\}$, $m \in \mathbf{N}$, and define Y^m for $m \leq 0$ by duality. Then, for arbitrary $m \in \mathbf{Z}$, ∇H_ω is bounded from Y^{m+1} to Y^m, and $(b + H_\omega)^{-1}$ is bounded from Y^m to Y^{m+2}. It implies that $\nabla H_\omega(b + H_\omega)^{-1+\epsilon} = 2(-i\nabla - A(x))(b + H_\omega)^{-1+\epsilon}$ is bounded on $L^2(\mathbf{R}^2)$ for $\epsilon > 0$ sufficiently small. On the other hand, the integral kernel of $(b + H_\omega)^{-1-\epsilon}$ has the exponential decay property [S], thus $(b + H_\omega)^{-1-\epsilon} \in L^2(\mathcal{A}, \mathcal{T})$. Since bounded operators on $L^2(\mathbf{R}^2)$ is ideal of $L^2(\mathcal{A}, \mathcal{T})$, we obtain $\nabla H_\omega(b + H_\omega)^{-2} \in L^2(\mathcal{A}, \mathcal{T})$. Similar argument shows $(b + H_\omega)^{-2}\nabla H_\omega = (b + H_\omega)^{-2}\nabla H_\omega(b + H_\omega)^{1+\epsilon}(b + H_\omega)^{-1-\epsilon}$ and $(b + H_\omega)^{-1}\nabla(b + H_\omega)^{-1} = (b + H_\omega)^{-1}\nabla H_\omega(b + H_\omega)^{\epsilon}(b + H_\omega)^{-1-\epsilon}$ belong to $L^2(\mathcal{A}, \mathcal{T})$.

(2) By taking dual, we can assume $m \geq 0$. When $m = 0$, the real part of $(\delta - \mathcal{L}_{H_\omega} - \mathcal{E}\cdot\nabla)$ is bounded below by $\delta > 0$ since $\mathcal{L}_{H_{\mathcal{E},\omega}}$ is anti self-adjoint. When $m = 1$, we compute for $A \in X^1$:

$$(b + H_\omega)(\delta - \mathcal{L}_{H_\omega} - \mathcal{E}\cdot\nabla)A = (\delta - \mathcal{L}_{H_\omega} - \mathcal{E}\cdot\nabla)(b + H_\omega)A + \mathcal{E}\cdot(\nabla H_\omega)(b + H_\omega)^{-1}(b + H_\omega)A.$$

Therefore, there exists a constant $C > 0$ such that $\|(b+H_\omega)(\delta-\mathcal{L}_{H_\omega}-\mathcal{E}\cdot\nabla)A\|_0 \geq (\delta - C|\mathcal{E}|)\|(b + H_\omega)A\|_0$ for $|\mathcal{E}|$ sufficiently small. Similarly, we obtain $\|\{(\delta-\mathcal{L}_{H_\omega}-\mathcal{E}\cdot\nabla)A\}(b+H_\omega)\|_0 \geq (\delta-C'|\mathcal{E}|)\|A(b+H_\omega)\|_0$. When $m \geq 2$, we can also obtain $\|(\delta - \mathcal{L}_{H_\omega} - \mathcal{E}\cdot\nabla)A\|_m \geq (\delta - C_m|\mathcal{E}|)\|A\|_m$ by the same argument.

(3) We show (3) for $f_{T,\epsilon_F}(H_\omega)$ only. The proof of the statement for P_ω follows similarly. Lemma 2.1 (2) implies that $f_{T,\epsilon_F}(H_\omega) \in X^n$ and $\nabla^m((b + H_\omega)^n f_{T,\epsilon_F}(H_\omega)) \in L^2(\mathcal{A}, \mathcal{T})$ for any $n \in \mathbf{Z}$, $m \geq 0$. For $i + j = n$, $i \geq 0$, $j \geq 0$, $n \geq 2$, we compute:

$$\begin{aligned}(b + H_\omega)^i(\nabla f_{T,\epsilon_F}(H_\omega))(b + H_\omega)^j &= \nabla[(b + H_\omega)^i f_{T,\epsilon_F}(H_\omega)(b + H_\omega)^j] \\ &\quad -(\nabla(b + H_\omega)^i)(b + H_\omega)^{-i}(b + H_\omega)^i f_{T,\epsilon_F}(H_\omega)(b + H_\omega)^j \\ &\quad -(b + H_\omega)^i f_{T,\epsilon_F}(H_\omega)(b + H_\omega)^j(b + H_\omega)^{-j}(\nabla(b + H_\omega)^j). \quad (2.16)\end{aligned}$$

The second and third term of (2.16) belong to $L^2(\mathcal{A},\mathcal{T})$ since $f_{T,\epsilon_F}(H_\omega) \in X^n$ and $(\nabla(b+H_\omega)^i)(b+H_\omega)^{-i}$ is bounded on $L^2(\mathbf{R}^2)$. Thus $f_{T,\epsilon_F}(H_\omega) \in Z^\infty$. The proof for $\nabla f_{T,\epsilon_F}(H_\omega)$ is similar: we expand $\nabla^2[(b+H_\omega)^i f_{T,\epsilon_F}(H_\omega)(b+H_\omega)^j]$ and use $f_{T,\epsilon_F}(H_\omega) \in Z^\infty$.

(4) It is obvious that the equation (2.14) holds for A, $B \in \mathcal{A}_0$ by Lemma 2.3. Then it can be extended to $A \in X^{-m}$ and $B \in X^m$ by Lemma 2.4 (2).

(5) The boundedness of $e^{t\mathcal{L}_{H_{\mathcal{E},\omega}}}A$ for $A \in X^m$ $(m \leq 0)$ is easily proved by Lemma 2.4 (2) and the formula

$$e^{-itA} = s - \lim_{n\to\infty}(1 + i\frac{t}{n}A)^{-n} \tag{2.17}$$

for a self-adjoint operator A(the formula (2.17) can be extended to X^m).

(6) For $A \in Z^n (n \geq 2)$, we consider the following element of $L^2(\mathcal{A},\mathcal{T})$:

$$\begin{aligned}\mathcal{C}(A) := -i(\delta - \mathcal{L}_{H_\omega})^{-1}[\{(\delta - \mathcal{L}_{H_\omega})^{-1}A\}(\nabla H_\omega)\\ - (\nabla H_\omega)\{(\delta - \mathcal{L}_{H_\omega})^{-1}A\}.\end{aligned}$$

Since $A \in X^{n+1}, \mathcal{C}(A) \in X^n$. We notice that $[\nabla, \mathcal{L}_{H_\omega}]A = -i(A(\nabla H_\omega) - (\nabla H_\omega)A)$ and conclude $[\nabla, (\delta - \mathcal{L}_{H_\omega})^{-1}]A = \mathcal{C}(A) \in X^n$. This and the fact that $\nabla A \in X^n$ imply $\nabla(\delta - \mathcal{L}_{H_\omega})^{-1}A \in X^n$. Let $X^\infty := \cap_{n\geq 2} X^n$. Then it follows that $(\delta - \mathcal{L}_{H_\omega} - \mathcal{E}\cdot\nabla)^{-1}A \in X^\infty$, $\nabla(\delta - \mathcal{L}_{H_\omega})^{-1}A \in X^\infty$ for $A \in Z^\infty$, and

$$\begin{aligned}&(\delta - \mathcal{L}_{H_\omega} - \mathcal{E}\cdot\nabla)^{-1}A - (\delta - \mathcal{L}_{H_\omega})^{-1}A\\ &= (\delta - \mathcal{L}_{H_\omega} - \mathcal{E}\cdot\nabla)^{-1}(\delta - \mathcal{L}_{H_\omega})(\delta - \mathcal{L}_{H_\omega})^{-1}A\\ &\quad -(\delta - \mathcal{L}_{H_\omega} - \mathcal{E}\cdot\nabla)^{-1}(\delta - \mathcal{L}_{H_\omega} - \mathcal{E}\cdot\nabla)(\delta - \mathcal{L}_{H_\omega})^{-1}A\\ &\quad = (\delta - \mathcal{L}_{H_\omega} - \mathcal{E}\cdot\nabla)^{-1}\mathcal{E}\cdot\nabla(\delta - \mathcal{L}_{H_\omega})^{-1}A. \quad \square\end{aligned}$$

3. PROOF OF THEOREM

In this section, we complete the proof of Theorem using the results in section 2. From now on, we shall omit the ω-dependence of $H_{\mathcal{E},\omega}$, H_ω and P_ω for simplicity but we may not forget that the existence of $\mathcal{T}$ holds only for $\mathbf{P}$-a.e.ω in view of Birkhoff's ergodic theorem.

At first, we confirm that quantities defined in Introduction are well-defined.

LEMMA 3.1. – *For $S > 0$, $T > 0$, $\mathcal{E} > 0$, and $\delta > 0$ fixed and for $\mathbf{P}$-a.e.ω, $j(T, \mathcal{E}, S)$, $\sigma_H(T, \mathcal{E}, S)$, and $\sigma_H^\delta(T, \mathcal{E})$ defined in (1.4), (1.8) and (1.10) respectively, are all well-defined.*

Proof. – We compute:

$$
\begin{aligned}
j^{\omega}(T,\mathcal{E},S) &= \int_0^S dt\ \mathcal{T}(e^{itH_{\mathcal{E},\omega}} i[H_\omega, x] e^{-itH_{\mathcal{E},\omega}} f_{T,\epsilon_F}(H_\omega)) \\
&= \int_0^S dt\ < e^{t\mathcal{L}_{H_{\mathcal{E},\omega}}}(\nabla H_\omega) | f_{T,\epsilon_F}(H_\omega) >,
\end{aligned}
$$

and we use Lemma 2.4 (3), (5). This argument also shows the well-definedness of $\sigma_H(T,\mathcal{E},S)$ and $\sigma_H^\delta(T,\mathcal{E})$ for $S > 0, T > 0, \mathcal{E} > 0$, and $\delta > 0$ fixed and for $\mathbf{P} - a.e.\omega$. □

We proceed the calculation of $\sigma_H^\delta(T,\mathcal{E})$ to study the $\mathcal{E} \downarrow 0$, $T \downarrow 0$, and $\delta \downarrow 0$ limit.

LEMMA 3.2. – *$\sigma_H^\delta(T,\mathcal{E})$ has the following form for $\mathbf{P}$-a.e.ω.*

$$
\sigma_H^\delta(T,\mathcal{E}) =< (\delta - \mathcal{L}_H - \mathcal{E}\partial_1)^{-1}(\partial_2 H) | \partial_1 f_{T,\epsilon_F}(H) >, \tag{3.1}
$$

where $< \cdot | \cdot >$ is the dual coupling between X^{-2} and X^2.

Proof. – We use the formula:

$$
\int_0^\infty dt\ e^{-\delta t} < e^{t\mathcal{L}_H} A | B > =< (\delta - \mathcal{L}_H)^{-1} A | B >, \quad A, B \in \mathcal{A}_0, \tag{3.2}
$$

and extend to $A \in X^{-2}$ and $B \in X^2$ by usual density argument. The result is

$$
\sigma_H^\delta(T,\mathcal{E}) = \frac{1}{\mathcal{E}} \delta < (\delta - \mathcal{L}_H - \mathcal{E}\partial_1)^{-1}(\partial_2 H) | f_{T,\epsilon_F}(H) > .
$$

Then, we use Lemma 2.4 (3), (4), (6):

$$
\begin{aligned}
\sigma^\delta(T,\mathcal{E}) &= -\frac{1}{\mathcal{E}} \delta < (\partial_2 H) | (\delta - \mathcal{L}_H)^{-1} f_{T,\epsilon_F}(H) > \\
&\quad - \frac{1}{\mathcal{E}} \delta < (\partial_2 H) | (\delta - \mathcal{L}_H - \mathcal{E}\partial_1)^{-1} \mathcal{E}\partial_1 (\delta - \mathcal{L}_H)^{-1} \partial_1 f_{T,\epsilon_F}(H) > \\
&=: I + II.
\end{aligned}
$$

From the trivial equality:

$$
f_{T,\epsilon_F}(H) = \delta^{-1}(\delta - \mathcal{L}_H) f_{T,\epsilon_F}(H), \tag{3.3}
$$

it holds that $I = -\frac{1}{\mathcal{E}} < (\partial_2 H) | f_{T,\epsilon_F}(H) >$, and $I = 0$ by Proposition 3 of [B2]. As for the second term II, we use (3.3) and Lemma 2.4 (4) again. Thus we obtain (3.1). □

Next, we turn to $\mathcal{E} \downarrow 0$ and $T \downarrow 0$ limit.

LEMMA 3.3. – *For $\delta > 0$ fixed and for* $\mathbf{P}$*-a.e.ω,*

$$\lim_{\mathcal{E}\downarrow 0, T\downarrow 0} < (\delta - \mathcal{L}_H - \mathcal{E}\partial_1)^{-1}(\partial_2 H)|\partial_1 f_{T,\epsilon_F}(H) > \\ =< (\delta - \mathcal{L}_H)^{-1}(\partial_2 H)|\partial_1 P >, \quad (3.4)$$

where $\lim_{\mathcal{E}\downarrow 0}$ *and* $\lim_{T\downarrow 0}$ *can be exchanged.*

Proof. – For $\mathcal{E} \downarrow 0$ limit, we have only to use Lemma 2.4 (3), (4), (6). For $T \downarrow 0$ limit, it suffices to show

$$(b+H)^2\partial_1 f_{T,\epsilon_F}(H) \to (b+H)^2\partial_1 P, \quad \text{as } T\downarrow 0, \qquad (3.5)$$

in $L^2(\mathcal{A}, \mathcal{T})$. By assumption of Theorem, we can assume the spectral gap $\Delta \subset \mathbf{R}$ where ϵ_F lies contains an open interval $(\epsilon_F - \epsilon, \epsilon_F + \epsilon)$, $\epsilon > 0$. Let $g(x) \in C_0^\infty(\mathbf{R})$ such that $g(x) = 1$ on $\sigma(H) \cap (-\infty, \epsilon_F - \epsilon)$ and $g(x) = 0$ on $(\epsilon_F + \epsilon, +\infty)$. Then $P = g(H)$. And let $h_T(x) \in C^\infty(\mathbf{R})$ such that $h_T(x) = f_{T,\epsilon_F}(x)$ on $(-\infty, \epsilon_F - \epsilon) \cap (\epsilon_F + \epsilon, +\infty)$ and $\sup_{x\in\mathbf{R}}(1+|x|)^{\epsilon+j}\sqrt{|h_T(x) - g(x)|}^{(j)} \to 0$ as $T \downarrow 0$ ($j = 0, 1, 2$, $\epsilon > 0$ is sufficiently small). On the other hand, for $f \in C^2(\mathbf{R}^2)$, we set

$$A(f) := (b+H)^2\partial_1 f(H) \\ = \partial_1\big((b+H)^2 f(H)\big) - \{\partial_1(b+H)^2\}(b+H)^{-2}(b+H)^2 f(H), \qquad (3.6)$$

and we shall see $A(h_T - g) \to 0$ in $L^2(\mathcal{A}, \mathcal{T})$ as $T \downarrow 0$. The operator $\{\partial_1(b+H)^2\}(b+H)^{-2}$ in the second term of (multiple) is bounded on $L^2(\mathbf{R}^2)$. Without loss of generality, we can assume $h_T(x) - g(x) \geq 0$, $x \in \mathbf{R}$. Then we write

$$(b+H)^2(h_T - g)(H) = (b+H)^2(\sqrt{h_T - g})(H)(\sqrt{h_T - g})(H). \quad (3.7)$$

By Lemma 2.1, $(b+H)^2(\sqrt{h_T - g})(H)$ is uniformly bounded in $L^2(\mathcal{A}, \mathcal{T})$ in T (we notice that the constant C_N in Lemma 2.1 (2) can be taken uniformly bounded w.r.t. the small variation of $f \in \mathcal{S}$). $(\sqrt{h_T - g})(H)$ can be written as

$$(\sqrt{h_T - g})(H) = \frac{1}{2\pi i}\int_{\mathbf{C}} \partial_{\bar{z}}\sqrt{h_T - g}^{\dagger}(z)(z - H)^{-1}dz \wedge d\bar{z}, \qquad (3.8)$$

where $\sqrt{h_T - g}^{\dagger}(z)$ is an almost analytic extension of $\sqrt{h_T - g}(z)$ [JN]. The fact that when $T \downarrow 0$, $\sup_{x\in\mathbf{R}}(1+|x|)^{\epsilon+j}\sqrt{|h_T(x) - g(x)|}^{(j)} \to 0$

($j = 0, 1, 2$) implies that $\sqrt{h_T - g}(H) \to 0$ in the operator norm as $T \downarrow 0$. As for the first term of (3.6),

$$\begin{aligned}\partial_1[(b+H)^2(h_T-g)(H)] \\ = \partial_1[(b+H)\sqrt{h_T-g}(H)](b+H)\sqrt{h_T-g}(H) \\ + (b+H)\sqrt{h_T-g}(H)\partial_1[(b+H)\sqrt{(h_T-g)}(H)]. \quad (3.9)\end{aligned}$$

We can show that $\partial_1\big((b+H)\sqrt{h_T-g}(H)\big)$ is uniformly bounded in $L^2(\mathcal{A}, \mathcal{T})$ in T and $(b+H)\sqrt{h_T-g}(H) \to 0$ as $T \downarrow 0$ in the operator norm. Therefore we obtain $A(h_T - g) \to 0$ in $L^2(\mathcal{A}, \mathcal{T})$ as $T \downarrow 0$. □

By above lemmas, we have arrived at

$$\lim_{\mathcal{E}\downarrow 0, T\downarrow 0} \sigma_H^\delta(T, \mathcal{E}) =< (\delta - \mathcal{L}_H)^{-1}\partial_2 H|\partial_1 P > . \quad (3.10)$$

Finally, we study the $\delta \downarrow 0$ limit. In this limit, we follow the argument in the Appendix A of [AG].

LEMMA 3.4. – *For* $\mathbf{P}$-*a.e.*ω, *it holds that*

$$\lim_{\delta\downarrow 0} < (\delta - \mathcal{L}_H)^{-1}\partial_2 H|\partial_1 P >= i\mathcal{T}(P[\partial_1 P,\ \partial_2 P]P).$$

Proof. – We notice that $\nabla P = (1-P)(\nabla P)P + P(\nabla P)(1-P)$ since $P^2 = P$. This fact and the cyclicity of trace per area implies

$$\begin{aligned}\text{the RHS of (3.10)} =< P\{(\delta - \mathcal{L}_H)^{-1}(\partial_2 H)\}(1-P)|(\partial_1 P) > \\ + < (1-P)\{(\delta - \mathcal{L}_H)^{-1}(\partial_2 H)\}P|(\partial_1 H) > . \quad (3.11)\end{aligned}$$

We notice that $[P, (\delta - \mathcal{L}_H)^{-1}] = 0$ (we consider P as a multiplication operator on $L^2(\mathcal{A}, \mathcal{T})$) and formally, $\partial_2 H = \mathcal{L}_H x_2$:

$$\begin{aligned}< (\delta - \mathcal{L}_H)^{-1}\mathcal{L}_H(Px_2(1-P) + (1-P)x_2 P)|(\partial_1 P) > \\ =< (\delta - \mathcal{L}_H)^{-1}\mathcal{L}_H([[x_2, P], P])|(\partial_1 P) > .\end{aligned}$$

Since $[[x_2, P], P] \in (Ker\mathcal{L}_H)^\perp$, we can use the spectral theorem which implies that

$$\lim_{\delta\downarrow 0}(\delta - \mathcal{L}_H)^{-1}\mathcal{L}_H(A) = -A, \quad in\ L^2(\mathcal{A}, \mathcal{T}), \quad A \in (Ker\mathcal{L}_H)^\perp.$$

Therefore,

$$\begin{aligned}\lim_{\delta\downarrow 0} < (\delta - \mathcal{L}_H)^{-1}\partial_2 H|\partial_1 P > = - < [[x_2, P], P]|(\partial_1 P) > \\ = i\mathcal{T}(P[\partial_1 P, \partial_2 P]P).\end{aligned}$$

□

Remark. – We cannot prove Lemma 3.4 for P replaced by $f_{T,\epsilon_F}(H)$, or when $\mathcal{E} > 0$. This is the reason why $\lim_{\delta\downarrow 0}$ and $\lim_{\mathcal{E}\downarrow 0, T\downarrow 0}$ cannot be exchanged.

By Lemma 3.2, 3.3, 3.4, we have completed the proof of Theorem.

Before ending this section, we shall comment something about the limiting argument. What we have proved is:

$$\lim_{\delta\downarrow 0}\lim_{\mathcal{E}\downarrow 0, T\downarrow 0}\sigma_H^\delta(T,\mathcal{E}) = \lim_{\delta\downarrow 0}\lim_{\mathcal{E}\downarrow 0, T\downarrow 0}\int_0^\infty dt\delta e^{-\delta t}\frac{1}{\mathcal{E}}\mathcal{T}\\ (e^{itH_{\mathcal{E}}}i[H,x_2]e^{-itH_{\mathcal{E}}}f_{T,\epsilon_F}(H))\\ = i\mathcal{T}(P[\partial_1 P,\ \partial_2 P]P).$$

We shall confirm that the exchange of $\lim_{\mathcal{E}\downarrow 0, T\downarrow 0}$ for $\int_0^\infty dt$ is permitted.

LEMMA 3.5. – *For $\delta > 0$ and for $\mathbf{P}$-a.e.ω, it holds that*

$$\lim_{\mathcal{E}\downarrow 0, T\downarrow 0}\int_0^\infty dt\ \delta e^{-t\delta}\frac{1}{\mathcal{E}}\mathcal{T}(e^{itH_{\mathcal{E}}}i[H,x_2]e^{-itH_{\mathcal{E}}}f_{T,\epsilon_F}(H))\\ = \int_0^\infty dt\ \delta e^{-t\delta}\lim_{\mathcal{E}\downarrow 0, T\downarrow 0}\frac{1}{\mathcal{E}}\mathcal{T}(e^{itH_{\mathcal{E}}}i[H,x_2]e^{-itH_{\mathcal{E}}}f_{T,\epsilon_F}(H)).$$

Proof. – We will show

$$\frac{1}{\mathcal{E}}\mathcal{T}(e^{itH_{\mathcal{E}}}i[H,x_2]e^{-itH_{\mathcal{E}}}f_{T,\epsilon_F}(H)) = O(t) \quad \text{as} \quad t\to\infty, \tag{3.12}$$

uniformly w.r.t. $\mathcal{E}\in\mathbf{R}$ small, and $T > 0$. Then Lemma 3.5 obeys by the dominated convergence theorem. We compute:

$$\text{the LHS of (3.12)} = \frac{1}{\mathcal{E}}\mathcal{T}(e^{itH_{\mathcal{E}}}(\partial_2 H)f_{T,\epsilon_F}(H)e^{-itH_{\mathcal{E}}})\\ + \frac{1}{\mathcal{E}}\mathcal{T}(e^{itH_{\mathcal{E}}}(\partial_2 H)[e^{-itH_{\mathcal{E}}}, f_{T,\epsilon_F}(H)]). \tag{3.13}$$

The first term of the RHS of (3.13) vanishes due to Proposition 3 of [B2]. We compute the commutator in the second term:

$$[e^{-itH_{\mathcal{E}}}, f_{T,\epsilon_F}(H)] = (e^{-itH_{\mathcal{E}}}f_{T,\epsilon_F}(H)e^{itH_{\mathcal{E}}} - f_{T,\epsilon_F}(H))e^{-itH_{\mathcal{E}}}\\ = \int_0^t ds\ e^{-isH_{\mathcal{E}}}i[f_{T,\epsilon_F}(H), H_{\mathcal{E}}]e^{isH_{\mathcal{E}}}e^{-itH_{\mathcal{E}}}\\ = -\int_0^t ds\ e^{-isH_{\mathcal{E}}}\mathcal{E}(\partial_1 f_{T,\epsilon_F}(H))e^{isH_{\mathcal{E}}}e^{-itH_{\mathcal{E}}}. \tag{3.14}$$

Thus,

$$\text{the LHS of } (3.12) = -\int_0^t ds\, \mathcal{T}((\partial_2 H)e^{-isH_{\mathcal{E}}}(\partial_1 f_{T,\epsilon_F}(H))e^{isH_{\mathcal{E}}}) \\ = -\int_0^t ds\, < e^{s\mathcal{L}_{H_{\mathcal{E}}}}(\partial_2 H)|\partial_1 f_{T,\epsilon_F}(H) >,$$

and from Lemma 2.4 (1),(3),(5), we obtain (3.12). □

Remark. – (3.12) implies that $\lim_{S\uparrow\infty}\lim_{\mathcal{E}\downarrow 0,T\downarrow 0}\sigma_H(T,\mathcal{E},S)$ in (1.7) does not necessarily exists since we cannot exclude the possibility that the LHS of (3.12) may oscillate for large $t > 0$. This is the reason why we could not justify the Abel limit argument.

ACKNOWLEDGMENTS

The author wishes to thank K. Yajima, J. Bellissard, S. Nakamura, T. Koma, T. Adachi, and M. Kaminaga for their useful discussions and valuable comments.

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(Manuscript received July 8th, 1996.)