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On the semi-classical approximation of the solution of the Heisenberg equation with spin

by

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Dedicated to Professor Sigeru Mizohata on the occasion of his 70th birthday

ABSTRACT. – We consider Hamiltonians describing the motion of some charged particles with spin in an electromagnetic field. Let $U_\hbar(t, s)$ be its propagator and $F_\hbar$ an observable. Then the solution of the Heisenberg equation with $F_\hbar$ at $t = s$ is given by $U_\hbar(t, s)^* F_\hbar U_\hbar(t, s)$. In this paper we compute the semi-classical approximation of $U_\hbar(t, s)^* F_\hbar U_\hbar(t, s)$ in terms of pseudo-differential operators. From this formula we get the classical limit as $\hbar \to 0$ of the time evolution of the mean value of $F_\hbar$ for initial states centered suitably in classical phase space. Then the relation between quantum and classical mechanics can be shown.

RÉSUMÉ. – On considère le hamiltonien décrivant le mouvement de quelques particules avec spin dans un champ électromagnétique. Soient $U_\hbar(t, s)$ son propagateur et $F_\hbar$ une observable. Alors la solution de l'équation de Heisenberg pour $F_\hbar$ à $t = s$ est donnée par $U_\hbar(t, s)^* F_\hbar U_\hbar(t, s)$. Dans cet article nous décrivons l'approximation semi-classique de $U_\hbar(t, s)^* F_\hbar \ U_\hbar(t, s)$ en terme d'opérateurs pseudo-différentiels. Cette formule nous fournit la limite classique quand $\hbar \to 0$ de l'évolution temporelle de la valeur moyenne de $F_\hbar$ pour des états initiaux convenablement centrés dans l'espace de phase. Ceci nous donne une description de la relation entre mécanique quantique et mécanique classique.

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1. INTRODUCTION

Consider some charged particles without spin in an electromagnetic field. For the sake of simplicity we suppose charge = one and mass = one. Then its Hamiltonian, expressed in terms of the electromagnetic potentials $A(t,x) = (A_1, \cdots, A_n), V(t,x)$ $(x \in R^n, t \in [0,T])$, is

$$H_{0\hbar}(t) = 1/2 \sum_{j=1}^{n} (\hbar D_{x_j} - A_j)^2 + V. \tag{1.1}$$

We denote by $U_{0\hbar}(t,s)$ $(t, s \in [0,T])$ the propagator of the Schrödinger equation, that is, the solution of

$$i\hbar \frac{\partial}{\partial t} U_{0\hbar}(t,s) = H_{0\hbar}(t) U_{0\hbar}(t,s), \quad U_{0\hbar}(s,s) = \text{Identity}. \tag{1.2}$$

Then it is well known that the solution of the Heisenberg equation

$$i\hbar \frac{d}{dt} G_\hbar(t) = [G_\hbar(t), U_{0\hbar}(t,s)^* H_{0\hbar}(t) U_{0\hbar}(t,s)], \quad G_\hbar(s) = F_\hbar \tag{1.3}$$

for an observable $F_\hbar$ is given formally by

$$G_\hbar(t) = U_{0\hbar}(t,s)^* F_\hbar U_{0\hbar}(t,s). \tag{1.4}$$

$U_{0\hbar}(t,s)^*$ is the adjoint operator of $U_{0\hbar}(t,s)$.

We use the following notations. For $x = (x_1, \cdots, x_n) \in R^n$ and a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ let $< x >= (1 + |x|^2)^{1/2}, \partial_{x_j} = \frac{\partial}{\partial x_j}, \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Let $L^2 = L^2(R^n)$ be the space of all square integrable functions on R^n with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. We denote by $\phi_s^t(y,\eta) = (q(t,s;y,\eta), p(t,s;y,\eta)) = (q_1, \cdots, q_n, p_1, \cdots, p_n)$ the classical orbit for (1.1) with (y,η) at $t = s$, that is, the solution of

$$\begin{aligned} &\frac{dq_j}{dt} = \frac{\partial \mathcal{H}_0}{\partial \xi_j}(t,q,p), \frac{dp_k}{dt} = -\frac{\partial \mathcal{H}_0}{\partial x_k}(t,q,p) \ (j,k = 1,2,\cdots,n), \\ &(q,p)|_{t=s} = (y,\eta), \end{aligned} \tag{1.5}$$

where

$$\mathcal{H}_0(t,x,\xi) = 1/2 \sum_{j=1}^{n} (\xi_j - A_j(t,x))^2 + V(t,x) \tag{1.6}$$

is the classical Hamiltonian.

Let $F_\hbar = (\exp ix\cdot\zeta)(\exp iz\cdot\hbar D_x)$ $(\zeta, z \in R^n)$. Then Hepp in [2] studied the semi-classical approximation of $U_{0\hbar}(t,s)^* F_\hbar U_{0\hbar}(t,s)$ for a large class of A and V and showed the following result. Let

$$v = \hbar^{-n/4}(\exp i\hbar^{-1}x\cdot\xi^{(0)})g((x-x^{(0)})/\hbar^{1/2}) \tag{1.7}$$

for $g \in L^2$ independent of $0 < \hbar \le 1$ with $\|g\| = 1$. We note that v is centered in classical phase space $R^{2n}_{x,\xi}$ around $(x^{(0)}, \xi^{(0)})$ and that $\|v\| = 1$ (Remark 3.3 in the present paper). Then the mean value $(U_{0\hbar}(t,s)^* F_\hbar U_{0\hbar}(t,s)v, v)$ converges to $(\exp iq(t,s;x^{(0)},\xi^{(0)})\cdot\zeta)(\exp iz\cdot p(t,s;x^{(0)},\xi^{(0)}))$ as $\hbar > 0$ tends to zero. In case $F_\hbar$ is the position x_j and the momentum $\hbar D_{x_j} - A_j$ of particles Zucchini in [10] studied this problem.

More general $F_\hbar$ was studied in [8] for sufficiently smooth $A(t,x)$ and $V(t,x)$ in $x \in R^n$ in terms of pseudo-differential operators. Let $< x;\xi > = (1 + |x|^2 + |\xi|^2)^{1/2}$ and $S(< x;\xi >^m) = S(< x;\xi >^m; dx^2 + d\xi^2) = \{a(x,\xi) \in C^\infty; |\partial_\xi^\alpha \partial_x^\beta a| \le C_{\alpha\beta} < x;\xi >^m \text{ for all } \alpha,\beta\}$ $(-\infty < m < \infty)$ Hörmander's symbol class ([3]). We denote by $B^m(\hbar)$ $(m \ge 0)$ the weighted Sobolev space $\{r \in L^2; < x >^m r \in L^2, < \xi >^m \hat{r} \in L^2\}$ with norm $\|r\|_{B^m(\hbar)} = \|r\| + \| < x >^m r\| + \| < \hbar\xi >^m \hat{r}\|$ as in [8]. $\hat{r}$ denotes the Fourier transform $\int e^{-ix\cdot\xi} r(x)dx$. Let $F_\hbar$ be a pseudo-differential operator $f^w(x,\hbar D_x)$ with the Weyl symbol $f(x,\xi) \in S(< x;\xi >^m)$ $(m \ge 0)$ defined by

$$f^w(x,\hbar D_x)r(x) = (2\pi)^{-n}\iint e^{i(x-y)\cdot\xi} f\left(\frac{x+y}{2}, \hbar\xi\right) r(y)dyd\xi \quad (r \in \mathcal{S}). \tag{1.8}$$

$\mathcal{S}$ is the space of all rapidly decreasing functions on R^n. We denote by $f^w(\phi_s^t(x,\hbar D_x))$ the pseudo-differential operator with the Weyl symbol $f(\phi_s^t(x,\xi))$. Then Wang in [8] showed that $U_{0\hbar}(t,s)^* F_\hbar U_{0\hbar}(t,s)$ is approximated semi-classically by $f^w(\phi_s^t(x,\hbar D_x))$. From this result he got the following. Let $g \in B^{m/2}(1)$ with $\|g\| = 1$ be independent of $0 < \hbar \le 1$ and define an initial state v by (1.7). Then we have

$$\lim_{\hbar\to+0}(U_{0\hbar}(t,s)^* F_\hbar U_{0\hbar}(t,s)v, v) = f(\phi_s^t(x^{(0)},\xi^{(0)})). \tag{1.9}$$

It is evident that the right-hand side above is the solution of the equation in classical mechanics

$$\frac{d}{dt}w(t) = \{\mathcal{H}_0(t), f\}(\phi_s^t(x^{(0)},\xi^{(0)})), \quad w(s) = f(x^{(0)},\xi^{(0)}), \tag{1.10}$$

where $\{\mathcal{H}_0(t), f\}(x, \xi)$ denotes the Poisson bracket $\sum_{j=1}^n \left(\frac{\partial \mathcal{H}_0}{\partial \xi_j}\frac{\partial f}{\partial x_j} - \frac{\partial \mathcal{H}_0}{\partial x_j}\frac{\partial f}{\partial \xi_j}\right)$. These results in [2], [10], and [8] go back to Ehrenfest's theorem ([6]).

In the present paper we consider some charged particles with spin. Its Hamiltonian is

$$H_\hbar(t) = H_{0\hbar}(t)I_N + \hbar K_\hbar(t) \tag{1.11}$$

on the product space $L^2(R^n)^N$ of N copies of $L^2(R^n)$. I_N is an identity matrix. When no confusion can arise, we use the same notations $(\cdot, \cdot)$ and $\|\cdot\|$ of the inner product and the norm in $L^2(R^n)^N$ as in $L^2(R^n)$. Suppose that the (i, j)-component of $K_\hbar(t)$ $(i, j = 1, \cdots, N)$ is the Weyl operator with symbol $k_{ij}(t, x, \xi)$. Throughout the present paper we assume

$$\partial_\xi^\alpha \partial_x^\beta k_{ij}(t, x, \xi) \in S(< x; \xi >)\ (i, j = 1, \cdots, N),\ |\alpha + \beta| = 1 \tag{1.12}$$

and that $k(t, x, \xi) = (k_{ij}(t, x, \xi))_{i,j=1}^N$ is a Hermitian matrix. Then $K_\hbar(t)$ with domain $\mathcal{S}^N$ is essentially self-adjoint on $L^2(R^n)^N$([4]). Denote by $U_\hbar(t, s)$ the propagator for $H_\hbar(t)$ and let $F_\hbar = (f_{ij}^w(x, \hbar D_x))_{i,j=1}^N$ be an observable, where $f_{ij}(x, \xi) \in S(< x; \xi >^m)$ for some $m \geq 0$. Then as in the case of particles without spin, the solution of the Heisenberg equation with $F_\hbar$ at $t = s$ is given by $U_\hbar(t, s)^* F_\hbar U_\hbar(t, s)$. Our aim in the present paper is to give the formula of the semi-classical approximation of $U_\hbar(t, s)^* F_\hbar U_\hbar(t, s)$ and study the classical limit $\lim_{\hbar \to +0}(U_\hbar(t, s)^* F_\hbar U_\hbar(t, s)v, v)$ of the mean value for initial states like v in (1.7). A typical example of $K_\hbar(t)$ is

$$\begin{aligned} &B_{23}(t, x)\sigma_1 + B_{31}(t, x)\sigma_2 + B_{12}(t, x)\sigma_3 \\ &\qquad + V'(t, x)(L_1\sigma_1 + L_2\sigma_2 + L_3\sigma_3)\ (n = 3) \end{aligned}$$

([6]), where (B_{23}, B_{31}, B_{12}) is the magnetic strength, L_j the angular momentum, and σ_j the Pauli matrix. Let $\lambda(x, \xi) \in S(< x; \xi >^m)$ be a scalar function. A typical example of $F_\hbar$ is $\lambda^w(x, \hbar D_x)I_N$. Another one is given by $f_{ll}^w(x, \hbar D_x) = \lambda^w(x, \hbar D_x)$ for some l and $f_{ij}^w(x, \hbar D_x) = 0$ for $(i, j) \neq (l, l)$.

Our results will be stated in section 3 and there some remarks will be given. In section 4 we will give the proof of results.

2. A SIMPLE REMARK ON YAJIMA'S CONDITION

We first recall the definition of the electromagnetic potentials A, V (*cf.* [1], [6]). Let $(B_{jk}(t, x))_{1 \leq j < k \leq n}$ be the magnetic strength tensor and

$E(t,x) = (E_1, \cdots, E_n)$ the electric strength. It follows from the Maxwell equation that

$$d\Big(\sum_{1\le j<k\le n} B_{jk}dx_j \wedge dx_k\Big) = 0,$$
$$d\Big(\sum_{j=1}^{n} E_j dx_j\Big) = -\sum_{1\le j<k\le n} \partial_t B_{jk} dx_j \wedge dx_k$$

on R^n. The vector potential A is defined by

$$d\Big(\sum_{j=1}^{n} A_j dx_j\Big) = \sum_{1\le j<k\le n} B_{jk}dx_j \wedge dx_k. \qquad (2.1)$$

So we have $B_{jk} = \partial_{x_j}A_k - \partial_{x_k}A_j$. From this we have $d\{\sum_{j=1}^{n}(E_j + \partial_t A_j)dx_j\} = 0$. The scalar potential V is defined by

$$dV = -\sum_{j=1}^{n}(E_j + \partial_t A_j)dx_j. \qquad (2.2)$$

So $E_j = -\partial_t A_j - \partial_{x_j}V$ holds.

Let $(q(t,s;y,\eta), p(t,s;y,\eta))$ be the solution of (1.5). In [9] Yajima showed that $\partial_\eta^\alpha \partial_y^\beta q_j(t,s;y,\eta)$ and $\partial_\eta^\alpha \partial_y^\beta p_k(t,s;y,\eta)$ $(j,k = 1,2,\cdots,n)$ are bounded in $t,s \in [0,T]$ and $y,\eta \in R^n$ for any α,β such that $|\alpha + \beta| \ge 1$ under some condition. His condition depends on B_{jk}, A, and V. In this section we give a simple modification of his condition. Our one fundamentally depends on B_{jk} and E.

We set

$$B_{jk} = \begin{cases} -B_{kj}, & 1 \le k < j \le n, \\ 0, & 1 \le j = k \le n. \end{cases} \qquad (2.3)$$

Let $(x(t,s;y,\zeta), v(t,s;y,\zeta)) = (x_1,\cdots,x_n,v_1,\cdots,v_n)$ be the solution of the Lagrange equation corresponding to (1.5)

$$\frac{dx_j}{dt} = v_j,\ \frac{dv_k}{dt} = \sum_{l=1}^{n} B_{kl}v_l + E_k\ (j,k = 1,2,\cdots,n),\ (x,v)|_{t=s} = (y,\zeta). \qquad (2.4)$$

Then we have

$$q(t,s;y,\eta) = x(t,s;y,\zeta),$$
$$p(t,s;y,\eta) = v(t,s;y,\zeta) + A(t,x(t,s;y,\zeta)), \quad \zeta = \eta - A(s,y) \qquad (2.5)$$

(*cf.* [6]).

In [9] Yajima showed the following.

PROPOSITION 2.1. – *Suppose that* $B_{jk}(t,x)$ $(1 \le j < k \le n)$ *and* $E_j(t,x)$ $(j = 1, 2, \cdots, n)$ *are continuous in* $[0,T] \times R^n$ *and are infinitely differentiable in* R^n. *Assume the below. There exist an* $\epsilon > 0$ *and constants* C_α *such that*

$$|\partial_x^\alpha B_{jk}(t,x)| \le C_\alpha < x >^{-(1+\epsilon)}, \ |\partial_x^\alpha E_j(t,x)| \le C_\alpha,$$
$$|\alpha| \ge 1, \quad (t,x) \in [0,T] \times R^n. \tag{2.6}$$

Then $\partial_\zeta^\alpha \partial_y^\beta x_j(t,s;y,\zeta)$ *and* $\partial_\zeta^\alpha \partial_y^\beta v_k(t,s;y,\zeta)$ $(j,k = 1,2,\cdots,n)$ *are bounded in* $t, s \in [0,T]$ *and* $y, \zeta \in R^n$ *for any* α, β *such that* $|\alpha + \beta| \ge 1$.

LEMMA 2.2. – *Suppose besides the assumption in Proposition* 2.1 *that* $\partial_t B_{jk}(t,x)$ $(1 \le j < k \le n)$ *are continuous in* $[0,T] \times R^n$. *Then there exist the electromagnetic potentials* A, V *such that* (i) $\partial_t A_j$ $(j = 1,2,\cdots,n)$ *and* V *are continuous in* $[0,T] \times R^n$, (ii) A_j *and* V *are infinitely differentiable in* R^n, *and* (iii)

$$|\partial_x^\alpha A_j(t,x)| \le C_\alpha, \quad |\partial_x^\alpha V(t,x)| \le C_\alpha < x >,$$
$$|\alpha| \ge 1, \quad (t,x) \in [0,T] \times R^n. \tag{2.7}$$

Remark 2.1. – As will be seen in the proof below, we can choose $V(x) = 0$ in Lemma 2.2.

Proof. – Using (2.3), we set

$$A'_j(t,x) = -\sum_{k=1}^{n} \int_0^1 B_{jk}(t,sx) s x_k ds \quad (j = 1,\cdots,n).$$

It follows from the Poincaré lemma that $A' = (A'_1, \cdots, A'_n)$ is the vector potential, that is, A' satisfies (2.1) ([1]). Let $|\alpha| \ge 1$. We can easily get from (2.6)

$$\left| \int_0^1 (\partial_x^\alpha B_{jk})(t,sx) s^{|\alpha|+1} x_k ds \right| \le C_\alpha \int_0^1 < sx >^{-(1+\epsilon)} |x| ds$$
$$\le C'_\alpha \int_0^\infty < s >^{-(1+\epsilon)} ds < \infty.$$

So $\partial_x^\alpha A'_j(t,x)$ for any $\alpha \ne 0$ is bounded in $[0,T] \times R^n$. For this A' determine the scalar potential V' from (2.2). Let us define A, V by the Gauge transformation

$$A_j(t,x) = A'_j(t,x) + \partial_{x_j} \int_0^t V'(\theta,x) d\theta,$$
$$V(t,x) = V'(t,x) - \partial_t \int_0^t V'(\theta,x) d\theta = 0.$$

Then $\partial_t A_j$ $(j = 1, 2, \cdots, n)$ is continuous in $[0, T] \times R^n$ because so is $\partial_t A'_j(t, x)$. We also have $E = -\partial_t A$ by (2.2) and so

$$A_j(t,x) = A_j(0,x) + t \int_0^1 (\partial_t A_j)(\theta t, x) d\theta$$
$$= A'_j(0,x) - t \int_0^1 E_j(\theta t, x) d\theta.$$

Hence we can see by (2.6) that $\partial_x^\alpha A_j(t,x)$ for any $\alpha \neq 0$ is bounded in $[0, T] \times R^n$. Thus we could complete the proof. Q.E.D.

The proposition below follows from Proposition 2.1, Lemma 2.2, and (2.5).

PROPOSITION 2.3. – *Suppose the same assumption as in Lemma* 2.2. *Choose the electromagnetic potentials A, V satisfying* (2.7). *Then $\partial_\eta^\alpha \partial_y^\beta q_j(t, s; y, \eta)$ and $\partial_\eta^\alpha \partial_y^\beta p_k(t, s; y, \eta)$ $(j, k = 1, 2, \cdots, n)$ are bounded in $t, s \in [0, T]$ and $y, \eta \in R^n$ for any α, β such that $|\alpha + \beta| \geq 1$.*

3. RESULTS

Let $B^m(\hbar)$ $(m \geq 0)$ be the weighted Sobolev space introduced in section 1. We denote its dual space and norm by $B^{-m}(\hbar)$ and $\| \cdot \|_{B^{-m}(\hbar)}$ respectively.

Let's denote the direct product space of N copies of $B^m(\hbar)$ $(-\infty < m < \infty)$ by $B^m(\hbar)^N$ with norm $\|(f_1, \cdots, f_N)\|_{B^m(\hbar)^N} = (\sum_{j=1}^N \|f_j\|^2_{B^m(\hbar)})^{1/2}$. The space of all $B^m(\hbar)^N$-valued j times continuously differentiable functions in $t, s \in [0, T]$ is denoted by $\mathcal{E}^j_{t,s}([0, T]; B^m(\hbar)^N)$. We define the semi-norms of $a(x, \xi) \in S(< x; \xi >^m)$ by

$$|a|_l = \max_{|\alpha+\beta| \leq l} \sup_{x,\xi} < x; \xi >^{-m} |\partial_\xi^\alpha \partial_x^\beta a(x, \xi)|, \quad l = 0, 1, 2, \cdots.$$

We proved the following in [4] (*cf.* [3], [7], [8]).

LEMMA 3.1. – (i) *Let $m \geq 0$ and $\Gamma_m = \gamma_m(x, \hbar D_x)$ the pseudo-differential operator with symbol $\gamma_m(x, \xi) = (< x > + < \xi >)^m$, that is, $\Gamma_m r(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \gamma_m(x, \hbar\xi) \hat{r}(\xi) d\xi$. Then there exist constants $\mu(m), C_{mB}$, and C_{mB} independent of $0 < \hbar \leq 1$ such that we have for any $r \in \mathcal{S}$*

$$C_{mB}^{-1} \|(\mu(m) + \Gamma_m) r\| \leq \|r\|_{B^m(\hbar)} \leq C_{mB} \|(\mu(m) + \Gamma_m) r\|$$

and

$$\dot{C}_{mB}^{-1}\|(\mu(m)+\Gamma_m)^{-1}r\| \le \|r\|_{B^{-m}(\hbar)} \le \dot{C}_{mB}\|(\mu(m)+\Gamma_m)^{-1}r\|.$$

In addition, there exists a bounded family $\{l_m(x,\xi;\hbar)\}_{0<\hbar\le 1}$ *in* $S(<x;\xi>^{-m})$ *such that* $l_m(x,\hbar D_x;\hbar) = (\mu(m)+\Gamma_m)^{-1}$ *on* $\mathcal{S}$.

(ii) *Let* $a(x,\xi) \in S(<x;\xi>^m)$ $(-\infty < m < \infty)$ *and* $A = a^w(x,\hbar D_x)$ *or* $a(x,\hbar D_x)$. *Then for any* $-\infty < m' < \infty$ *there exist constants* l *and* $C_{m,m'}$ *independent of* $0<\hbar\le 1$ *such that we have for any* $r \in B^{m'}(\hbar)$

$$\|Ar\|_{B^{m'-m}(\hbar)} \le C_{m,m'}|a|_l\|r\|_{B^{m'}(\hbar)}.$$

Let $H_\hbar(t)$ be the Hamiltonian with spin defined by (1.11). We consider the equation

$$i\hbar\frac{\partial u}{\partial t}(t) = H_\hbar(t)u(t),\ u(s) = v \quad (t,s\in[0,T]), \tag{3.1}$$

where $u = {}^t(u_1,\cdots,u_N)$. Suppose (2.7). Then we have

$$\partial_\xi^\alpha\partial_x^\beta\mathcal{H}_0(t,x,\xi) \in S(<x;\xi>),\ |\alpha+\beta| = 1. \tag{3.2}$$

Consequently we get the following from Theorem in [4].

LEMMA 3.2. – *Assume* (1.12) *and* (2.7). *Then for any* $v \in B^m(\hbar)^N$ $(-\infty < m < \infty)$ *there exists a unique solution* $u(t) \in \mathcal{E}^0_{t,s}([0,T];B^m(\hbar)^N) \cap \mathcal{E}^1_{t,s}([0,T];B^{m-2}(\hbar)^N)$ *of* (3.1). *In addition, there exists a constant* $C_m(T)$ *independent of* $0<\hbar\le 1$ *such that*

$$\|u(t)\|_{B^m(\hbar)^N} \le C_m(T)\|v\|_{B^m(\hbar)^N} \quad (0\le t\le T). \tag{3.3}$$

In particular we have for $v \in (L^2)^N$

$$\|u(t)\| = \|v\| \quad (0\le t\le T). \tag{3.4}$$

The propagator $U_\hbar(t,s)$ of (3.1) is defined by $u(t) = U_\hbar(t,s)v$.

Let us define an N by N matrix $z(t,s;x,\xi)$ by

$$\frac{dz}{dt} = -ik(t,\phi_s^t(x,\xi))z, \quad z|_{t=s} = I_N. \tag{3.5}$$

We denote the adjoint matrix of z by $z^\dagger$ as in [6]. Since we assumed that $k(t,x,\xi)$ is Hermitian, we can easily have $\frac{d}{dt}z(t,s)^\dagger z(t,s) = 0$ and $z(t,s)^\dagger z(t,s)|_{t=s} = I_N$ and so

$$z(t,s)^\dagger z(t,s) = I_N. \tag{3.6}$$

That is, $z(t,s)$ is a unitary matrix. Denote the (i,j)-component of $z(t,s)$ by $z_{ij}(t,s)$.

LEMMA 3.3. – *Suppose the same assumption as in Lemma* 2.2 *and choose* A *and* V *satisfying* (2.7). *In addition, we assume*

$$\sup_{x,\xi}\int_0^T |(\partial_\xi^\alpha \partial_x^\beta k_{ij})(\theta, \phi_0^\theta(x,\xi))|d\theta \le C_{\alpha,\beta} < \infty, \quad |\alpha+\beta| \ge 1 \tag{3.7}$$

for $i,j = 1,2,\cdots,N$. *Then* $\partial_\xi^\alpha \partial_x^\beta z_{ij}(t,s;x,\xi)$ $(i,j = 1,2,\cdots,N)$ *is bounded in* $t,s \in [0,T]$ *and* $x,\xi \in R^n$ *for any* α,β.

Remark 3.1. – Consider the typical example $K_\hbar(t) = B_{23}(t,x)\sigma_1 + B_{31}(t,x)\sigma_2 + B_{12}(t,x)\sigma_3$ $(n = 3)$. We suppose the same assumption as in Lemma 2.2. Then it follows from (2.6) that (1.12) and (3.7) are automatically satisfied.

Proof. – $|z_{ij}(t,s)| \le 1$ is clear, because $z(t,s)$ is unitary. We can easily have from (3.5)

$$\frac{d}{dt}\frac{\partial z}{\partial x_j} = -ik(t,\phi_s^t(x,\xi))\frac{\partial z}{\partial x_j} - i\left(\frac{\partial}{\partial x_j}k(t,\phi_s^t(x,\xi))\right)z$$

and so

$$\begin{aligned}\frac{\partial z}{\partial x_j}&(t,s;x,\xi)\\ &= -i\int_s^t z(t,s;x,\xi)z(\theta,s;x,\xi)^{-1}\left(\frac{\partial}{\partial x_j}k(\theta,\phi_s^\theta(x,\xi))\right)z(\theta,s;x,\xi)d\theta.\end{aligned}$$

Since z is unitary, it follows from Proposition 2.3 and (3.7) that $\partial_{x_j} z(t,s;x,\xi)$ is bounded in $t,s \in [0,T]$ and $x,\xi \in R^n$. In the same way we can complete the proof by induction. Q.E.D.

We suppose the same assumption as in Lemma 3.3. Then we had $z_{ij}(t,s;x,\xi) \in S(1)$. Set

$$Z_\hbar(t,s) = (z_{ij}^w(t,s;x,\hbar D_x))_{i,j=1}^N. \tag{3.8}$$

Then we see from (ii) in Lemma 3.1 that $Z_\hbar(t,s)$ is a bounded operator on $L^2(R^n)^N$. We denote by $Z_\hbar(t,s)^*$ its adjoint operator on $L^2(R^n)^N$. In the same way the adjoint operator $U_\hbar(t,s)^*$ of $U_\hbar(t,s)$ can be defined from Lemma 3.2.

Let $f_{ij}(x,\xi) \in S(<x;\xi>^m)$ $(m \geq 0, i,j = 1,2,\cdots,N)$. Then we see from Proposition 2.3

$$f_{ij}(\phi_s^t(x,\xi)) \in S(<x;\xi>^m), \tag{3.9}$$

because we have $|q(t,s;x,\xi)|, |p(t,s;x,\xi)| \leq \text{Const.} <x;\xi>$. We set

$$F(\phi_s^t)_\hbar = (f_{ij}^w(\phi_s^t(x,\hbar D_x)))_{i,j=1}^N. \tag{3.10}$$

That is, the (i,j)-component of $F(\phi_s^t)_\hbar$ is the pseudo-differential operator with the Weyl symbol $f_{ij}(\phi_s^t(x,\xi))$. We obtain the result below including that in [8].

THEOREM 3.4. – *Suppose the same assumption as in Lemma* 3.3 *and* (1.12). *Let* $f_{ij} \in S(<x;\xi>^m)$ $(m \geq 0, i,j = 1,\cdots,N)$. *Then for any* $-\infty < m' < \infty$ *there exists a constant* $C_{m,m'}(T)$ *independent of* $0 < \hbar \leq 1$ *such that we have for any* $v \in B^{m'}(\hbar)^N$

$$\begin{aligned} &\|(U_\hbar(t,s)^* F_\hbar U_\hbar(t,s) - Z_\hbar(t,s)^* F(\phi_s^t)_\hbar Z_\hbar(t,s))v\|_{B^{m'-m-1}(\hbar)^N} \\ &\leq \hbar C_{m,m'}(T)\|v\|_{B^{m'}(\hbar)^N},\ t,s \in [0,T]. \end{aligned} \tag{3.11}$$

In particular let $F_\hbar = \lambda^w(x,\hbar D_x)I_N$ *where* $\lambda(x,\xi)$ *is scalar. Then we have*

$$\begin{aligned} &\|(U_\hbar(t,s)^* F_\hbar U_\hbar(t,s) - \lambda^w(\phi_s^t(x,\hbar D_x)))v\|_{B^{m'-m-1}(\hbar)^N} \\ &\leq \hbar C_{m,m'}(T)\|v\|_{B^{m'}(\hbar)^N},\ t,s \in [0,T]. \end{aligned} \tag{3.12}$$

Remark 3.2. – Suppose that $|t-s|$ is small. Then following [5], we can construct the asymptotic solution in $\hbar$ of

$$i\hbar\frac{\partial u}{\partial t}(t) = H_\hbar(t)u(t), \quad u(s) = (\exp i\hbar^{-1}x\cdot\xi)v(x).$$

Let $\phi(t,s;x,\xi)$ be the solution of $\partial_t\phi + \mathcal{H}_0(t,x,\partial_x\phi) = 0$ with $\phi|_{t=s} = x\cdot\xi$, where $\partial_x\phi = (\partial_{x_1}\phi,\cdots,\partial_{x_n}\phi)$. Set $\frac{\partial q}{\partial y} = (\frac{\partial q_i}{\partial y_j})_{i,j=1}^n$. We define $u_\hbar(t,x)$ by

$$u_\hbar(t,x) = (\exp i\hbar^{-1}\phi(t,s;x,\xi))(\det\frac{\partial q}{\partial y}(t,s;y,\xi))^{-1/2}z(t,s;y,\xi)v(y)$$

where $x = q(t,s;y,\xi)$. Then we have

$$i\hbar\frac{\partial u_\hbar}{\partial t}(t) - H_\hbar(t)u_\hbar(t) = O(\hbar^2), \quad u_\hbar(s) = (\exp i\hbar^{-1}x\cdot\xi)v(x).$$

Thus $z(t,s;x,\xi)$ defined by (3.5) naturally appears.

Let

$$v = \hbar^{-n\tau/2}(\exp i\hbar^{-1}x\cdot\xi^{(0)})g((x-x^{(0)})/\hbar^{\tau}) \tag{3.13}$$

be an initial state, where $0 \le \tau \le 1$ is a constant and $g = {}^t(g_1,\cdots,g_N)$. Then we have $\|v\| = \|g\|$.

THEOREM 3.5. – *Suppose the same assumption as in Theorem* 3.4. *Let* $g \in B^{(m+1)/2}(1)^N$ *with* $\|g\| = 1$ *be independent of* $0 < \hbar \le 1$ *and define* v *by* (3.13). *Set* $f(x,\xi) = (f_{ij}(x,\xi))_{i,j=1}^N$. *Then the mean value* $(U_\hbar(t,s)^*F_\hbar U_\hbar(t,s)v,v)$ *is well defined. In addition, as* $\hbar$ *tends to zero, the mean value above converges to*

$$\begin{cases} (f(\phi_s^t(x^{(0)},\xi^{(0)}))z(t,s;x^{(0)},\xi^{(0)})g, z(t,s;x^{(0)},\xi^{(0)})g), & 0<\tau<1, \\ (f(\phi_s^t(\cdot+x^{(0)},\xi^{(0)}))z(t,s;\cdot+x^{(0)},\xi^{(0)})g, z(t,s;\cdot+x^{(0)},\xi^{(0)})g), & \tau=0, \\ (f(\phi_s^t(x^{(0)},D_x+\xi^{(0)}))z(t,s;x^{(0)},D_x+\xi^{(0)})g, z(t,s;x^{(0)},D_x+\xi^{(0)})g), & \tau=1. \end{cases} \tag{3.14}$$

In particular let $F_\hbar = \lambda^w(x,\hbar D_x)I_N$ *where* $\lambda(x,\xi)$ *is scalar. Then* $(U_\hbar(t,s)^*F_\hbar U_\hbar(t,s)v,v)$ *converges to*

$$\begin{cases} \lambda(\phi_s^t(x^{(0)},\xi^{(0)})), & 0<\tau<1, \\ (\lambda(\phi_s^t(\cdot+x^{(0)},\xi^{(0)}))g,g), & \tau=0, \\ (\lambda(\phi_s^t(x^{(0)},D_x+\xi^{(0)}))g,g), & \tau=1 \end{cases} \tag{3.15}$$

as $\hbar$ *tends to zero. So when* $0<\tau<1$, *the classical limit is the solution of the classical equation* (1.10).

Theorems 3.4 and 3.5 will be proved in the next section.

Remark 3.3. – We can easily see that v in (3.13) is represented in the momentum space by

$$\begin{aligned} &(2\pi\hbar)^{-n/2}\int(\exp -i\hbar^{-1}x\cdot\xi)v(x)dx \\ &= (2\pi\hbar^{1-\tau})^{-n/2}(\exp -i\hbar^{-1}x^{(0)}\cdot(\xi-\xi^{(0)}))\hat{g}((\xi-\xi^{(0)})/\hbar^{1-\tau}). \end{aligned}$$

Let $0 < \tau < 1$ and $\hbar$ sufficiently small. Then v is centered around $(x^{(0)},\xi^{(0)})$ in classical phase space $R^{2n}_{x,\xi}$. On the other hand in case of $\tau = 0$ v is done around $\xi^{(0)}$ only in the momentum space R^n_ξ. In case of $\tau = 1$ v is done around $x^{(0)}$ only in R^n_x. Our result in Theorem 3.5 corresponds to these.

Remark 3.4. – In Theorem 3.5 replace $K_\hbar(t)$ by the multiplication operator $\hat{K}_\hbar(t) = k(t,\phi_s^t(x^{(0)},\xi^{(0)}))$. We set $\hat{H}_\hbar(t) = H_{0\hbar}(t)I_N + \hbar\hat{K}_\hbar(t)$ and denote the propagator for it by $\hat{U}_\hbar(t,s)$. Let $0 < \tau < 1$. Then applying Theorem 3.5 to $\hat{H}_\hbar(t)$, we can see that the mean value $(\hat{U}_\hbar(t,s)^* F_\hbar \hat{U}_\hbar(t,s)v, v)$ converges to the same function as for $H_\hbar(t)$. We also remark that $\hat{U}_\hbar(t,s)$ is given by $z(t,s;x^{(0)},\xi^{(0)})U_{0\hbar}(t,s)$.

Remark 3.5. – Let $N = 1, K_\hbar(t) = 0$, and $\tau = 1/2$ in Theorem 3.5. Then our result generalizes his in [8]. In this case the classical limit of the mean value is the solution of the classical equation. But this is not true in case of $\tau = 0$. In fact consider

$$H_\hbar = -\frac{1}{2}\hbar^2\partial_x^2 + V(x),\ V(x) = \frac{1}{3}\chi(x)x^3,\ x \in R^1,$$

where $\chi(x)$ is an infinitely differentiable and real-valued function with compact support such that $\chi(x) = 1$ for $|x| \le 1$. Let $g(x)$ be an infinitely differentiable function with $\|g\| = 1$ such that $|g(-x)| = |g(x)|$ for $x \in R^1$ and $g(x) = 0$ for $|x| \ge 1$. Setting $\tau = 0$ and $(x^{(0)},\xi^{(0)}) = (0,0)$, define v by (3.13). We choose the position operator x as $F_\hbar$. Then it follows from Theorem 3.5 that the mean value converges to $Q(t) = (q(t,s;\cdot,0)g,g)$. So we have $\frac{d^2Q}{dt^2}(s) = -(\frac{\partial V}{\partial x}(\cdot)g,g) = -(x^2g,g) < 0$ from the assumption on χ and g. We also have $\frac{\partial V}{\partial x}(Q(s)) = 0$ because of $Q(s) = (xg,g) = 0$. So $\frac{d^2Q}{dt^2}(s) \ne -\frac{\partial V}{\partial x}(Q(s))$. This indicates that the classical limit $(q(t,s;\cdot,0)g,g)$ doesn't satisfy the classical equation.

4. PROOF OF THEOREMS

LEMMA 4.1. – *Let* $z(t,s;x,\xi)$ *be the solution of* (3.5). *Then we have:*

(i) $z(t,s;x,\xi)^\dagger = z(s,t;\phi_s^t(x,\xi))$.

(ii) $\frac{\partial}{\partial s}z(t,s;x,\xi)^\dagger = -ik(s,x,\xi)z(t,s;x,\xi)^\dagger + \{z(t,s)^\dagger,\mathcal{H}_0(s)\}(x,\xi)$.

Here $\{z(t,s)^\dagger,\mathcal{H}_0(s)\}$ *denotes the matrix whose* (i,j)*-component is defined by* $\{\zeta_{ij},\mathcal{H}_0(s)\}$, *letting* ζ_{ij} *be the* (i,j)*-component of* $z(t,s)^\dagger$.

Proof. – (i) We have

$$z(t,s;x,\xi)z(s,\theta;\phi_s^\theta(x,\xi)) = z(t,\theta;\phi_s^\theta(x,\xi)). \tag{4.1}$$

In fact both sides are the solutions of

$$\frac{d}{dt}w(t) = -ik(t,\phi_s^t(x,\xi))w(t), \quad w(s) = z(s,\theta;\phi_s^\theta(x,\xi))$$

because of $\phi_\theta^t(\phi_s^\theta(x,\xi)) = \phi_s^t(x,\xi)$. So we get (4.1). Setting $\theta = t$ in (4.1), we have (i) because $z(t,s)$ is unitary.

(ii) Let $a(x, \xi)$ be a scalar function. Then we know

$$\frac{\partial}{\partial t} a(\phi_s^t(x,\xi)) = -\{a, \mathcal{H}_0(t)\}(\phi_s^t(x,\xi)),$$
$$\frac{\partial}{\partial s} a(\phi_s^t(x,\xi)) = \{a(\phi_s^t), \mathcal{H}_0(s)\}(x,\xi), \tag{4.2}$$

where $a(\phi_s^t)(x,\xi) = a(\phi_s^t(x,\xi))$ (*cf.* [8]). Using this, we have from (i)

$$\begin{aligned}\frac{\partial}{\partial s} z(t,s;x,\xi)^\dagger &= \frac{dz}{dt}(s,t;\phi_s^t(x,\xi)) + \{z(s,t;\phi_s^t), \mathcal{H}_0(s)\}(x,\xi) \\ &= \frac{dz}{dt}(s,t;\phi_s^t(x,\xi)) + \{z(t,s)^\dagger, \mathcal{H}_0(s)\}(x,\xi).\end{aligned}$$

Since we have from (3.5)

$$\frac{dz}{dt}(s,t;\phi_s^t(x,\xi)) = -ik(s,\phi_t^s(\phi_s^t(x,\xi))z(t,s)^\dagger = -ik(s,x,\xi)z(t,s)^\dagger,$$

we see that (ii) holds. Q.E.D.

The lemma below follows from section 18.5 in [3].

LEMMA 4.2. – *Let* $a_j(x,\xi) \in S(< x;\xi >^{m_j})$ $(-\infty < m_j < \infty, j = 1,2)$ *be a scalar function. We set*

$$\begin{aligned}a_1 \sharp a_2(x,\xi;\hbar) = \pi^{-2n} \iiiint e^{2i\eta'\cdot y - 2iy'\cdot\eta} a_1(x+y, \xi+\hbar\eta) \\ \times a_2(x+y', \xi+\hbar\eta') dy d\eta dy' d\eta'.\end{aligned}$$

Then we have:

(i) $a_1\sharp a_2{}^w(x,\hbar D_x;\hbar) = a_1^w(x,\hbar D_x) a_2^w(x,\hbar D_x)$.

(ii) $\{a_1\sharp a_2(x,\xi;\hbar)\}_{0<\hbar\le 1}$ *is a bounded family in* $S(< x;\xi >^{m_1+m_2})$.

(iii) *So are* $\{(a_1\sharp a_2(x,\xi;\hbar) - a_1(x,\xi)a_2(x,\xi))/\hbar\}_{0<\hbar\le 1}$ *and* $\{(a_1\sharp a_2 (x,\xi;\hbar) - a_2\sharp a_1(x,\xi;\hbar) - \frac{\hbar}{i}\{a_1,a_2\}(x,\xi))/\hbar^2\}_{0<\hbar\le 1}$. *Moreover we assume that one of* a_1 *and* a_2 *satisfies*

$$\partial_\xi^\alpha \partial_x^\beta a_j(x,\xi) \in S(< x;\xi >^{m_j - 1}), \quad |\alpha+\beta| = 1.$$

Then we have (iii) *where* $S(< x;\xi >^{m_1+m_2})$ *is replaced by* $S(< x;\xi >^{m_1+m_2-1})$.

Now we will prove Theorem 3.4. We see from (1.12) and (3.2) that $\mathcal{H}_0(t,x,\xi)$ and each component of $k(t,x,\xi)$ belong to $S(< x;\xi >^2)$. It is not difficult to prove

$$Z_\hbar(t,s)^* = z^w(t,s;x,\hbar D_x)^\dagger. \tag{4.3}$$

The right hand side above denotes the pseudo-differential operator with the Weyl symbol $z(t,s;x,\xi)^{\dagger}$.

Let us apply Lemma 4.2 to the commutator $[Z_\hbar(t,s)^*, H_{0\hbar}(s)]$, noting (3.2), Lemma 3.3, and (4.3). Then there exists a bounded family $\{r_1(t,s,x,\xi;\hbar)\}_{0<\hbar\leq 1}$ in $S(<x;\xi>)^{N^2}$ such that

$$[Z_\hbar(t,s)^*, H_{0\hbar}(s)] = \frac{\hbar}{i}\{z(t,s)^{\dagger}, \mathcal{H}_0(s)\}^w(x,\hbar D_x) + \hbar^2 r_1^w(t,s,x,\hbar D_x;\hbar).$$

Using (1.12), we also have from Lemma 4.1

$$\begin{aligned}\frac{\partial}{\partial s}Z_\hbar(t,s)^* = &- iK_\hbar(s)Z_\hbar(t,s)^* + \{z(t,s)^{\dagger}, \mathcal{H}_0(s)\}^w(x,\hbar D_x)\\ &+ \hbar r_2^w(t,s,x,\hbar D_x;\hbar),\end{aligned}$$

where $\{r_2(t,s,x,\xi;\hbar)\}_{0<\hbar\leq 1}$ is a bounded family in $S(<x;\xi>)^{N^2}$. Hence there exists a bounded family $\{r_3(t,s,x,\xi;\hbar)\}_{0<\hbar\leq 1}$ in $S(<x;\xi>)^{N^2}$ such that

$$\begin{aligned}\frac{\partial}{\partial s}Z_\hbar(t,s)^* = &- iK_\hbar(s)Z_\hbar(t,s)^* + \frac{i}{\hbar}[Z_\hbar(t,s)^*, H_{0\hbar}(s)]\\ &+ \hbar r_3^w(t,s,x,\hbar D_x;\hbar).\end{aligned} \tag{4.4}$$

We have from this

$$\frac{\partial}{\partial s}Z_\hbar(t,s) = iZ_\hbar(t,s)K_\hbar(s) + \frac{i}{\hbar}[Z_\hbar(t,s), H_{0\hbar}(s)] + \hbar r_3^w(t,s,x,\hbar D_x;\hbar)^*. \tag{4.5}$$

In the same way we can prove the following because we have from (4.2)

$$\frac{\partial}{\partial s}f(\phi_s^t(x,\xi)) = \{f(\phi_s^t), \mathcal{H}_0(s)\}(x,\xi)$$

and had $f_{ij}(\phi_s^t) \in S(<x;\xi>^m)$. There exists a bounded family $\{r_4(t,s,x,\xi;\hbar)\}_{0<\hbar\leq 1}$ in $S(<x;\xi>^{m+1})^{N^2}$ such that

$$\frac{\partial}{\partial s}F(\phi_s^t)_\hbar = \frac{i}{\hbar}[F(\phi_s^t)_\hbar, H_{0\hbar}(s)] + \hbar r_4^w(t,s,x,\hbar D_x;\hbar). \tag{4.6}$$

It is easy to see from Lemma 3.2 that $U_\hbar(t,s)^* = U_\hbar(s,t)$ and

$$i\hbar\frac{\partial}{\partial t}U_\hbar(t,s)^* = -U_\hbar(t,s)^* H_\hbar(t). \tag{4.7}$$

We are now ready to mimic the proof in [8]. Set

$$\Omega(\theta) = U_\hbar(\theta, s)^* Z_\hbar(t, \theta)^* F(\phi_\theta^t)_\hbar Z_\hbar(t, \theta) U_\hbar(\theta, s). \tag{4.8}$$

Considering Lemma 3.3 and (3.9), we have from (4.4)-(4.7)

$$\begin{aligned}
& i\hbar \frac{d\Omega}{d\theta}(\theta) \\
&= - U_\hbar(\theta, s)^* \big(H_{0\hbar}(\theta) + \hbar K_\hbar(\theta)\big) Z_\hbar(t, \theta)^* F(\phi_\theta^t)_\hbar Z_\hbar(t, \theta) U_\hbar(\theta, s) \\
&\quad + U_\hbar(\theta, s)^* \big(\hbar K_\hbar(\theta) Z_\hbar(t, \theta)^* \\
&\qquad\qquad - [Z_\hbar(t, \theta)^*, H_{0\hbar}(\theta)]\big) F(\phi_\theta^t)_\hbar Z_\hbar(t, \theta) U_\hbar(\theta, s) \\
&\quad + U_\hbar(\theta, s)^* Z_\hbar(t, \theta)^* \big(-[F(\phi_\theta^t)_\hbar, H_{0\hbar}(\theta)]\big) Z_\hbar(t, \theta) U_\hbar(\theta, s) \\
&\quad + U_\hbar(\theta, s)^* Z_\hbar(t, \theta)^* F(\phi_\theta^t)_\hbar \big(-\hbar Z_\hbar(t, \theta) K_\hbar(\theta) \\
&\qquad\qquad - [Z_\hbar(t, \theta), H_{0\hbar}(\theta)]\big) U_\hbar(\theta, s) \\
&\quad + U_\hbar(\theta, s)^* Z_\hbar(t, \theta)^* F(\phi_\theta^t)_\hbar Z_\hbar(t, \theta) \big(H_{0\hbar}(\theta) + \hbar K_\hbar(\theta)\big) U_\hbar(\theta, s) \\
&\quad + \hbar^2 U_\hbar(\theta, s)^* r_5^w(t, \theta, s, x, \hbar D_x; \hbar) U_\hbar(\theta, s),
\end{aligned}$$

where $\{r_5(t, \theta, s, x, \xi; \hbar)\}_{0<\hbar\leq 1}$ is bounded in $S(< x; \xi >^{m+1})^{N^2}$. So we get

$$\begin{aligned}
& i\hbar U_\hbar(\theta, s) \frac{d\Omega}{d\theta}(\theta) U_\hbar(\theta, s)^* \\
&= \big(-[H_{0\hbar}(\theta), Z_\hbar(t, \theta)^* F(\phi_\theta^t)_\hbar Z_\hbar(t, \theta)] \\
&\qquad + [H_{0\hbar}(\theta), Z_\hbar(t, \theta)^*] F(\phi_\theta^t)_\hbar Z_\hbar(t, \theta) \\
&\qquad + Z_\hbar(t, \theta)^* [H_{0\hbar}(\theta), F(\phi_\theta^t)_\hbar] Z_\hbar(t, \theta) \\
&\qquad + Z_\hbar(t, \theta)^* F(\phi_\theta^t)_\hbar [H_{0\hbar}(\theta), Z_\hbar(t, \theta)]\big) \\
&\quad + \hbar\big([K_\hbar(\theta), Z_\hbar(t, \theta)^* F(\phi_\theta^t)_\hbar Z_\hbar(t, \theta)] \\
&\qquad - K_\hbar(\theta) Z_\hbar(t, \theta)^* F(\phi_\theta^t)_\hbar Z_\hbar(t, \theta) + Z_\hbar(t, \theta)^* F(\phi_\theta^t)_\hbar Z_\hbar(t, \theta) K_\hbar(\theta)\big) \\
&\quad + \hbar^2 r_5^w(t, \theta, s, x, \hbar D_x; \hbar) \\
&= \hbar^2 r_5^w(t, \theta, s, x, \hbar D_x; \hbar).
\end{aligned}$$

Thus we obtain

$$\frac{d\Omega}{d\theta}(\theta) = -i\hbar U_\hbar(\theta, s)^* r_5^w(t, \theta, s, x, \hbar D_x; \hbar) U_\hbar(\theta, s) \tag{4.9}$$

and so

$$\begin{aligned}
& U_\hbar(t, s)^* F_\hbar U_\hbar(t, s) - Z_\hbar(t, s)^* F(\phi_s^t)_\hbar Z_\hbar(t, s) \\
&\qquad = -i\hbar \int_s^t U_\hbar(\theta, s)^* r_5^w(t, \theta, s, x, \hbar D_x; \hbar) U_\hbar(\theta, s) d\theta.
\end{aligned} \tag{4.10}$$

Applying Lemmas 3.1 and 3.2 to (4.10), we can complete the proof of (3.11).

We will prove (3.12). Apply Lemma 4.2 to $Z_\hbar(t,s)^* F(\phi_s^t)_\hbar Z_\hbar(t,s)$. Then there exists a bounded family $\{r_6(t,s,x,\xi;\hbar)\}_{0<\hbar\leq 1}$ in $S(<x;\xi>^m)^{N^2}$ such that

$$Z_\hbar(t,s)^* F(\phi_s^t)_\hbar Z_\hbar(t,s) = \lambda^w(\phi_s^t(x,\hbar D_x)) + \hbar r_6^w(t,s,x,\hbar D_x;\hbar) \quad (4.11)$$

because $z(t,s;x,\xi)$ is unitary and $f(\phi_s^t)z(t,s) = z(t,s)f(\phi_s^t)$. So we can prove (3.12) from (3.11). Thus we could complete the proof of Theorem 3.4.

Next we will prove Theorem 3.5. Let $a(x,\xi) \in S(<x;\xi>^{m'})$ $(-\infty < m' < \infty)$ be scalar. It is easy to see that

$$\begin{aligned} &a(x,\hbar D_x)(e^{i\hbar^{-1}x\cdot\xi^{(0)}}r) = e^{i\hbar^{-1}x\cdot\xi^{(0)}}a(x,\hbar D_x + \xi^{(0)})r, \\ &a(x,\hbar D_x)r(\cdot/\hbar^\tau) = (a(\hbar^\tau x, \hbar^{1-\tau}D_x)r)(\cdot/\hbar^\tau). \end{aligned} \quad (4.12)$$

Apply this to v defined by (3.13). Then setting $m' = (m+1)/2$ (≥ 0), we have from (i) in Lemma 3.1

$$\|v\|_{B^{m'}(\hbar)^N} \leq C_{m'B}\|\{\mu(m') + \gamma_{m'}(\hbar^\tau x, \hbar^{1-\tau}D_x + \xi^0)\}g(\cdot - x^0)\|.$$

Applying (ii) in Lemma 3.1 to the right-hand side above, we get the following. There exists a constant C independent of $0 < \hbar \leq 1$ such that

$$\|v\|_{B^{(m+1)/2}(\hbar)^N} \leq C\|g\|_{B^{(m+1)/2}(1)^N}. \quad (4.13)$$

We can easily show from Theorem 3.4 and (4.13) that $(U_\hbar(t,s)^* F_\hbar U_\hbar(t,s)v, v)$ is well defined and that

$$\lim_{\hbar\to+0}\big((U_\hbar(t,s)^* F_\hbar U_\hbar(t,s)v,v) - (Z_\hbar(t,s)^* F(\phi_s^t)_\hbar Z_\hbar(t,s)v,v)\big) = 0. \quad (4.14)$$

Set

$$\begin{aligned} &\delta(t,s;x,\xi) = z(t,s;x,\xi)^\dagger f(\phi_s^t(x,\xi))z(t,s;x,\xi), \\ &\Delta_\hbar(t,s) = \delta^w(t,s;x,\hbar D_x). \end{aligned} \quad (4.15)$$

Then noting (4.3), we also have from Lemma 4.2

$$\lim_{\hbar\to+0}\big((Z_\hbar(t,s)^* F(\phi_s^t)_\hbar Z_\hbar(t,s)v,v) - (\Delta_\hbar(t;s)v,v)\big) = 0. \quad (4.16)$$

We denote the Weyl operator $(\exp i\hbar^{-1}x \cdot \xi^{(0)})(\exp -ix^{(0)} \cdot D_x)$ by $W_\hbar(x^{(0)}, \xi^{(0)})$ as in [8]. Then we can write

$$v = \hbar^{-n\tau/2} W_\hbar(x^{(0)}, \xi^{(0)}) g(\cdot/\hbar^\tau). \tag{4.17}$$

By direct calculations we have

$$W_\hbar(x^{(0)}, \xi^{(0)})^* \Delta_\hbar(t,s) W_\hbar(x^{(0)}, \xi^{(0)}) = \delta^w(t,s; x + x^{(0)}, \hbar D_x + \xi^{(0)}) \tag{4.18}$$

(*cf.* [3], [8]). So using (4.17), it holds that

$$\begin{aligned}(\Delta_\hbar(t,s)v, v) &= \hbar^{-n\tau}\big(\delta^w(t,s;x+x^{(0)}, \hbar D_x + \xi^{(0)}) g(\cdot/\hbar^\tau), g(\cdot/\hbar^\tau)\big) \\ &= \hbar^{-n\tau}\big((\delta^w(t,s;\hbar^\tau x + x^{(0)}, \hbar^{1-\tau} D_x \\ &\qquad + \xi^{(0)})g)(\cdot/\hbar^\tau), g(\cdot/\hbar^\tau)\big) \\ &= (\delta^w(t,s;\hbar^\tau x + x^{(0)}, \hbar^{1-\tau} D_x + \xi^{(0)})g, g). \end{aligned} \tag{4.19}$$

Let us apply (ii) in Lemma 3.1. Then as $\hbar$ tends to zero, $(\Delta_\hbar(t,s)v, v)$ converges to

$$\begin{cases} (\delta^w(t,s;x^{(0)}, \xi^{(0)})g, g), & 0 < \tau < 1, \\ (\delta^w(t,s;x + x^{(0)}, \xi^{(0)})g, g), & \tau = 0, \\ (\delta^w(t,s;x^{(0)}, D_x + \xi^{(0)})g, g), & \tau = 1. \end{cases} \tag{4.20}$$

Hence we can see from (4.14)-(4.16) and (4.20) that $\lim_{\hbar \to +0}(U_\hbar(t,s)^* F_\hbar U_\hbar(t,s)v, v)$ is equal to (3.14).

Let $F_\hbar = \lambda^w(x, \hbar D_x) I_N$. Then since $z(t,s;x,\xi)$ is unitary, (3.14) is equal to (3.15). Thus we could complete the proof of Theorem 3.5.

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