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Solitary waves for Maxwell-Dirac and Coulomb-Dirac models

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ABSTRACT. – Using variational methods, we show the existence of a stationary state for the Maxwell-Dirac model in the case $\omega \in]-m, m[$ extending a previous result by Esteban, Georgev and Séré. We also show the existence of infinitely many solutions in the case of the Coulomb-Dirac model. © Elsevier, Paris

Key words: Closed solutions of Dirac equations, solitary waves, nonlinear partial differential equations, variational methods, quantum theory.

Mots clés : Rom

RÉSUMÉ. – On utilise des méthodes variationnelles pour démontrer l'existence d'un état stationnaire pour l'équation de Maxwell-Dirac dans le cas où $\omega \in]-m, m[$ en généralisant aussi un résultat récent de Esteban, Georgiev et Séré. On démontre l'existence d'un nombre infini des solutions stationnaires dans le système de Coulomb-Dirac. © Elsevier, Paris

The Maxwell-Dirac equations, which describe the interaction of an electron with its own magnetic field, have been widely considered in literature (see for instance [20], [10]-[12], [16], [18], [4]).

Existence of solitary waves for such a system has been an open problem for a long time (see [19] for instance). Using variational methods, Esteban *et al.* [13] proved the existence of regular solutions in (3+1)-Minkowski space, stationary in time, localized in space, of the form

$$\psi(x_0, x) = e^{i\omega x_0}\varphi(x); \qquad \varphi : \mathbb{R}^3 \to \mathbb{C}^4,\ \omega \in]0, m[, \tag{0.1}$$

leaving open the question of the existence of solutions of this form for $\omega \leq 0$. Indeed their method fails in such a case. Moreover the Dirac equations [26], [2], [3], [7], [8], [23], [24], [14], [15] with nonlinear interaction depending on $\langle\varphi, \gamma^0\varphi\rangle$ have no stationary states with $\omega = 0$ and also for $\omega < 0$ in some interesting models (see [8] for instance). On the other hand, in the case of the Maxwell-Dirac model, Garrett Lisi [17] gave numerical evidence of the existence of bound states for $\omega \in]-m, m[$ of the form

$$\varphi(x) = \begin{pmatrix} u_1(r,z)e^{i(m_z-\frac{1}{2})\phi} \\ u_2(r,z)e^{i(m_z+\frac{1}{2})\phi} \\ -iu_3(r,z)e^{i(m_z-\frac{1}{2})\phi} \\ -iu_4(r,z)e^{i(m_z+\frac{1}{2})\phi} \end{pmatrix} \tag{0.2}$$

where (r, z, ϕ) are the cylindrical coordinates in $\mathbb{R}^3$. Here $m_z = \pm\frac{1}{2}$ represents the total angular momentum up or down.

In this paper, using variational methods along the lines of [13], we extend their result for the Maxwell-Dirac model. Indeed, we show the existence of stationary states of the form (0.2) in the range $\omega \in]-m, m[$. On the subspace of states of the form (0.2), the electromagnetic field has axial symmetry. As a consequence, the nonlinear part of the functional is positive definite (compare with (2.1) in [13]). Then the proof of the existence of stationary states for $\omega \in]-m, 0]$ is treatable and, at the same time, some estimates for the case $\omega \in]0, m[$ are straightforward. In any case, the scheme of the proof is quite similar to that in [13] and for brevity we give here only a sketch of the proof pointing out the differences with respect to the paper of Esteban et al. Moreover, since many estimates and technical lemmata are the same, for simplicity we adopt notations consistent with [13].

Finally we prove that there exists an infinite number of solutions for the Coulomb-Dirac model. This model is obtained neglecting the magnetic field in the Maxwell-Dirac equations and has no physical significance since it is not Lorenz invariant. In any case we think interesting to consider it here, since, using ansatz (0.2), the Coulombian term in the nonlinear part of the Maxwell-Dirac functional is dominant (compare the functional F^ω in section 2 with the Coulomb-Dirac functional in section 4 and see also [29]).

In the following section we introduce notations and list the main results. In section 2 we briefly outline the linking structure of the functional. We consider the Maxwell-Dirac functional restricted to the subspace of axially symmetric stationary states in $H^{1/2}$. The functional restricted to this space is still non compact and non convex, as in [13]. Hence, we prove existence of a critical point using Hofer-Wysocki linking theorem [21] (for other applications see [14], [15], [25], [27], [28]). On the other side, in our setup, the Maxwell-Dirac functional has positive definite nonlinear part: this property allows to show the existence of min-max levels for $\omega \in]-m, 0]$ and to simplify the arguments in [13]. Indeed the possibility of proving the existence of min - max levels also for $\omega \leq 0$ is a consequence of formula (2.3) and inequality (2.2) (compare them with (2.4) and (2.2) in [13]).

In section 3, we prove the existence of a non trivial axially symmetric critical point which corresponds to a solution of the Maxwell-Dirac equations. Here the major simplification w.r.t. the proof in [13] comes from the possibility of giving uniform bounds on the norms of certain sequences of axially symmetric states and so of proving directly their local convergence in $H^{1/2}$. Finally, the existence of stationary states is a consequence of concentration-compactness Lemma [22] as in [13].

In section 4 we consider the case of Coulomb-Dirac equations (see for instance [29], [17]) and we show the existence of infinitely many solutions. For this problem, we study the existence of stationary states in a spherically symmetric subspace of $H^{1/2}$ already introduced in [11] and used in [13] in the case of the Klein-Gordon-Dirac model. The existence proof is an easy consequence of compact embedding properties of this space in convenient L^p spaces and of the regularizing properties of the inverse of the Dirac operator. We establish the multiplicity result proving that, for any $\omega \in]-m, m[$ there exists an infinite number of critical levels c_n such that $\lim_{n\to+\infty} c_n = +\infty$.

1. MAIN RESULTS AND NOTATIONS

The Maxwell-Dirac equations [6] are

$$\begin{cases} (i\gamma^\mu \partial_\mu - \gamma^\mu A_\mu)\psi - m\psi = 0 & \text{in } \mathbb{R} \times \mathbb{R}^3 \\ \partial_\mu A^\mu = 0, \qquad 4\pi \partial_\mu \partial^\mu A^\nu = J^\nu & \text{in } \mathbb{R} \times \mathbb{R}^3 \end{cases} \tag{1.1}$$

where $\nu, \mu \in \{0, 1, 2, 3\}$, $m > 0$, $(\,,\,)$ is the usual hermitean product in $\mathbb{C}^4$, $\psi(x_0, x) \in \mathbb{C}^4$ for $(x_0, x) \in \mathbb{R} \times \mathbb{R}^3$ and $\gamma^0 = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right) \in \mathcal{M}^{4\times4}(\mathbb{C})$, $\gamma^k = \left(\begin{smallmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{smallmatrix}\right) \in \mathcal{M}^{4\times4}(\mathbb{C})$, $\overline{\psi} = \gamma^0\psi$, $J^\mu = (\overline{\psi}, \gamma^\mu\psi)$, $J_0 = J^0$, $J_k = -J^k$, $k = 1, 2, 3$, and σ^k are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

We look for bound states, that is solutions of (1.1) of the form

$$\begin{cases}\psi(x_0,x)=e^{i\omega x_0}\varphi(x)\,, & \varphi:\mathbb{R}^3\to\mathbb{C}^4\\ A^\mu(x)=J^\mu*\dfrac{1}{|x|}=\displaystyle\int_{\mathbb{R}^3}\frac{dy}{|x-y|}\,J^\mu(y), & \end{cases}\tag{1.3}$$

where $\omega\in]-m,m[$. Following [17], we consider the case where φ takes the form

$$\varphi(x)=\begin{pmatrix}u_1(r,z)\\ u_2(r,z)e^{i\phi}\\ -iu_3(r,z)\\ -iu_4(r,z)e^{i\phi}\end{pmatrix}\tag{1.4}$$

where (r,z,ϕ) are the cylindrical coordinates of $x\in\mathbb{R}^3$. In (1.4) we have fixed the angular momentum $m_z=\frac{1}{2}$ and the quantum number $k=1$, the other cases may be considered along the same lines. Substituting (1.3) in (1.1), the equations become

$$\begin{cases}i\gamma^k\partial_k\varphi-m\varphi-\omega\gamma^0\varphi-\gamma^\mu A_\mu\varphi=0 & \text{in }\ \mathbb{R}^3\\ -4\pi\Delta A_0=J^0\equiv|\varphi|^2,\qquad -4\pi\Delta A_k=-J^k & \text{in }\ \mathbb{R}^3\end{cases}\tag{1.5}$$

and stationary solutions are given by the critical points $\varphi\in H^{\frac{1}{2}}(\mathbb{R}^3,\mathbb{C}^4)$ of the functional

$$\begin{aligned}I^\omega_{\rm md}(\varphi)&=\int_{\mathbb{R}^3}\frac{1}{2}(i\gamma^0\gamma^k\partial_k\varphi,\varphi)-\frac{m}{2}(\overline{\varphi},\varphi)-\frac{\omega}{2}|\varphi|^2\\ &\quad-\frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{J^\mu(x)\,J_\mu(y)}{|x-y|}\,dx\,dy\\ &=\int_{\mathbb{R}^3}\frac{1}{2}(\varphi,D\varphi)-\frac{\omega}{2}||\varphi||_2^2-\frac{1}{4}\int J^\mu A_\mu\\ &=\frac{1}{2}(||P_+\varphi||^2-||P_-\varphi||^2)-\frac{\omega}{2}||\varphi||_2^2-\frac{1}{4}\int J^\mu A_\mu,\end{aligned}\tag{1.6}$$

where $E=H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$, $D=i\gamma^0\gamma^k\partial_k-m\gamma^0$, $|D|=(D^2)^{\frac{1}{2}}$, $(f|g)_E=\int(f,|D|g)$, $||\cdot||\equiv||\cdot||_E$, $||\cdot||_p\equiv||\cdot||_{L^p}$, $P_\pm=\frac{|D|^{-1}}{2}(|D|\pm D)$, $\overline{\varphi}=\gamma^0\varphi$, $J^\mu=(\overline{\varphi},\gamma^\mu\varphi)$, $A^\mu=J^\mu*\frac{1}{|x|}$, $J^\mu A_\mu=J^0A_0-\sum_{k=1}^3 J^kA_k$.

In sections 2 and 3 we prove

THEOREM A. – *For any $\omega\in]-m,m[$ there exists a non-zero critical point φ^ω of $I^\omega_{\rm md}$ in $H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$. φ^ω is of the form given in equation (1.4) and is a smooth function of x exponentially decreasing at infinity with all its* *derivatives. Finally,* $\psi(x_0, x) = e^{i\omega x_0}\varphi^\omega$, $A_\mu(x_0, x) = J_\mu * \frac{1}{|x|}$ *are solutions of the Maxwell-Dirac system.*

We also consider stationary solutions for the Coulomb-Dirac model [28]

$$\begin{cases} (i\gamma^\mu\partial_\mu - \gamma^0 A_0)\psi - m\psi = 0 & \text{in} \quad \mathbb{R}\times\mathbb{R}^3 \\ 4\pi\,\partial_k\partial^k\, A^0 = J^0 & \text{in} \quad \mathbb{R}\times\mathbb{R}^3, \end{cases} \tag{1.7}$$

where $k = 1, ..., 3$, m, γ^0 are as above. We look for solutions of the form

$$\begin{cases} \psi(x_0, x) = e^{i\omega x_0}\varphi(x)\,, \qquad \varphi : \mathbb{R}^3 \to \mathbb{C}^4 \\ A^0(x) = J^0 * \dfrac{1}{|x|} = \displaystyle\int_{\mathbb{R}^3} \frac{dy}{|x-y|}\, J^0(y). \end{cases} \tag{1.8}$$

where $\omega \in]-m, m[$ and

$$\varphi(x) = \begin{pmatrix} w_1(r) \\ 0 \\ iw_2(r)\cos\theta \\ iw_2(r)e^{i\phi}\sin\theta \end{pmatrix} \tag{1.9}$$

where (r, θ, ϕ) are the spherical coordinates of $x \in \mathbb{R}^3$. Stationary solutions of

$$\begin{cases} i\gamma^k\partial_k\varphi - m\varphi - \omega\gamma^0\varphi - \gamma^0 A_0\varphi = 0 & \text{in} \quad \mathbb{R}^3 \\ -4\pi\Delta A_0 = J^0 = |\varphi|^2 & \text{in} \quad \mathbb{R}^3. \end{cases} \tag{1.10}$$

are given by the critical points $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ of the functional

$$\begin{aligned} I^\omega_{cd}(\varphi) &= \int_{\mathbb{R}^3} \frac{1}{2}\,(i\gamma^0\gamma^k\partial_k\varphi, \varphi) - \frac{m}{2}\,(\overline{\varphi}, \varphi) - \frac{\omega}{2}\,|\varphi|^2 \\ &\quad - \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{J^0(x)\,J_0(y)}{|x-y|}\,dx\,dy \\ &= \int_{\mathbb{R}^3} \frac{1}{2}\,(\varphi, D\varphi) - \frac{\omega}{2}\,||\varphi||_2^2 - \frac{1}{4}\int J^0 A_0 \\ &= \frac{1}{2}\big(||P_+\varphi||^2 - ||P_-\varphi||^2\big) - \frac{\omega}{2}\,||\varphi||_2^2 - \frac{1}{4}\int J^0 A_0, \end{aligned} \tag{1.11}$$

where we use the same notations as before. In sections 4 and 5 we prove

THEOREM B. – *For any* $\omega \in]-m, m[$ *there are infinitely many critical levels* c^ω_N, $N \in \mathbb{N}$ *such that* $\lim_{N\to+\infty} c^\omega_N = +\infty$. *To each* c^ω_N *there corresponds a critical point* φ^ω_N *of* I^ω_{cd} *in* E *of the form (1.7).* φ^ω_N *are* *smooth functions, exponentially decreasing at infinity together with all theirs derivatives. Moreover the fields* $\psi(x_0, x) = e^{i\omega x_0}\varphi(x)$, $A^0(x)$ *are stationary solutions of the Coulomb-Dirac equations.*

Remark. – Both for Maxwell-Dirac and Coulomb-Dirac models, we prove the existence of stationary states in $H^{\frac{1}{2}}$. The proofs of regularity and fast decreasing properties are omitted, since they follow from a bootstrap argument and estimates on solutions essentially along the same lines as in [15] and [13].

2. LINKING PROCEDURE FOR THE MAXWELL-DIRAC EQUATIONS

In this section we describe the linking in the case of the stationary Maxwell-Dirac equations (1.3). Let us denote by $E^{s'}$ the subspace of E given by φ of the form (1.4) and $E_\pm = P_\pm E$. $E^{s'}_\pm \equiv P_\pm E^{s'} \subset E^{s'}$; since $E^{s'}$ is a stable subspace of E w.r.t. D, we work directly in $E^{s'}$. In the space $E^{s'}$ we construct a linking of the kind introduced by Hofer-Wysocki [21] and already applied to the Maxwell-Dirac model directly in E in [13], when $0 < \omega < m$. First of all let us observe that in the space $E^{s'}$:

$$\begin{aligned}
J^0(x) &= \sum_{i=1}^{4} |u_i(r,z)|^2 \\
J^1(x) &= 2(u_1u_4 - u_2u_3)(r,z)\sin(\phi) \\
J^2(x) &= 2(u_1u_4 - u_2u_3)(r,z)\cos(\phi) \\
J^3(x) &= 0,
\end{aligned} \tag{2.1}$$

where (r, z, ϕ) are the cylindrical coordinates of x. Then $A_3 \equiv 0$ so that the magnetic field is oriented along the z-axis. The key point, which allows the extension of the arguments of [13] to the case $\omega \leq 0$, is the following inequality

$$\begin{aligned}
\int J_\mu A^\mu(\varphi) &= \int J_0 A^0 - J_k A^k \\
&\geq \mathcal{Q}(\varphi) \equiv \int \sum_{i=1}^{4} |u_i(r_1,z_1)|^2 \sum_{j=1}^{4} |u_j(r_2,z_2)|^2 \\
&\quad \times \frac{r_1 r_2 (1-\cos(\phi_1-\phi_2)) dr_1 dr_2 dz_1 dz_2 d\phi_1 d\phi_2}{\sqrt{r_1^2 + r_2^2 + (z_1-z_2)^2 - 2r_1 r_2 \cos(\phi_1-\phi_2)}}.
\end{aligned} \tag{2.2}$$

(2.2) easily follows from (2.1) and $\sum_{i=1}^{4} u_i^2 \geq 2u_1u_4 - 2u_2u_3$, since

$$\begin{aligned}
&\int J_0 A^0 = \int |\varphi|^2(x)|\varphi|^2(y)\frac{dxdy}{|x-y|}\\
&\int J_k A^k = 4\int r_1 r_2 (u_4u_1 - u_2u_3)(r_1, z_1)\\
&\quad\times (u_4u_1 - u_2u_3)(r_2, z_2)dr_1dr_2dz_1dz_2\\
&\int_{[-\pi,\pi]\times[-\pi,\pi]} \frac{\cos(\phi_1-\phi_2)d\phi_1 d\phi_2}{\sqrt{r_1^2+r_2^2+(z_1-z_2)^2-2r_1r_2\cos(\phi_1-\phi_2)}}. \qquad (2.3)
\end{aligned}$$

We now prove

LEMMA 1. – *Let $\varphi = \varphi_- + e_+$, where $\varphi_- \in E_-$ and $e_+ \in E_+^N$, where E_+^N is any N dimensional subspace in $E^{s'} \cap E_+$. Then $\exists C > 0$, depending on E_+^N such that*

$$\mathcal{Q}(\varphi) \geq C||\varphi||^4 \qquad (2.4)$$

Proof. – The proof is by contradiction. In fact $\mathcal{Q}$ is homogeneous of degree 4, so, if the lemma is false, there exists a sequence $\varphi_-^{(n)} \in E_-$ with $||\varphi_-^{(n)}|| \leq 1$ and a sequence $e_+^{(n)} \in E_+^N$ with $||e_+^{(n)}|| = 1$, such that $\mathcal{Q}(\varphi_-^{(n)} + e_+^{(n)}) \to 0$. After extraction, the sequence $e_+^{(n)}$ converges to $\tilde{e}_+ \in E_+^N$ and the sequence $\varphi_-^{(n)}$ weakly converges to $\tilde{\varphi}_-$. Moreover,

$$||\tilde{e}_+|| = 1$$

and $||\tilde{\varphi}_-|| \leq 1$. Since $\mathcal{Q}$ is convex and continuous, it is also weakly lower semi-continuous, so that $\mathcal{Q}(\tilde{\varphi}_- + \tilde{e}_+) = 0$. But from equation (2.2), we conclude that $\tilde{\varphi}_- + \tilde{e}_+ = 0$ for a.e. $x \in \mathbb{R}^3$. Since, by construction $\tilde{\varphi}_-$ and $\tilde{e}_+$ are orthogonal in E, then $\tilde{e}_+ = 0$ a.e. $x \in \mathbb{R}^3$, giving a contradiction with (2.5). □

Using the lemma above, we prove

PROPOSITION 2. – $\forall \omega^* \in]0, m[$, $\exists R^* = R(\omega^*), r^* = r(\omega^*), \rho^* = \rho(\omega^*) > 0$, $R^* > r^*$ *and* $e_+ \in E_+ \cap E^{s'} \equiv E_+^{s'}$, *with* $||e_+|| = R^*$, *such that, if we denote*

$$\begin{aligned}
\mathcal{N}_- &= \{\varphi = \varphi_- + \lambda e_+,\ \varphi_- \in E_-^{s'},\ ||\varphi_-|| \leq R^*,\ \lambda \in [0,1]\},\\
\partial\mathcal{N}_- &= \{\varphi = \varphi_- + \lambda e_+ \in \mathcal{N}_- \mid \textit{either } ||\varphi_-|| = R^*, \textit{ or } \lambda = 0, 1\},\\
\Sigma^+ &= \{\varphi \in E^{s'} \cap E_+ \ :\ ||\varphi|| = r^*\}, \qquad (2.5)
\end{aligned}$$

then

$$\begin{aligned} &I^{\omega}_{\rm md}(\varphi) \le 0 && \forall \varphi \in \partial\mathcal{N}_- \ , \ \forall \omega \in [-\omega^*, m[\\ &I^{\omega}_{\rm md}(\varphi) \ge \rho^* > 0 && \forall \varphi \in \Sigma^+, \ \forall \omega \le \omega^*. \end{aligned} \tag{2.6}$$

Proof. – We give just the sketch of the proof, pointing only the differences with the scheme of [13]. First of all let us notice that $E^{s'}_+ \neq \emptyset$, since the vector constructed in [13] belongs to $E^{s'}_+$. Let $\phi_+ \in E^{s'}_+$ with $||\phi_+|| = 1$ and $Q(\phi_+) > 0$. Then there exists $R > 0$ such that, if $e_+ \equiv R\phi_+$,

$$I^{\omega}_{\rm md}(e_+) \le 0. \tag{2.7}$$

Let us now prove that $I^{\omega}_{\rm md}(\varphi) \le 0$ for any $\varphi \in \partial\mathcal{N}_-$. We show that the following auxiliary functional

$$F^{\omega}(\varphi) = \frac{1}{2}\left[\, ||P_+\varphi||^2 - ||P_-\varphi||^2 - \omega||\varphi||_2^2 \right] - \frac{1}{4}\mathcal{Q}(\varphi) \le 0 \quad \forall \varphi \in \partial\mathcal{N}_-. \tag{2.8}$$

Since (2.2) implies that

$$I^{\omega}_{\rm md}(\varphi) \le F^{\omega}(\varphi),$$

the claim follows immediately.

From Lemma 1, if $\varphi = \varphi_- + e_+$, with $\varphi_- \in E_-$, e_+ as above, $||\varphi_-|| \le ||e_+|| = R$, then we may find a constant $C > 0$ such that $\mathcal{Q}(\varphi) \ge C||\varphi||^4 \ge CR^4$. Then

$$\begin{aligned} \omega \ge 0: &\qquad F^{\omega}(\varphi) \le \frac{1}{2}R^2\left[1 - \frac{1}{2}CR^2\right] \\ \omega < 0: &\qquad F^{\omega}(\varphi) \le \frac{1}{2}R^2\left[1 - 2\frac{\omega}{m} - \frac{1}{2}CR^2\right]. \end{aligned}$$

Moreover if $\varphi \in E_-$ and $\omega \ge -m$, then

$$F^{\omega}(\varphi) = -\frac{1}{2}\left(||\varphi||^2 + \omega||\varphi||_2^2\right) - \frac{1}{4}\mathcal{Q}(\varphi) \le -\frac{1}{2}\left(||\varphi||^2 - m||\varphi||_2^2\right) \le 0,$$

since $m||\varphi||_2^2 \le ||\varphi||_E^2$.

We have then just to check (2.8) when $\varphi = \varphi_- + \lambda e_+$, where $\varphi \in E_-$, e_+ as above, $||\varphi_-|| = R = ||e_+||$ and $\lambda \in]0, 1[$.

In the case $\omega \ge 0$ we have

$$F^{\omega}(\varphi) \le \frac{\lambda^2}{2}||e_+||^2 - \frac{1}{2}||\varphi_-||^2.$$

If $\omega \in [-\omega^*, 0]$ with $\omega^* \in]0, m[$,

$$F^\omega(\varphi) \leq \frac{1}{2}R^2\Big(\lambda^2 - 1 - \frac{\omega}{m}(\lambda^2+1)\Big) - \frac{1}{4}\mathcal{Q}(\varphi).$$

We consider two cases. If $\lambda \in [0, \lambda^*]$ where $\lambda^* = \sqrt{\left[1-\frac{\omega^*}{m}\right]\left[1+\frac{\omega^*}{m}\right]}$, (2.8) follows trivially. If $\lambda \in [\lambda^*, 1[$, for all $\varphi = \varphi_- + \lambda e_+$, from Lemma 1, $\mathcal{Q}(\varphi) \geq C||\varphi||^4 \geq CR^4$, so that (2.7) is satisfied for R sufficiently big.

Collecting the above results, we may choose $R^* < \infty$ such that

$$I^\omega_{\mathrm{md}}(\varphi) \leq F^\omega(\varphi) \leq 0 \qquad \forall \varphi \in \partial\mathcal{N}_-$$

The proof of the second of (2.6) is straightforward. □

Let us now denote by

$$\begin{aligned}\nabla I^\omega_{\mathrm{md}}(\varphi) &= |D|^{-1}(I^\omega_{\mathrm{md}})'(\varphi)\\ &= P_+\varphi - P_-\varphi - |D|^{-1}\Big[\omega\varphi - \big(A_\mu(\varphi)\gamma^\mu\gamma^0\varphi\big)\Big]\\ &= P_+\varphi - P_-\varphi - B^\omega(\varphi).\end{aligned} \tag{2.9}$$

If $\varphi \in E^{s'}$ is a critical point of I^ω_{md} w.r.t. $E^{s'}$ then φ is a critical point of I^ω_{md} w.r.t. E, since $\nabla I^\omega_{\mathrm{md}}(\varphi) \in E^{s'}$. As in [13], B^ω is odd, nonlinear and continuous, but does not map bounded sets into relatively compact sets, since the problem is still invariant by translation along the z axis.

Let us define the following flow $h^\omega_t(\varphi)$ associated to $\nabla I^\omega_{\mathrm{md}}$ restricted to $E^{s'}$:

$$\begin{cases}\dfrac{\partial}{\partial t}h^\omega_t = -\nabla I^\omega_{\mathrm{md}} \circ h^\omega_t\\ \quad h^\omega_0 = Id_{E^{s'}}\end{cases} \tag{2.10}$$

The flow is well defined for $t \geq 0$ since $\nabla I^\omega_{\mathrm{md}}$ is Lipschitz continuous and I^ω_{md} is non-increasing along the flow. In particular $\forall t \geq 0$, $h^\omega_t(\partial\mathcal{N}_-) \cap \Sigma^+ = \emptyset$.

PROPOSITION 3. – *$\forall \omega^* \in]0, m[$, let $\mathcal{N}_-$, $\partial\mathcal{N}_-$, $\Sigma^+ \subset E^{s'}$, r^*, ρ^*, R^* be as in Proposition 2. Then*

$$h^\omega_t(\mathcal{N}_-) \cap \Sigma^+ \neq \emptyset \qquad \forall t \geq 0, \quad \forall \omega \in [-\omega^*, \omega^*]. \tag{2.11}$$

The proof of Proposition is as in [13] (see also [21], [15], [30]).

COROLLARY 4.

1) $c^\omega = \inf_{t\geq 0} \sup_{\mathcal{N}_-} I^\omega_{\mathrm{md}} \circ h^\omega_t(\varphi) > 0$;

2) *Given* $\omega \in]-m,m[$, *there exists* $\{\varphi^{(n)}\} \subset E^{s'}$ *Cerami sequence at the critical level* c^ω *such that*

$$\begin{aligned} &a)\ I^\omega_{\mathrm{md}}(\varphi^{(n)}) \ \mapsto\ c^\omega \\ &b)\ (1+||\varphi^{(n)}||)\nabla I^\omega_{\mathrm{md}}(\varphi^{(n)}) \ \mapsto\ 0 \end{aligned} \quad as \quad n \to +\infty \tag{2.12}$$

Proof. – Since $h^\omega_t(\mathcal{N}_-) \cap \Sigma^+ \neq \emptyset$, $c^\omega \geq \rho^* > 0$; since $\mathcal{N}_-$ is bounded and the flow decreasing, $c^\omega \leq \Gamma < +\infty$. Finally, we get the existence of the Cerami sequence [9] using standard arguments. □

3. EXISTENCE OF A NON TRIVIAL CRITICAL POINT FOR MAXWELL-DIRAC MODEL

In this section we first show that the Cerami sequences introduced above are uniformly bounded from below and above. From these estimates we are then able to show the existence of a non trivial critical point at the critical level $\tilde{c}^\omega$, where $0 < \tilde{c}^\omega \leq c^\omega$.

If φ is a solution of problem (1.5), it obviously satisfies $I^\omega_{\mathrm{md}}(\varphi) = I^\omega_{\mathrm{md}}(\varphi) - \frac{1}{2}\langle \nabla I^\omega_{\mathrm{md}}(\varphi), \varphi\rangle$ and the Pohozaev identity

$$\int (i\gamma^0\gamma^k\partial_k\varphi, \varphi) = \frac{3}{2}\int \Big(m(\bar{\varphi},\varphi) + \omega|\varphi|^2 + \frac{5}{6}J^\mu A_\mu\Big).$$

Using these identities, we can prove

LEMMA 5. – *Let* $\varphi \in E^{s'}$ *be a non trivial critical point of* $I^\omega_{\mathrm{md}} = c$. *Then*

$$\begin{aligned} &\frac{1}{4}\int J^\mu A_\mu(\varphi) = c \\ &-\frac{1}{2}\int (m(\bar{\varphi}\varphi) + \omega|\varphi|^2) = c \\ &\frac{1}{2}\int (i\gamma^0\gamma^k\partial_k\varphi, \varphi) = c \qquad \square \end{aligned} \tag{3.1}$$

PROPOSITION 6. – *Let* $\{\varphi^{(n)}\}$ *be a Cerami sequence at the level c, then there exist constants* $C' \geq C'' > 0$ *such that*

$$0 < C'' < ||\varphi^{(n)}|| \leq C' < +\infty. \tag{3.2}$$

The proof of Proposition 6 follows the same scheme as in Lemmas 3.2 and 3.5 and Theorem 1 in [13] directly for the Cerami sequences of

Lemma 2.4. Indeed in [13] it is unknown whether the corresponding $\varphi^{(n)}$ are bounded and the authors introduce a sequence of auxiliary problems for which the analogue of Proposition 6 holds true.

We give just a sketch of the proof. The Cerami sequences at the level c are obviously bounded from below since

$$I^{\omega}_{\mathrm{md}}(\varphi^{(n)}) = c\,(1 + o(1)) \;\leq\; \alpha\,||\varphi^{(n)}||^2 \qquad n >> 0 \qquad (3.3)$$

where $\alpha = \frac{1}{2}$ if $\omega \geq 0$ and $\alpha = 1$ if $\omega < 0$.

The proof that the Cerami sequence is bounded from above is by contradiction. Indeed if $||\varphi^{(n)}|| \to +\infty$, let

$$\tilde{\varphi}^{(n)} = \frac{\varphi^{(n)}}{||\varphi^{(n)}||}$$

Since $\nabla I^{\omega}_{\mathrm{md}}(\varphi^{(n)}) \to 0$, we have that

$$\begin{aligned}\nabla I^{\omega}_{\mathrm{md}}(\varphi^{(n)}) &= |D|^{-1}(D\varphi^{(n)} - \omega\varphi^{(n)} - \gamma^0\gamma^\mu A_\mu(\varphi^{(n)})\varphi^{(n)}) \equiv |D|^{-1}Q^{(n)} \\ &= o(1) \in E.\end{aligned}$$

Let $\tilde{Q}^{(n)} = Q^{(n)}||\varphi^{(n)}||^{-2} = D\tilde{\varphi}^{(n)} - \omega\tilde{\varphi}^{(n)} - \gamma^0\gamma^\mu A_\mu(\varphi^{(n)})\tilde{\varphi}^{(n)}$, then $\tilde{Q} \to 0$ in E'.

The sequence $\{\tilde{\varphi}^{(n)}\}$ is relatively compact in $H^{1/2}_{loc}(\mathbb{R}^3)$ since, using estimates analogue to those in [13],

$$\psi^{(n)} \equiv \tilde{\varphi}^{(n)} - (D - \omega I)^{-1}\tilde{Q}^{(n)} = (D - \omega I)^{-1}\gamma^0\gamma^\mu A_\mu(\varphi^{(n)})\tilde{\varphi}^{(n)}$$

is relatively compact in $H^{1/2}_{loc}$.

As in [13], we then apply the concentration-compactness Lemma [22] to the sequence

$$\rho^{(n)} = \langle\tilde{\varphi}^{(n)}, |D|\tilde{\varphi}^{(n)}\rangle.$$

Vanishing is excluded, so we may only have compactness or dichotomy. Moreover, from (2.3), it is possible to show that lack of compactness may only occur in the z-direction. Therefore there exist $q \geq 1$, $\xi_i \in E^{s'}$ and $\bar{y}_i^{(n)} = (0, 0, y_i^{(n)}) \in \mathbb{R}^3$, $i = 1, ..., q$, such that $\left|y_j^{(n)} - y_k^{(n)}\right| \to +\infty$, as $n \to +\infty$, if $j \neq k$ and such that

$$\left|\left|\tilde{\varphi}^{(n)} - \sum_{i=1}^{q} \xi_i(\cdot - \bar{y}_i^{(n)})\right|\right| \to 0, \quad \text{as } n \to +\infty.$$

On the other side since $A^\mu J_\mu(\tilde{\varphi}^{(n)}) \to 0$, we have that

$$0 \sum_{i=1}^{q} \int A^\mu J_\mu(\xi_i) \geq \sum_{i=1}^{q} \mathcal{Q}(\xi_i).$$

Therefore $|\xi_i|^2 = 0$ a.e. $x \in \mathbb{R}$, $\forall i$, but this is in contradiction with the fact that ξ_i are not all identically zero.

PROPOSITION 7. – *Let $\omega \in]-m, m[$ be fixed. Let $\varphi^{(n)} \in E^{s'}$ be a Cerami sequence at the critical level $c > 0$ such that*

$$0 < \inf_n ||\varphi^{(n)}|| \leq \sup_n ||\varphi^{(n)}|| < +\infty.$$

Then there exist $1 \leq p < +\infty$ non-trivial solutions of the Maxwell-Dirac equations $\phi_1, ..., \phi_p$ and $\bar{x}_i^{(n)} \equiv (0, 0, x_i^{(n)}) \subset \mathbb{R}^3$, $i = 1, ..., p$ such that $|x_j^{(n)} - x_k^{(n)}| \to +\infty$ if $j \neq k$ and

$$||\varphi^{(n)} - \sum_{i=1}^{p} \phi_i(\cdot - \bar{x}_i^{(n)})|| \to 0, \qquad as\ n \to +\infty.$$

Proof. – As in the proof of Proposition 6 above and in Proposition 3.6 in [13], $\varphi^{(n)}$ is relatively compact and using concentration compactness one can show the existence of p solutions of the Maxwell-Dirac equations.

p is finite since, as in [13], there exists a constant $K > 0$ such that, if φ is a non-trivial critical point, then $I^\omega_{\text{md}}(\varphi) \geq K$ and $||\varphi||^2 \geq 2K$.

Moreover the lack of compactness, due to the symmetry properties of the functional restricted to $E^{s'}$, can only occur along the axis in which the magnetic field points (the z-axis in our case). □

COROLLARY 8. – *Let $\omega \in]-m, m[$ be fixed. Then $\exists \varphi \in E^{s'}$ such that $(I^\omega_{\text{md}})'(\varphi) = 0$ and $0 < I^\omega_{\text{md}}(\varphi) \leq c^\omega$.*

Proof. – It is sufficient to apply Proposition 6 and 7 to the Cerami sequence of Corollary 4, and set $\varphi = \phi_1$. □

4. THE COULOMB CASE

Let E^s be the subspace of E given by φ of the form (1.9). We denote $E_\pm = P_\pm E$ and $E^s_\pm = P_\pm E^s \subset E^s$. E^s is stable w.r.t. E, so we look for stationary states using the restriction of the Coulomb functional to E^s. In this setting, the linking follows from classical arguments (see for instance [1], [5], [15]) and we only give some details here for completeness.

PROPOSITION 9. – *$\forall N \in \mathbb{N}^*$, $\forall \omega^* \in]0, m[$, $\exists R_* = R(\omega^*, N)$, $r_* = r(\omega^*)$, $\rho_* = \rho(\omega^*) > 0$, $R_* > r_*$ and $E_+^N \subset E_+ \cap E^s$ such that, if we denote*

$$\begin{aligned}\mathcal{M}_-^N &= \{\varphi = \varphi_- + \lambda e_+,\ \varphi_- \in E_-^s,\ \|\varphi_-\| \le R_*,\ e_+ \in E_+^N,\\ &\qquad \|e_+\| = R_*\ \lambda \in [0,1]\}\\ \partial\mathcal{M}_-^N &= \{\varphi = \varphi_- + \lambda e_+ \in \mathcal{N}_- \mid \ \text{either } \|\varphi_-\| = R_*, \text{ or } \lambda = 0, 1\}\\ \Delta^+ &= \{\varphi \in E_+^s : \|\varphi\| = r_*\},\end{aligned} \tag{4.1}$$

then

$$\begin{aligned} I_{\text{cd}}^\omega(\varphi) &\le 0 &\quad& \forall \varphi \in \partial\mathcal{M}_-^N, \quad \forall \omega \in [-\omega^*, m[\\ I_{\text{cd}}^\omega(\varphi) &\ge \rho_* > 0 && \forall \varphi \in \Delta^+, \quad \forall \omega \le \omega^* \end{aligned} \tag{4.2}$$

We just give a sketch of the proof. Let E_+^N be any finite dimensional subset of E_+^s. Notice that E_+^s is an infinite dimensional subspace of E_+ since the set of states $P_+\phi$ introduced in Proposition 2 are contained in E_+^s. Moreover $E^s \subset E^{s'}$.

We may again apply Lemma 1, since

$$\mathcal{S}(\varphi) = \int \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|}dxdy \ \ge\ \int J^\mu A_\mu(\varphi)\ \ge\ C\|\varphi\|^4, \tag{4.3}$$

so that the rest of the proof is similar to that of Proposition 2. □

Let us denote by

$$\begin{aligned}\nabla I_{\text{cd}}^\omega(\varphi) &= |D|^{-1}(I_{\text{cd}}^\omega)'\\ &= P_+\varphi - P_-\varphi - \omega|D|^{-1}\varphi - |D|^{-1}\left(|\varphi^2| * \frac{1}{|x|}\right)\varphi\\ &= P_+\varphi - P_-\varphi - K^\omega(\varphi).\end{aligned} \tag{4.4}$$

If $\varphi \in E^s$ is a critical point of I_{cd}^ω w.r.t. E^s then φ is a critical point of I_{cd}^ω w.r.t. E since $\nabla I_{\text{cd}}^\omega(\varphi) \in E^s$.

Moreover K^ω is odd, nonlinear, and maps bounded sets into relatively compact sets. If $\varphi^{(n)} \in E^s$ and $\|\varphi^{(n)}\| \le C$, then $K^\omega(\varphi^{(n)})$ is relatively compact in $H^{\frac{1}{2}}$. Let

$$\begin{cases} \dfrac{\partial}{\partial t}\eta_t^\omega = \nabla I_{\text{cd}}^\omega \circ \eta_t^\omega \\ \eta_0^\omega = Id_{E^s}. \end{cases} \tag{4.5}$$

Then, the flow η_t^ω is well defined for $t \ge 0$ and it is non-decreasing.

PROPOSITION 10. – *$\forall N \in \mathbb{N}^*$, $\forall \omega^* \in]0, m[$, let $\mathcal{M}_-^N$, $\partial\mathcal{M}_-^N$, $\Delta^+ \subset E^s$, r_*, ρ_*, R_* as above. Then*

$$\gamma\left[\eta_t^\omega(\Delta^+) \cap \mathcal{M}_-^N\right] \geq N, \qquad \forall \omega \in [-\omega^*, \omega^*], \quad \forall t \geq 0, \tag{4.6}$$

where γ is the $\mathbb{Z}_2$-degree of symmetric sets.

The proof is classical and is omitted (see for instance [1]).

PROPOSITION 11. – *Under the above hypotheses, let $\omega \in [-\omega^*, \omega^*]$ be fixed, and let*

$$c_N^\omega = \inf_{X \in \Gamma^N} \sup_{\varphi \in X} I_{\mathrm{cd}}^\omega(\varphi), \tag{4.7}$$

where

$$\Gamma^N = \left\{ X \subset E^s : X = -X \ \text{and} \ \forall t \geq 0 \quad \gamma(\eta_t^\omega(\Delta^+) \cap X) \geq N \right\}.$$

Then

1) $0 < \rho \leq c_N^\omega \leq \rho_N < +\infty$;
2) $c_{N_1}^\omega \leq c_{N_2}^\omega$, $\forall N_1, N_2 \in \mathbb{N}$, $N_1 < N_2$;
3) $\forall N > 0$ *there exists* $\left\{\varphi^{(n)}\right\} \subset E^s$ *s.t.*

$$I_{\mathrm{cd}}^\omega(\varphi^{(n)}) \to c_N^\omega; \quad ||\nabla I_{\mathrm{cd}}^\omega(\varphi^{(n)})|| \left(1 + ||\varphi^{(n)}||\right) \to 0 \quad as \quad n \to +\infty \tag{4.8}$$ □

The proof of Proposition 11 is standard and is omitted. The existence of a bounded state at each critical value c_N^ω easily follows. Indeed any Cerami sequence is bounded. The proof is again by contradiction and is simpler than in Proposition 6. The convergence of bounded Cerami sequences to non-trivial critical points is then straightforward and follows from the properties of compact embedding of E^s in appropriate L^p spaces (see for instance [13]). Finally, we get existence of critical points in E^s, applying the above results to the sequences of Proposition 11.

For $\omega \in]-m, m[$ fixed, we get the existence of an infinite number of solutions proving $\lim_{N \to \infty} c_N = +\infty$, where we omit the index ω for simplicity in the sequences c_N^ω of Proposition 11. Indeed, by construction, there exists $\lim_{N \to +\infty} c_N$. Suppose by contradiction that the limit is finite and denote it with $\bar{c}$.

It is possible to show that the sets

$$\{\varphi \in E^s \ : \ I_{\mathrm{cd}}(\varphi) \in [a_1, a_2], \nabla I_{\mathrm{cd}}(\varphi) = 0\} \qquad \forall a_1, a_2 \in \mathbb{R}, \ a_1 \leq a_2$$

are compact. In particular the critical set associated to the $\bar{c}$ is also compact.

Moreover, using classical arguments (see for instance [1], [15]), it is possible to prove that

$$c_N < \bar{c} \qquad \forall N \geq 1.$$

Then, using the properties of the degree, we get a contradiction since $\gamma(\mathcal{K}^{\bar{c}})$ cannot be finite and $\mathcal{K}^{\bar{c}}$ is compact if $\bar{c}$ is finite. This is of course the required contradiction.

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