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On the geometry and dynamics of crystalline continua

by

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ABSTRACT. – We introduce in continuum mechanics the concept of a material manifold. To describe, in the continuum limit, a crystalline solid containing an arbitrary distribution of dislocations, we endow the material manifold with a structure a k in to that of a Lie group. Starting with a state function defined on the space of local thermodynamic equilibrium states, we formulate the dynamics of crystalline continua in accordance with the least action principle, within the framework of general relativity, in terms of a mapping of the space time manifold into the material manifold. We include in our formulation electromagnetic effects. © Elsevier, Paris

Key words: Continuum mechanics; crystalline solids; dislocations

RÉSUMÉ. – Nous introduisons le concept de variété matérielle dans le cadre de la mécanique des milieux continus. Pour décrire, dans la limite continue, un solide cristallin contenant une distribution arbitraire de dislocations, nous dotons la variété matérielle d'une structure similaire à celle d'un groupe de Lie. En partant d'une fonction d'état définie sur l'espace des états d'équilibre thermodynamique local, nous formulons la dynamique des milieux cristallins continus conformément avec le principe de moindre action, dans le cadre de la relativité générale, en termes d'une application de la variété spatio-temporelle dans la variété matérielle. Notre formulation comprend aussi des effets électromagnétiques. © Elsevier, Paris

1. INTRODUCTION

This paper formulates the dynamics of crystalline solids, in the continuum limit, when arbitrary distributions of elementary dislocations are present in the crystal lattice. The interactions with the gravitational and electromagnetic fields are included in the formulation.

Crystal dislocations have been treated extensively in the litterature, both at the atomic level as well as in the continuum limit. The book by F.R.N. Nabarro [N1] is a standard reference on the subject as a whole, that by E. Kröner [K] on the continuum theory in particular. Also, the collections of articles in [B-K-P] and [N2], containing more recent contributions, should give the reader an impression of the state of the art in this actively pursued subject.

The present paper represents a departure from previous continuum mechanical formulations. Our approach is based on the concept of the *material manifold*, which represents the material continuum with those of its properties which are *intrinsic* to it , being *independent* of its relation to the spacetime continuum. In the case of a fluid, the material manifold is simply an oriented differentiable manifold with a volume form, whose integral on a material domain represents the number of particles contained in the domain. In the case of a crystalline solid however, the material manifold is endowed with a richer structure which we call *crystalline* (section 2). A material manifold with a crystalline structure is a generalization of the notion of a Lie group, reducing to the latter in the case where the dislocation density is constant. The crystalline structure itself corresponds to the Lie algebra. Uniform distributions of the two basic kinds of elementary dislocations in a crystal lattice give rise, in the continuum limit, to the two simplest non-Abelian Lie groups.

The *thermodynamic state space* (section 3) is an open set in a tensor space defined on the crystalline structure, which represents the set of possible local thermodynamic states of the material. On this space is defined the *state function* which determines the laws governing the dynamics.

The dynamics is described by a mapping of the spacetime manifold into the material manifold (section 4). The spacetime manifold is that of the general theory of relativity. The equations of motion are generated from a Lagrangian, constructed, through the mapping, from the state function. The energy-momentum-stress tensor is defined, in accordance with the principles of general relativity, by considering the response of the action of matter to variations of the spacetime metric. The equations of motion are then a consequence of the field equations of gravitation.

It should be noted that apart from the gain in geometric and physical insight, the general relativistic formulation of the theory finds application in the description of the crust and core of neutron stars.

The electromagnetic interactions of crystalline solids are the subject of the last section (section 5). The case of perfect insulators is considered first. The theory contains in this case the low frequency limit of nonlinear crystal optics. The paper ends with the treatment of the case of perfect conductors.

In order to make the mathematical concepts stand out as clearly as possible, the material manifold is taken to be of dimension n, throughout. The physical case is, of course, $n = 3$.

2. THE CRYSTALLINE STRUCTURE

Let $\mathcal{N}$ be a differentiable manifold of dimension n which is oriented. In what is to follow $\mathcal{N}$ shall be the *material manifold*, each point y of which represents a material particle. Let $\mathcal{X}(\mathcal{N})$ be the space of C^∞ vectorfields on $\mathcal{N}$. For each $y \in \mathcal{N}$ we denote by ε_y the evaluation map $\mathcal{X}(\mathcal{N}) \to T_y\mathcal{N}$ by:

$$\varepsilon_y(X) = X(y)$$

DEFINITION. – A *crystalline structure* on $\mathcal{N}$ is a distinguished linear subspace $\mathcal{V}$ of $\mathcal{X}(\mathcal{N})$ such that the evaluation map restricted to $\mathcal{V}$, $\varepsilon_y : \mathcal{V} \to T_y\mathcal{N}$, is an isomorphism for each $y \in \mathcal{N}$.

The orientation of $\mathcal{N}$ induces an orientation in $\mathcal{V}$ which makes ε_y orientation preserving at each $y \in \mathcal{N}$.

PROPOSITION 2.1. – *$\mathcal{N}$ admits a crystalline structure if and only if $\mathcal{N}$ is parallelizable.*

Proof. – Let $\mathcal{V}$ be a crystalline structure on $\mathcal{N}$. Pick a point $y_* \in \mathcal{N}$ and let $(E_{1*}, ..., E_{n*})$ be a frame at y_*. Define $E_a \in \mathcal{V}$ by: $E_a = \varepsilon_{y_*}^{-1}(E_{a*})$; $a = 1, ..., n$. Then $(E_1, ..., E_n)$ is a global frame field for $\mathcal{N}$. Otherwise there would be a point $y \in \mathcal{N}$ such that the vectors $E_1(y), ..., E_n(y)$ are linearly dependent, i.e. there are constants $c^1, ..., c^n$, not all zero, such that

$$\sum_{a=1}^{n} c^a E_a(y) = 0$$

Consider the vectorfield

$$X = \sum_{a=1}^{n} c^a E_a \in \mathcal{V}$$

We have $X(y) = 0$. But $\varepsilon_y : \mathcal{V} \to T_y\mathcal{N}$ is an isomorphism. It follows that $X = 0$ everywhere. This yields a contradiction at y_*. Conversely, let $\mathcal{N}$ be parallelizable and let $(E_1, ..., E_n)$ be a global frame field for $\mathcal{N}$. Define $\mathcal{V}$ to be the linear span of $E_1, ..., E_n$. Then $\mathcal{V}$ is a crystalline structure. For, given $X \in \mathcal{V}$ there is a unique element $(c^1, ..., c^n)$ of $\Re^n$ such that

$$X = \sum_{a=1}^{n} c^a E_a$$

The map $i : (c^1, ..., c^n) \mapsto X$ is an isomorphism of $\Re^n$ onto $\mathcal{V}$. Consider an arbitrary point $y \in \mathcal{N}$. Then $j_y : \Re^n \to T_y\mathcal{N}$ by:

$$(c^1, ..., c^n) \mapsto \sum_{a=1}^{n} c^a E_a(y)$$

is an isomorphism of $\Re^n$ onto $T_y\mathcal{N}$. Therefore $\varepsilon_y = j_y \circ i^{-1}$ is an isomorphism of $\mathcal{V}$ onto $T_y\mathcal{N}$.

Given a crystalline structure $\mathcal{V}$ on $\mathcal{N}$ there is an isomorphism i of $\mathcal{X}(\mathcal{N})$ onto the space of functions on $\mathcal{N}$ with values in $\mathcal{V}$ which takes the vectorfield X to the function $i(X)$ given by:

$$i(X)(y) = \varepsilon_y^{-1}(X(y)) \in \mathcal{V} \tag{2.1}$$

at each $y \in \mathcal{N}$. The image by i of $\mathcal{V} \subset \mathcal{X}(\mathcal{N})$ is the subspace of constant $\mathcal{V}$ - valued functions on $\mathcal{N}$.

If f is a function on $\mathcal{N}$ with values in any space $\mathcal{T}(\mathcal{V})$ of tensors on $\mathcal{V}$, then df, the differential of f, is a 1-form on $\mathcal{N}$ with values in $\mathcal{T}(\mathcal{V})$. The differential of f corresponds to δf, a function on $\mathcal{N}$ with values in $\mathcal{L}(\mathcal{V}, \mathcal{T}(\mathcal{V}))$, defined by:

$$\delta f(y)(Z) = df(y)(\varepsilon_y(Z)) \in \mathcal{T}(\mathcal{V}) \quad : \forall y \in \mathcal{N}, \forall Z \in \mathcal{V} \tag{2.2}$$

We call a crystalline structure $\mathcal{V}$ on a manifold $\mathcal{N}$ *complete* if each $X \in \mathcal{V}$ is a complete vectorfield on $\mathcal{N}$. If $\mathcal{V}$ is complete, each element of $\mathcal{V}$ generates a 1-parameter group of diffeomorphisms of $\mathcal{N}$. These groups represent physically the continuum limit of the groups of translations of a crystal lattice. The parametrization of the group orbits, integral curves of elements of $\mathcal{V}$, is to be thought of as proportional to the number of atoms traversed.

Given a crystalline structure $\mathcal{V}$ on $\mathcal{N}$ we can define a mapping

$$\Lambda : \mathcal{N} \to \mathcal{L}(\mathcal{V} \wedge \mathcal{V}, \mathcal{V})$$

by:

$$\Lambda(y)(X,Y) = \varepsilon_y^{-1}([X,Y](y)) \quad : \forall y \in \mathcal{N}, \forall X, Y \in \mathcal{V} \tag{2.3}$$

We call Λ *dislocation density*. Suppose that $X, Y \in \mathcal{V}$ generate the 1-parameter groups $\{\phi_t : t \in \Re\}$, $\{\chi_t : t \in \Re\}$ of diffeomorphisms of $\mathcal{N}$, respectively. Then for a given point $y \in \mathcal{N}$, the curve

$$t \mapsto \chi_{-t}(\phi_{-t}(\chi_t(\phi_t(y))))$$

coincides to order t^2, as $t \to 0$, with the curve $t \mapsto \psi_{t^2}$, where $\{\psi_t : t \in \Re\}$ is the 1-parameter group of diffeomorphisms of $\mathcal{N}$ generated by $\Lambda(y)(X,Y) \in \mathcal{V}$.

The dislocation density is a concept that arises in the continuum limit when one considers a distribution of elementary dislocations in a crystal lattice. An elementary lattice dislocation has the property that if we start at an atom and move according to one group of lattice translations a certain number of atoms p, then move according to a different group of translations a number of atoms q, then according to the first $-p$ and finally according to the second $-q$, then on completing the circuit we arrive at an atom which does not coincide with the atom from which we started, but, provided that the circuit encloses a single elementary dislocation, is arrived at in a single step corresponding to a third lattice translation. The lattice vector corresponding to this step is called *Burgers vector*.

Returning to the continuum description, the differential of Λ, $d\Lambda$, a 1-form on $\mathcal{N}$ with values in $\mathcal{L}(\mathcal{V} \wedge \mathcal{V}, \mathcal{V})$, corrresponds to a mapping

$$\delta\Lambda : \mathcal{N} \to \mathcal{L}(\mathcal{V}, \mathcal{L}(\mathcal{V} \wedge \mathcal{V}, \mathcal{V}))$$

defined according to 2.2:

$$\delta\Lambda(y)(Z) = d\Lambda(y)(\varepsilon_y(Z)) \quad : \forall y \in \mathcal{N}, \forall Z \in \mathcal{V} \tag{2.4}$$

We then define a mapping

$$I : \mathcal{N} \to \mathcal{L}(\mathcal{V} \wedge \mathcal{V} \wedge \mathcal{V}, \mathcal{V})$$

by:

$$I(y)(X,Y,Z) = \delta\Lambda(y)(X)(Y,Z) + \delta\Lambda(y)(Y)(Z,X) + \delta\Lambda(y)(Z)(X,Y)$$
$$: \forall y \in \mathcal{N}, \forall X, Y, Z \in \mathcal{V} \tag{2.5}$$

We can also define another mapping of the same type

$$J : \mathcal{N} \to \mathcal{L}(\mathcal{V} \wedge \mathcal{V} \wedge \mathcal{V}, \mathcal{V})$$

by:

$$\begin{aligned} J(y)(X,Y,Z) = \Lambda(y)(X, \Lambda(y)(Y,Z)) + \Lambda(y)(Y, \Lambda(y)(Z,X)) \\ + \Lambda(y)(Z, \Lambda(y)(X,Y)) \\ : \forall y \in \mathcal{N}, \forall X, Y, Z \in \mathcal{V} \end{aligned} \tag{2.6}$$

PROPOSITION 2.2. – *We have:*

$$I + J = 0$$

Proof. – Choose a basis $(E_1, ..., E_n)$ in $\mathcal{V}$. Then there are unique constants X^a, Y^a, Z^a ; $a = 1, ..., n$ such that

$$X = \sum_{a=1}^{n} X^a E_a, \quad Y = \sum_{a=1}^{n} Y^a E_a, \quad Z = \sum_{a=1}^{n} Z^a E_a.$$

By linearity the statement then reduces to:

$$\begin{aligned} &I(y)(E_a, E_b, E_c) + J(y)(E_a, E_b, E_c) = 0 \\ &\forall y \in \mathcal{N} \; : \; a, b, c = 1, .., n \end{aligned}$$

Since $(E_1, ..., E_n)$ is a global frame field for $\mathcal{N}$, there are functions Λ^a_{bc} on $\mathcal{N}$ such that

$$[E_b, E_c] = \sum_{a=1}^{n} \Lambda^a_{bc} E_a$$

According to the definition of the dislocation density, 2.3, we then have:

$$\Lambda(y)(E_b, E_c) = \sum_{a=1}^{n} \Lambda^a_{bc}(y) E_a \quad : \forall y \in \mathcal{N}$$

Consequently,

$$d\Lambda(y)(E_b, E_c) = \sum_{a=1}^{n} d\Lambda^a_{bc}(y) E_a$$

and:

$$\delta\Lambda(y)(E_a)(E_b, E_c) = \sum_{d=1}^{n} E_a(\Lambda_{bc}^d)(y)E_d$$

We also have:

$$[E_a, [E_b, E_c]] = [E_a, \sum_{d=1}^{n} \Lambda_{bc}^d E_d]$$

$$= \sum_{d=1}^{n} \Lambda_{bc}^d [E_a, E_d] + \sum_{d=1}^{n} E_a(\Lambda_{bc}^d)E_d$$

At each $y \in \mathcal{N}$, the second sum coincides with the vectorfield

$$\delta\Lambda(y)(E_a)(E_b, E_c) \in \mathcal{V}$$

evaluated at y, while the first sum coincides with the vectorfield

$$\Lambda(y)(E_a, \Lambda(y)(E_b, E_c)) \in \mathcal{V}$$

evaluated at y. Therefore, at each $y \in \mathcal{N}$, by the Jacobi identity,

$$\varepsilon_y(I(y)(E_a, E_b, E_c) + J(y)(E_a, E_b, E_c)) =$$
$$[E_a, [E_b, E_c]](y) + [E_b, [E_c, E_a]](y) + [E_c, [E_a, E_b]](y) = 0$$

In view of the fact that $\varepsilon_y : \mathcal{V} \to T_y\mathcal{N}$ is an isomorphism, the proposition follows.

If the dislocation density is constant on $\mathcal{N}$ then for any $X, Y \in \mathcal{V}$ there is a $Z \in \mathcal{V}$ such that $[X, Y] = Z$, thus $\mathcal{V}$ constitutes in this case a Lie algebra. We then have $I = 0$ hence $J = 0$. By the fundamental theorems of Lie group theory $\mathcal{N}$ can then be given the structure of a Lie group, upon choosing an identity element $e \in \mathcal{N}$, so that $\mathcal{V}$ is the space of vectorfields on $\mathcal{N}$ which generate the right action of the group on itself; $\mathcal{V}$ is then at the same time the space of vectorfields on $\mathcal{N}$ which are invariant under left group multiplications.

At the atomic level two basic kinds of elementary dislocations are found. The first kind is called *edge dislocation*. It appears in a 2- dimensional lattice in which an extra half-line of atoms has been inserted along the negative 1st axis. A circuit of translations in the directions of the 1st and 2nd axes, alternately, which encloses the origin, ends at an atom which is reached in a single step by a translation in the direction of the 2nd axis. On the other hand, circuits not enclosing the origin close. A uniform distribution

of edge dislocations gives rise in the continuum limit to the *affine group*. The group manifold $\mathcal{N}$ is $\Re^2$ and the group multiplication is given by:

$$(y_1, y_2)(y_1', y_2') = (y_1 + y_1', y_2 + e^{y_1} y_2')$$

It is the group of affine transformations on the real line:

$$x \mapsto e^{y_1} x + y_2$$

The generators of right multiplications are the vectorfields

$$E_1 = \frac{\partial}{\partial y_1}, \quad E_2 = e^{y_1} \frac{\partial}{\partial y_2}$$

The Lie algebra $\mathcal{V}$ is the linear span of (E_1, E_2) and we have the commutation relation:

$$[E_1, E_2] = E_2$$

The second basic kind of elementary dislocation is called a *screw dislocation*. It appears in a 3-dimensional lattice in the following way. A circuit of translations in the directions of the 1st and 2nd axes, alternately, which encloses the 3rd axis, ends at an atom which is reached in a single step by a translation in the direction of the 3rd axis, while circuits not enclosing the 3rd axis close. A uniform distribution of screw dislocations gives rise in the continuum limit to the *Heisenberg group*. The group manifold $\mathcal{N}$ is $\Re^3$ with the group multiplication:

$$(y_1, y_2, y_3)(y_1', y_2', y_3') = (y_1 + y_1', y_2 + y_2', y_3 + y_3' + y_1 y_2')$$

It acts as a unitary transformation group on the space of square-integrable complex valued functions ψ on the real line by:

$$((y_1, y_2, y_3), \psi) \mapsto \psi'$$

where:

$$\psi'(x) = e^{i(y_2 x + y_3)} \psi(x + y_1)$$

The generators of right multiplications are the vectorfields:

$$E_1 = \frac{\partial}{\partial y_1}, \quad E_2 = \frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial y_3}, \quad E_3 = \frac{\partial}{\partial y_3}$$

the Lie algebra $\mathcal{V}$ is the linear span of (E_1, E_2, E_3) and we have the commutation relations:

$$[E_1, E_2] = E_3, \quad [E_1, E_3] = [E_2, E_3] = 0$$

A complete crystalline structure $\mathcal{V}$ on $\mathcal{N}$ defines an exponential map

$$\text{Exp} : \mathcal{N} \times \mathcal{V} \to \mathcal{N}$$

as follows. $\text{Exp}(y, X)$, is the point at parameter value 1 along the integral curve of X initiating at y. For each $y \in \mathcal{N}$, let

$$\text{Exp}_y : \mathcal{V} \to \mathcal{N}$$

be the map:

$$\text{Exp}_y(X) = \text{Exp}(y, X) \quad : \forall X \in \mathcal{V}$$

We have:

$$\text{Exp}_y(0) = y, \quad d\text{Exp}_y(0) = \varepsilon_y$$

Thus $d\text{Exp}_y(0)$ is an isomorphism, for each $y \in \mathcal{N}$. By the implicit function theorem it follows that, for each $y \in \mathcal{N}$ there is a neighborhood $\mathcal{U}_y$ of the zero vector in $\mathcal{V}$ such that Exp_y restricted to $\mathcal{U}_y$ is a diffeomorphism onto its image in $\mathcal{N}$.

We now choose a totally antisymmetric n-linear form ω on $\mathcal{V}$ which is positive when evaluated on a positive basis. Any other choice $\tilde{\omega}$ satisfying this condition differs from ω by a positive multiplicative constant. The form ω defines a volume form $d\mu_\omega$ on $\mathcal{N}$ by:

$$\begin{aligned} d\mu_\omega(Y_{y,1}, ..., Y_{y,n}) =& \omega(\varepsilon_y^{-1}(Y_{y,1}), ..., \varepsilon_y^{-1}(Y_{y,n})) \\ & : \forall y \in \mathcal{N}, \forall Y_{y,1}, ..., Y_{y,n} \in T_y\mathcal{N} \end{aligned} \tag{2.7}$$

The volume assigned by $d\mu_\omega$ to a domain $\mathcal{E} \subset \mathcal{N}$,

$$\int_{\mathcal{E}} d\mu_\omega,$$

represents the number of particles contained in $\mathcal{E}$.

3. THE THERMODYNAMIC STATE SPACE

We consider the space $S_2^+(\mathcal{V})$ of positive definite symmetric bilinear forms on $\mathcal{V}$. The *thermodynamic state space* is the space $S_2^+(\mathcal{V}) \times \Re^+$. An element (γ, σ) of the thermodynamic state space shall be referred to as a *thermodynamic state*. The variables which specify a thermodynamic state are $\gamma \in S_2^+(\mathcal{V})$, the *thermodynamic configuration*, and $\sigma \in \Re^+$, the *thermodynamic entropy*. Each element $\gamma \in S_2^+(\mathcal{V})$ defines a totally antisymmetric n- linear form ω_γ on $\mathcal{V}$ by the condition that if $(E_1, ..., E_n)$ is a positive basis for $\mathcal{V}$ which is orthonormal relative to γ, then:

$$\omega_\gamma(E_1, ..., E_n) = 1$$

It follows that there is a positive function v on $S_2^+(\mathcal{V})$ such that:

$$\omega_\gamma = v(\gamma)\omega \tag{3.1}$$

The positive real number $v(\gamma)$ is the *thermodynamic volume* corresponding to the thermodynamic configuration γ.

The *thermodynamic state function* κ is a real valued function on the space $S_2^+(\mathcal{V}) \times \Re^+$.

The *thermodynamic stress* corresponding to a thermodynamic state (γ, σ) is the element $\pi(\gamma, \sigma)$ of $(S_2(\mathcal{V}))^*$ defined by:

$$\frac{\partial(\kappa(\gamma, \sigma)v(\gamma))}{\partial\gamma} = -\frac{1}{2}\pi(\gamma, \sigma)v(\gamma) \tag{3.2}$$

We have:

$$\frac{\partial v(\gamma)}{\partial\gamma} = \frac{1}{2}\gamma^{-1}v(\gamma) \tag{3.3}$$

Here γ^{-1} is the inverse of γ considered as an element of $(S_2(\mathcal{V}))^*$:

$$\gamma^{-1}(\dot{\gamma}) = \mathrm{tr}(\gamma^{-1} \cdot \dot{\gamma}) \quad : \forall \dot{\gamma} \in S_2(\mathcal{V})$$

with $\gamma^{-1} \cdot \dot{\gamma} \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ given by:

$$\gamma((\gamma^{-1} \cdot \dot{\gamma})(X), Y) = \dot{\gamma}(X, Y) \quad : \forall X, Y \in \mathcal{V}$$

In view of 3.3 we can express:

$$\pi(\gamma, \sigma) = -2\frac{\partial\kappa(\gamma, \sigma)}{\partial\gamma} - \gamma^{-1}\kappa(\gamma, \sigma) \tag{3.4}$$

The *thermodynamic temperature* corresponding to a thermodynamic state (γ, σ) is the real number $\vartheta(\gamma, \sigma)$ given by:

$$\vartheta(\gamma, \sigma) = \frac{\partial(\kappa(\gamma, \sigma)v(\gamma))}{\partial\sigma} \tag{3.5}$$

We require that $\vartheta(\gamma, \sigma)$ positive and tending to zero as σ tends to zero.

4. THE DYNAMICS

According to the general theory of relativity the *spacetime manifold* is a $n+1$-dimensional oriented differentiable manifold $\mathcal{M}$ which is endowed with a *Lorentzian metric* g, that is, a continuous assignment of g_x, a symmetric bilinear form of index 1 in $T_x\mathcal{M}$, at each $x \in \mathcal{M}$. The Lorentzian metric divides $T_x\mathcal{M}$ into three subsets, I_x, N_x, S_x, the set of *timelike, null, spacelike* vectors at x, according as to whether the quadratic form g_x is respectively negative, zero, or positive. The subset N_x is a double cone $N_x^+ \bigcup N_x^-$, the *null cone* at x. The subset I_x is the interior of this cone, an open set consisting of two components I_x^+ and I_x^-, the *future* and *past* components respectively. The boundaries of these components are the corresponding components of N_x. The subset S_x is the exterior of the null cone, a connected open set if $n > 1$. A curve in $\mathcal{M}$ is called *causal* if its tangent vector at each point belongs to the set $I \bigcup N$ corresponding to that point. A curve is called *timelike* if its tangent vector at each point belongs to I. We assume that $(\mathcal{M}, g)$ is *time oriented*, that is a continuous choice of future component of I_x at each $x \in \mathcal{M}$ can and has been made. A timelike curve is then either *future directed* or *past directed* according as to whether its tangent vector at a point belongs to the subset I^+ or I^- corresponding to that point. A hypersurface $\mathcal{H}$ in $\mathcal{M}$ is called spacelike if at each $x \in \mathcal{H}$ the restriction of g_x to $T_x\mathcal{H}$ is positive definite. A spacelike hypersurface in $\mathcal{M}$ is called a *Caychy hypersurface* if each causal curve in $\mathcal{M}$ intersects $\mathcal{H}$ at one and only one point. We assume that $(\mathcal{M}, g)$ posesses such a Cauchy hypersurface.

The motion of the material continuum is described by a mapping $f : \mathcal{M} \to \mathcal{N}$ of the spacetime manifold into the material manifold. This mapping tells us which material particle is at a given event in spacetime. The mapping f is subject to the following requirements. First, the restriction of f to a Caychy hypersurface must be one to one. Second, at each $x \in \mathcal{M}$, $df(x)$ must have a 1-dimensional kernel which is contained in I_x. Then, for each $y \in f(\mathcal{M}) \subset \mathcal{N}$, $f^{-1}(y)$ is a timelike curve in $\mathcal{M}$. The *material velocity* u is the corresponding future directed unit tangent vectorfield:

$$\text{span}(u_x) = \ker(df(x)) = T_x f^{-1}(y), \quad f(x) = y; \quad g(u,u) = -1 \qquad (4.1)$$

The *simultaneous space* at x is the orthogonal complement of the linear span of u_x:

$$\Sigma_x = (\text{span}(u_x))\perp \qquad (4.2)$$

We note that g_{Σ_x}, the restriction of g_x to Σ_x, is positive definite. Also, the restriction of $df(x)$ to Σ_x is an isomorphism of Σ_x onto $T_y\mathcal{N}$, $f(x) = y$.

The third and final requirement on f is that this isomorphism be orientation preserving.

To include thermal effects we introduce the *entropy function* s, a positive function on $\mathcal{M}$.

The equations of motion, a system of partial differential equations for the mapping f and the entropy function s, are to be derived from a Lagrangian L, a function on $\mathcal{M}$, constructed from f and s. The action in a domain $\mathcal{D} \subset \mathcal{M}$ is the integral:

$$\mathcal{A}[\mathcal{D}] = \int_{\mathcal{D}} L d\mu_g \tag{4.3}$$

where $d\mu_g$ is the volume form of $(\mathcal{M}, g)$.

Any mapping f fulfilling the three requirements stated above, defines at each $x \in \mathcal{M}$ an orientation preserving isomorphism $j_{f,x}$ of the crystalline structure $\mathcal{V}$ of $\mathcal{N}$ onto Σ_x by:

$$j_{f,x} = \left(df(x)_{|\Sigma_x}\right)^{-1} \circ \varepsilon_{f(x)} \tag{4.4}$$

The isomorphism $j_{f,x}$ induces an isomorphism $j^*_{f,x}$ of the space $\mathcal{T}_p(\Sigma_x)$ of p-linear forms on Σ_x into $\mathcal{T}_p(\mathcal{V})$, the space of p-linear forms on $\mathcal{V}$, taking $t_x \in \mathcal{T}_p(\Sigma_x)$ to $j^*_{f,x} t_x \in \mathcal{T}_p(\mathcal{V})$, where:

$$(j^*_{f,x} t_x)(Y_1, ..., Y_p) = t_x(j_{f,x}(Y_1), ..., j_{f,x}(Y_p)) \quad : \forall Y_1, ..., Y_p \in \mathcal{V} \tag{4.5}$$

The isomorphism $j^*_{f,x}$ extends to a linear mapping

$$j^*_{f,x} : \mathcal{T}_p(T_x\mathcal{M}) \to \mathcal{T}_p(\mathcal{V}),$$

given by:

$$j^*_{f,x} t_x = j^*_{f,x} t_{\Sigma_x},$$

for any $t_x \in \mathcal{T}_p(T_x\mathcal{M})$. Here t_{Σ_x} denotes the restriction of t_x to Σ_x. In particular we have:

$$j^*_{f,x} g_x = j^*_{f,x} g_{\Sigma_x} \in S_2^+(\mathcal{V})$$

The volume form $d\mu_\omega$ of $(\mathcal{N}, \mathcal{V})$ defines, at each $x \in \mathcal{M}$, through the mapping f, a totally antisymmetric n- linear form on Σ_x, the restriction to

Σ_x of the pullback $f^* d\mu_\omega$. If $d\mu_{\Sigma_x}$ is the volume form induced by g on Σ_x, then, according to the above, there is a positive function N on $\mathcal{M}$ such that:

$$(f^* d\mu_\omega)_{|\Sigma_x} = N(x) d\mu_{\Sigma_x} \quad : \forall x \in \mathcal{M} \tag{4.6}$$

At each $x \in \mathcal{M}$, the positive real number $N(x)$ represents the *number of particles per unit volume* at x. We have:

$$\frac{1}{N(x)} = v(j^*_{f,x} g_x) \tag{4.7}$$

The Lagrangian function L is defined by:

$$L(x) = \kappa(j^*_{f,x} g_x, s(x)) \quad : \forall x \in \mathcal{M} \tag{4.8}$$

where κ is the thermodynamic state function on $S_2^+(\mathcal{V}) \times \Re^+$.

We note that $L(x)$ depends on g only through g_x. The *energy tensor* at x is the element T_x of the dual space $(S_2(T_x\mathcal{M}))^*$ defined by:

$$\frac{\partial(L(x) d\mu_g(x))}{\partial g_x} = -\frac{1}{2} T_x d\mu_g(x) \tag{4.9}$$

We have:

$$\frac{\partial d\mu_g(x)}{\partial g_x} = \frac{1}{2} g_x^{-1} d\mu_g(x) \tag{4.10}$$

Here g_x^{-1} is the inverse of g_x considered as an element of $(S_2(T_x\mathcal{M}))^*$:

$$g_x^{-1}(\dot{g}_x) = \mathrm{tr}(g_x^{-1} \cdot \dot{g}_x) \quad : \forall \dot{g}_x \in S_2(T_x\mathcal{M})$$

with $g_x^{-1} \cdot \dot{g}_x \in \mathcal{L}(T_x\mathcal{M}, T_x\mathcal{M})$ given by:

$$g_x((g_x^{-1} \cdot \dot{g}_x)(X_x), Y_x) = \dot{g}_x(X_x, Y_x) \quad : \forall X_x, Y_x \in T_x\mathcal{M}$$

In view of 4.10, we can express:

$$T_x = -2\frac{\partial L(x)}{\partial g_x} - g_x^{-1} L(x) \tag{4.11}$$

PROPOSITION 4.1. – *The energy tensor at $x \in \mathcal{M}$ is given by:*

$$T_x(\dot{g}_x) = \kappa(j^*_{f,x} g_x, s(x)) \dot{g}_x(u_x, u_x) + \pi(j^*_{f,x} g_x, s(x))(j^*_{f,x} \dot{g}_x)$$
$$: \forall \dot{g}_x \in S_2(T_x\mathcal{M})$$

Proof. – We first determine, for a given element $Y \in \mathcal{V}$ and at a given point $x \in \mathcal{M}$, the dependence on g_x of the vector

$$X_x = j_{f,x}(Y) \in T_x\mathcal{M}$$

According to the definition 4.4, X_x is the solution of

$$df(x)(X_x) = \varepsilon_{f(x)}(Y)$$

belonging to Σ_x, the g_x- orthogonal complement of $\ker(df(x))$. Thus if the Lorentzian metric at x is changed from g_x to $\tilde{g}_x$, the vector X_x changes to $\tilde{X}_x$, which differs from X_x by an element of $\ker(df(x))$. It follows that there is a

$$\lambda_x \in \mathcal{L}(S_2(T_x\mathcal{M}), \mathcal{V}^*)$$

such that:

$$\left(\frac{\partial j_{f,x}(Y)}{\partial g_x}\right)(\dot{g}_x) = \lambda_x(\dot{g}_x)(Y)u_x \tag{4.12}$$

In conjunction with the definition 4.5, this implies that, for any $Y_1, Y_2 \in \mathcal{V}$,

$$\left(\frac{\partial (j^*_{f,x}g_x)(Y_1,Y_2)}{\partial g_x}\right)(\dot{g}_x) = (j^*_{f,x}\dot{g}_x)(Y_1,Y_2)$$
$$+ g_x(\lambda_x(\dot{g}_x)(Y_1)u_x, j_{f,x}(Y_2)) + g_x(j_{f,x}(Y_1), \lambda_x(\dot{g}_x)(Y_2)u_x)$$

Now, $j_{f,x}(Y_1), j_{f,x}(Y_2) \in \Sigma_x$ are g_x- orthogonal to u_x. Therefore the above reduces to:

$$\left(\frac{\partial j^*_{f,x}g_x}{\partial g_x}\right)(\dot{g}_x) = j^*_{f,x}\dot{g}_x \tag{4.13}$$

In view of 4.13 and the definitions 4.11, 4.8 and 3.4 we obtain:

$$T_x(\dot{g}_x) = \kappa(j^*_{f,x}g_x, s(x))\nu_x + \pi(j^*_{f,x}g_x, s(x))(j^*_{f,x}\dot{g}_x) \tag{4.14}$$

where:

$$\nu_x = (j^*_{f,x}g_x)^{-1}(j^*_{f,x}\dot{g}_x) - g_x^{-1}(\dot{g}_x) \tag{4.15}$$

We have:

$$(j^*_{f,x}g_x)^{-1}(j^*_{f,x}\dot{g}_x) = (j^*_{f,x}g_{\Sigma_x})^{-1}(j^*_{f,x}\dot{g}_{\Sigma_x}) = \mathrm{tr}((j^*_{f,x}g_{\Sigma_x})^{-1} \cdot j^*_{f,x}\dot{g}_{\Sigma_x})$$

Now, $j^*_{f,x}$ is an isomorphism of $S_2(\Sigma_x)$ onto $S_2(\mathcal{V})$ and the trace is invariant under isomorphisms. Consequently,

$$\mathrm{tr}((j^*_{f,x}g_{\Sigma_x})^{-1}\cdot j^*_{f,x}\dot{g}_{\Sigma_x}) = \mathrm{tr}(g^{-1}_{\Sigma_x}\cdot \dot{g}_{\Sigma_x}) = g^{-1}_{\Sigma_x}(\dot{g}_{\Sigma_x})$$

Hence:

$$\nu_x = g^{-1}_{\Sigma_x}(\dot{g}_{\Sigma_x}) - g^{-1}_x(\dot{g}_x) \tag{4.16}$$

Let $A_x = g^{-1}_x\cdot \dot{g}_x \in \mathcal{L}(T_x\mathcal{M}, T_x\mathcal{M})$, $A_{\Sigma_x} = g^{-1}_{\Sigma_x}\cdot \dot{g}_{\Sigma_x} \in \mathcal{L}(\Sigma_x, \Sigma_x)$. Then A_{Σ_x} is the restriction of A_x to Σ_x, and we have:

$$\nu_x = \mathrm{tr}(A_{\Sigma_x}) - \mathrm{tr}(A_x) \tag{4.17}$$

Relative to the g_x- orthogonal decomposition

$$T_x\mathcal{M} = \mathrm{span}(u_x)\oplus \Sigma_x,$$

we can write:

$$A_x = \begin{pmatrix} B & \\ & A_{\Sigma_x} \end{pmatrix}$$

where

$$B_x = -g_x(u_x, A_x u_x) = -\dot{g}_x(u_x, u_x)$$

Hence:

$$\mathrm{tr}(A_x) = B_x + \mathrm{tr}(A_{\Sigma_x})$$

and:

$$\nu_x = -B_x = \dot{g}_x(u_x, u_x) \tag{4.18}$$

In view of 4.14, 4.18, the proposition follows.

We can write:

$$T_x = \rho(x)u_x\otimes u_x + S_x \tag{4.19}$$

where

$$\rho(x) = \kappa(j^*_{f,x}g_x, s(x)) \tag{4.20}$$

is the *density of mass-energy* at x, and S_x, given by

$$S_x(\dot{g}_x) = \pi(j^*_{f,x}g_x, s(x))(j^*_{f,x}\dot{g}_x) \tag{4.21}$$

is the *stress tensor* at x. Since $j^*_{f,x}\dot{g}_x = j^*_{f,x}\dot{g}_{\Sigma_x}$, S_x can be viewed as belonging to $(S_2(\Sigma_x))^*$.

The *energy tensorfield* T is the assignment of T_x at each $x \in \mathcal{M}$, a symmetric 2-contravariant tensorfield on $\mathcal{M}$.

The *temperature function* is the function θ on $\mathcal{M}$ given by:

$$\theta(x) = \vartheta(j_{f,x}^* g_x, s(x)) \tag{4.22}$$

The spacetime manifold $(\mathcal{M}, g)$ is endowed with a unique symmetric connection Γ compatible with the metric g. The associated covariant derivative operator, acting on sections of tensor bundles over $\mathcal{M}$, we denote by ∇. The covariant derivative of the energy tensorfield T, ∇T, is a tensorfield of type $\mathcal{T}_1^2$ on $\mathcal{M}$. Its trace is a vectorfield on $\mathcal{M}$, the covariant divergence $\nabla \cdot T$ of T.

The *equations of motion* of the material continuum are the differential energy-momentum conservation laws:

$$\nabla \cdot T = 0 \tag{4.23}$$

These equations have $n + 1$ components, corresponding to the entropy function s and to the n component functions which describe the mapping f in terms of an atlas of local charts of $\mathcal{N}$.

The equations of motion 4.23 are a consequence of structure of the general theory of relativity. The connection Γ represents, within the framework of general relativity, the *gravitational force*. Let R be the corresponding curvature and let us denote by Ric the trace of R, the Ricci curvature, a symmetric 2-covariant tensorfield on $\mathcal{M}$. Let us also denote by S the scalar curvature: $S = g^{-1}(Ric)$. Then the Einstein equations for the spacetime manifold $(\mathcal{M}, g)$ read:

$$Ric - \frac{1}{2} Sg = 2T \tag{4.24}$$

The left hand side satisfies the twice contracted Bianchi identities:

$$\nabla \cdot (Ric - \frac{1}{2} Sg) = 0 \tag{4.25}$$

In view of these identities the equations of motion 4.23 are consequences of the Einstein equations 4.24.

The Einstein equations are obtained by adding to the Lagrangian of matter the Lagrangian of gravitation, $-(1/4)S$, and requiring that the resulting total action in any domain $\mathcal{D} \subset \mathcal{M}$ be stationary with respect to arbitrary variations of the metric g supported in any subdomain $\mathcal{D}'$ with compact closure in $\mathcal{D}$. The identities 4.25 can be viewed as resulting from

the invariance of the action of gravitation in $\mathcal{D}$ under all diffeomorphisms of $\mathcal{M}$ which leave $\mathcal{D}$ invariant.

We shall now demonstrate the relationship between the equations of motion 4.23 and the Euler-Lagrange equations obtained by varying the action of matter with respect to the mapping f.

Let $\{f_t : t \in (-1,1)\}$ be a differentiable family of maps $\mathcal{M} \to \mathcal{N}$, satisfying the three requirements stated in the previous section, and agreeing in $\mathcal{M} \setminus \mathcal{D}'$ with $f_0 = f$. Then for each $x \in \mathcal{M}$, $t \mapsto f_t(x)$ is a curve in $\mathcal{N}$ through $f(x)$. Let $\dot{f}(x)$ be the tangent vector of this curve at $f(x)$. Then $\dot{f}$, the variation of the map f, is the assignment of the vector $\dot{f}(x) \in T_{f(x)}\mathcal{N}$ at each $x \in \mathcal{M}$. Thus $\dot{f}$ is a section of the pullback bundle $f^*(T\mathcal{N})$, a vector bundle over $\mathcal{M}$. Consider the action of matter corresponding to f_t, with the entropy function s and the Lorentzian metric g on $\mathcal{M}$ fixed. Since $\dot{f}$ has compact support in $\mathcal{D}$, we can express:

$$\left(\frac{d\mathcal{A}[f_t]}{dt}\right)_{t=0} = \int_{\mathcal{D}} M(\dot{f}) d\mu_g \tag{4.26}$$

where M is a section of the dual bundle $f^*(T^*\mathcal{N})$. On the other hand, if $\{s_t : t \in (-1,1)\}$ is a differentiable family of entropy functions on $\mathcal{M}$, $s_0 = s, (\partial s_t/\partial t)_{t=0} = \dot{s}$, and we consider the action of matter corresponding to s_t, with f and g fixed, we have, in view of 3.5, 4.7, 4.8, 4.22

$$\left(\frac{d\mathcal{A}[s_t]}{dt}\right)_{t=0} = \int_{\mathcal{D}} N\theta\dot{s} d\mu_g \tag{4.27}$$

Finally, if $\{g_t : t \in (-1,1)\}$ is a differentiable family of Lorentzian metrics on $\mathcal{M}$, $g_0 = g, (\partial g_t/\partial t)_{t=0} = \dot{g}$, and we consider the action of matter corresponding to g_t, with f and s fixed, we have, in view of 4.9,

$$\left(\frac{d\mathcal{A}[g_t]}{dt}\right)_{t=0} = -\frac{1}{2}\int_{\mathcal{D}} T(\dot{g}) d\mu_g \tag{4.28}$$

Let X be a vectorfield on $\mathcal{M}$ with support compact contained in $\mathcal{D}$, and let $\{\phi_t : t \in \Re\}$ be the 1- parameter group of diffeomorphisms of $\mathcal{M}$ generated by X. Then, for each $t \in \Re$, ϕ_t restricts to a diffeomorphism of $\mathcal{D}$ onto itself. Let us set $f_t = \phi_t^* f = f \circ \phi_t$, $s_t = \phi_t^* s = s \circ \phi_t$, $g_t = \phi_t^* g$, for each $t \in (-1,1)$. Then we have:

$$\dot{f} = \mathcal{L}_X f = df(X), \quad \dot{s} = \mathcal{L}_X s = ds(X), \quad \dot{g} = \mathcal{L}_X g \tag{4.29}$$

Now the action is invariant under the group $\{\phi_t\}$: $\mathcal{A}[f_t, s_t, g_t] = \mathcal{A}[f, s, g]$. Thus, we have:

$$\dot{\mathcal{A}} := \left(\frac{d\mathcal{A}[f_t, s_t, g_t]}{dt} \right)_{t=0} = 0 \tag{4.30}$$

By 4.26, 4.27, 4.28, and 4.29,

$$\dot{\mathcal{A}} = \int_{\mathcal{D}} \left\{ M(df(X)) + N\theta ds(X) - \frac{1}{2} T(\mathcal{L}_X g) \right\} d\mu_g \tag{4.31}$$

Integrating by parts, the equations of motion 5.1 imply:

$$-\frac{1}{2} \int_{\mathcal{D}} T(\mathcal{L}_X g) d\mu_g = \int_{\mathcal{D}} g(\nabla \cdot T, X) d\mu_g = 0 \tag{4.32}$$

In fact the equations of motion are equivalent to 4.32 for all domains $\mathcal{D} \subset \mathcal{M}$ and for all vectorfields X with compact support in $\mathcal{D}$. From 4.32 in conjunction with 4.31 and 4.30 we then conclude that the equations of motion 4.23 are equivalent to the equations:

$$M \cdot df + N\theta ds = 0 \tag{4.33}$$

Here $M \cdot df$ is the 1- form on $\mathcal{M}$ defined by:

$$(M \cdot df)(X_x) = M(df(X_x)) \quad : \forall X_x \in T_x\mathcal{M}, \forall x \in \mathcal{M}$$

If df is continuous at x, then evaluating 4.33 on u_x yields, in view of the fact that $u_x \in \ker(df(x))$, the *adiabatic condition*:

$$u_x(s) = 0 \quad : \text{at every point } x \text{ of continuity of } df \tag{4.34}$$

5. ELECTROMAGNETIC EFFECTS

We now consider the interaction of a crystalline solid with the electromagnetic field. The *electromagnetic field* is a differential 2-form F on the spacetime manifold $\mathcal{M}$ which is exact, that is there exists a differential 1-form A, the *electromagnetic potential*, such that

$$F = dA \tag{5.1}$$

At each $x \in \mathcal{M}$, the antisymmetric bilinear form F_x in $T_x\mathcal{M}$ decomposes relative to the simultaneous space Σ_x into $E_x \in \Sigma_x^*$, the *electric field* at x, given by

$$E_x(X_x) = F(X_x, u_x) \ \forall X_x \in \Sigma_x \tag{5.2}$$

and $B_x \in \wedge_2(\Sigma_x)$, the *magnetic field*, the restriction of F_x to Σ_x:

$$B_x = F_{\Sigma_x} \tag{5.3}$$

The electric and magnetic fields at x extend to $T_x\mathcal{M}$ by:

$$\begin{aligned} E_x(X_x) &= F(X_x, u_x) \ : \forall X_x \in T_x\mathcal{M} \\ B_x(X_{x,1}, X_{x,2}) &= F(\Pi_x X_{x,1}, \Pi_x X_{x,2}) \ : \forall X_{x,1}, X_{x,2} \in T_x\mathcal{M} \end{aligned} \tag{5.4}$$

Here $\Pi_x \in \mathcal{L}(T_x\mathcal{M}, T_x\mathcal{M})$ is the operator of projection to Σ_x:

$$\Pi_x X_x = X_x + g_x(u_x, X_x)u_x \tag{5.5}$$

To include electromagnetic effects, the thermodynamic state space is extended to:

$$S_2^+(\mathcal{V}) \times \Re^+ \times \mathcal{V}^* \times \wedge_2(\mathcal{V})$$

To specify a thermodynamic state requires in addition to the variables (γ, σ), the electromagnetic variables (α, β), $\alpha \in \mathcal{V}^*$, $\beta \in \wedge_2(\mathcal{V})$. The thermodynamic state function κ is a real valued function on the extended thermodynamic state space and the thermodynamic stress and temperature are defined as before. We now define, in addition,

$$\eta = -\frac{\partial \kappa}{\partial \alpha}, \quad \zeta = 2\frac{\partial \kappa}{\partial \beta} \tag{5.6}$$

Then at a given thermodynamic state $(\gamma, \sigma, \alpha, \beta)$, η is an element of $\mathcal{V}$ and ζ is an element of $(\wedge_2(\mathcal{V}))^*$.

The Lagrangian function L is given by:

$$L(x) = \kappa(j_{f,x}^* g_x, s(x), j_{f,x}^* E_x, j_{f,x}^* B_x) \tag{5.7}$$

The *electric displacement* at x is the element $D_x \in \Sigma_x$ given by:

$$D_x = -\frac{\partial L}{\partial E_x} \tag{5.8}$$

The *magnetic displacement* at x is the element $H_x \in (\wedge_2(\Sigma))^*$ given by:

$$H_x = 2\frac{\partial L}{\partial B_x} \tag{5.9}$$

In view of the definitions 5.6 we have:

$$\xi_x(D_x) = \eta(j^*_{f,x}\xi_x) \quad : \forall \xi_x \in \Sigma^*_x \tag{5.10}$$

$$H_x(\chi_x) = \zeta(j^*_{f,x}\chi_x) \quad : \forall \chi_x \in \wedge_2(\Sigma_x) \tag{5.11}$$

The *electromagnetic displacement* at x is the element G_x of $(\wedge_2(T_x\mathcal{M}))^*$ given by:

$$G_x = 2\frac{\partial L}{\partial F_x} \tag{5.12}$$

From 5.2, 5.3, 5.8, 5.9 we obtain:

$$G_x = u_x \wedge D_x + H_x \tag{5.13}$$

The *electromagnetic displacement* G is the assignment of G_x at each $x \in \mathcal{M}$, an antisymmetric 2-contravariant tensorfield on $\mathcal{M}$.

The energy tensor at x is defined as before, by 4.9. However, we now have:

PROPOSITION 5.1. – *The energy tensor at $x \in \mathcal{M}$ has the form:*

$$T_x = \rho(x)u_x \otimes u_x + u_x \otimes P_x + P_x \otimes u_x + S_x$$

Here, $\rho(x)$, the density of mass-energy at x, is given by:

$$\rho(x) = \kappa(j^*_{f,x}g_x, s(x), j^*_{f,x}E_x, j^*_{f,x}B_x) + E_x(D_x)$$

$P_x \in \Sigma_x$ is the *electromagnetic momentum density* at x, given by:

$$P_x = H_x \cdot E_x,$$

with

$$\xi_x(H_x \cdot E_x) = \frac{1}{2}H_x(\xi_x \wedge E_x) \quad : \forall \xi_x \in \Sigma^*_x$$

Finally, $S_x \in (S_2(\Sigma_x))^*$, the stress tensor at x, is given by:

$$S_x(\dot{g}_x) = \pi(j^*_{f,x}g_x, s(x), j^*_{f,x}E_x, j^*_{f,x}B_x)(j^*_{f,x}\dot{g}_x)$$

Proof. – We must determine the dependence on g_x of E_x, B_x. We consider E_x, B_x extended to $T_x\mathcal{M}$ according to 5.4. We first determine the partial

derivatives with respect to g_x of u_x and Π_x. Now the linear span of u_x is independent of g_x. It follows that there is a $l_x \in (S_2(T_x\mathcal{M}))^*$ such that:

$$\frac{\partial u_x}{\partial g_x}(\dot{g}_x) = l_x(\dot{g}_x)u_x \tag{5.14}$$

Differentiating the normalization condition

$$g_x(u_x, u_x) = -1$$

with respect to g_x and substituting 5.14 yields:

$$\dot{g}_x(u_x, u_x) + 2g_x(l_x(\dot{g}_x)u_x, u_x) = 0$$

Hence:

$$l_x(\dot{g}_x) = \frac{1}{2}\dot{g}_x(u_x, u_x) \tag{5.15}$$

Differentiating 5.5 with respect to g_x and using 5.14, 5.15 we then obtain:

$$\left(\frac{\partial \Pi_x X_x}{\partial g_x}\right)(\dot{g}_x) = k_x(\dot{g}_x)(X_x)u_x \tag{5.16}$$

where $k_x \in \mathcal{L}(S_2(T_x\mathcal{M}), T_x^*\mathcal{M})$ is given by:

$$k_x(\dot{g}_x)(X_x) = \dot{g}_x(u_x, \Pi_x X_x) \quad : \forall X_x \in T_x\mathcal{M}, \forall \dot{g}_x \in S_2(T_x\mathcal{M}) \tag{5.17}$$

Since $k_x(\dot{g}_x)(X_x) = k_x(\dot{g}_x)(\Pi_x X_x)$, we can consider $k_x(\dot{g}_x)$ to belong to Σ_x^*. From the first of 5.4, by 5.14 we deduce:

$$\frac{\partial E_x}{\partial g_x}(\dot{g}_x) = l_x(\dot{g}_x)E_x \tag{5.18}$$

while from the second of 5.4, by 5.16 we deduce:

$$\frac{\partial B_x}{\partial g_x}(\dot{g}_x) = E_x \wedge k_x(\dot{g}_x) \tag{5.19}$$

It then follows that:

$$\frac{\partial (j_{f,x}^* E_x)}{\partial g_x} = l_x(\dot{g}_x) j_{f,x}^* E_x \tag{5.20}$$

and:

$$\frac{\partial (j_{f,x}^* B_x)}{\partial g_x} = j_{f,x}^*(E_x \wedge k_x(\dot{g}_x)) \tag{5.21}$$

According to the expression 4.11 and in view of the definitions 5.6, in the presence of electromagnetic effects the following additional contributions to $T_x(\dot{g}_x)$ are obtained:

$$2\eta \cdot \frac{\partial(j^*_{f,x}E_x)}{\partial g_x} - \zeta \cdot \frac{\partial(j^*_{f,x}B_x)}{\partial g_x} \tag{5.22}$$

By 5.20, 5.15 and 5.10,

$$2\eta \cdot \frac{\partial(j^*_{f,x}E_x)}{\partial g_x} = 2l_x(\dot{g}_x)E_x(D_x) = E_x(D_x)\dot{g}_x(u_x, u_x)$$
$$= E_x(D_x)(u_x \otimes u_x)(\dot{g}_x) \tag{5.23}$$

while by 5.21, 5.17 and 5.11,

$$-\zeta \cdot \frac{\partial(j^*_{f,x}B_x)}{\partial g_x} = H_x(k_x(\dot{g}_x) \wedge E_x) = 2k_x(\dot{g}_x)(P_x) = 2\dot{g}_x(u_x, P_x)$$
$$= (u_x \otimes P_x + P_x \otimes u_x)(\dot{g}_x) \tag{5.24}$$

The proposition thus follows.

The equations of motion are, as before, the differential energy-momentum conservation laws 4.23. These are the equations for the mapping f and the entropy function s, while the equations for the electromagnetic field F are the *Maxwell equations* which consist of the condition that F is exact, together with the equations:

$$\nabla \cdot G = 0 \tag{5.25}$$

These equations are obtained by requiring that the action in any domain $\mathcal{D} \subset \mathcal{M}$ be stationary with respect to variations of the electromagnetic potential A supported in any subdomain $\mathcal{D}'$ with compact closure in $\mathcal{D}$. For, if $\{A_t : t \in (-1, 1)\}$ is a differentiable family of electromagnetic potentials on $\mathcal{M}$, $A_0 = A$, $(\partial A_t/\partial t)_{t=0} = \dot{A}$, and we consider the action corresponding to A_t with f, s and g fixed, we have, in view of 5.12,

$$\left(\frac{d\mathcal{A}[A_t]}{dt}\right)_{t=0} = \int_{\mathcal{D}} \frac{1}{2} G(d\dot{A}) d\mu_g = \int_{\mathcal{D}} \nabla \cdot G(\dot{A}) d\mu_g \tag{5.26}$$

(The second entry of G is traced in defining the divergence). The argument of the previous section then shows that, modulo the Maxwell equations, the equations of motion are again equivalent to the equations 4.33. The adiabatic condition 4.34 then follows, as before.

What we have been describing above is the interaction of the electromagnetic field with a crystalline solid which is a *perfect insulator*. The *electric current* J, which constitutes the right hand side of the general form of the Maxwell equations:

$$\nabla \cdot G = J \tag{5.27}$$

vanishes in perfect insulators.

On the other hand, in *perfect conductors* it is the electric field which vanishes. This together with the condition that F is exact implies that $\mathcal{L}_u F = 0$, hence there is an exact 2-form Φ on the material manifold $\mathcal{N}$ such that:

$$F = f^* \Phi \tag{5.28}$$

The electromagnetic field thus ceases to be an independent dynamical variable, its dynamics being determined by the mapping f. The vanishing of the electric field reduces the thermodynamic state space to the subspace

$$S_2^+(\mathcal{V}) \times \Re^+ \times \wedge_2(\mathcal{V})$$

corresponding to $\alpha = 0$, and the Lagrangian function is given by:

$$L(x) = \kappa(j_{f,x}^* g_x, s(x), i^* \Phi(f(x))) \tag{5.29}$$

where $i^*\Phi$ is the mapping

$$i^* \Phi : \mathcal{N} \to \wedge_2(\mathcal{V})$$

by:

$$i^* \Phi(y)(X, Y) = \Phi_y(\varepsilon_y(X), \varepsilon_y(Y)) \quad : \forall y \in \mathcal{N}, \forall X, Y \in \mathcal{V} \tag{5.30}$$

The energy tensor at x is then given by Prop 5.1 with $E_x = 0$ and $B_x = (f^*\Phi)_x$, that is:

$$T_x = \rho(x) u_x \otimes u_x + S_x \tag{5.31}$$

with:

$$\rho(x) = \kappa(j_{f,x}^* g_x, s(x), i^* \Phi(f(x))) \tag{5.32}$$

and:

$$S_x(\dot{g}_x) = \pi(j_{f,x}^* g_x, s(x), i^* \Phi(f(x)))(j_{f,x}^* \dot{g}_x) \tag{5.33}$$

The equations of motion 4.23 then constitute a complete system and the electric current is simply *defined* by 5.27.

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