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The 2D Schrödinger equation for a neutral pair in a constant magnetic field

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ABSTRACT. – We consider the two-body Schrödinger operator for a neutral pair of particles in two-dimensional space with a constant magnetic field. The two particles interact through a potential. We prove absolute continuity of the spectrum under several different assumptions on the potential. These assumptions cover both the Coulomb and the Yukawa potential. We discuss some explicitly solvable models.

Key words: Schrödinger operator, magnetic field, absolutely continuous spectrum.

RÉSUMÉ. – Nous considérons l'opérateur de Schrödinger d'une paire neutre de particules bidimensionnelles soumise à un champ magnétique constant. Les deux particules intéragissent par l'intermédiaire d'un potentiel. Nous démontrons que le spectre est absolument continu sous plusieurs conditions sur le potentiel. Ces conditions sont satisfaites par les potentiels de Coulomb et de Yukawa. Nous considérons quelques modèles explicitement solubles.

1. INTRODUCTION

We consider the Schrödinger operator for a neutral pair of particles in two-dimensional space with a constant magnetic field. This operator is given by

$$H = \frac{1}{2m_1}(p_x - eA(x))^2 + \frac{1}{2m_2}(p_y + eA(y))^2 + V(x-y), \qquad (1.1)$$

acting on $\mathcal{H} = L^2(\mathbf{R}^4))$. Here $(x, y) \in \mathbf{R}^4 = \mathbf{R}^2 \times \mathbf{R}^2$, m_j, $j = 1, 2$ denotes the masses, e the (positive) charge, $p_x = -i\partial_x$ the momentum, $A(x)$ the vector potential, and V the interaction between the pair. The magnetic field $B = \mathrm{rot}A(x)$ is assumed to be constant. In this paper the spectrum of H is studied.

It is well-known that the spectrum of the one particle Hamiltonian $\frac{1}{2m}(p_x - eA(x))^2$ on $L^2(\mathbf{R}^2)$ consists of the Landau levels $\{\frac{eB}{m}(n+\frac{1}{2}) \mid n = 0, 1, 2, \ldots\}$. The spectrum is pure point and infinitely degenerate. Hence, if we take $V \equiv 0$ in (1.1), then the spectrum of H is also pure point and infinitely degenerate. Physically this corresponds to the classical motion of the particles in the constant magnetic field in two dimensions. If we further assume that the total charge is zero, which will be done throughout the paper, then the particles might be able to move freely in the magnetic field. Thus we arrive at our problem:

QUESTION. – *Is $\sigma(H)$ absolutely continuous, if $V \not\equiv 0$?*

The answer is *yes* for a fairly large class of potentials, including decaying potentials of Coulomb and Yukawa type, and growing potentials, *i.e.* $V(x) \to \infty$ as $|x| \to \infty$. These results are described in detail in Section 3, and then some solvable models and extensions are discussed in Section 4. In Section 5 we give some results on the analogue of effective mass in our model.

We expect to find absolutely continuous spectrum partly from analogy with the corresponding classical system of two particles with charges $\pm e$ in a constant magnetic field. The classical equations of motion admit solutions of the form $(\vec{x}(t), \vec{y}(t)) = (t\vec{v}, \vec{d} + t\vec{v})$, where $\vec{v}$ and $\vec{d}$ are chosen such the Lorenz force on each particle $\pm e\vec{v} \times \vec{B}$ cancels the force from the potential $-\nabla V(\pm\vec{d})$. This cancellation is easily shown to take place for e.g. the Coulomb interaction $V = -e^2/|\vec{x} - \vec{y}|$, and in both dimension two and three, in the latter case $\vec{v}$ and $\vec{d}$ are vectors in the plane orthogonal to $\vec{B}$. See [11] for discussion of the problems this type of motion causes in the study of the many-body problem.

By analogy with solid state physics one could call the pair with the interaction considered here for an exciton. See [1, page 626ff] for a discussion of excitons.

One may ask whether there exist non-constant potentials such that the spectrum of H is not purely absolutely continuous. We have not been able to answer this question. It seems to be a non-trivial question in inverse spectral theory.

The idea of the proofs is the following: We follow the argument of Avron, Herbst, and Simon [2] to separate the center of mass motion. This yields a family of operators $K(\xi)$, $\xi \in \mathbf{R}^2$, where ξ is the total pseudo-momentum. We show that each $K(\xi)$ has purely discrete spectrum, and the eigenvalues depend analytically on ξ. Then we use an analogue of the Floquet-Bloch theory for periodic Schrödinger operators to show absolute continuity of the spectrum of H. The key element of the proof is to show that the eigenvalues of $K(\xi)$ are non-constant with respect to ξ. We give several conditions on V for this property to hold, thus obtaining positive answers to the question above.

The proofs give more detailed information than mentioned above. For example, if V is a non-trivial potential, such that $V(z) \leq 0$ for all $z \in \mathbf{R}^2$, $V(z) \to 0$ as $|z| \to \infty$, and $\|V\|_{L^\infty} < eB/m_1$, then in the case $m_1 = m_2$ the spectrum of H consists of an infinite sequence of non-overlapping intervals (bands), with the right endpoints at eBk/m_1, $k = 1, 2, \ldots$.

Finally some remarks on results in the literature. We refer to [5, 6, 7, 14] for general results and references on magnetic Schrödinger operators. Recently there has been considerable interest in the many-body Schrödinger operator with a constant magnetic field in space dimension three (for each particle), and several results on the scattering theory have been obtained by Gérard and Laba, see the review paper [11] and references therein. The problem considered in this paper seems not to have been treated before.

2. NOTATION. PRELIMINARY RESULTS

We study the operator

$$H = H_0 + V = \frac{1}{2m_1}(p_x - eA(x))^2 + \frac{1}{2m_2}(p_y + eA(y))^2 + V(x-y), \quad (2.1)$$

on $\mathcal{H} = L^2(\mathbf{R}^4)$, where we write $(x, y) \in \mathbf{R}^4 = \mathbf{R}^2 \times \mathbf{R}^2$ and $x = (x_1, x_2)$, and for the momentum operators $p_x = -i\partial_x = (-i\partial_{x_1}, -i\partial_{x_2})$, etc. We assume the magnetic field is constant with value $B > 0$, and use the Coulomb gauge, so the vector potential is given by

$$A(x) = \frac{B}{2}(-x_2, x_1). \tag{2.2}$$

To state our assumptions on the potential V we recall the definition of the Kato class K_2 in dimension 2: $V \in K_2$, if $V \in L^1_{\mathrm{loc}}(\mathbf{R}^2)$ and

$$\lim_{r \downarrow 0}\left(\sup_{x \in \mathbf{R}^2} \int_{|x-y| \le r} (-\log|x-y|)|V(y)|\,dy \right) = 0.$$

Throughout the paper we suppose that V satisfies the following (rather weak) assumption.

ASSUMPTION 2.1. – *V is a real-valued function, decomposed as $V = V_+ - V_-$, $V_\pm \ge 0$, such that $V_+ \in L^1_{\mathrm{loc}}(\mathbf{R}^2)$, and $V_- \in K_2$.*

Under this assumption it is well-known (see for example [14]) that H is defined as a quadratic form on $\mathcal{Q}(H_0) \cap \mathcal{Q}(V_+)$, with $C_0^\infty(\mathbf{R}^4)$ as a form core. Thus H is well defined as a self-adjoint operator on $\mathcal{H}$. If furthermore $V \in L^2_{\mathrm{loc}}(\mathbf{R}^2)$, then $C_0^\infty(\mathbf{R}^4)$ is an operator core for H.

Since H describes a neutral pair, the center of mass motion can be removed. Let

$$k = p_x + eA(x) + p_y - eA(y) \tag{2.3}$$

denote the total pseudo-momentum, which is the generator of the magnetic translation group. For a neutral system the components commute, *i.e.* $[k_1, k_2] = 0$, which can be verified by direct computation. On the other hand, it is well-known (and easy to verify) that k commutes with H. Thus $\{H, k_1, k_2\}$ form a commuting family of self-adjoint operators, and therefore can be diagonalized simultaneously, see [2, 8]. In order to carry out concrete computations, we introduce an analogue of the Fourier transform corresponding to k. Let us write

$$z = x - y, \quad w = \frac{1}{2}(x + y). \tag{2.4}$$

Then k is written as

$$k = p_w + eA(z). \tag{2.5}$$

Generalized eigenfunctions of this operator are given by

$$\psi_\xi(z, w) = e^{iw\cdot(\xi - eA(z))}, \quad \xi \in \mathbf{R}^2, \tag{2.6}$$

and this leads to the definition of the transform

$$(\Phi f)(z, \xi) = \frac{1}{2\pi}\int e^{-iw\cdot(\xi - eA(z))} f(z, w)\,dw. \tag{2.7}$$

LEMMA 2.2. – *Φ is a unitary operator on $L^2(\mathbf{R}^4)$ and satisfies*

$$\Phi p_w \Phi^{-1} = \xi - eA(z), \quad \Phi w \Phi^{-1} = -p_\xi, \tag{2.8}$$

$$\Phi(p_z - eA(w))\Phi^{-1} = p_z, \Phi z \Phi^{-1} = z. \tag{2.9}$$

Moreover,

$$\Phi H \Phi^{-1} = \frac{1}{2m_1}\left(p_z + \frac{1}{2}\xi - eA(z)\right)^2 + \frac{1}{2m_2}\left(p_z - \frac{1}{2}\xi + eA(z)\right)^2 + V(z). \tag{2.10}$$

This Lemma follows by simple computations. Note that the expression (2.10) depends on the choice of gauge. Now in $\Phi H \Phi^{-1}$ the variable ξ appears only as a parameter, hence we can write

$$H = \int_{\mathbf{R}^2}^{\oplus} K(\xi)d\xi \quad \text{on} \quad L^2(\mathbf{R}^4) = \int_{\mathbf{R}^2}^{\oplus} L^2(\mathbf{R}^2)d\xi, \tag{2.11}$$

where the operator $K(\xi)$ is given by the right hand side of (2.10), acting on $L^2(\mathbf{R}^2)$. Since

$$eA(z) - \frac{1}{2}\xi = eA(z+\beta) \quad \text{with} \quad \beta = \frac{1}{eB}(-\xi_2, \xi_1), \tag{2.12}$$

we can translate $K(\xi)$ to get

$$K(\xi) \cong K(\beta) = \frac{1}{2m_1}(p_z - eA(z))^2 + \frac{1}{2m_2}(p_z + eA(z))^2 + V(z - \beta), \tag{2.13}$$

where $\cong$ denotes unitary equivalence. We first consider the spectrum of the free term

$$K_0 = \frac{1}{2m_1}(p_z - eA(z))^2 + \frac{1}{2m_2}(p_z + eA(z))^2 \tag{2.14}$$

LEMMA 2.3. – *The spectrum of K_0 is discrete, and is given by*

$$\sigma(K_0) = \left\{ \frac{eB}{m_1}\left(n_1 + \frac{1}{2}\right) + \frac{eB}{m_2}\left(n_2 + \frac{1}{2}\right) \middle| n_1, n_2 \in \mathbf{N}_0 \right\}, \tag{2.15}$$

with multiplicities. Furthermore, K_0 has compact resolvent. The ground state is simple and given by the Gaussian wave function $\psi_0(z) = \exp(-(eB/4)z^2)$. In general, the eigenfunctions have the form

$$\psi_n(z) = h_n(z)\exp(-(eB/4)z^2), \tag{2.16}$$

where $h_n(z)$ is a polynomial in z.

Proof. – Let $\gamma = eB/2$ and let

$$a_1 = \frac{1}{2}\{\gamma^{-1/2}(p_{z_1} + ip_{z_2}) - i\gamma^{1/2}(z_1 + iz_2)\}, \tag{2.17}$$

$$a_2 = \frac{1}{2}\{\gamma^{-1/2}(p_{z_2} + ip_{z_1}) - i\gamma^{1/2}(z_2 + iz_1)\}. \tag{2.18}$$

Then it is easy to show that

$$K_0 = \frac{eB}{m_1}\left(a_1^* a_1 + \frac{1}{2}\right) + \frac{eB}{m_2}\left(a_2^* a_2 + \frac{1}{2}\right), \tag{2.19}$$

and $\{a_j^*, a_k\}$, $j = 1, 2$, $k = 1, 2$, form a commuting family of creation-annihilation operators, *i.e.* for $j = 1, 2$, $k = 1, 2$,

$$[a_j, a_k^*] = \delta_{jk}, \quad [a_j, a_k] = 0, \quad [a_j^*, a_k^*] = 0.$$

Thus we learn from standard arguments that the spectrum of K_0 is given by the right hand side of (2.15), with multiplicities. Then it is clear that K_0 has compact resolvent. The ground state is given by the solution to

$$a_j \psi_0 = 0, \quad j = 1, 2,$$

and these lead to

$$(p_{z_j} + i\gamma z_j)\psi_0 = 0, \quad j = 1, 2.$$

The solution is the Gaussian function (up to a constant), and thus the ground state is simple. Then the eigenfunction corresponding to the eigenvalue

$$\frac{eB}{m_1}\left(n_1 + \frac{1}{2}\right) + \frac{eB}{m_2}\left(n_2 + \frac{1}{2}\right), \quad n_1, n_2 \geq 0,$$

is given by

$$\psi_{n_1, n_2} = c(a_1^*)^{n_1}(a_2^*)^{n_2}\psi_0, \quad c > 0,$$

which is of the form (2.16). □

Let $\{E_n(\infty) \mid n \in \mathbf{N}\}$ denote a non-decreasing enumeration of $\sigma(K_0)$, with repetition according to multiplicity. We note that

$$E_2(\infty) - E_1(\infty) \geq \frac{eB}{\max\{m_1, m_2\}} > 0. \tag{2.20}$$

We also note that for $m_1 = m_2$ the distance between distinct eigenvalues is eB/m_1. If $m_1 \neq m_2$, and m_1/m_2 is rational, then the distance between distinct eigenvalues is bounded from below by a positive constant. If m_1/m_2 is irrational, then the distance can be arbitrarily small for n sufficiently large. The following expression for K_0 will be useful in subsequent computations. Let $\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}$ denote the reduced mass, and $L_3 = z_1 p_{z_2} - z_2 p_{z_1}$ the third component of angular momentum. Then

$$K_0 = \frac{1}{2m} p_z^2 + \frac{e^2 B^2}{8m} z^2 + \frac{eB}{2}\left(\frac{1}{m_2} - \frac{1}{m_1}\right) L_3. \tag{2.21}$$

This formula can be rewritten as

$$K_0 = \frac{1}{2m}(p_z - \tilde{e}A(z))^2 + \frac{e^2 B^2}{2M} z^2, \tag{2.22}$$

where

$$\frac{\tilde{e}}{m} = e\left(\frac{1}{m_1} - \frac{1}{m_2}\right), \quad M = m_1 + m_2. \tag{2.23}$$

3. ABSOLUTELY CONTINUOUS SPECTRUM FOR H

In this section we give our main results on the spectrum of H given by (2.1). The spectrum is shown to be purely absolutely continuous under several different assumptions on V. The idea of the proof is similar to the Floquet-Bloch theory for periodic Schrödinger operators. We will show that the family of operators $\{K(\xi)\}$ is analytic with respect to ξ, and each operator has compact resolvent. If we can show that the eigenvalues are not constant with respect to ξ, absolute continuity of the spectrum of H follows. For this purpose we introduce the following assumption.

ASSUMPTION 3.1. – *V satisfies Assumption 2.1 and one of the following three conditions:*

(i) *$V(z) \to 0$ as $|z| \to \infty$, $V \not\equiv 0$, and $V(z) \leq 0$ for all $z \in \mathbf{R}^2$.*

(ii) *$V(z) \to 0$ as $|z| \to \infty$, and for each $\alpha > 0$ there exists $R > 0$ such that*

$$V(z) \geq e^{-\alpha|z|^2} \quad \text{for all } |z| \geq R. \tag{3.1}$$

(iii) *$V(z) \to \infty$ as $|z| \to \infty$.*

THEOREM 3.2. – *Let V satisfy Assumption 3.1. Then H has purely absolutely continuous spectrum.*

This Theorem is proved in a series of Lemmas. We begin by noting that each $K(\xi)$ has purely discrete spectrum.

LEMMA 3.3. – *Let V satisfy Assumption 2.1. Then $K(\xi)$ has purely discrete spectrum for each $\xi \in \mathbf{R}^2$, and the resolvent is compact.*

Proof. – We use the unitarily equivalent expression $K(\beta) = K_0 + V(\cdot - \beta)$ from (2.13). This operator is defined as a quadratic form, hence $\mathcal{Q}(K_0 + V(\cdot - \beta)) = \mathcal{Q}(K_0 + V_+(\cdot - \beta)) \subseteq \mathcal{Q}(K_0)$. Thus we have, using Theorem A.1,

$$\begin{aligned}
\mu_n(K_0 + &V_+(\cdot - \beta)) \\
&= \sup_{\varphi_1,\ldots,\varphi_{n-1}} \inf_{\substack{\psi \in [\varphi_1,\ldots,\varphi_{n-1}]^\perp \\ \psi \in \mathcal{Q}(K_0 + V_+(\cdot - \beta)),\ \|\psi\|=1}} (\psi, (K_0 + V_+(\cdot - \beta))\psi) \\
&\geq \sup_{\varphi_1,\ldots,\varphi_{n-1}} \inf_{\substack{\psi \in [\varphi_1,\ldots,\varphi_{n-1}]^\perp \\ \psi \in \mathcal{Q}(K_0),\ \|\psi\|=1}} (\psi, K_0 \psi) \\
&= E_n(\infty)
\end{aligned}$$

where $\mu_n(A)$ denotes the nth singular value of the operator A. It follows that $K_0 + V_+(\cdot - \beta)$ has purely discrete spectrum, since alternative (a) in Theorem A.1. holds. Now V_- is $K_0 + V_+(\cdot - \beta)$-form-bounded with relative bound zero, hence $K_0 + V(\cdot - \beta)$ has compact resolvent and purely discrete spectrum, see [13, Theorem XIII.68]. □

In order to apply Theorem A.5, we fix one parameter ξ_2, or equivalently, β_1. The nth eigenvalue of $K(\xi)$ is denoted $E_n(\xi)$ (repeated according to multiplicity). We note that under Assumption 2.1 $K(\xi)$ is an analytic family of type B (see [9]), thus the next result is well-known.

LEMMA 3.4. – *Let $\xi^0 \in \mathbf{R}^2$. If $E_n(\xi^0)$ is a non-degenerate eigenvalue, then there exists $\delta > 0$ such that for $|\xi_1 - \xi_1^0| < \delta$ $E_n(\xi_1, \xi_2^0)$ is analytic in ξ_1. If $E_n(\xi^0)$ is a j-fold degenerate eigenvalue,* i.e. *we have $E_{n-1}(\xi^0) < E_n(\xi^0) = \ldots = E_{n+j-1}(\xi^0) < E_{n+j}(\xi^0)$, there exist $\delta > 0$ and j analytic functions $f_1(\xi_1), \ldots, f_j(\xi_1)$, such that*

$$\{E_n(\xi_1, \xi_2^0), \ldots, E_{n+j-1}(\xi_1, \xi_2^0)\} = \{f_1(\xi_1), \ldots, f_j(\xi_1)\}$$

for all ξ_1 with $|\xi_1 - \xi_1^0| < \delta$.

Thus we can relabel the eigenvalues $E_n(\xi)$ to get a family with each function analytic in the ξ_1-variable. These analytic functions are denoted $\mathcal{E}_n(\xi)$, hence $\{E_n(\xi) \,|\, n = 1, 2, \ldots\} = \{\mathcal{E}_n(\xi) \,|\, n = 1, 2, \ldots\}$, for each $\xi \in \mathbf{R}^2$. In order to apply Theorem A.5 we need to show that each $\mathcal{E}_n(\xi)$ is non-constant. For the cases (i) and (ii) in Assumption 3.1 we use the following property.

LEMMA 3.5. – *Suppose that V satisfies Assumption 2.1 and $V(z) \to 0$ as $|z| \to \infty$. Then for each n, $E_n(\xi) \to E_n(\infty)$ as $|\xi| \to \infty$.*

Proof. – We use the representations (2.13) and (2.22) here. The proof is based on the cut-and-paste technique (or geometric perturbation theory), where we decompose $\mathbf{R}^2$ into two pieces. Let $\chi \in C^\infty([0,\infty))$ be a smooth cut-off function such that $0 \leq \chi(r) \leq 1$ for all $r \geq 0$, and

$$\chi(r) = \begin{cases} 1 & \text{if } r \leq 1, \\ 0 & \text{if } r \geq 2. \end{cases} \tag{3.2}$$

We choose another smooth cut-off function $\bar{\chi} \in C^\infty([0,\infty))$ such that $\chi(r)^2 + \bar{\chi}(r)^2 = 1$ for all $r \geq 0$. Let

$$\chi_R(z) = \chi(|z|/R), \quad \bar{\chi}_R(z) = \bar{\chi}(|z|/R), \quad \text{for} \quad R > 0 \quad \text{and all} \quad z \in \mathbf{R}^2.$$

Furthermore, let

$$W_{R,\beta}(z) = \begin{cases} V(z-\beta) & \text{if } |z| \leq 2R, \\ 0 & \text{if } |z| > 2R, \end{cases} \tag{3.3}$$

$$L^1_{R,\beta} = K_0 + W_{R,\beta} \quad \text{on } L^2(\mathbf{R}^2). \tag{3.4}$$

and

$$L^2_{R,\beta} = \frac{1}{2m}(p_z - \tilde{e}A(z))^2 + \frac{e^2B^2}{2M}\max\{R^2, |z|^2\} + V(z-\beta). \tag{3.5}$$

We write

$$L_{R,\beta} = L^1_{R,\beta} \oplus L^2_{R,\beta} \quad \text{on} \quad L^2(\mathbf{R}^2) \oplus L^2(\mathbf{R}^2). \tag{3.6}$$

The map $J_R : L^2(\mathbf{R}^2) \to L^2(\mathbf{R}^2) \oplus L^2(\mathbf{R}^2)$ denotes the isometry given by

$$J_R\varphi(z) = \chi_R(z)\varphi(z) \oplus \bar{\chi}_R(z)\varphi(z) \quad \text{for } \varphi \in L^2(\mathbf{R}^2).$$

Then we can approximate $K(\beta)$ by $J_R^* L_{R,\beta} J_R$. It is easy to see that

$$(K(\beta) - \zeta)^{-1} = J_R^*(L_{R,\beta} - \zeta)^{-1} J_R + J_R^*(L_{R,\beta} - \zeta)^{-1} T_R (K(\beta) - \zeta)^{-1} \tag{3.7}$$

with $T_R = L_{R,\beta} J_R - J_R K(\beta)$. A simple computation shows that for $c > 0$

$$\|T_R(K_0 + c)^{-1/2}\| \leq C\,R^{-1}, \quad R > 0, \tag{3.8}$$

where C is a constant independent of R. Now we note that

$$L^2_{R,\beta} \geq \frac{e^2B^2}{2M}R^2 - C, \quad \text{for all } R > 0 \text{ and } \beta \in \mathbf{R}^2, \tag{3.9}$$

for some constant $C > 0$, due to the $(p_z - \tilde{e}A(z))^2$-form-boundedness of V_-. Note that the form bound is independent of β.

Let $F_n(R,\beta)$ denote the nth eigenvalue of $L^1_{R,\beta}$. If R is sufficiently large, the inequality (3.9) implies that $F_n(R,\beta)$ is the nth eigenvalue of $L_{R,\beta}$. But then (3.7) and (3.8), together with the Riesz integral representation for the projection onto the eigenspace, imply that we have

$$|F_n(R,\beta) - E_n(\beta)| \leq C\,R^{-1} \tag{3.10}$$

for sufficiently large R, independently of $\beta \in \mathbf{R}^2$. We take an arbitrary $\varepsilon > 0$ and fix R so large that the right hand side of (3.10) is less than $\varepsilon/3$.

Now the same arguments can be applied to the case $V \equiv 0$, which we denote by $\beta = \infty$. On the other hand, it follows from the definitions that

$$\|L^1_{R,\beta} - L^1_{R,\infty}\| \leq \sup_{|z|\leq 2R} |V(z-\beta)| \leq \sup_{|z|\geq|\beta|-2R} |V(z)|. \tag{3.11}$$

As a consequence of these observations we get

$$|F_n(R,\beta) - F_n(R,\infty)| \leq \sup_{|z|\geq|\beta|-2R} |V(z)|. \tag{3.12}$$

and the right hand side converges to zero as $|\beta| \to \infty$. We choose $b_0 > 0$ so large that the right hand side of (3.12) is less than $\varepsilon/3$ for $|\beta| \geq b_0$. Then we have

$$\begin{aligned}|E_n(\beta) - E_n(\infty)| &\leq |E_n(\beta) - F_n(R,\beta)| + |F_n(R,\beta) - F_n(R,\infty)| \\ &\quad + |F_n(R,\infty) - E_n(\infty)| \\ &< \varepsilon\end{aligned}$$

for $|\beta| > b_0$. □

LEMMA 3.6. – *Let V satisfy Assumption 3.1-*(i)*. Then for each $\xi \in \mathbf{R}^2$, $n \in \mathbf{N}$, we have $E_n(\xi) < E_n(\infty)$.*

Proof. – We use the Poincaré Principle, Theorem A.2. Let φ_j denote the jth eigenfunction of K_0, and let $\widetilde{\mathcal{X}}_n = \operatorname{span}\{\varphi_1,\ldots,\varphi_n\} \subset L^2(\mathbf{R}^2)$. We have for any $\varphi \in \widetilde{\mathcal{X}}_n$

$$\begin{aligned}(\varphi, K(\beta)\varphi) &= (\varphi, K_0\varphi) + (\varphi, V(\cdot-\beta)\varphi) \\ &\leq E_n(\infty)\|\varphi\|^2 + (\varphi, V(\cdot-\beta)\varphi) \\ &< E_n(\infty)\|\varphi\|^2,\end{aligned}$$

since due to (2.16) the zero-set of φ has measure zero, and by assumption $V \not\equiv 0$.

Since $\widetilde{\mathcal{X}}_n$ is finite-dimensional, we have

$$\max_{\varphi \in \widetilde{\mathcal{X}}_n, \, \|\varphi\|=1} (\varphi, K(\beta)\varphi) < E_n(\infty),$$

and the claim follows from the Poincaré Principle. □

LEMMA 3.7. – *Let V satisfy Assumption 3.1-*(ii). *Then for each $n \in \mathbf{N}$ there exists $b_0 > 0$ such that $E_n(\beta) > E_n(\infty)$ for any β with $|\beta| \geq b_0$.*

Proof. – We use the notation from the proof of Lemma 3.5, and try to estimate the difference $F_n(R, \beta) - E_n(\beta)$. If we use the Agmon method for exponential decay estimates, we can show that for any $\gamma < eB/4$,

$$|F_n(R, \beta) - E_n(\beta)| \leq Ce^{-\gamma R^2}. \tag{3.13}$$

The proof is standard, and will be omitted, see for example [3, 4]. In the same manner we can obtain

$$|F_n(R, \infty) - E_n(\infty)| \leq Ce^{-\gamma R^2}, \tag{3.14}$$

with the same constant C as in (3.13). On the other hand, we can change the definition of $W_{R,\beta}$ slightly to obtain the estimate (3.16) below, namely, we let

$$W_{R,\beta}(z) = \begin{cases} V(z-\beta) & \text{if } |z| \leq 2R, \\ R^{-1} & \text{if } |z| > 2R. \end{cases} \tag{3.15}$$

We take β with $|\beta| = 4R$ and use the assumption (3.1) such that we have

$$V(z-\beta) \geq e^{-\alpha(6R)^2} = e^{-36\alpha R^2} \quad \text{for} \quad |z| < 2R,$$

if R is sufficiently large. Hence, we also have

$$W_{R,\beta} \geq e^{-36\alpha R^2},$$

and

$$F_n(R, \beta) - F_n(R, \infty) \geq e^{-36\alpha R^2}. \tag{3.16}$$

We then choose α so that $0 < 36\alpha < \gamma < eB/4$. We get

$$\begin{aligned} E_n(\beta) - E_n(\infty) &\geq F_n(R, \beta) - F_n(R, \infty) - |F_n(R, \beta) - E_n(\beta)| \\ &\qquad - |F_n(R, \infty) - E_n(\infty)| \\ &\geq e^{-36\alpha R^2} - 2Ce^{-\gamma R^2} \\ &> 0, \end{aligned}$$

if R is sufficiently large. □

REMARK 3.8. – In the proof above we use (3.1) from 3.1-(ii) for $\alpha < eB/144$, but this is not optimal. In fact, we can show $\alpha < eB/8$ is sufficient by modifying the proof (which then becomes somewhat complicated).

LEMMA 3.9. – *Let V satisfy Assumption 3.1-*(iii). *Then for each $n \in \mathbf{N}$ we have $E_n(\beta) \to \infty$ as $|\beta| \to \infty$.*

Proof. – Choose $R > 0$ such that $V(z) > 0$ for $|z| > R$. Let χ_R denote the characteristic function of the set $\{z \,|\, |z| \leq R\}$, and let $\bar{\chi}_R = 1 - \chi_R$. We decompose $V = V_1 + V_2 = \chi_R V + \bar{\chi}_R V$. Then V_1 is $(p_z - \tilde{e}A(z))^2$-form-bounded and $V_2 \geq 0$. Moreover, due to 3.1-(iii) we have

$$\inf_z \left(V_2(z-\beta) + \frac{e^2B^2}{2M}|z|^2 \right) \to \infty \quad \text{as} \quad |\beta| \to \infty$$

Thus we obtain as quadratic forms for some constant c

$$\begin{aligned} K(\beta) &= \left(\frac{1}{2m}(p_z - \tilde{e}A(z))^2 + V_1(z-\beta) \right) + \left(V_2(z-\beta) + \frac{e^2B^2}{2M}|z|^2 \right) \\ &\geq c + \inf_z \left(V_2(z-\beta) + \frac{e^2B^2}{2M}|z|^2 \right) \quad \to \infty \quad \text{as} \quad |\beta| \to \infty. \end{aligned}$$

It follows that each eigenvalue of $K(\beta)$ diverges to infinity as $|\beta| \to \infty$. □

Proof of Theorem 3.2. – We write

$$H = \int^{\oplus} \left(\int^{\oplus} K(\xi_1, \xi_2) d\xi_1 \right) d\xi_2 = \int^{\oplus} \widehat{K}(\xi_2)\, d\xi_2,$$

where

$$\widehat{K}(\xi_2) = \int^{\oplus} K(\xi_1, \xi_2)\, d\xi_1 \quad \text{on} \quad L^2(\mathbf{R} \times \mathbf{R}^2).$$

For each fixed ξ_2 each eigenvalue of $K(\xi_1, \xi_2)$ is (locally) an analytic function of ξ_1 by Lemma 3.4. By Lemmas 3.5-3.7, 3.8 each eigenvalue cannot be constant in ξ_1, hence all conditions in Theorem A.5 are satisfied. Thus $\widehat{K}(\xi_2)$ has purely absolutely continuous spectrum for each ξ_2. Then H is also purely absolutely continuous, by Theorem A.3. □

For the interval near the bottom of the spectrum of H we can show absolute continuity under different assumptions on V.

ASSUMPTION 3.10. – *Let V satisfy Assumption 2.1, and further $V(z) \to 0$ as $|z| \to \infty$, and $V(z) > 0$ for all $z \in \mathbf{R}^2$.*

THEOREM 3.11. – *Let V satisfy Assumption 3.10. Then* $\inf \sigma(H) = E_1(\infty)$, *and there exists $\lambda > E_1(\infty)$ such that $[E_1(\infty), \lambda] \subset \sigma(H)$, and H is absolutely continuous on $[E_1(\infty), \lambda]$.*

Proof. – It suffices to show that the function $E_1(\xi)$ is not constant. Note that $E_1(\infty)$ is non-degenerate by Lemma 2.3. We again use the representation $K(\beta)$ together with (2.22). Since $V \geq 0$, we have $E_1(\beta) \geq E_1(\infty)$ for all $\beta \in \mathbf{R}^2$. Now let us assume $E_1(\beta) = E_1(\infty)$ for some β. Let ψ be a normalized eigenfunction of $K(\beta)$ corresponding to the eigenvalue $E_1(\beta)$. Then

$$\begin{aligned} E_1(\infty) = E_1(\beta) &= (\psi, (K_0 + V(\cdot - \beta))\psi) \\ &\geq E_1(\infty) + (\psi, V(\cdot - \beta)\psi), \end{aligned}$$

and hence $(\psi, V(\cdot - \beta)\psi) \leq 0$. Since $V(z) > 0$ for all z, this implies $\psi = 0$. Therefore we have $E_1(\xi) > E_1(\infty)$ for all $\xi \in \mathbf{R}^2$. Together with Lemma 3.5 this implies that $E_1(\xi)$ is non-constant. The result now follows as in the proof of Theorem 3.3, if we note (2.20). □

We now give two results which require more restrictive assumptions on the potential. The proofs will only be sketched.

ASSUMPTION 3.12. – *Let V satisfy Assumption 2.1, and furthermore $V \in C^1(\mathbf{R}^2)$ such that for some $\nu \in \mathbf{R}^2$, $|\nu| = 1$,*

$$(\nu \cdot \nabla V)(z) > 0 \quad for\ all\ z \in \mathbf{R}^2. \tag{3.17}$$

THEOREM 3.13. – *Let V satisfy Assumption 3.12. Then the spectrum of H is purely absolutely continuous.*

Proof. – We only outline the proof. The detailed arguments are similar to the arguments in Section 5. Let $\beta_0 \in \mathbf{R}^2$ and let $E(\beta_0)$ denote one of the eigenvalues of $K(\beta_0)$, which is assumed to be regular in a small neighborhood of β_0. Let Γ denote a circle centered at $E(\beta_0)$ with a sufficiently small radius, such that no other eigenvalues are on or inside Γ for all β with $|\beta - \beta_0| < \delta$. Let

$$P(\beta) = \frac{1}{2\pi i} \oint_\Gamma (\zeta - K(\beta))^{-1} d\zeta.$$

Let $\psi \in L^2(\mathbf{R}^2)$, such that $P(\beta_0)\psi \neq 0$. Standard arguments from perturbation theory then give us the formula

$$(\nu \cdot \nabla_\beta E)(\beta_0) = -\frac{(P(\beta_0)\psi,\, (\nu \cdot \nabla_z V)(\cdot - \beta_0) P(\beta_0)\psi)}{(\psi,\, P(\beta_0)\psi)}.$$

It follows from the assumption (3.17) that $E(\beta)$ is not constant close to β_0. This argument can be applied to all eigenvalues and almost all values of β. The proof is then concluded as above. □

ASSUMPTION 3.14. – *Let V satisfy Assumption 2.1. Assume there exist constants $V_\pm^0 \in \mathbf{R}$, $V_+^0 \neq V_-^0$, and $\nu \in \mathbf{R}^2$, $|\nu| = 1$, such that*

$$V(z - t\nu) \to V_\pm^0 \quad as \quad t \to \pm\infty$$

uniformly in z on compact subsets of $\mathbf{R}^2$.

THEOREM 3.15. – *Let V satisfy Assumption 3.14. Then the spectrum of H is purely absolutely continuous.*

Proof. – The proof is based on the geometric perturbation theory and is a variant of the proof of Lemma 3.5. We briefly comment on the changes necessary. We define

$$W_{R,t}(z) = \begin{cases} V(z - t\nu) & \text{if } |z| \leq 2R, \\ 0 & \text{if } |z| > 2R, \end{cases} \tag{3.18}$$

as a replacement for (3.3) and introduce

$$W_{R,+\infty}(z) = \begin{cases} V_+^0 & \text{if } |z| \leq 2R, \\ 0 & \text{if } |z| > 2R. \end{cases}$$

Using (3.18) we let

$$L_{R,t}^1 = K_0 + W_{R,t}$$

and define

$$L_{R,+\infty}^1 = K_0 + W_{R,+\infty}.$$

With these and other obvious modifications the arguments in the proof of Lemma 3.5 can be repeated. We note that the eigenvalues of $L_{R,+\infty}^1$ satisfy $F_n(R, +\infty) \to E_n(\infty) + V_+^0$ as $R \to \infty$. Similar arguments apply in the case $t < 0$, with an obvious change in the definition (3.19) and the final result is

$$E_n(t\nu) \to E_n(\infty) + V_\pm^0 \quad \text{as} \quad t \to \pm\infty.$$

Since $V_+^0 \neq V_-^0$, we see that each eigenvalue is non-constant, and the proof is completed as above. □

Finally we note for later use the following result.

PROPOSITION 3.16. – *Assume V real-valued, $V \in L^\infty(\mathbf{R}^2)$ with*

$$\|V\|_{L^\infty} < \frac{eB}{\max\{m_1, m_2\}},$$

and $V(z) \to 0$ as $|z| \to \infty$. Then the ground state $E_1(\xi)$ of $K(\xi)$ is non-degenerate for all $\xi \in \mathbf{R}^2$.

Proof. – Standard arguments from perturbation theory show that

$$|E_n(\xi) - E_n(\infty)| \leq \|V\|_{L^\infty}.$$

for all $n \in \mathbf{N}$ and all $\xi \in \mathbf{R}^2$. This estimate together with the assumption (3.20), Lemma 3.5, and (2.20) yield the result in the Proposition. □

4. SOLVABLE MODELS AND SOME EXTENSIONS

In this section we discuss some explicitly solvable models, and we give some extension of the results in the previous section.

We first consider the case where we add an external constant electric field $E \in \mathbf{R}^2 \setminus \{0\}$. For $V \equiv 0$ we get an explicitly solvable problem, and in general for bounded V we show that the spectrum is absolutely continuous.

Since the two particles have equal and opposite charges, we have without potential the Hamiltonian

$$H_0(E) = \frac{1}{2m_1}(p_x - eA(x))^2 + \frac{1}{2m_2}(p_y + eA(y))^2 + eE \cdot (x - y), \quad (4.1)$$

which fits into the framework of Section 2, and we can carry out the decomposition as above. This argument also shows that $H_0(E)$ is a well-defined self-adjoint operator. We state the result in the β-formulation, see (2.13), and use the form (2.22). We have

$$H_0(E) \cong \int_{\mathbf{R}^2}^{\oplus} K_0(E, \beta)\, d\beta, \quad (4.2)$$

where

$$\begin{aligned} K_0(E,\beta) &= \frac{1}{2m}(p_z - \tilde{e}A(z))^2 + \frac{e^2B^2}{2M}z^2 + eE \cdot z - eE \cdot \beta \\ &= \frac{1}{2m}(p_z - \tilde{e}A(z))^2 + \frac{e^2B^2}{2M}\left(z + \frac{2M}{eB^2}E\right)^2 - \frac{M}{2B^2}E^2 - eE \cdot \beta \\ &= K_1(E) - \frac{M}{2B^2}E^2 - eE \cdot \beta. \end{aligned} \quad (4.3)$$

The spectrum of $K_1(E)$ can be found explicitly, since this operator is unitarily equivalent to K_0 (see (2.22)) via a magnetic translation. This observation was also made in [15], where scattering problems in space-dimension three were considered.

We conclude that $K_0(E,\beta)$ has compact resolvent. The spectrum is given by

$$\sigma(K_0(E,\beta)) = \{E_n(\infty) - \frac{M}{2B^2}E^2 - eE\cdot\beta \,|\, n \in \mathbf{N}\}, \tag{4.4}$$

where $E_n(\infty)$ is a non-decreasing enumeration of the eigenvalues of K_0, as in Section 2 . Obviously, each eigenvalue is non-constant in β. Furthermore, each eigenvalue covers $\mathbf{R}$ as β varies through $\mathbf{R}^2$. Repeating the arguments in Section 3, we conclude that $H_0(E)$ has purely absolutely continuous spectrum equal to $\mathbf{R}$.

This explicit result can be generalized as follows.

PROPOSITION 4.1. – *Let $V \in L^\infty(\mathbf{R}^2)$ be real-valued and let $E \in \mathbf{R}^2\setminus\{0\}$. Then the operator*

$$H(E) = \frac{1}{2m_1}(p_x - eA(x))^2 + \frac{1}{2m_2}(p_y + eA(y))^2 + eE\cdot(x-y) + V(x-y) \tag{4.5}$$

has purely absolutely continuous spectrum equal to $\mathbf{R}$.

Proof. – As above we have

$$H(E) \cong \int_{\mathbf{R}^2}^{\oplus} K(E,\beta)\,d\beta$$

with

$$K(E,\beta) = K_1(E) + V(z-\beta) - \frac{M}{2B^2}E^2 - eE\cdot\beta.$$

Since $K_1(E)$ has compact resolvent and $\sigma_{\mathrm{d}}(K_1(E)) = \{E_n(\infty)\,|\, n \in \mathbf{N}\}$, and since V is bounded, the operator $K_1(E) + V(z-\beta)$ has compact resolvent. We write

$$\sigma_{\mathrm{d}}(K_1(E) + V(z-\beta)) = \{F_n(\beta)\,|\, n \in \mathbf{N}\}.$$

Standard perturbation theory yields

$$|E_n(\infty) - F_n(\beta)| \le \|V\|_{L^\infty} \tag{4.6}$$

for all $n \in \mathbf{N}$ and all $\beta \in \mathbf{R}^2$. Thus $K(E, \beta)$ has compact resolvent and its eigenvalues are given by

$$E_n(K(E,\beta)) = F_n(\beta) - \frac{M}{2B^2}E^2 - eE \cdot \beta. \tag{4.7}$$

Using (4.6) we see that the eigenvalues are non-constant in β, and cover $\mathbf{R}$ as β varies through $\mathbf{R}^2$. Thus the spectrum of $H(E)$ is absolutely continuous and equals $\mathbf{R}$. $\square$

REMARK 4.2. – One can also apply Theorem 3.15. Under the assumptions $V \in C^1(\mathbf{R}^2)$ and $E \cdot \nabla V(z) + E^2 > 0$ for all $z \in \mathbf{R}^2$ we conclude that the spectrum of H is purely absolutely continuous. We should also add a condition that allows us to define $H(E)$ (see (4.5)) as a self-adjoint operator.

Let us now consider the particular case $V(z) = c_0 z^2$, the harmonic oscillator potential. We also include an external constant electric field E, but this time we allow E to equal 0. We take

$$H(E, c_0) = \frac{1}{2m_1}(p_x - eA(x))^2 + \frac{1}{2m_2}(p_y + eA(y))^2 + eE\cdot(x-y) + c_0(x-y)^2, \tag{4.8}$$

and decompose as above

$$H(E, c_0) \cong \int_{\mathbf{R}^2}^{\oplus} K(E, c_0; \beta) d\beta.$$

Some straightforward algebraic manipulations give

$$\begin{aligned} K(E, c_0; \beta) &= \frac{1}{2m}(p_z - \tilde{e}A(z))^2 + \frac{\omega_0^2}{2M}\left(z + \frac{2M}{\omega_0^2}(\frac{1}{2}eE - c_0\beta)\right)^2 \\ &\quad + \frac{e^2B^2}{c_0\omega_0^2}\left(\frac{1}{2}eE - c_0\beta\right)^2 - \frac{1}{4}\frac{e^2E^2}{c_0} \\ &= K_1(c_0) + \frac{e^2B^2}{c_0\omega_0^2}\left(\frac{1}{2}eE - c_0\beta\right)^2 - \frac{1}{4}\frac{e^2E^2}{c_0}, \end{aligned} \tag{4.9}$$

where $\omega_0^2 = e^2B^2 + 2Mc_0$. The operator $K_1(c_0)$ is unitarily equivalent to the operator

$$\frac{1}{2m}(p_z - \tilde{e}A(z))^2 + \frac{\omega_0^2}{2M}z^2 \tag{4.10}$$

via a magnetic translation. Therefore $K_1(c_0)$ has compact resolvent. The operator in (4.10) is quadratic in p_z and z. It is well known that its spectrum can be computed explicitly, see for example [12] for details. One finds that the eigenvalues of $K(E, c_0; \beta)$ are given by

$$\begin{aligned} F_{n_1,n_2}(\beta) &= \frac{1}{2m}\left(n_1 + \frac{1}{2}\right)\left[\left(4\frac{m\omega_0^2}{M} + \frac{\tilde{e}^2 B^2}{4}\right)^{1/2} + \frac{\tilde{e}B}{2}\right] \\ &+ \frac{1}{2m}\left(n_2 + \frac{1}{2}\right)\left[\left(4\frac{m\omega_0^2}{M} + \frac{\tilde{e}^2 B^2}{4}\right)^{1/2} - \frac{\tilde{e}B}{2}\right] \\ &+ \frac{e^2B^2}{c_0\omega_0^2}\left(\frac{1}{2}eE - c_0\beta\right)^2 - \frac{e^2E^2}{c_0}. \end{aligned} \tag{4.11}$$

They are obviously non-constant in β.

We state the results as follows.

PROPOSITION 4.3. – *Let $E \in \mathbf{R}^2$ and let $c_0 > 0$. Then the operator $H(E, c_0)$ given by* (4.8) *has purely absolutely continuous spectrum $\sigma(H(E, c_0)) = [\mathcal{E}_0, \infty)$, where*

$$\mathcal{E}_0 = \frac{1}{2m}\left(4\frac{m\omega_0^2}{M} + \frac{\tilde{e}^2B^2}{4}\right)^{1/2} - \frac{1}{4}\frac{e^2E^2}{c_0}$$

with

$$\omega_0^2 = e^2B^2 + 2Mc_0 \quad \textit{and} \quad \frac{\tilde{e}}{m} = e\left(\frac{1}{m_1} - \frac{1}{m_2}\right).$$

Proof. – The proof follows from the explicit formula (4.11) for the eigenvalues and arguments as above. □

REMARK 4.4. – Since $V(z) = c_0 z^2 + E \cdot z$ satisfies Assumption 3.1-(iii), we know from Theorem 3.2 that the spectrum of $H(E, c_0)$ is purely absolutely continuous. Lemma 3.9 further implies that the spectrum is a half-line. The above result gives an example where the bottom can be determined explicitly.

5. EFFECTIVE MASS

In this section we give some results on the analogue of effective mass computation for periodic Schrödinger operators. Here we study the effective mass for the bottom of the spectrum. Namely, we look at the Hessian matrix:

$$(e_{ij}) = \left(\frac{\partial^2 E_1(\xi)}{\partial\xi_i\partial\xi_j}\right)\Bigg|_{\xi=\xi_0}, \quad i, j = 1, 2,$$

where $E_1(\xi)$ is the lowest eigenvalue of $K(\xi)$, and ξ_0 is the minimal point of $E_1(\xi)$. By the definition, (e_{ij}) is nonnegative. If (e_{ij}) is strictly positive, the inverse of the eigenvalues are called *effective masses*, by analogy to periodic Schrödinger operators, see [1] for the physics background and [10] for some rigorous results. These values are expected to dominate the long-time behavior of the the time-evolution of states with energy near the bottom of the spectrum.

Example 5.1. – Let $V(z) = c_0 z^2$. Then, taking $E = 0$, $n_1 = 0$, $n_2 = 0$ in (4.11) and using (2.12), we find

$$E_1(\xi) = \frac{1}{2m}\left(4\frac{m\omega_0^2}{M} + \frac{\tilde{e}^2 B^2}{4}\right)^{1/2} + \frac{c_0}{\omega_0^2}\,\xi^2, \quad \xi \in \mathbf{R}^2,$$

where $\omega_0^2 = e^2 B^2 + 2Mc_0$. Thus (e_{ij}) is given by $(2c_0\omega_0^{-2}\delta_{ij})$, and the effective mass is $\omega_0^2/(2c_0)$.

In the following, we study the properties of (e_{ij}) and give a sufficient condition for the strict positivity of this matrix, *i.e.*, the finiteness of the effective masses. Without loss of generality, we may suppose $\xi_0 = 0$. Moreover, we suppose

ASSUMPTION 5.2. – *$E_1(\xi)$ is non-degenerate for $\xi \in \mathbf{R}^2$.*

A sufficient condition for Assumption 5.2 is given by Proposition 3.16. As before, we always suppose Assumption 2.1. We use the following notation: ψ is the ground state of $K(0)$, *i.e.*, $K(0)\psi = E_1(0)\psi$ with $\|\psi\| = 1$. $P(\xi)$ is the projection onto the ground state of $K(\xi)$:

$$P(\xi) = \frac{1}{2\pi i}\oint_\Gamma (\zeta - K(\xi))^{-1} d\zeta,$$

where Γ is a sufficiently small circle around $E_1(\xi)$. If ξ is in a small neighborhood of 0, it is easy to see that

$$E_1(\xi) = \frac{(K(\xi)P(\xi)\psi, \psi)}{(P(\xi)\psi, \psi)}.$$

We write

$$f(\xi) = (P(\xi)\psi, \psi); \quad g(\xi) = (K(\xi)P(\xi)\psi, \psi); \quad Q = 1 - P(0).$$

LEMMA 5.3

$$\begin{aligned} e_{ij} = {} & \big((\partial_{\xi_i}\partial_{\xi_j} K(0))\psi, \psi\big) \\ & - 2\big(Q(QK(0)Q - E_1(0))^{-1} Q(\partial_{\xi_i} K(0))\psi, (\partial_{\xi_j} K(0))\psi\big) \end{aligned} \tag{5.1}$$

Proof. – For the sake of simplicity, we set $E_1(0) = 0$ without loss of generality. By differentiating $E_1(\xi)f(\xi) = g(\xi)$, we obtain

$$(\partial_{\xi_i}\partial_{\xi_j}E_1)f + (\partial_{\xi_i}E_1)(\partial_{\xi_j}f) + (\partial_{\xi_j}E_1)(\partial_{\xi_i}f) + E_1(\partial_{\xi_i}\partial_{\xi_j}f) = \partial_{\xi_i}\partial_{\xi_j}g.$$

Noting $E_1(0) = 0$ and $\partial_\xi E_1(0) = 0$, we learn

$$\partial_{\xi_i}\partial_{\xi_j}E_1(0) = \partial_{\xi_i}\partial_{\xi_j}g(0).$$

By direct computations, we see that the right hand side is given by

$$\partial_{\xi_i}\partial_{\xi_j}g(0) = \big((\partial_{\xi_i}\partial_{\xi_j}K)\psi,\, \psi\big) + \frac{1}{2\pi i}\oint_\Gamma \big((\partial_{\xi_i}K)(\zeta - K)^{-1}(\partial_{\xi_j}K)\psi,\, \psi\big)\frac{d\zeta}{\zeta}$$
$$+ \frac{1}{2\pi i}\oint_\Gamma \big((\partial_{\xi_j}K)(\zeta - K)^{-1}(\partial_{\xi_i}K)\psi,\, \psi\big)\frac{d\zeta}{\zeta}$$

where Γ is a small circle around 0. Then we use

$$(\zeta - K(0))^{-1} = \zeta^{-1}P(0) + (\zeta - K(0))^{-1}Q,$$

and the claim follows by simple computations. We note that each term in the right hand side of (5.1) is symmetric in i and j. □

In order to compute the right hand side of (5.1), we employ the following expression for $K(\xi)$:

$$K(\xi) = \Big(\frac{\alpha}{2m}p_z^2 + V(z)\Big) + \frac{1}{2m}\Big(\frac{\tilde{e}}{e}p_z - eA(z) + \frac{1}{2}\xi\Big)^2, \tag{5.2}$$

where m and $\tilde{e}$ are given at the end of Section 2, and

$$\alpha = 1 - \Big(\frac{\tilde{e}}{e}\Big)^2 \in (0, 1].$$

It is easy to see that

$$\partial_\xi K(0) = \frac{1}{2m}\Big(\frac{\tilde{e}}{e}p_z - eA(z)\Big),$$
$$\partial_{\xi_i}\partial_{\xi_j}K(0) = \frac{1}{4m}\,\delta_{ij}.$$

We set

$$h = \frac{1}{2m}p_z^2 + V(z), \quad F = \inf\sigma(h).$$

LEMMA 5.4.

$$\Big\|\Big(\frac{\tilde{e}}{e}p_z - eA(z)\Big)\psi\Big\| \le \sqrt{2m(E_1(0) - F)} \tag{5.4}$$

Proof. – By direct computations, we have

$$\begin{aligned}\frac{1}{2m}\Big\|\Big(\frac{\tilde e}{e}p_z - eA(z)\Big)\psi\Big\|^2 &= \Big(\frac{1}{2m}\Big(\frac{\tilde e}{e}p_z - eA(z)\Big)^2\psi,\psi\Big)\\ &= \big((K(0)-h)\psi,\psi\big)\\ &\le \big((K(0)-F)\psi,\psi\big) = E_1(0)-F,\end{aligned}$$

and the claim follows. □

By (5.3) and (5.4), we learn

$$\|\partial_\xi K(0)\psi\| \le \sqrt{\frac{E_1(0)-F}{2m}}.$$

On the other hand, it is easy to see

$$\|(QK(0)Q - E_1(0))^{-1}\| \le (E_2(0)-E_1(0))^{-1}.$$

Combining these with Lemma 5.3, we obtain our main result of this section:

THEOREM 5.5. – *With the notation introduced above,*

$$(e_{ij}) \ge \frac{1}{4m}\Big(1_{2\times 2} - 4\Big(\frac{E_1(0)-F}{E_2(0)-E_1(0)}\Big)\Big) \qquad (5.5)$$

holds in the operator sense on $\mathbf{C}^2$, *where* $1_{2\times 2}$ *is the* 2 *by* 2 *unit matrix.*

The proof is a straightforward computation, which we omit.

COROLLARY 5.6. – *If* $4(E_1(0)-F) < E_2(0)-E_1(0)$, *then* (e_{ij}) *is non-degenerate,* i.e., *the effective masses at the bottom of the spectrum are finite.*

In particular, if $m_1 = m_2$, then $\tilde e = 0$, and we have a simpler picture. Namely, $K(0)$ is given by

$$K(0) = \Big(\frac{1}{2m}p_z^2 + V(z)\Big) + \frac{(eB)^2}{2m}z^2.$$

Hence, by the Courant-Weyl principle, the discrete eigenvalues of $K(0)$ converges to those of h as $B \to 0$. If, moreover, F is discrete in $\sigma(h)$, then

$$\lim_{B\to 0} E_2(0) = \inf(\sigma(h)\setminus\{F\}) > F = \lim_{B\to 0} E_1(0).$$

Thus we have

$$\lim_{B\to 0}\frac{E_1(0)-F}{E_2(0)-E_1(0)} = 0.$$

These imply the following:

COROLLARY 5.7. – *Suppose* $m_1 = m_2$, *and suppose* F *is discrete in* $\sigma(h)$. *Then* (e_{ij}) *is non-degenerate if* $|B|$ *is sufficiently small.*

A. AUXILIARY RESULTS

This appendix contains the precise statement of some results needed in our study, and in a few cases outlines of proofs.

A.1. Min-max and max-min

For reference we state precisely the min-max and max-min principles used in our study.

We recall the following result for quadratic forms from [13]. It is there called the min-max principle. Here we prefer to follow [16] and call it the Courant-Weyl principle.

THEOREM A.1. – *Assume H is self-adjoint and bounded below. Let*

$$\mu_n(H) = \sup_{\varphi_1,\dots,\varphi_{n-1}} \inf_{\substack{\psi\in[\varphi_1,\dots,\varphi_{n-1}]^\perp \\ \psi\in\mathcal{Q}(H),\ \|\psi\|=1}} (\psi, H\psi). \tag{A.1}$$

Then for each fixed n either

(a) *there are n eigenvalues below the bottom of the essential spectrum, and $\mu_n(H)$ is the nth eigenvalue (counted with multiplicity)*

or

(b) *$\mu_n(H)$ is the bottom of the essential spectrum.*

An alternative formulation is what in [16] is called the Poincaré principle, and for consistency one could call it the max-min principle.

THEOREM A.2. – *Assume H is self-adjoint and bounded below. Let*

$$\mu_n(H) = \inf_{\substack{\mathcal{X}_n\subset\mathcal{Q}(H) \\ \dim\mathcal{X}_n = n}} \sup_{\substack{\varphi\in\mathcal{X}_n \\ \|\varphi\|=1}} (\varphi, H\varphi).$$

Then the alternative in Theorem A.1 holds.

A.2. Results on decomposable operators

In this part of the appendix we give some results on the spectrum of a decomposable operator. The results are well-known and are stated here for easy reference.

THEOREM A.3. – *Let (M, μ) be a σ-finite measure space and let $T = \int_M^{\oplus} T(m)d\mu(m)$ be a decomposed self-adjoint operator on $\mathcal{H} = \int_M^{\oplus} \mathcal{H}' d\mu(m)$. Then the following results hold:*

(i) *For any bounded Borel function F*

$$F(T) = \int_M^{\oplus} F(T(m))d\mu(m).$$

(ii) $\lambda \in \sigma(T)$ *if and only if for all* $\varepsilon > 0$ $\mu(\{m \mid \sigma(T(m)) \cap (\lambda-\varepsilon, \lambda+\varepsilon) \neq \emptyset\}) > 0$.

(iii) $\lambda \in \sigma_{\mathrm{pp}}(T)$ *if and only if* $\mu(\{m \mid \lambda \in \sigma_{\mathrm{pp}}(T(m))\}) > 0$.

(iv) *Let* $I \cap \sigma(T(m)) \subseteq \sigma_{\mathrm{ac}}(T(m))$ *for a.e.* $m \in M$, *where* $I \subseteq \mathbf{R}$ *is an interval. Then* $I \cap \sigma(T) \subseteq \sigma_{\mathrm{ac}}(T)$.

Proof. – For the results (i)–(iii) we refer to [13, Theorem XIII.85]. For (iv) we note that the proof in [13] can be localized to the interval I. □

DEFINITION A.4. – A function $f : \mathbf{R} \to \mathbf{R}$ is said to be *piecewise strictly monotone*, if there exists a partition $\mathbf{R} = \cup_{j\in\mathbf{N}} I_j$ into disjoint intervals whose endpoints have no finite point of accumulation, such that the restriction of f to each I_j is strictly increasing or strictly decreasing.

THEOREM A.5. – *Let* $T = \int_{\mathbf{R}}^{\oplus} T(m)dm$ *be a decomposed self-adjoint operator on the space* $\mathcal{H} = \int_{\mathbf{R}}^{\oplus} \mathcal{H}' dm$. *Assume*

(1) *For each* $m \in \mathbf{R}$ $T(m)$ *has compact resolvent.*

(2) *For each* $m \in \mathbf{R}$ *and* $n \in \mathbf{N}$ $T(m)\psi_n(m) = E_n(m)\psi_n(m)$.

(3) *For each* $n \in \mathbf{N}$ *the function* $m \mapsto E_n(m)$ *is bounded, and piecewise strictly monotone, and furthermore the piecewise inverse is Lipschitz continuous.*

(4) *The map* $m \mapsto \psi_n(m)$ *is measurable, and for each* $m \in \mathbf{R}$ $\{\psi_n(m) \mid n \in \mathbf{N}\}$ *is an orthonormal basis for* $\mathcal{H}'$.

Then T *has purely absolutely continuous spectrum.*

Proof. – The proof is a variant of the proof of [13, Theorem XIII.86]. We outline it for the sake of completeness. Define

$$\mathcal{H}_n = \{\psi \in \mathcal{H} \mid \psi(m) = f(m)\psi_n(m),\ f \in L^2(\mathbf{R})\}.$$

The subspaces $\mathcal{H}_n$ are closed, pairwise orthogonal, and due to assumption (4)

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n.$$

Furthermore, $\mathcal{H}_n \subseteq \mathcal{D}(T)$ and $T(\mathcal{H}_n) \subseteq \mathcal{H}_n$. Let $\psi \in \mathcal{H}_n$, $\psi(m) = f(m)\psi_n(m)$. A computation shows that $\|\psi\|_{\mathcal{H}} = \|f\|_{L^2(\mathbf{R})}$. Thus $U_n : \psi \mapsto f$, $U_n : \mathcal{H}_n \to L^2(\mathbf{R})$ defines a unitary map.

Let $T_n = U_n T U_n^{-1}$. Then a computation shows that

$$(T_n f)(m) = E_n(m) f(m).$$

Thus T_n is multiplication by the function E_n, which satisfies the conditions in assumption (3), and therefore has purely absolutely continuous spectrum. □

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