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Decay of spatial correlations in thermal states

by

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ABSTRACT. – We study the cluster properties of thermal equilibrium states in theories with a maximal propagation velocity (such as relativistic QFT). Our analysis, carried out in the setting of algebraic quantum field theory, shows that there is a tight relation between spectral properties of the generator of time translations and the decay of spatial correlations in thermal equilibrium states, in complete analogy to the well understood case of the vacuum state. © Elsevier, Paris

Key words: Clustertheorem, thermofield theory, modular theory, correlations, KMS condition.

RÉSUMÉ. – Nous étudions la propriété de cluster des états thermaux d'équilibre dans des théories possédant une vitesse maximale de propagation (comme les théories des champs relativistes). Notre analyse, menée dans le cadre de la théorie algébrique des champs, montre qu'il existe une relation étroite entre les propriétés spectrales du générateur des translations temporelles et la décroissance des corrélations spatiales pour les états thermaux d'équilibre, en complète analogie avec le cas bien connu de l'état de vide. © Elsevier, Paris

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1. INTRODUCTION

The cluster theorem of relativistic quantum field theory states that correlations of local observables in the vacuum state decrease at least like δ^{-2} if δ is their space-like separation [AHR]. In the presence of a mass-gap one has even an exponential decay like $e^{-M\delta}$, where M is the minimal mass in the theory [AHR], [F].

Little is known about the decay of correlations in the case of thermal equilibrium states. But a few remarks are in order: a) The spectrum of the generators of translations is all of $\mathbb{R}^4$, i.e., one cannot distinguish between theories with short respectively long range forces by looking at the shape of the spectrum. Yet, as we shall see, spectral properties of the generators are still of importance. b) The decay of correlations may be weaker than in the vacuum case. Simple examples are KMS states in the theory of free massless bosons, where correlations between space-like separated observables decay only like δ^{-1}. c) Due to the KMS condition, which is characterizing for equilibrium states, the main contribution to the spatial correlations comes from low energy excitations; high energy excitations are suppressed in an equilibrium state. The KMS condition paves the way for a model independent analysis: If we have some information on the spectral properties of the generator $\mathbf{H}_\beta$ of the time evolution in the thermal representation, then we can specify the decay of spatial correlation functions.

We start with a list of assumptions and properties relevant for the thermal states of a local quantum field theory. We refer to [H], [HK] for further discussion and motivation.

(i) *(Local observables)*. The central object in the mathematical description of a theory is an inclusion preserving map

$$\mathcal{O} \to \mathcal{A}(\mathcal{O}), \tag{1}$$

which assigns to any open bounded region $\mathcal{O}$ in Minkowski space $\mathbb{R}^4$ a unital C^*-algebra. The Hermitian elements of $\mathcal{A}(\mathcal{O})$ are interpreted as the observables which can be measured at times and locations in $\mathcal{O}$. Thus the net is *isotonous*,

$$\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2). \tag{2}$$

Isotony allows one to embed $\mathcal{A}(\mathcal{O})$ in the *algebra of quasilocal observables*

$$\mathcal{A} = \overline{\bigcup_{\mathcal{O} \subset \mathbb{R}^4} \mathcal{A}(\mathcal{O})}^{C^*} . \tag{3}$$

(ii) *(Einstein-Causality or Locality).* The net $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ is local: Let $\mathcal{O}_1$ and $\mathcal{O}_2$ denote two arbitrary space-like separated open, bounded regions. Then

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)' \quad \text{for} \quad \mathcal{O}_1 \subset \mathcal{O}_2', \tag{4}$$

where $\mathcal{O}'$ denotes the space-like complement of $\mathcal{O}$. Here $\mathcal{A}(\mathcal{O})'$ denotes the set of operators in $\mathcal{A}$ which commute with all operators in $\mathcal{A}(\mathcal{O})$. Thus

$$[a, b] := ab - ba = 0 \tag{5}$$

for all $a \in \mathcal{A}(\mathcal{O}_1)$, $b \in \mathcal{A}(\mathcal{O}_2)$.

(iii) *(Time-translations).* The time evolution acts as a strongly continuous group of automorphisms on $\mathcal{A}$, and it respects the local structure, i.e.,

$$\tau_t(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O} + te), \tag{6}$$

for all $t \in \mathbb{R}$. Here e is a unit vector denoting the time direction with respect to a given Lorentz-frame.

We now recall how equilibrium states are characterised within the set of states [BR], [E], [S], [T], [R]. Heuristically, one expects an equilibrium state to have the following properties: a) Time-evolution invariance. b) Stability against small adiabatic perturbations. Together with a few (physically motivated) technical assumptions, these properties lead [HKTP] to the subsequent criterion [HHW], named after Kubo [Ku], Martin and Schwinger [MS]†

(iv) *(KMS-condition).* A state ω_β satisfies the KMS-condition at inverse temperature $\beta > 0$ iff for every pair of elements $a, b \in \mathcal{A}$ there exists an analytic function $F_{a,b}$ in the strip $S_\beta = \{z \in \mathbb{C} | 0 < \Im z < \beta\}$ with continuous boundary values at $\Im z = 0$ and $\Im z = \beta$ given by

$$F_{a,b}(t) = \omega_\beta(b\tau_t(a)), \quad F_{a,b}(t + i\beta) = \omega_\beta(\tau_t(a)b), \qquad \forall t \in \mathbb{R}. \tag{7}$$

We recall that given a KMS state ω_β, the GNS-construction provides a representation π_β of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\beta$, together with a unit vector such that $\omega_\beta(a) = (\Omega_\beta, \pi_\beta(a)\Omega_\beta)$, for all $a \in \mathcal{A}$. Moreover, Ω_β is cyclic and separating for $\pi_\beta(\mathcal{A})''$. If ω_β is locally normal w.r.t. the vacuum, then

† Recently it was shown in [BB] that in a relativistic theory the thermal correlation functions have stronger analyticity properties in configuration space than those imposed by the KMS condition. These analyticity properties may be understood as a remnant of the relativistic spectrum condition in the vacuum sector and lead to a Lorentz-covariant formulation of the KMS-condition.

the KMS condition implies that Ω_β is cyclic even for the strictly local algebras $\pi_\beta(\mathcal{A}(\mathcal{O}))$, i.e.,

$$\mathcal{H}_\beta = \overline{\pi_\beta(\mathcal{A}(\mathcal{O}))\Omega_\beta}, \tag{8}$$

for arbitrary open space-time regions $\mathcal{O} \in \mathbb{R}^4$ [J]. The function $F_{1,b}(z)$, $z \in \mathcal{S}_\beta$ satisfies $F_{1,b}(t) = F_{1,b}(t + i\beta)$. Thus $F_{1,b}(t)$ extends to a periodic, bounded entire function. By Liouville's theorem $F_{1,b}(z)$ is constant, $\omega_\beta \circ \tau_t = \omega_\beta$. Thus

$$U(t)\pi_\beta(a)\Omega_\beta = \pi_\beta(\tau_t(a))\Omega_\beta, \qquad \forall a \in \mathcal{A}, \tag{9}$$

defines a strongly continuous one-parameter group of unitary operators $\{\mathbf{U}(t)\}_{t\in\mathbb{R}}$ implementing the time-evolution. By Stone's theorem there exists a self-adjoint generator $\mathbf{H}_\beta$ such that $\mathbf{U}(t) = e^{i\mathbf{H}_\beta t}$ for all $t \in \mathbb{R}$. For $0 \le \beta < \infty$ the operator $\mathbf{H}_\beta$ is not bounded from below, its spectrum is symmetric and consists typically of the whole real line. Restricting attention to pure phases we assume that Ω_β is the unique — up to a phase — time-invariant vector in $\mathcal{H}_\beta$.

2. ANALYTIC PROPERTIES OF THERMAL CORRELATION FUNCTIONS

An equilibrium state distinguishes a rest frame [N] [O] and thereby destroys relativistic covariance. But the key feature of a relativistic theory, known as "Nahwirkungsprinzip" or locality, survives: Given two regions $\mathcal{O}_1$ and $\mathcal{O}_2$, separated by a space-like distance $\delta > 0$ such that $\mathcal{O}_1 + te \subset \mathcal{O}_2$, $|t| < \delta$, the commutator of two local observables $a \in \mathcal{A}(\mathcal{O}_1)$, $b \in \mathcal{A}(\mathcal{O}_2)'$ vanishes for $|t| < \delta$, i.e.,

$$[a, \tau_t(b)] = 0, \tag{10}$$

for $|t| < \delta$. Thus locality together with the KMS condition implies that the function

$$F_{a,b}: t \to \omega_\beta(b\tau_t(a)) \tag{11}$$

can be analytically continued into the infinitely often cut plane

$$\mathcal{I}_\delta = \mathbb{C}\backslash\{z \in \mathbb{C} | \Im z = k\beta, k \in \mathbb{Z}, |\Re z| \ge \delta\}. \tag{12}$$

Set $\mathbf{A} = \pi_\beta(a)$, $\mathbf{B} \in \pi_\beta(b)$, then $F_{a,b}: \mathcal{I}_\delta \to \mathbb{C}$ is given by

$$F_{a,b}(t+i\alpha) := \begin{Bmatrix} (\Omega_\beta, \mathbf{B}e^{i(t+i\alpha-i\beta)\mathbf{H}_\beta}\mathbf{A}\Omega_\beta), \\ (\Omega_\beta, \mathbf{B}e^{i(t+i\alpha)\mathbf{H}_\beta}\mathbf{A}\Omega_\beta), \end{Bmatrix} \text{ for } \begin{Bmatrix} \beta < \alpha < 2\beta, \\ 0 \le \alpha \le \beta, \end{Bmatrix} \tag{13}$$

and periodic extension in $\Im z$ with period 2β.

The periodic structure of the cuts can now be explored† . Let us consider the spectral projections $\mathbf{P}^+$, $\mathbf{P}^-$ and $\mathbf{P}_\beta = |\Omega_\beta)(\Omega_\beta|$ onto the strictly positive, the strictly negative, and the discrete spectrum $\{0\}$ of $\mathbf{H}_\beta$. The function $f_+: \{z \in \mathbb{C} | \Im z \ge 0\} \to \mathbb{C}$,

$$f_+(z) = (\Omega_\beta\,,\, \mathbf{B}e^{iz\mathbf{H}_\beta}\mathbf{P}^+\mathbf{A}\Omega_\beta) - (\Omega_\beta\,,\, \mathbf{A}e^{-iz\mathbf{H}_\beta}\mathbf{P}^-\mathbf{B}\Omega_\beta) \tag{14}$$

is analytic in the upper half plane $\Im z > 0$, and continuous for $\Im z \searrow 0$, the function $f_-: \{z \in \mathbb{C} | \Im z \le 0\} \to \mathbb{C}$,

$$f_-(z) = -(\Omega_\beta\,,\, \mathbf{B}e^{iz\mathbf{H}_\beta}\mathbf{P}^-\mathbf{A}\Omega_\beta) + (\Omega_\beta\,,\, \mathbf{A}e^{-iz\mathbf{H}_\beta}\mathbf{P}^+\mathbf{B}\Omega_\beta) \tag{15}$$

is analytic in the lower half plane $\Im z < 0$, and continuous for $\Im z \nearrow 0$. The discrete spectral value $\{0\}$ gives no contribution, $f_{\{0\}}: \mathbb{C} \to \mathbb{C}$,

$$\begin{aligned} f_{\{0\}}(z) &= (\Omega_\beta\,,\, \mathbf{B}e^{iz\mathbf{H}_\beta}\Omega_\beta)(\Omega_\beta\,,\, \mathbf{A}\Omega_\beta) - (\Omega_\beta\,,\, \mathbf{A}e^{-iz\mathbf{H}_\beta}\Omega_\beta)(\Omega_\beta\,,\, \mathbf{B}\Omega_\beta) \\ &= 0, \qquad \forall z \in \mathbb{C}. \end{aligned} \tag{16}$$

Thus we can decompose the commutator $\omega_\beta([b\,,\, \tau_t(a)])$ into two pieces:

$$(\Omega_\beta, \pi_\beta([b\,,\, \tau_t(a)])\Omega_\beta) = f_+(t) - f_-(t) \qquad \forall t \in \mathbb{R}. \tag{17}$$

The l.h.s. vanishes for $|t| < \delta$, i.e., the boundary values of the function defined in (14) and (15) (from the upper and the lower half plane, respectively) coincide for $|\Re z| < \delta$. Using the Edge-of-the-Wedge Theorem [SW] one concludes that there is a function $f_{a,b}$ which is analytic on the twofold cut plane $\mathcal{P}_\delta = \mathbb{C} \backslash \{z \in \mathbb{C} | \Im z = 0, |\Re z| \ge \delta\}$ such that

$$f_{a,b}(z) = \begin{Bmatrix} (\Omega_\beta, \mathbf{B}e^{iz\mathbf{H}_\beta}\mathbf{P}^+\mathbf{A}\Omega_\beta) - (\Omega_\beta, \mathbf{A}e^{-iz\mathbf{H}_\beta}\mathbf{P}^-\mathbf{B}\Omega_\beta) \\ (\Omega_\beta, \mathbf{A}e^{-iz\mathbf{H}_\beta}\mathbf{P}^+\mathbf{B}\Omega_\beta) - (\Omega_\beta, \mathbf{B}e^{iz\mathbf{H}_\beta}\mathbf{P}^-\mathbf{A}\Omega_\beta) \end{Bmatrix}$$

† The author acknowledges stimulating discussions with J. Bros on this point, the present solution is due to D. Buchholz.

$$\text{for}\left\{\begin{matrix} \Im z > 0, \\ \left\{\begin{matrix} \Im z = 0, |\Re z| < \delta, \\ \Im z < 0. \end{matrix}\right\} \end{matrix}\right\} \tag{18}$$

Since $\mathbf{P}^+ \notin \pi_\beta(\mathcal{A})$ the KMS condition does not apply. But we can approximate $\mathbf{P}^+\mathbf{A}\Omega_\beta$ with the help of analytic elements $\mathbf{A}_\epsilon \in \pi_\beta(\mathcal{A}_\tau)$ such that♯

$$\lim_{\epsilon\to 0} \mathbf{A}_\epsilon\Omega_\beta = \mathbf{P}^+\mathbf{A}\Omega_\beta, \qquad \lim_{\epsilon\to 0} \mathbf{A}_\epsilon^*\Omega_\beta = \mathbf{P}^-\mathbf{A}^*\Omega_\beta. \tag{19}$$

Set

$$\mathbf{A}_\epsilon = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}t\, g_\epsilon(t)\, \mathrm{e}^{i\mathbf{H}_\beta t}\mathbf{A}\mathrm{e}^{-i\mathbf{H}_\beta t}, \qquad \forall \mathbf{A} \in \pi_\beta(\mathcal{A}), \tag{20}$$

where $g_\epsilon \in S(\mathbb{R})$ are complex test functions with uniformly bounded Fourier transform and

$$\tilde{g}_\epsilon(\nu) = \left\{\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right\} \text{ for } \left\{\begin{matrix} \nu \in (-\infty, \epsilon/2], \\ \nu \in [\epsilon, 1/\epsilon] \\ \nu \in [\epsilon + 1/\epsilon, \infty). \end{matrix}\right\} \tag{21}$$

The Bochner integral (20) exists* for all $\epsilon > 0$ and $\mathbf{A}_\epsilon \in \pi_\beta(\mathcal{A}_\tau)$. The spectral resolution of $\mathbf{H}_\beta$ implies

$$\mathbf{A}_\epsilon\Omega_\beta = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}t\, g_\epsilon(t)\, \mathrm{e}^{i\mathbf{H}_\beta t}\mathbf{A}\Omega_\beta = \tilde{g}_\epsilon(\mathbf{H}_\beta)\mathbf{A}\Omega_\beta. \tag{22}$$

The sequence $\tilde{g}_\epsilon$ converges uniformly on compact sets in $\mathbb{R}\backslash\{0\}$ to the Heaviside step-function. By assumption, the Fourier transforms $\tilde{g}_\epsilon$ are uniformly bounded. Thus the spectral theorem yields $\lim_{\epsilon\to 0} \|\tilde{g}_\epsilon(\mathbf{H}_\beta)\mathbf{A}\Omega_\beta - \mathbf{P}^+\mathbf{A}\Omega_\beta\| = 0$, for all $\mathbf{A} \in \pi_\beta(\mathcal{A})$. For $\mathbf{A}_\epsilon^*\Omega_\beta$ we find

$$\mathbf{A}_\epsilon^*\Omega_\beta = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}t\, \overline{g_\epsilon}(t)\, \mathrm{e}^{i\mathbf{H}_\beta t}\mathbf{A}^*\Omega_\beta = \tilde{g}_\epsilon(-\mathbf{H}_\beta)\mathbf{A}^*\Omega_\beta, \tag{23}$$

and $\lim_{\epsilon\to 0} \|\tilde{g}_\epsilon(-\mathbf{H}_\beta)\mathbf{A}^*\Omega_\beta - \mathbf{P}^-\mathbf{A}^*\Omega_\beta\| = 0$. We can now exploit the KMS condition so as to obtain

$$\begin{aligned} f_{a,b}(z) &= \lim_{\epsilon\to 0}\Big((\Omega_\beta\,, \mathbf{B}\mathrm{e}^{iz\mathbf{H}_\beta}\mathbf{A}_\epsilon\Omega_\beta) - (\mathrm{e}^{i\bar{z}\mathbf{H}_\beta}\mathbf{A}_\epsilon^*\Omega_\beta\,, \mathbf{B}\Omega_\beta)\Big) \\ &= \lim_{\epsilon\to 0}(\Omega_\beta\,, \mathbf{B}(\mathbb{1} - \mathrm{e}^{-\beta\mathbf{H}_\beta})\mathrm{e}^{iz\mathbf{H}_\beta}\mathbf{A}_\epsilon\Omega_\beta) \\ &= (\Omega_\beta\,, \mathbf{B}(\mathbb{1} - \mathrm{e}^{-\beta\mathbf{H}_\beta})\mathrm{e}^{iz\mathbf{H}_\beta}\mathbf{P}^+\mathbf{A}\Omega_\beta), \end{aligned} \tag{24}$$

♯ Here $\mathcal{A}_\tau$ denotes the set of entire analytic elements for τ [BR, 2.5.20]. Note that $\mathcal{A}_\tau$ is a norm dense *-subalgebra of $\mathcal{A}$ [BR, 2.5.22].

* Since τ_t is strongly continuous and $g_\epsilon \in S(\mathbb{R})$, there exists a sequence of countably valued functions $t \to a_n(t)$, $n \in \mathbb{N}$ converging almost everywhere to $t \to g_\epsilon(t)\tau_t(a)$.

for $\Im z > 0$. A similar argument holds for $\Im z < 0$, thus

$$f_{a,b}(z) = \begin{Bmatrix} (\Omega_\beta \, , \, \mathbf{B}(\mathbb{1} - \mathrm{e}^{-\beta \mathbf{H}_\beta}) \mathrm{e}^{iz\mathbf{H}_\beta} \mathbf{P}^+ \mathbf{A}\Omega_\beta) \\ (\Omega_\beta \, , \, \mathbf{B}(\mathrm{e}^{-\beta \mathbf{H}_\beta} - \mathbb{1}) \mathrm{e}^{iz\mathbf{H}_\beta} \mathbf{P}^- \mathbf{A}\Omega_\beta) \end{Bmatrix}$$

$$\text{for} \begin{Bmatrix} \Im z > 0, \\ \begin{Bmatrix} \Im z = 0, |\Re z| < \delta, \\ \Im z < 0. \end{Bmatrix} \end{Bmatrix} \tag{25}$$

The function $f_{a,b}$ is closely related to the original function $\omega_\beta(b\tau_t(a))$. The following result expresses $\omega_\beta(b\tau_t(a))$ for $|t| < \delta$ as a finite sum of values of the function $f_{a,b}(z)$ evaluated at $z = t + il\beta$ with $l \in \mathbb{Z}$.

LEMMA 2.1. – *Let ω_β denote a (τ, β)-KMS state, and let Ω_β be the unique — up to a phase — time-invariant vector in $\mathcal{H}_\beta$. Then, for $a \in \mathcal{A}(\mathcal{O}_1)$, $b \in \mathcal{A}(\mathcal{O}_2)$ and $|t| < \delta$,*

$$\begin{aligned} \omega_\beta(b\tau_t(a)) - \omega_\beta(b)\omega_\beta(a) = \sum_{l=0}^{n-1} f_{a,b}(t + il\beta) + \sum_{l=1}^{n} f_{a,b}(t - il\beta) \\ + (\Omega_\beta, \pi_\beta(b)\mathrm{e}^{-n\beta\mathbf{H}_\beta}\mathbf{P}^+\pi_\beta(\tau_t(a))\Omega_\beta) \\ + (\Omega_\beta, \pi_\beta(b)\mathrm{e}^{n\beta\mathbf{H}_\beta}\mathbf{P}^-\pi_\beta(\tau_t(a))\Omega_\beta), \end{aligned} \tag{26}$$

holds for all $n \in \mathbb{N}$.

Proof. – Set $\pi_\beta(a) = \mathbf{A}$, $\pi_\beta(b) = \mathbf{B}$. We decompose $\mathbb{1}$ into $\mathbf{P}^+$, $\mathbf{P}^-$, and $\mathbf{P}_\beta = |\Omega_\beta)(\Omega_\beta|$,

$$\omega_\beta(b\tau_t(a)) = \omega_\beta(b)\omega_\beta(a) + (\Omega_\beta, \mathbf{BP}^+\mathrm{e}^{it\mathbf{H}_\beta}\mathbf{A}\Omega_\beta) + (\Omega_\beta, \mathbf{BP}^-\mathrm{e}^{it\mathbf{H}_\beta}\mathbf{A}\Omega_\beta). \tag{27}$$

Lemma 2.1 follows by iteration of the identities

$$\begin{aligned} (\Omega_\beta, \mathbf{BP}^+\mathrm{e}^{it\mathbf{H}_\beta}\mathbf{A}\Omega_\beta) = (\Omega_\beta, \mathbf{B}(\mathbb{1} - \mathrm{e}^{-\beta\mathbf{H}_\beta})\mathbf{P}^+\mathrm{e}^{it\mathbf{H}_\beta}\mathbf{A}\Omega_\beta) \\ + (\Omega_\beta, \mathbf{B}\mathrm{e}^{-\beta\mathbf{H}_\beta}\mathbf{P}^+\mathrm{e}^{it\mathbf{H}_\beta}\mathbf{A}\Omega_\beta), \end{aligned} \tag{28}$$

and

$$\begin{aligned} (\Omega_\beta, \mathbf{BP}^-\mathrm{e}^{it\mathbf{H}_\beta}\mathbf{A}\Omega_\beta) = (\Omega_\beta, \mathbf{B}(\mathrm{e}^{-\beta\mathbf{H}_\beta} - \mathbb{1})\mathrm{e}^{(it+\beta)\mathbf{H}_\beta}\mathbf{P}^-\mathbf{A}\Omega_\beta) \\ + (\Omega_\beta, \mathbf{B}\mathrm{e}^{(it+\beta)\mathbf{H}_\beta}\mathbf{P}^-\mathbf{A}\Omega_\beta) \end{aligned} \tag{29}$$

from relation (25). Both identities hold for $|t| < \delta$. □

Bounds for $|f_{a,b}(il\beta)|$, $l \in \mathbb{Z}$ will be derived from

LEMMA 2.2. – *Let* $\mathcal{Q} = \{z \in \mathbb{C} \mid |\Re z| < \delta, |\Im z| < \delta\}$ *be an open square centered at the origin. Let* $f: \mathcal{Q} \to \mathbb{C}$ *be a bounded function analytic in* $\mathcal{Q}$ *and assume that there exists numbers* $C_1 > 0$ *and* $m > 0$ *such that for* $\Im z \neq 0$, *we have*

$$|f(z)| \leq C_1 |\Im z|^{-m}. \tag{30}$$

Let $r \in \mathbb{R}$, $|r| \leq \delta$. *Then*

$$|f(ir)| \leq C_1 \Big(\frac{\delta}{\sqrt{5}}\Big)^{-m}. \tag{31}$$

Proof. – The function

$$g(z) = f(z)(\delta + z)^m (\delta - z)^m \tag{32}$$

is analytic in the open square $\mathcal{Q}$ and continuous at the boundary of the square $\partial\mathcal{Q}$. Hence assumption (30) implies bounds for $g(z)$ at the boundary of the square:

$$|g(z)| \leq C_1 (\sqrt{5}\delta)^m, \qquad z \in \partial\mathcal{Q}. \tag{33}$$

By the maximum modulus principle these bounds also hold inside the square. Finally $f(ir) = g(ir)(\delta^2 + r^2)^{-m}$ for $r \in \mathbb{R} \cap \mathcal{Q}$ and (31) follows from (33). □

3. DECAY OF SPATIAL CORRELATIONS

In the last section we have seen that locality together with the KMS condition implies that the function

$$t \to \omega_\beta(b\tau_t(a)) - \omega_\beta(b)\omega_\beta(a) \tag{34}$$

can be analytically continued into the infinitely often cut plane

$$\mathcal{I}_\delta = \mathbb{C} \backslash \{z \in \mathbb{C} \mid \Im z = k\beta, k \in \mathbb{Z}, |\Re z| \geq \delta\}. \tag{35}$$

For $|t| < \delta$ the function (34) can be written as an infinite sum,

$$\omega_\beta(b\tau_t(a)) - \omega_\beta(b)\omega_\beta(a) = \sum_{l=-\infty}^{\infty} f_{a,b}(t + il\beta), \tag{36}$$

involving the function $f_{a,b}$, which is analytic on the twofold cut plane

$$\mathcal{P}_\delta = \mathbb{C}\backslash\{z \in \mathbb{C} | \Im z = 0, |\Re z| \geq \delta\}. \tag{37}$$

In fact, Lemma 2.1 even provides an explicit expression for the remainder, if we prefer to deal with a finite sum:

$$\begin{aligned}\Big|\omega_\beta(ba) - \omega_\beta(b)\omega_\beta(a)\Big| &\leq \sum_{l=-n}^{n-1} |f_{a,b}(il\beta)| \\ &+ \|e^{-\frac{n\beta}{2}\mathbf{H}_\beta}\mathbf{P}^+\pi_\beta(b^*)\Omega_\beta\| \, \|e^{-\frac{n\beta}{2}\mathbf{H}_\beta}\mathbf{P}^+\pi_\beta(a)\Omega_\beta\| \\ &+ \|e^{\frac{n\beta}{2}\mathbf{H}_\beta}\mathbf{P}^-\pi_\beta(b^*)\Omega_\beta\| \, \|e^{\frac{n\beta}{2}\mathbf{H}_\beta}\mathbf{P}^-\pi_\beta(a)\Omega_\beta\|, \end{aligned} \tag{38}$$

for all $n \in \mathbb{N}$. Bounds for $|f_{a,b}(il\beta)|$, $l \in \mathbb{Z}$, will be derived from information on the energy level density of the local excitations of a KMS state. As expected, the decay of spatial correlations depends on the infrared properties of the model and the essential ingredients for the following theorem are the continuity properties of the spectrum of $\mathbf{H}_\beta$ near zero.

For most applications it is sufficient to consider the following geometric situation: Let $\mathcal{O} \subset \mathbb{R}^4$ denote an open and bounded space-time region. Consider two space-like separated space time regions $\mathcal{O}_1$, $\mathcal{O}_2$, which can be embedded into $\mathcal{O}$ by translation; i.e., $\mathcal{O}_1 + te \subset \mathcal{O}_2'$ for all $|t| \leq \delta$, and $\mathcal{O}_i + x_i \subset \mathcal{O}$, $i = 1, 2$, for some $x_i \in \mathbb{R}^4$. Under these geometric assumptions the following result holds:

THEOREM 3.1. – *Let Ω_β denote the unique — up to a phase — normalized eigenvector with eigenvalue $\{0\}$ of $\mathbf{H}_\beta$ in the GNS representation $(\mathcal{H}_\beta, \pi_\beta, \Omega_\beta)$ associated with a (τ, β)-KMS state. If there exist positive constants $m > 0$ and $C_1(\mathcal{O})$ such that*

$$\|e^{-\frac{\lambda}{2}|\mathbf{H}_\beta|}(\pi_\beta(a) - \omega_\beta(a)\mathbb{1})\Omega_\beta\| \leq C_1(\mathcal{O})\lambda^{-m}\,\|a\|, \quad \forall a \in \mathcal{A}(\mathcal{O}), \tag{39}$$

holds, then the correlations of two local observables $a \in \mathcal{A}(\mathcal{O}_1)$, $b \in \mathcal{A}(\mathcal{O}_2)$, — $\mathcal{O}_1$, $\mathcal{O}_2$ as described above — decrease like

$$|\omega_\beta(ba) - \omega_\beta(b)\omega_\beta(a)| \leq C_2(\beta, \mathcal{O})(k\beta)^{-2m}\|a\|\,\|b\|, \tag{40}$$

for $\delta > \beta$ with $k < \delta/\beta \leq (k+1)$, $k \in \mathbb{N}$. The constant $C_2(\beta, \mathcal{O}) \in \mathbb{R}^+$ is independent of δ, a and b.

Remark. – For δ sufficiently large compared to the inverse temperature $\beta = 1/k_BT$, the correlations decrease like δ^{-2m}. For $\delta < \beta$, we find

$$\Big|\omega_\beta(ba) - \omega_\beta(b)\omega_\beta(a)\Big| \leq 4C(\mathcal{O}, \beta)\Big(\frac{\delta}{\sqrt{5}}\Big)^{-(2m+1)}\|a\|\,\|b\| + \epsilon\|a\|\,\|b\|, \tag{41}$$

where $\epsilon = 2(C_1(\mathcal{O}))^2\beta^{-2m}$. This fact indicates that one has better decay properties of the correlations in the limit of zero temperature, $\beta^{-1} \to 0$. The constant $C_1(\mathcal{O})$ may, as indicated, depend on the size of $\mathcal{O}$, but not on the particular choice of $a, b \in \mathcal{A}(\mathcal{O})$. The constant $m > 0$ may depend on the size of $\mathcal{O}$, but we expect that m becomes independent of the size of $\mathcal{O}$ for $\mathcal{O}$ sufficiently large. Both m and $C_1(\mathcal{O})$ will in general depend on β.

Proof. – We proceed in several steps.

(i) By definition $\mathbf{P}^{\pm}$ projects onto the *strictly* positive resp. negative spectrum of $\mathbf{H}_\beta$. Thus $\mathbf{P}^{\pm}\Omega_\beta = 0$, and the thermal expectation values of the local observables can be subtracted; for instance

$$\begin{aligned}\|e^{\mp\frac{\lambda}{2}\mathbf{H}_\beta}\mathbf{P}^{\pm}\pi_\beta(a)\Omega_\beta\| &= \|e^{\mp\frac{\lambda}{2}\mathbf{H}_\beta}\mathbf{P}^{\pm}\big(\pi_\beta(a) - \omega_\beta(a)\mathbb{1}\big)\Omega_\beta\| \\ &\leq \|\mathbf{P}^{\pm}\|\|e^{-\frac{\lambda}{2}|\mathbf{H}_\beta|}\big(\pi_\beta(a) - \omega_\beta(a)\mathbb{1}\big)\Omega_\beta\| \\ &\leq C_1(\mathcal{O})\lambda^{-m}\,\|a\|, \qquad \forall a \in \mathcal{A}(\mathcal{O}). \end{aligned} \tag{42}$$

This provides a bound for the remainder in (38), namely

$$\Big|\omega_\beta(ba) - \omega_\beta(b)\omega_\beta(a)\Big| \leq \sum_{l=-n}^{n-1} |f_{a,b}(il\beta)| + 2\big(C_1(\mathcal{O})(n\beta)^{-m}\big)^2\|a\|\,\|b\|. \tag{43}$$

Note that the number $2n$ of terms in this sum has not been fixed yet.

(ii) The spectral properties of $\mathbf{H}_\beta$ imply bounds for $\sup_{t\in\mathbb{R}} |f_{a,b}(t + i\lambda)|$, namely

$$\begin{aligned}&\sup_{t\in\mathbb{R}} |f_{a,b}(t + i\lambda)| \\ &\quad = \sup_{t\in\mathbb{R}}\Big|\big(e^{-\frac{\lambda}{2}\mathbf{H}_\beta}\mathbf{P}^{+}\pi_\beta(b^*)\Omega_\beta\,,\, e^{it\mathbf{H}_\beta}(\mathbb{1} - e^{-\beta\mathbf{H}_\beta})e^{-\frac{\lambda}{2}\mathbf{H}_\beta}\mathbf{P}^{+}\pi_\beta(a)\Omega_\beta\big)\Big| \\ &\quad \leq \|e^{-\frac{\lambda}{2}\mathbf{H}_\beta}(\mathbb{1} - e^{-\beta\mathbf{H}_\beta})\mathbf{P}^{+}\pi_\beta(a)\Omega_\beta\|\,\|e^{-\frac{\lambda}{2}\mathbf{H}_\beta}\mathbf{P}^{+}\pi_\beta(b^*)\Omega_\beta\| \\ &\quad \leq C(\mathcal{O},\beta)\lambda^{-(2m+1)}\|a\|\,\|b\|. \end{aligned} \tag{44}$$

This can be seen as follows. Set $\Psi = \pi_\beta(a)\Omega_\beta$. The spectral representation of $\mathbf{H}_\beta$ yields

$$\begin{aligned}\|e^{-\frac{\lambda}{2}\mathbf{H}_\beta}(\mathbb{1} - e^{-\beta\mathbf{H}_\beta})\mathbf{P}^{+}\pi_\beta(a)\Omega_\beta\|^2 &= \int_0^\infty d\mu_\Psi(\nu)\; e^{-\lambda\nu}(1 - e^{-\beta\nu})^2 \\ &\leq \sup_{\nu'} e^{-\lambda\nu'/2}(1 - e^{-\beta\nu'})^2 \int_0^\infty d\mu_\Psi(\nu)\; e^{-\lambda\nu/2} \\ &\leq c_\beta\lambda^{-2}\|e^{-\frac{\lambda}{4}\mathbf{H}_\beta}\mathbf{P}^{+}\pi_\beta(a)\Omega_\beta\|^2 \\ &\leq \big(C(\mathcal{O},\beta)\lambda^{-(m+1)}\|a\|\big)^2. \end{aligned} \tag{45}$$

Bounds similar to (44) hold for $\sup_{t\in\mathbb{R}} |f_{a,b}(t - i\lambda)|$.

(iii) Refined bounds on $|f_{a,b}(ir)|$, $r \in \mathbb{R}$, $|r| \leq \delta$, follow from Lemma 2.2:

$$|f_{a,b}(ir)| \leq C(\mathcal{O},\beta)5^{m+1/2}\delta^{-(2m+1)}||a||\,||b||, \qquad \forall 0 \leq r \leq \delta. \tag{46}$$

Given $\mathcal{O}$, β, and m, the constants $C_1(\mathcal{O})$ and $C(\mathcal{O},\beta)$ are fixed and

$$\Big|\omega_\beta\big((b-\omega_\beta(b))a\big)\Big| \leq \inf_{n\beta\leq\delta} 2||a||\,||b||\Big(2nC(\mathcal{O},\beta)\Big(\frac{\delta}{\sqrt{5}}\Big)^{-(2m+1)} + \big(C_1(\mathcal{O})(n\beta)^{-m}\big)^2\Big). \tag{47}$$

For $\delta > \beta$ we set $k = \{n \in \mathbb{N} | n < \delta/\beta \leq n+1\}$. Thus

$$\Big|\omega_\beta(ba) - \omega_\beta(b)\omega_\beta(a)\Big| \leq C_2(\mathcal{O},\beta)(k\beta)^{-2m}||a||\,||b||. \tag{48}$$

The choice $n = 1$ in (47) reproduce (41). □

Remark. – As mentioned in the introduction, the correlations for free massless bosons in a KMS state decay only like δ^{-1}. From explicit calculations one expects that†

$$||e^{-\frac{\lambda}{2}|\mathbf{H}_\beta|}(\pi_\beta(a) - \omega_\beta(a)\mathbb{1})\Omega_\beta|| \leq \text{const}\cdot\lambda^{-1/2}||a||, \qquad a \in \mathcal{A}(\mathcal{O}). \tag{49}$$

Thus Theorem 3.1 gives the optimal exponent $2m = 1$ for δ large (compared to β).

4. UNIFORM DECAY OF SPATIAL CORRELATIONS

We start this section with a few remarks on the vacuum sector; we refer to [BW], [BD'AL], [H] for further discussion and motivation. If the model under consideration has decent vacuum phase space properties, these properties manifest [BW] themselves in the nuclearity of the map

$$\begin{array}{llll} \Theta^{vac}_{\lambda,\mathcal{O}}: & \pi(\mathcal{A}(\mathcal{O}))'' & \to & \mathcal{H} \\ & \mathbf{A} & \mapsto & e^{-\lambda\mathbf{H}}(\mathbf{A} - (\Omega\,,\,\mathbf{A}\Omega)\mathbb{1})\,\Omega, \end{array} \tag{50}$$

for $\lambda > 0$. Here Ω denotes the vacuum vector. If the nuclear norm obeys

$$||\Theta^{vac}_{\lambda,\mathcal{O}}||_1 \leq \text{const}\cdot\lambda^{-m}, \tag{51}$$

† The author thanks D. Buchholz for this information.

Lemma 3.1 of [D'ADFL] applies and

$$\Big|(\Omega, \sum_{j=1}^{j_\circ} \mathbf{A}_j \mathbf{B}_j \Omega) - \sum_{j=1}^{j_\circ} (\Omega, \mathbf{A}_j \Omega)(\Omega, \mathbf{B}_j \Omega)\Big|$$
$$\leq \text{const}' \cdot \delta^{-(2m+1)} \Big\| \sum_{j=1}^{j_\circ} \mathbf{A}_j \mathbf{B}_j \Big\|, \tag{52}$$

for any $j_\circ \in \mathbb{N}$ and any two families of operators $\mathbf{A}_j \in \pi(\mathcal{A}(\mathcal{O}_1))''$ and $\mathbf{B}_j \in \pi(\mathcal{A}(\mathcal{O}_2))''$; $\mathcal{O}_1$ and $\mathcal{O}_2$ as described in front of Theorem 3.1.

In contrast to the non-relativistic case the local von Neumann algebras in the vacuum representation $\pi(\mathcal{A}(\mathcal{O}))''$ are in general not of type I. Nevertheless, if the nuclearity condition (51) holds, then (52) implies that the local algebras $\pi(\mathcal{A}(\mathcal{O}))''$ can be embedded into factors of type I without essential loss of the local structure, i.e., for every pair $\mathcal{O}_1 \subset\subset \mathcal{O}_2$ there exists a type I factor $\mathcal{M}(\mathcal{O}_1, \mathcal{O}_2)$ such that

$$\pi(\mathcal{A}(\mathcal{O}_1))'' \subset \mathcal{M}(\mathcal{O}_1, \mathcal{O}_2) \subset \pi(\mathcal{A}(\mathcal{O}_2))''. \tag{53}$$

By $\mathcal{O}_1 \subset\subset \mathcal{O}_2$ we mean that the closure of the open bounded set $\mathcal{O}_1$ lies in the interior of $\mathcal{O}_2$.

Uniform bounds on the thermal correlation functions of a KMS state can now be derived from thermal phase space properties. We recall from [BD'AL] the discussion of these phase space properties: The state induced by the vector $\pi_\beta(a)\Omega_\beta$, $a \in \mathcal{A}(\mathcal{O})$, represents a localised excitation of the KMS state. The energy transferred to ω_β can be restricted by taking time-averages

$$\frac{1}{\sqrt{2\pi}} \int \mathrm{d}t\, f(t) \pi_\beta(\tau_t(a))\Omega_\beta = \tilde{f}(\mathbf{H}_\beta)\pi_\beta(a)\Omega_\beta, \qquad a \in \mathcal{A}(\mathcal{O}), \tag{54}$$

with suitable testfunctions $f(t)$, whose Fourier transforms $\tilde{f}(\nu)$ decrease exponentially. The assumption that the theory has decent phase space properties can be cast into the condition that the maps $\Theta^{therm}_{\lambda,\mathcal{O}}: \mathcal{A}(\mathcal{O}) \to \mathcal{H}_\beta$,

$$a \to \mathrm{e}^{-\lambda|\mathbf{H}_\beta|}(\pi_\beta(a) - \omega_\beta(a)\mathbb{1})\,\Omega_\beta, \quad \lambda > 0, \tag{55}$$

are nuclear for all open, bounded space-time regions $\mathcal{O} \subset \mathbb{R}^4$. Quantitative information on the decay of the correlations can be extracted from the nuclear norm of $\Theta^{therm}_{\lambda,\mathcal{O}}$.

THEOREM 4.1. – *Let Ω_β be the unique — up to a phase — time-invariant vector in $\mathcal{H}_\beta$ corresponding to the (τ, β)-KMS state ω_β. Given the same* *geometric situation as in Theorem 3.1, we pick a $j_\circ \in \mathbb{N}$ and consider two families of operators $a_j \in \mathcal{A}(\mathcal{O}_1)$ and $b_j \in \mathcal{A}(\mathcal{O}_2)$, $j \in \{1, \ldots, j_\circ\}$. If the nuclear norm $\|\Theta^{therm}_{\lambda,\mathcal{O}}\|_1$ is bounded by*

$$\|\Theta^{therm}_{\lambda,\mathcal{O}}\|_1 \leq \mathcal{C}_1(\mathcal{O})(2\lambda)^{-m}, \tag{56}$$

for some $\mathcal{C}_1(\mathcal{O}), m > 0$, then for $\delta > \beta$

$$\Big|\omega_\beta\Big(\sum_{j=1}^{j_\circ} a_j b_j\Big) - \sum_{j=1}^{j_\circ} \omega_\beta(a_j)\omega_\beta(b_j)\Big| \leq (k\beta)^{-2m}\mathcal{C}_2(\mathcal{O},\beta)\Big\|\sum_{j=1}^{j_\circ} a_j b_j\Big\|, \tag{57}$$

where $k < \delta/\beta \leq (k+1)$, $k \in \mathbb{N}$. $\mathcal{C}_2(\mathcal{O},\beta)$ does not depend on δ or $j_\circ$.

Proof. – The uniform bounds are based on the algebraic independence [Ro] of the operators $a_j \in \mathcal{A}(\mathcal{O}_1)$ and $b_j \in \mathcal{A}(\mathcal{O}_2)$. It implies

$$\Big\|\sum_{j=1}^{j_\circ} \phi(a_j)\psi(b_j)\Big\| \leq \|\phi\|\|\psi\| \cdot \Big\|\sum_{j=1}^{j_\circ} a_j b_j\Big\|, \tag{58}$$

for arbitrary linear functionals ϕ, ψ. Note that $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ still denote the C^*-algebras, and not their weak closures in the representation π_β. Lemma 2.1 generalizes to

$$\begin{aligned}
&\Big|\omega_\beta\Big(\sum_{j=1}^{j_\circ}(b_j - \omega_\beta(b_j))a_j\Big)\Big| \\
&\leq \Big|\sum_{j=1}^{j_\circ}\big(\Omega_\beta, \pi_\beta(b_j)\big(\mathrm{e}^{-n\beta\mathbf{H}_\beta}\mathbf{P}^+ + \mathrm{e}^{n\beta\mathbf{H}_\beta}\mathbf{P}^-\big)\pi_\beta(a_j)\Omega_\beta\big)\Big| \\
&\quad + \sum_{l=-n}^{n-1}\Big|\sum_{j=1}^{j_\circ} f_{a_j,b_j}(il\beta)\Big|.
\end{aligned} \tag{59}$$

Introducing sequences of vectors $\Phi_i, \Psi_k \in \mathcal{H}_\beta$ and of linear functionals $\phi_i \in \pi_\beta(\mathcal{A}(\mathcal{O}_1))^*$, $\psi_k \in \pi_\beta(\mathcal{A}(\mathcal{O}_2))^*$ corresponding [P] to the nuclear maps $\Theta_1 := \Theta^{therm}_{n\beta/2,\mathcal{O}_1}$ resp. $\Theta_2 := \Theta^{therm}_{n\beta/2,\mathcal{O}_2}$, we find for the first term in the sum on the r.h.s. of equation (59)

$$\begin{aligned}
\Big|\sum_{j=1}^{j_\circ}(\Theta_2(b_j^*)\,,\,\mathbf{P}^+\Theta_1(a_j))\Big| &= \Big|\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=1}^{j_\circ}\overline{\psi_i(b_j^*)}\phi_k(a_j)(\Psi_i\,,\,\mathbf{P}^+\Phi_k)\Big| \\
&\leq \sum_i \|\phi_i\|\|\Phi_i\| \sum_k \|\psi_k\|\|\Psi_k\| \Big\|\sum_{j=1}^{j_\circ} a_j b_j\Big\|
\end{aligned} \tag{60}$$

Taking the infimum over all suitable sequences of vectors and linear functionals we obtain

$$\Big|\sum_{j=1}^{j_\circ}(\Theta_2(b_j^*)\,,\,\mathbf{P}^+\Theta_1(a_j))\Big| \le \big(\mathcal{C}_1(\mathcal{O})(n\beta)^{-m}\big)^2\Big\|\sum_{j=1}^{j_\circ} a_j b_j\Big\|. \tag{61}$$

A similar bound holds for the term containing $\mathbf{P}^-$ in (59). We thus arrive at

$$\begin{aligned}&\Big|\omega_\beta\Big(\sum_{j=1}^{j_\circ}(b_j-\omega_\beta(b_j))a_j\Big)\Big| \\ &\quad\le \sum_{l=-n}^{n-1}\Big|\sum_{j=1}^{j_\circ} f_{a_j,b_j}(il\beta)\Big| + 2\big(\mathcal{C}_1(\mathcal{O})(n\beta)^{-m}\big)^2\Big\|\sum_{j=1}^{j_\circ} a_j b_j\Big\|.\end{aligned} \tag{62}$$

Introducing sequences of vectors $\hat{\Phi}_i, \hat{\Psi}_k \in \mathcal{H}_\beta$ and of linear functionals $\hat{\phi}_i \in \pi_\beta(\mathcal{A}(\mathcal{O}_1))^*$, $\hat{\psi}_k \in \pi_\beta(\mathcal{A}(\mathcal{O}_2))^*$ corresponding to the nuclear maps $\Theta_3 := \Theta^{therm}_{\lambda/4,\mathcal{O}_1}$ resp. $\Theta_4 := \Theta^{therm}_{\lambda/2,\mathcal{O}_2}$, we find

$$\begin{aligned}&\sup_{t\in\mathbb{R}}\Big|\sum_{j=1}^{j_\circ} f_{a_j,b_j}(t+i\lambda)\Big| \\ &= \sup_{t\in\mathbb{R}}\Big|\sum_{j=1}^{j_\circ}\big(\Theta_4(b_j^*)\,,\,\mathrm{e}^{it\mathbf{H}_\beta}(\mathbb{1}-\mathrm{e}^{-\beta|\mathbf{H}_\beta|})\mathrm{e}^{-\frac{\lambda}{4}|\mathbf{H}_\beta|}\mathbf{P}^+\Theta_3(a_j)\big)\Big| \\ &\le \sup_{t\in\mathbb{R}}\sum_{i,k}\Big|\sum_{j=1}^{j_\circ}\overline{\hat{\psi}_k(b_j^*)}\hat{\phi}_i(a_j)\big(\hat{\Psi}_k\,,\,\mathrm{e}^{it\mathbf{H}_\beta}(\mathbb{1}-\mathrm{e}^{-\beta|\mathbf{H}_\beta|})\mathrm{e}^{-\frac{\lambda}{4}|\mathbf{H}_\beta|}\mathbf{P}^+\hat{\Phi}_i\big)\Big| \\ &\le \|(\mathbb{1}-\mathrm{e}^{-\beta|\mathbf{H}_\beta|})\mathrm{e}^{-\frac{\lambda}{4}|\mathbf{H}_\beta|}\|\sum_i\|\hat{\phi}_i\|\|\hat{\Phi}_i\|\sum_k\|\hat{\psi}_k\|\|\hat{\Psi}_k\|\Big\|\sum_{j=1}^{j_\circ} a_j b_j\Big\|.\end{aligned} \tag{63}$$

Taking the infimum over all suitable sequences of vectors and linear functionals we obtain

$$\sup_{t\in\mathbb{R}}\Big|\sum_{j=1}^{j_\circ} f_{a_j,b_j}(t+i\lambda)\Big| \le \mathcal{C}(\mathcal{O},\beta)\lambda^{-1}\Big(\frac{\lambda}{2}\Big)^{-m}\lambda^{-m}\Big\|\sum_{j=1}^{j_\circ} a_j b_j\Big\|, \tag{64}$$

and similarly

$$\sup_{t\in\mathbb{R}}\Big|\sum_{j=1}^{j_\circ} f_{a_j,b_j}(t-i\lambda)\Big| \le 2^m\mathcal{C}(\mathcal{O},\beta)\lambda^{-(2m+1)}\Big\|\sum_{j=1}^{j_\circ} a_j b_j\Big\|.$$

Theorem 4.1 has now been reduced to the situation discussed in the proof of Theorem 3.1. □

Remark. – The constant $m > 0$ depends on the dynamics of the model under consideration. Up to now m has not been computed in a thermal state for any model. For the vacuum, Buchholz and Jacobi [BJ] computed for free massless bosons $m = 3$, for $r \geq \lambda$, where r denotes the diameter of $\mathcal{O}$, and $m = 1$, for $r < \lambda$. For the free electromagnetic field they found $m = 3$, for $r \geq \lambda$, and $m = 1$, for $r < \lambda$. Note that in these computations the vacuum expectation values were not subtracted in the definition of the maps $\Theta^{vac}_{\lambda,\mathcal{O}}$.

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