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Contact transformations in Wheeler-Feynman electrodynamics

by

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ABSTRACT. – The effect of a change of variables depending on higher derivatives (contact transformation) on dynamics of an N-body Lagrangian system is examined. The conditions of the existence of invertible contact transformations are formulated, and the theorem about the dynamical equivalence between Lagrangian N-body systems connected by an invertible contact transformation is proven. The problem of the balance between the numbers of degrees of freedom in the original theory and the transformed theory is discussed on the Hamiltonian level. A method of contact transformations is applied to the two-body nonlocal system admitting single-time local perturbative expansions (e.g., Wheeler-Feynman electrodynamics). First-order Lagrangian system with double number of the variables, that are related to each other by the operator of time inversion, is obtained.

Key words: Infinite jet bundle, contact transformation, Lagrangian and Hamiltonian formalisms, constraints.

RÉSUMÉ. – Dans cet article on étudie l'influence du changement de variables qui dépend des dérivées d'ordre supérieur (transformation de contact) sur la dynamique du système lagrangien de N particules. On a formulé les conditions d'existence des transformations réversibles de contact et on a prouvé le théorème de l'équivalence dynamique des systèmes lagrangiens liés par une transformation de contact réversible. Dans le cadre du formalisme hamiltonien on a étudié le problème de la correspondance entre les nombres de degrés de liberté de la théorie initiale et de sa transformée. On a appliqué la méthode des transformations de contact au système non-local de deux particules qui permet le développement en séries locales unitemporelles (de l'électrodynamique de Wheeler-Feynman). On a obtenu un système lagrangien sans dérivées d'ordre supérieur mais avec le double de variables qui sont liées entre elles par l'opérateur d'inversion de temps.

1. INTRODUCTION

When considering a relativistic N-body system in which particle creation and annihilation is not allowed, it is better to apply relativistic dynamics for directly interacting particles. The notion of the intermediate field as an independent object with its own degrees of freedom is not used in similar theories. Wheeler-Feynman electrodynamics for two point charges [1] is an example. This theory is nonlocal because an action at a distance with finite propagation speed occurs. In this model a charge a moves in an electromagnetic field which is determined by the half-sum of retarded and advanced Lienard-Wiechert potentials produced by charge b.

The nonlocal Lagrangian for Wheeler-Feynman electrodynamics admits a local perturbative expansion given by Kerner in ref. [2]. It is written as an infinite power series of c^{-1} with the terms depending on simultaneous position variables and their higher derivatives with respect to the common evolution parameter (time) t. The sum of the zeroth-order and first-order terms constitute the lower-derivative quasirelativistic approximation of the non-relativistic Lagrangian function. This is the well known Darwin's Lagrangian [3]. Any other finite-order term includes higher derivatives.

The higher-derivative terms added to a lower-derivative Lagrangian as a correction with a small coefficient make the new theory quite different from the original one [4]. The reason is that additional degrees of freedom correlated with higher derivatives appear. Generally an infinite set of initial data is required to specify motion. Wheeler-Feynman electrodynamics is considered to lead to an excessively wide set of motions. Thus an additional principle of selection of admissible motions must be added. It consists in the requirement of analyticity of solutions of the equations of motion with respect to the expansion parameter [4]-[8]. The runaway solutions are excluded as unphysical motions. In refs. [6], [7] the above requirement yields the constraints which must be imposed in order to eliminate the redundant degrees of freedom. Finally we have to obtain a finite-parameter family of solutions that are analytical with respect to the small parameter of expansion.

In ref. [8] a method was developed which permits to exclude higher derivatives by using equations of motion in the higher-order parts of perturbative Lagrangians. This method consists in an iterative procedure based on contact transformations of position variables. It means that the expressions depending on some variables and their time-derivatives are substituted for the position variables. Finally we have to obtain the lower-derivative Lagrangian in terms of new variables. Thus only initial coordinates and velocities are required to specify motion. The method proposed in ref. [8] guarantees analyticity of solutions of the reduced (second-order) equations of motion.

The present paper is mainly concerned with the problem of how the change of Lagrangian variables including higher derivatives transforms the dynamics of an arbitrary N-body system. In ref. [9] the substitutions containing particle coordinates and their first-order derivatives (non-point transformations) were investigated. Such transformation in the model of a one-dimensional harmonic oscillator was given as an example. The set of new motions is wider than that of the original simple harmonic oscillations. Indeed, both the order of the transformed Lagrangian and the order of the corresponding equation of motion are higher than those of the original lower-derivative theory. It is difficult to interpret the new degrees of freedom which are caused by the derivatives in the substitution expression. Thus in ref. [9] much attention was paid to the problem of constructing a one-to-one correspondence between the set of new motions and the original one. In particular the non-point transformations which correspond to the canonical transformations have been found.

The present paper is organized as follows. Section 2 is devoted to some aspects of the general formal theory of contact transformations. In Section 3 we consider the effect of the contact transformations on the dynamics of an N-body system. We show that Lagrangian systems connected by an invertible contact transformation are dynamically equivalent. In Section 4 we examine the influence of the contact transformation on Hamiltonian dynamics. A constrained Hamiltonian formalism corresponds to Lagrangian theory obtained by an invertible contact transformation. In Section 5 we present the application of the method of contact transformations to two-body nonlocal systems admitting local perturbative expansions. In the specific case of a time-asymmetric theory we propose the invertible contact transformation which allows to construct the first-order Lagrangian function. The time-symmetric theory turns into a lower-derivative Lagrangian system with double number of variables, related to each other by the time reflection.

2. CONTACT TRANSFORMATIONS ON AN INFINITE JET BUNDLE

In this Section we shall consider the changes of coordinates of points belonging to the infinite jet bundle $J^{\infty}\pi$ of jets of local sections of a bundle (E, π, M). The total space of π is $(N+1)$-dimensional manifold E and the base M is one-dimensional. In the next Section these substitutions will be used in a Lagrangian formalism of classical mechanics.

We deal with a local trivialisation of π around $t \in I$ where diffeomorphism $o_t : \pi^{-1}(I) \to I \times Q$ is defined. Here $I \subset M$ is an open interval, and a typical fibre of π is an N-dimensional manifold Q. In some adapted coordinate system, which is constructed from local trivialisations, projection $\pi : E \to M$ relates the point $(t, q_a) \in U \subset \pi^{-1}(I)$, $a = 1, \ldots, N := \overline{1, N}$, with the point t in the time interval I. We also use the induced coordinate system (U^{∞}, q^{∞}) [10] on the infinite jet bundle $J^{\infty}\pi$ which is defined locally by $q^{\infty} = (t, q_a, q_a^1, \ldots, q_a^s, \ldots)$. Summation over repeated indices is understood throughout the paper; Latin indices a, b, c run from 1 to N, and the Latin index s from 0 to infinity.

Let us consider a bundle morphism $(f, id_M) : \pi_n \to \pi$ from the bundle $(J^n\pi, \pi_n, M)$ to the bundle (E, π, M), so that

$$
\begin{aligned}
f :& J^n\pi \to E, \\
& j_t^n\sigma' \mapsto \sigma,
\end{aligned}
\tag{2.1}
$$

(*cf.* [10, pg.203]). Here a section $\sigma : I \to E$ is an inverse to π map, $\pi \circ \sigma = id_I$, given by $t \mapsto (t, x_a(t))$ where $x_a = q_a \circ \sigma$. Another local section $\sigma' \in \Gamma_I(\pi)$ is represented in the form $\sigma'(t) = (t, y_b(t))$ and $j_t^n\sigma'$ is the n-jet of a section σ' at a point $t \in I$. Locally the map (2.1) may be written as the following coordinate substitution:

$$
\begin{aligned}
t &= t, \\
x_a &= f_a(t, y_b, {y_b}^1, \ldots, {y_b}^n).
\end{aligned}
\tag{2.2}
$$

We postulate further that f is differentiable of class C^{∞} and the condition

$$
\det \left\| \frac{\partial f_a}{\partial y_b} \right\| \neq 0 \tag{2.3}
$$

is satisfied.

Having given this mapping, we wish to find the transformational law for the first-order derivative coordinates ${x_a}^1$. Following ref. [10], we construct the first prolongation of smooth mapping $f : J^n\pi \to E$:

$$\begin{aligned} j^1 f : & J^1\pi_n \to J^1\pi, \\ & j_t^1(j^n\sigma') \mapsto j_t^1\sigma. \end{aligned} \tag{2.4}$$

Locally, we obtain N expressions

$${x_a}^1 = \frac{\partial f_a}{\partial t} + \sum_{r=0}^{n} y_b^{r;1} \frac{\partial f_a}{\partial {y_b}^r} \tag{2.5}$$

in addition to the relations (2.2). To construct the map $f^1 : J^{n+1}\pi \to J^1\pi$ we use a canonical embedding $\iota_{1,n} : J^{n+1}\pi \to J^1\pi_n$. It leads to the relations ${x_a}^1 = d_T f_a$ instead of eqs. (2.5). Here d_T is the Tulczyjew operator [11] (the derivation of type d_* which acts on the 0-forms as a total time derivative). It is reasonable to say that the differentiable mapping $f^1 = j^1 f \circ \iota_{1,n}$ is the holonomic prolongation of f.

In analogy with f^1 we construct s-order holonomic prolongation $f^s : J^{n+s}\pi \to J^s\pi$ of the map (2.1) as a composition $j^s f \circ \iota_{s,n}$ of the s-th prolongation $j^s f$ with the canonical embedding $\iota_{s,n} : J^{n+s}\pi \to J^s\pi_n$ (*cf.* ref. [10, pg.205] where prolongations of a differential equation are determined). If we avoid the need to keep track of the order of the jets, we define the smooth mapping

$$\begin{aligned} \mathcal{F} : & J^\infty\pi \to J^\infty\pi, \\ & j_t^\infty\sigma' \mapsto j_t^\infty\sigma \end{aligned} \tag{2.6}$$

in the form of the following coordinate transformation:

$$\begin{aligned} x_a &= f_a(t, y_b, {y_b}^1, \ldots, {y_b}^n), \\ {x_a}^s &= d_T^s f_a, \qquad s \geq 0. \end{aligned} \tag{2.7}$$

The time variable does not change.

The transformation (2.7) is analogous to the change of coordinates in an infinite jet prolongation of the extended configuration space of an N-body Lagrangian system, examined in ref. [8]. An infinite Cartan distribution

$C^*\pi_\infty$ is invariant with respect to the substitution (2.7) and, therefore, this replacement is a contact transformation [10]. Locally $C^*\pi_\infty$ is spanned by the contact one-forms ω_a^s written as

$$\omega_a^s = dq_a{}^s - q_a{}^{s+1}dt, \quad s \geq 0, \tag{2.8}$$

(*cf.* ref. [12] where finite-dimensional case is considered). A smooth mapping (2.6), derived from differentiable mapping (2.1), will be called n-order contact transformation, or contact transformation in short.

Composition of the contact transformation $\mathcal{F} : J^\infty\pi \longrightarrow J^\infty\pi$ with similar substitution $\mathcal{G} : J^\infty\pi \longrightarrow J^\infty\pi$ given by

$$\begin{aligned} y_b &= g_b(t, z_c, z_c{}^1, \ldots, z_c{}^m), \\ y_b{}^k &= d_T^k g_b, \qquad k \geq 1, \end{aligned} \tag{2.9}$$

is a contact transformation $\mathcal{G} \circ \mathcal{F} : J^\infty\pi \longrightarrow J^\infty\pi$ written as

$$\begin{aligned} x_a &= f_a(t, g_b(t, z_c, \ldots, z_c{}^m), \ldots, d_T^n g_b(t, z_c, \ldots, z_c{}^m)) \\ &\equiv h_a(t, z_c, z_c{}^1, \ldots, z_c{}^{n+m}), \\ x_a{}^k &= d_T^k h_a, \qquad k \geq 1. \end{aligned} \tag{2.10}$$

The associative property $(\mathcal{H} \circ \mathcal{G}) \circ \mathcal{F} = \mathcal{H} \circ (\mathcal{G} \circ \mathcal{F})$ is true. Thus we define the category [13], say $\mathbf{C}$, of contact transformations on an infinite jet bundle $J^\infty\pi$. An identity map $id_{J^\infty\pi} : J^\infty\pi \longrightarrow J^\infty\pi$ is given by the relations $x_a = x_a$ and $x_a{}^s = d_T^s x_a$. The problem of existence of the inverse substitution to the replacement (2.7) requires careful consideration.

The characteristics of a contact transformation of type (2.6) are determined by the properties of the originating smooth mapping (2.1). To show this we examine the differential equation determined by the differential operator $\mathcal{D}_f : \Gamma_{loc}(\pi) \rightarrow \Gamma_{loc}(\pi)$ (see ref. [10, pg.203]) and a local section $\sigma \in \Gamma_I(\pi)$, *i.e.* the submanifold

$$S_{f;\sigma} = \{j_t^n\sigma' : f_a(t, y_b, \dot{y}_b, \ldots, \overset{n}{y}_b) = x_a(t)\} \subset J^n\pi. \tag{2.11}$$

Here symbols $\overset{r}{y}_b$ denote derivative coordinates $q_b{}^r(j_t^n\sigma') = d^r(q_b \circ \sigma')/dt^r \equiv d^r y_b(t)/dt^r$. A solution of this differential equation is a local section $\sigma' \in \Gamma_I(\pi)$ satisfying $\mathcal{D}_f(\sigma') = \sigma$. Generally, n-order contact transformation is locally surjective. To solve specific problems, it is helpful to study an infrequent case of bijective mapping type (2.1) and corresponding contact diffeomorphism, *i.e.* invertible transformation type of (2.6). In such a case there exists a differentiable map

$$\begin{aligned} g :& J^n\pi \rightarrow E, \\ & j_t^n\sigma \mapsto \sigma', \end{aligned} \tag{2.12}$$

locally given by

$$\begin{aligned} t &= t, \\ y_b &= g_b(t, x_a, {x_a}^1, \ldots, {x_a}^n), \end{aligned} \tag{2.13}$$

so that composition $g \circ f = id_E$.

To establish the structure of invertible functions f_a and their inverse functions g_b, we define the linear mapping

$$\begin{aligned} \mathcal{F}^* :& T^* J^\infty\pi \rightarrow T^* J^\infty\pi, \\ & T^*_{j_t^\infty\sigma'} J^\infty\pi \mapsto T^*_{j_t^\infty\sigma} J^\infty\pi. \end{aligned} \tag{2.14}$$

on the cotangent bundle $(T^* J^\infty\pi, \tau^*_{J^\infty\pi}, J^\infty\pi)$ [10]. To obtain its coordinate representation, we act on eqs. (2.7) by the exterior derivative operator d. We use the commutation $d \circ d_T = d_T \circ d$ and the relations $d_T q^s = q^{s+1}, s \geq 0$. After some calculations we obtain the relations between the coordinates of elements of the cotangent bundle in the form

$$d{x_a}^k = \left\{ \sum_{r=0}^{n} \sum_{l=0}^{k} \mathrm{C}_k^l d_T^{(k-l)} \left(\frac{\partial f_a}{\partial {y_b}^r} \right) \delta_{r+l,j} \right\} d{y_b}^j + d_T^k \left(\frac{\partial f_a}{\partial t} \right) dt; \quad k = 0, 1, \ldots \tag{2.15}$$

Here $\mathrm{C}_k^l \equiv \binom{k}{l}$ is the binomial constant and $\delta_{r+l,j}$ is the Kronecker's symbol.

By analogy with this algorithm one can write coordinate representation of the linear mappings $\mathcal{G}^*$ and $(\mathcal{G} \circ \mathcal{F})^*$, which correspond to the contact transformations (2.9) and (2.10), respectively. Further computation shows that Jacobian matrices of an invertible contact transformation $\mathcal{F}$ and its inverse mapping $\mathcal{G}$ have to satisfy the following system of differential equations:

$$\sum_{k=\alpha}^{\beta} \sum_{r=j-k}^{n} \frac{\partial f_a}{\partial {y_b}^r} \mathrm{C}_r^{j-k} d_T^{(r-j+k)} \left(\frac{\partial g_b}{\partial {x_c}^k} \right) = \delta_{ac}\delta_{j0}, \tag{2.16}$$

$$\sum_{k=0}^{n} \frac{\partial f_a}{\partial {y_b}^k} d_T^k \left(\frac{\partial g_b}{\partial t} \right) + \frac{\partial f_a}{\partial t} = 0, \tag{2.17}$$

where integers $\alpha = j - n$ and $\beta = n$ for $j = n, n+1, \ldots, 2n$; $\alpha = 0$ and $\beta = j$ for $j = 0, 1, \ldots, n$. Similar consideration of the composition $f \circ g = \mathrm{id}_E$, where (2.7) are substituted in (2.13), yields a consistent system of equations which can be obtained from (2.16) and (2.17) by the simultaneous transpositions $f \leftrightarrow g$ and $x \leftrightarrow y$.

In this paper we do not discuss the problems concerned with groups of contact transformations (see refs. [14], [15] where so-called groups of Lie-Bäcklund tangent transformations are investigated). We note only that an invertible contact transformation can be built on the basis of point change of variables

$$x_a = f_a(y_b\,; \lambda_\beta), \quad \beta = \overline{1, \mathcal{R}}, \tag{2.18}$$

belonging to an $\mathcal{R}$-parametric Lie group G. This group may contain the invariants I_k, *i.e.* functions such as

$$I_k(x_a)|_{x_a = f_a} = I_k(y_b), \quad k = \overline{1, r}. \tag{2.19}$$

We substitute the arbitrary functions $\varphi_\beta(I_k, d_T I_k)$ for the group parameters λ_β in eqs. (2.18). Finally, we obtain a new representation of the group G consisting of the following contact transformations:

$$x_a = f_a(y_b\,; \varphi_\beta(I_k(y_b), d_T I_k(y_b))), \quad \beta = \overline{1, \mathcal{R}}. \tag{2.20}$$

Inverse transformations have the form

$$y_b = f_b(x_a\,; -\varphi_\alpha(I_k(x_a), d_T I_k(x_a))), \quad \alpha = \overline{1, \mathcal{R}}. \tag{2.21}$$

3. CONTACT TRANSFORMATIONS IN LAGRANGIAN FORMALISM OF CLASSICAL MECHANICS

In this Section we treat the main topic of our paper. We will show how the transformations of Lagrangian variables depending on higher-order derivatives modify the dynamics of an N-body system. The theorem about the dynamical equivalence between Lagrangian N-body systems connected by an invertible contact transformation will be proven by working completely on the Lagrangian level.

Higher-derivative Lagrangian dynamics is based on the smooth Lagrangian function $L : J^k\pi \to \mathbb{R}$, say

$$L = L(t, x_a(t), \dot{x}_a(t), \ldots, \overset{k}{x}_a(t)), \tag{3.1}$$

where $\overset{r}{x}_a(t) = d^r(q_a \circ \sigma)/dt^r \equiv d^r x_a(t)/dt^r$. Non-autonomous Lagrangian systems are meant throughout the Section. We shall call a "motion" [8] a section $s(t) = (t, x_a(t))$ where the coordinates $x_a(t)$ are solutions of the Euler-Lagrange equations

$$\frac{\delta S}{\delta x_a} \equiv \sum_{r=0}^{k} \left(-\frac{d}{dt}\right)^r \frac{\partial L}{\partial \overset{r}{x}_a} = 0, \tag{3.2}$$

obtained by applying the variational principle to the action integral $S = \int_I dtL$. A set of all motions of this particle system will be denoted as $M_I(s)$. Eqs. (3.2) are then nothing but a closed embedded submanifold $[L] \subset J^{2k}\pi$, written as

$$\left[\frac{\delta S}{\delta x_a}\right] \equiv \sum_{r=0}^{k} (-d_T)^r \frac{\partial L}{\partial x_a{}^r} = 0, \tag{3.3}$$

so that $2k-$order prolongation $j^{2k}s$ of motion $s \in M_I(s)$ takes its values in $[L] = \bigcap\limits_a [\delta S/\delta x_a]$ (see ref. [10]).

Having carried out the n-order contact transformation (2.7) in k-order Lagrangian (3.1) we construct the Lagrangian function

$$\widetilde{L}(t, y_b, \dot{y}_b, \ldots, \overset{k+n}{y_b}) = L(t, f_a(t, y_b, \ldots, \overset{n}{y}_b), \ldots, \frac{d^k}{dt^k} f_a(t, y_b, \ldots, \overset{n}{y}_b)), \tag{3.4}$$

which is defined on the bundle $J^{k+n}\pi$ of $(k+n)$-jets $j_t^{k+n}\sigma'$ of local sections $\sigma'(t) = (t, y_b(t))$. The Lagrangian (3.1) behaves as a scalar [8] because the time variable does not change. In this paper we indicate the initial Lagrangian functions, Euler-Lagrange equations, etc., by the adjective "original" and those transformed by contact transformation by the adjective "new". As a rule, the set $M_I(s')$ of the motions $s'(t)$ which are solutions of new Euler-Lagrange equations $[\tilde{L}] \subset J^{2(k+n)}\pi$:

$$\left[\frac{\delta \widetilde{S}}{\delta y_b}\right] \equiv \sum_{l=0}^{k+n} (-d_T)^l \frac{\partial \widetilde{L}}{\partial y_b{}^l} = 0, \tag{3.5}$$

is wider than the original set $M_I(s)$. It is caused by the additional degrees of freedom related to higher derivatives [4]. Our purpose is to establish a correlation between the set $M_I(s)$ of original motions and the set $M_I(s')$ of new motions.

In ref. [8] the transformation law of the Euler-Lagrange equations with respect to the contact transformation (2.6) was obtained. Restricted on the finite-order Lagrangian dynamics, this law is

$$\left[\frac{\delta\widetilde{S}}{\delta y_b}\right] = \sum_{l=0}^{n}(-d_T)^l\frac{\partial f_a}{\partial {y_b}^l}\left[\frac{\delta S}{\delta x_a}\right]_{\mathcal{F}}, \tag{3.6}$$

(see Appendix A). It suggests the formulation of the Euler-Lagrange equations (3.5) in the following form:

$$T_{ab}\chi_a \equiv \sum_{l=0}^{n}(-d_T)^l\frac{\partial f_a}{\partial {y_b}^l}\chi_a = 0 \quad (\mathrm{a}), \quad \sum_{r=0}^{k}(-d_T)^r\frac{\partial L}{\partial {x_a}^r}\bigg|_{\mathcal{F}} = \chi_a \quad (\mathrm{b}). \tag{3.7}$$

Here the functions $\chi_a \in C^n(J^n\pi), a = \overline{1,N}$, are components of a vector function $\vec{\chi}$ belonging to the kernel of the differential operator $\widehat{T}$ with matrix elements T_{ab}. Problem of correspondence between the original motions and the new ones can be solved by investigation of $\ker\widehat{T}$ which has the structure of a real vector space.

Let the vectors $(\vec{\chi}_1,\ldots,\vec{\chi}_M)$ form a basis for $\ker\widehat{T}$. Decomposition of new Euler-Lagrange equations (3.5), which is given by eqs. (3.7a) and (3.7b), allows to define a fibred manifold over $\ker\widehat{T}$, where fibre $\tau^{-1}(\vec{\chi})$ over $\vec{\chi} = A^\gamma\vec{\chi}_\gamma$ is the submanifold of $J^{2k+n}\pi$ determined by eqs. (3.7b) with fixed numbers A^γ. We denote $M_I^\chi(s')$ a set of solutions of differential equation $\tau^{-1}(\vec{\chi})$. Union of these sets, *i.e.* $\bigcup\limits_{\chi} M_I^\chi(s')$, is isomorphic to $M_I(s')$.

A fibre $\tau^{-1}(\vec{0})$ over zero vector is related only with an original Euler-Lagrange equation (3.3). As a rule the map

$$\varphi : M_I^0(s') \to M_I(s) \tag{3.8}$$

is surjective because a set of solutions of the differential equation (2.11) corresponds to any original motion $s(t) = (t, x_a(t))$. Prior to examining a bijective mapping of type (3.8) (invertible contact transformation), we construct the category of Euler-Lagrange equations.

First of all we define the category $\mathbf{L}$ which consists of the set of Lagrangian functions (objects of $\mathbf{L}$) and contact transformations (morphisms of **L**). The following diagram is commutative. It shows that the category

$$
\begin{tikzcd}
L(J^k\pi) \arrow[rr, "\mathcal{F}"] \arrow[dr, "\mathcal{H}"] \arrow[ddd, "E.\text{-}L."'] & & \tilde{L}(J^{k+n}\pi) \arrow[dl, "\mathcal{G}"'] \arrow[ddd, "E.\text{-}L."] \\
& \bar{L}(J^{k+n+m}\pi) \arrow[d, "E.\text{-}L."] & \\
& \dfrac{\delta\bar{S}}{\delta z_c} & \\
\dfrac{\delta S}{\delta x_a} \arrow[ur, "\widehat{T}_h"] \arrow[rr, "\widehat{T}_f"] & & \dfrac{\delta\tilde{S}}{\delta y_b} \arrow[ul, "\widehat{T}_g"']
\end{tikzcd}
$$

E which consists of the set of expressions of Euler-Lagrange equations (objects of **E**) and differential matrix operators $\widehat{T}$ (morphisms of **E**) can be introduced. The "Euler-Lagrange derivative" *E.-L.* is the functor [13] from the category **L** to the category **E**. The symbols $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ denote contact transformations (2.6), (2.9) and (2.10), respectively; differential matrix operators $\widehat{T}_f$, $\widehat{T}_g$ and $\widehat{T}_h$ relate expressions of Euler-Lagrange equations according to the rule (3.6). The equality $\widehat{T}_g\widehat{T}_f = \widehat{T}_h$ is satisfied. This can be proven by using the method of mathematical induction.

THEOREM. – *A necessary and sufficient condition for $\varphi : M_I(s') \to M_I(s)$ to be a bijective mapping is that new and original Lagrangian functions are related by an invertible contact transformation.*

Sufficiency. – In case of the invertible functions $f_a(t, y_b, \ldots, y_b{}^n)$ and their inverse functions $g_b(t, x_a, \ldots, x_a{}^n)$ which satisfy eqs. (2.16) and (2.17), we have $\widehat{T}_g\widehat{T}_f = I_N$ where I_N is the unit matrix. Any invertible operator $\widehat{T}$ has a trivial kernel: $\ker\widehat{T} = \{0\}$. Hence, transformed Euler-Lagrange equations (3.7) get simplified:

$$\sum_{r=0}^{k} (-d_T)^r \frac{\partial L}{\partial x_a{}^r}\bigg|_{\mathcal{F}} = 0. \tag{3.9}$$

It is evident that their solutions $y_b(t)$ satisfy the system of n-order differential equations (2.11), where functions $x_a(t)$ are the solutions of the original Euler-Lagrange equations (3.2). Having used inverse functions (2.13), we obtain the motions

$$y_b(t) = g_b(t, x_a(t), \dot{x}_a(t), \ldots, \overset{n}{x}_a(t)), \tag{3.10}$$

which are specified by the same initial data as the original motions $x_a(t)$. Thus, if Lagrangians are related by an invertible contact transformation, the set $M_I(s')$ of new motions is isomorphic to the set $M_I(s)$ of original motions.

Necessity. – If $\varphi : M_I(s') \rightarrow M_I(s)$ is a bijective mapping then $\ker \widehat{T}$ is trivial. Indeed, a fibre over zero is related with original motions only. Hence, operator $\widehat{T}$ is invertible as well as corresponding contact transformation. □

Let the original Lagrangian (3.1) be regular. In specific case of an invertible contact transformation $\dim[\tilde{L}] = \dim[L]$. We face the problem of how the balance between numbers of degrees of freedom in the original and the new dynamics is achieved. The relation between Hessian matrices $\widetilde{H} = \|\partial^2 \widetilde{L}/\partial y_b{}^{k+n} \partial y_c{}^{k+n}\|$ of the new Lagrangian (3.4) and $H = \|\partial^2 L/\partial x_a{}^k \partial x_b{}^k\|$ of the original Lagrangian (3.1) has the following form:

$$\widetilde{H} = F^T\, H\, F, \tag{3.11}$$

where matrix $F = \|\partial f_a/\partial y_b{}^n\|$, F^T is the transposed matrix. If we use an invertible contact transformation, functions f_a and their inverses g_b have to satisfy the following condition:

$$\frac{\partial f_a}{\partial y_b{}^n} \frac{\partial g_b}{\partial x_c{}^n} = 0, \tag{3.12}$$

(see eqs. (2.16) if integer j is equal to $2n$). Consequently, both matrices $F = \|\partial f_a/\partial y_b{}^n\|$ and $G = \|\partial g_b/\partial x_c{}^n\|$ are singular. Using this in eq. (3.11) will satisfy us that the new Lagrangian theory is degenerate. Moreover, the equations of motion (3.9) and their time derivatives up to order $n-1$, can be considered as Lagrangian constraints. In Section 4 we examine a problem of singularity of the new theory in a frame of a canonical formalism for higher-derivative theories.

In Appendix B we show that the gauge transformations of Lagrangian variables ([1]) are the invertible contact transformations. We prove that the gauge invariance of an action leads to relations between the original Euler-Lagrange equations [16].

Let the new Lagrangian (3.4) be regular just as the original one. It means that $\dim[\tilde{L}] = 2(k+n)N+1$. In such a case $\dim \ker \widehat{T} = nN$, since $\dim \tau^{-1}(\vec{\chi}) = (2k+n)N+1$ because the contact transformation is irreversible. The problem of existence of $n-$order contact transformation

([1]) First-class constraints are their counterparts in the Hamiltonian formalism [16].

with an intermediate value $0 < \dim\ker\widehat{T} < nN$ will not be considered in this paper.

In Appendix C we examine how the substitution

$$x = y + \alpha\dot{y} \tag{3.13}$$

transforms the dynamics of a simple harmonic oscillator. We obtain a "harmonic oscillator with the mass slightly modified, and an acceleration-squared piece" described in ref. [4]:

$$L = \frac{1}{2}(1 - \alpha^2\omega^2)\dot{y}^2 - \frac{1}{2}\omega^2 y^2 + \frac{1}{2}\alpha^2\ddot{y}^2. \tag{3.14}$$

Having carried out the separation of the set $M_I(s')$ into non-overlapping subsets $M_I^\chi(s')$, we specify the subset $M_I^0(s')$ corresponded to the original simple harmonic oscillation. The total time derivative terms $\alpha\dot{y}\ddot{y}$ and $-\omega^2\alpha y\dot{y}$ are omitted in eq. (3.14) because they don't change Euler-Lagrange equation. According to ref. [17], a canonical transformation is induced by an addition of a total time derivative term to the higher-order Lagrangian.

4. CONTACT TRANSFORMATIONS AND CANONICAL FORMALISM FOR HIGHER-DERIVATIVE THEORIES

The n-order contact transformation in the k-order Lagrangian function leads to a new Lagrangian defined on the bundle $J^{k+n}\pi$ of $(k+n)$-jets. As a result, the set $M_I(s')$ of new motions is larger than the original $M_I(s)$. At the same time, invertible contact transformations leave the dynamics invariant. For a careful consideration of these substitutions we apply the canonical formalism for higher-derivative theories developed by Ostrogradski [18] (see [19], [20], [21], [16] as well).

First of all we consider an autonomous situation where both an original Lagrangian and a new Lagrangian function are time-independent as well as the contact transformation connecting them. Let an original Lagrangian $L : T^kQ \to \mathbb{R}$ be regular. Corresponding Poincaré-Cartan two-form $\omega_L = -d\alpha_L$ is symplectic and then there exists an unique Euler-Lagrange vector field ξ_L which satisfies the global equation of motion $i_{\xi_L}\omega_L = dE_L$ [19]. In local coordinates the Poincaré-Cartan one-form α_L and the energy function E_L, associated with L, are defined by

$$\alpha_L = \sum_{r=0}^{k-1} \hat{p}_{a,r} d{x_a}^r, \tag{4.1}$$

$$E_L = \sum_{r=0}^{k-1} \hat{p}_{a,r} x_a{}^{r+1} - L. \tag{4.2}$$

Here functions $\hat{p}_{a,r}$, $r = \overline{0, k-1}$, defined on $T^{2k-1-r}Q$, are the original Jacobi-Ostrogradski momenta (see App. A, eqs. (A2)). One can construct a Hamiltonian system $(T^*(T^{k-1}Q), \omega, H)$ due to Ostrogradski-Legendre transformation $Leg : T^{2k-1}Q \to T^*(T^{k-1}Q)$ [19], [20], locally given by $Leg(x_a, \ldots, x_a{}^{2k-1}) = (x_a, \ldots, x_a{}^{k-1}; p_{a,0}, \ldots, p_{a,k-1})$. The dynamical trajectories of the system in phase space $T^*(T^{k-1}Q)$ are found as the integral curve of the Hamiltonian vector field X_H satisfying the equation $i_{X_H}\omega = dH$.

In Appendix A the relationships between the new Jacobi-Ostrogradski momenta, say $\hat{\pi}_{b,l}$, where index l runs from 0 to $k+n-1$, and the original ones are established. Having used these relations (see eqs. (A9)) together with eqs. (2.7), we rewrite the new Poincaré-Cartan one-form

$$\widetilde{\alpha}_L = \sum_{l=0}^{k+n-1} \hat{\pi}_{b,l} dy_b{}^l \tag{4.3}$$

as follows

$$\widetilde{\alpha}_L = \alpha_L + \sum_{j=0}^{n-1} \left\{ \sum_{l=j}^{n-1} (-d_T)^{l-j} \frac{\partial f_a}{\partial y_b{}^{l+1}} \left[\frac{\delta S}{\delta x_a} \right]_{\mathcal{F}} \right\} dy_b{}^j. \tag{4.4}$$

New energy function

$$\widetilde{E}_L = \sum_{l=0}^{k+n-1} \hat{\pi}_{b,l} y_b{}^{l+1} - \widetilde{L} \tag{4.5}$$

will be expressed in similar form

$$\widetilde{E}_L = E_L + \sum_{j=0}^{n-1} \left\{ \sum_{l=j}^{n-1} (-d_T)^{l-j} \frac{\partial f_a}{\partial y_b{}^{l+1}} \left[\frac{\delta S}{\delta x_a} \right]_{\mathcal{F}} \right\} y_b{}^{j+1}. \tag{4.6}$$

Both expressions (4.4) and (4.6) have a specific structure: a new quantity consists of its original counterpart and some additional terms. It is significant that these terms are proportional to the original Euler-Lagrange expressions for motion equations. If a new Lagrangian is obtained from an original one by an invertible contact transformation, then transformed Euler-Lagrange expressions $[\delta S/\delta x_a]_{\mathcal{F}}$ become motion equations. They are vanishing in both expressions (4.4) and (4.6). According to generalized Darboux theorem [22], in such a case the two-form $\widetilde{\omega}_L = -d\widetilde{\alpha}_L$ is presymplectic. Moreover, the original canonical coordinates are just required by this theorem, so that new canonical two-form is

$$\widetilde{\omega} = \sum_{r=0}^{k-1} dx_a{}^r \wedge dp_{a,r} \,. \tag{4.7}$$

A geometric algorithm, developed by Gotay *et al.* [22], enables us to treat a presymplectic dynamical system as well as a constrained symplectic system (see also refs. [19], [20], [22]). A local version of their scheme is the well-known Dirac-Bergmann theory of constraints. In the examined situation we have the set of primary constraints obtained by excluding the original momenta $p_{a,r}$ from the relations

$$\pi_{b,n+i} = \sum_{r=i}^{k-1} p_{a,r} \sum_{l=i}^{\beta} \mathrm{C}_r^l d_T^{r-l} \frac{\partial f_a}{\partial y_b{}^{n+i-l}}, \quad \beta = \begin{cases} r, r < n+i \\ n+i, r \geq n+i \end{cases}; \tag{4.8a}$$

$$\pi_{b,j} = \sum_{r=0}^{k-1} p_{a,r} \sum_{l=0}^{\gamma} \mathrm{C}_r^l d_T^{r-l} \frac{\partial f_a}{\partial y_b{}^{j-l}}, \quad \gamma = \begin{cases} r, r < j \\ j, r \geq j \end{cases}, \tag{4.8b}$$

where index i runs from 0 to $k - 1$ and index j runs from 0 to $n - 1$ (see App. A).

Recurrently, starting from stationarity conditions of primary constraints, we obtain all the secondary constraints. A final constrained manifold [23], [19], [22] is the original phase space $T^*(T^{k-1}Q)$.

Returning now to non-autonomous higher-order Lagrangian systems, we rewrite the new generalized Poincaré-Cartan one-form

$$\widetilde{\theta}_L = \sum_{j=0}^{k+n-1} \hat{\pi}_{b,j} \big(dy_b{}^j - y_b{}^{j+1} dt\big) + \widetilde{L} dt \tag{4.9}$$

as follows

$$\widetilde{\theta}_L = \theta_L + \sum_{j=0}^{n-1} \left\{ \sum_{l=j}^{n-1} (-d_T)^{l-j} \frac{\partial f_a}{\partial y_b{}^{l+1}} \left[\frac{\delta S}{\delta x_a}\right]_{\mathcal{F}} \right\} \big(dy_b{}^j - y_b{}^{j+1} dt\big). \tag{4.10}$$

Here

$$\theta_L = \sum_{r=0}^{k-1} \hat{p}_{a,r} \big(dx_a{}^r - x_a{}^{r+1} dt\big) + L dt \tag{4.11}$$

is the original generalized Poincaré-Cartan one-form. This result is coordinated with the one obtained in ref. [8] where infinite-order Lagrangian systems are investigated. If new Lagrangian $\widetilde{L}$ is degenerate so that the kernel of the operator $\widehat{T}$ is trivial, then $\widetilde{\theta}_L = \theta_L$ and the new two-form $\widetilde{\Omega}_L = -d\widetilde{\theta}_L$ is equal to the original generalized Poincaré-Cartan two-form $\Omega_L = -d\theta_L$. An original Lagrangian function L is regular. Hence the pair (Ω_L, dt) defines the cosymplectic structure [22], [21] on the jet bundle $J^{2k-1}\pi$, supplemented with the unique Reeb vector field ξ_L satisfying the motion equations $i_{\xi_L}\Omega_L = 0$, $i_{\xi_L}dt = 1$.

To demonstrate how the Hamiltonian constraint formalism is made up from the Lagrangian obtained by the invertible contact transformation, we consider a representative example of the original theory in first order. If parameter $k = 1$, the relations (4.8) are simplified:

$$\pi_{b,r} = p_a \frac{\partial f_a}{\partial {y_b}^r}, \quad r = \overline{0,n}. \tag{4.12}$$

Then there exist nN primary constraints

$$\Phi^{(1)}_{br} = \pi_{b,r} - \pi_{c,0}{\omega^c}_{br} \approx 0, \quad r = \overline{1,n}, \tag{4.12}$$

with ${\omega^c}_{br} = {\phi^c}_a\, \partial f_a/\partial {y_b}^r$, where the symbol $\approx$ means a weak equality. Here matrix ${\phi^c}_a$ is inverted to matrix $\partial f_a/\partial y_b$ satisfying condition (2.3). It can be easily proven that all the Poisson brackets $\{\Phi^{(1)}_{br}, \Phi^{(1)}_{ak}\}$ are identically equal to zero. The canonical transformation

$$x_a = f_a(t, y_b, \dots, {y_b}^n), \quad p_a = \pi_{c,0}{\phi^c}_a, \tag{4.14a}$$

$${y'_b}^r = {y_b}^r, \quad {\pi'}_{b,r} = \pi_{b,r} - \pi_{c,0}{\omega^c}_{br} \equiv \Phi^{(1)}_{br}; \quad r = \overline{1,n}, \tag{4.14b}$$

allows to eliminate the redundant degrees of freedom and leads to the non-constraint formalism with the Hamiltonian function $H(t, x_a, p_a)$. It can be obtained from the non-singular original Lagrangian by a simple Legendre transformation. As a consequence of the time-dependence of the transformation (4.14), we have the relation

$$H(t, x_a, p_a) = H(t, {y_b}^r, \pi_{b,r}) + \pi_{c,0}{\phi^c}_a \frac{\partial f_a}{\partial t} \tag{4.15}$$

between Hamiltonian functions. Therefore, the number of secondary constraints is equal to nN. Here we have second-class constraints only. Indeed, we assume that there are no relations between the original equations of motion (*cf.* Appendix B).

5. TWO-BODY PROBLEM IN THREE-DIMENSIONAL FORMALISM OF THE FOKKER-TYPE RELATIVISTIC DYNAMICS

In this Section we deal with the Fokker [24] action integral for two-body systems:

$$S = -\sum_{a=1}^{2} m_a \int_{\mathbb{R}} d\tau_a \sqrt{\dot{x}_a^2} - e_1 e_2 \iint_{\mathbb{R}} d\tau_1\, d\tau_2 \dot{x}_{1\mu} \dot{x}_2{}^{\mu} \delta(\sigma^2). \qquad (5.1)$$

Here m_a $(a = 1, 2)$ are the rest masses of the particles, τ_a-invariant parameter of their world-lines $x_a(\tau_a)$, $\dot{x}_a{}^{\mu} = dx_a{}^{\mu}/d\tau_a$ – four-velocities of the particles, and $\sigma^2 = (x_{2\mu} - x_{1\mu})(x_2{}^{\mu} - x_1{}^{\mu})$ – the square of the interval between points x_1 and x_2 lying on the world-lines of the particles. We choose a metric tensor of the Minkowski space $\eta_{\mu\nu} = (1, -1, -1, -1)$; velocity of light $c = 1$. The action integral (5.1) describes the interaction between two point charges e_1 and e_2 [1]. The nonlocal Wheeler-Feynman theory does not explain the particle creation and annihilation. Consequently, it is not valid for large particle velocities. An average velocity v will be implied as a small parameter in all expansions of this Section.

In ref. [25] the action (5.1) was transformed into a single-time form. Common time t for both particles was chosen as an instant parameter $t = x^0$ [26]. Whence the three-dimensional formalism for Wheeler-Feynman electrodynamics was obtained. It is based on the Lagrangian function being an expansion in v defined on the infinite jet bundle $J^{\infty}\varrho$ of jets of local sections of a trivial fibration $\varrho : \mathbb{R} \times Q \to \mathbb{R}$. Here Q is 6–dimensional configuration space of our two-body system spanned by the position variables $\vec{x}_a$. Note that we have to take into account a well known relativistic limitation on the modulo of particle velocities if we deal with the higher-order prolongations of the extended configuration space $\mathbb{R} \times Q$.

Our consideration will be based on the special form of the action (5.1) which is given in ref. [25]. It can be obtained from the expression (5.1) by replacement of the "instant" parameters $t_a = x_a{}^0$ for own parameters τ_a and substitution of the pair (t, θ) for integration variables (t_1, t_2), where

$$t_1 = t - (1 - \lambda)\theta, \qquad t_2 = t + \lambda\theta. \qquad (5.2)$$

Here λ is an arbitrary real number. The double sum (5.1) which describes interaction between particles becomes

$$S_{int} = \frac{1}{2} \iint_{\mathbb{R}} dt\, d\theta \Lambda_{int}(t, \theta)[\delta(\theta - r(t, \theta)) + \delta(\theta + r(t, \theta))], \qquad (5.3)$$

where

$$\Lambda_{int}(t,\theta) = -\frac{e_1 e_2}{r}\Big(1 - (\dot{\vec{x}}_1 \dot{\vec{x}}_2)\Big). \tag{5.4}$$

Here coordinates $\vec{x}_1(t_1) \equiv \vec{x}_1(t-(1-\lambda)\theta)$, $\vec{x}_2(t_2) \equiv \vec{x}_2(t+\lambda\theta)$, velocities $\dot{\vec{x}}_a(t_a) \equiv d\vec{x}_a(t_a)/dt_a$ and $r(t,\theta) = |\vec{x}_2(t_2) - \vec{x}_1(t_1)|$ is the nonlocal distance between charges.

Having integrated eq. (5.3) over the parameter θ, we obtain

$$S_{int} = \frac{1}{2}\int_{\mathbf{R}} dt \left(\frac{\Lambda_{int}(\vec{x}_a(t_a), \dot{\vec{x}}_a(t_a))}{1-(1-\lambda)\dfrac{(\vec{r}\dot{\vec{x}}_1)}{r} - \lambda\dfrac{(\vec{r}\dot{\vec{x}}_2)}{r}}\Bigg|_{C+} + \frac{\Lambda_{int}(\vec{x}_a(t_a), \dot{\vec{x}}_a(t_a))}{1+(1-\lambda)\dfrac{(\vec{r}\dot{\vec{x}}_1)}{r} + \lambda\dfrac{(\vec{r}\dot{\vec{x}}_2)}{r}}\Bigg|_{C-} \right). \tag{5.5}$$

The expression under the integral sign depends on the quantities $\vec{x}_a(t_a)|_{C\pm}$ and $\dot{\vec{x}}_a(t_a)\Big|_{C\pm}$, where symbols $|_{C+}$ and $|_{C-}$ mean that the parameter θ is a root of either algebraic equation

$$\theta - r(t,\theta) = 0 \quad (C^+) \quad \text{or} \quad \theta + r(t,\theta) = 0 \quad (C^-), \tag{5.6}$$

respectively. Let the integer κ be equal to $+1$ for the advanced cone C^+ and to -1 for the retarded cone C^-. Now we introduce the functions $\vec{y}_a^{(\kappa)}(t) = \vec{x}_a(t_a)|_{C^\kappa}$. In terms of these functions both algebraic equations (5.6) are unified:

$$\theta - \kappa y^{(\kappa)}(t) = 0, \tag{5.7}$$

where $y^{(\kappa)}(t) = |\vec{y}_2^{(\kappa)}(t) - \vec{y}_1^{(\kappa)}(t)|$. By using this relation in (5.2) and differentiating by t, we obtain the derivatives of the instant parameters t_a with respect to common time t

$$\frac{dt_1}{dt}\bigg|_{C^\kappa} = 1 - \kappa(1-\lambda)\dot{y}^{(\kappa)}, \qquad \frac{dt_2}{dt}\bigg|_{C^\kappa} = 1 + \kappa\lambda\dot{y}^{(\kappa)}. \tag{5.8}$$

With the help of (5.8) one can write the expression (5.5) in terms of functions $\vec{y}_a^{(\kappa)}(t)$ and their first-order derivatives.

We divide the free particle term of the action (5.1) into two equal parts. In one of these parts we introduce the common time t according to the rule for the "advanced" cone C^+ and in the other — for the "retarded" cone C^- (see (5.7) and (5.8) where $\kappa = +1$ for C^+ and $\kappa = -1$ for C^-). Finally, the single-time Lagrangian of the investigated two-body system has the following form

$$L = \frac{1}{2}\left(L^{(+)} + L^{(-)}\right), \tag{5.9}$$

where

$$\begin{aligned} L^{(\kappa)} =& - m_1 k_1^{(\kappa)} - m_2 k_2^{(\kappa)} \\ & - \frac{e_1 e_2}{y^{(\kappa)}} \frac{(1-\kappa(1-\lambda)\dot{y}^{(\kappa)})(1+\kappa\lambda\dot{y}^{(\kappa)}) - (\dot{\vec{y}}_1^{(\kappa)}\dot{\vec{y}}_2^{(\kappa)})}{J^{(\kappa)}}. \end{aligned} \tag{5.10}$$

Here

$$k_1^{(\kappa)} = \sqrt{(1-\kappa(1-\lambda)\dot{y}^{(\kappa)})^2 - \dot{\vec{y}}_1^{(\kappa)2}}, \quad k_2^{(\kappa)} = \sqrt{(1+\kappa\lambda\dot{y}^{(\kappa)})^2 - \dot{\vec{y}}_2^{(\kappa)2}}, \tag{5.11}$$

$$J^{(\kappa)} = 1 - \kappa\lambda\frac{(\vec{y}^{(\kappa)}\dot{\vec{y}}_1^{(\kappa)})}{y^{(\kappa)}} - \kappa(1-\lambda)\frac{(\vec{y}^{(\kappa)}\dot{\vec{y}}_2^{(\kappa)})}{y^{(\kappa)}}. \tag{5.12}$$

To compare the function (5.9) with the well-known Lagrangian given by Kerner in ref. [2], we have to find the explicit expressions $y_b{}^{i(\kappa)}(t) = f_b{}^{i(\kappa)}(t, x_a{}^i, \dot{x}_a{}^i, \ldots, \overset{s}{x}_a{}^i, \ldots)$. We write these functions in the integral form

$$y_1{}^{i(\kappa)}(t) = \int_{\mathbb{R}} d\theta\left(1 - \kappa\frac{\partial r(t,\theta)}{\partial\theta}\right) x_1{}^i(t-(1-\lambda)\theta)\delta(\theta-\kappa r(t,\theta)), \tag{5.13a}$$

$$y_2{}^{i(\kappa)}(t) = \int_{\mathbb{R}} d\theta\left(1 - \kappa\frac{\partial r(t,\theta)}{\partial\theta}\right) x_2{}^i(t+\lambda\theta)\delta(\theta-\kappa r(t,\theta)), \tag{5.13b}$$

where the partial derivative $\partial r/\partial\theta$ is

$$\frac{\partial r(t,\theta)}{\partial\theta} = (1-\lambda)\frac{(\vec{r}\dot{\vec{x}}_1)}{r} + \lambda\frac{(\vec{r}\dot{\vec{x}}_2)}{r}. \tag{5.14}$$

With the help of the time shift operators $\exp[-(1-\lambda)\theta D_1]$ and $\exp[\lambda\theta D_2]$, where D_a signifies differentiation with respect to t of the variable $\vec{x}_a$ only, we eliminate the parameter θ from arguments of the coordinates and velocities of the particles in the right sides of the relations (5.13). We develop the operator $\exp[\theta(-(1-\lambda)D_1 + \lambda D_2)]$ into series up to θ^k and integrate it. Hence we write the functions $y_a{}^{i(\kappa)}(t)$ as the power series defined on the bundle $J^\infty\varrho$ of the infinite-order jets $j_t^\infty\sigma$ of the sections $\sigma = (t, x_1{}^i(t), x_2{}^i(t))$:

$$y_a{}^{i(\kappa)} = \sum_{s=0}^{\infty} \frac{(\lambda d_T - D_1)^s}{s!}(\kappa r)^s \left(1 - \kappa(1-\lambda)\frac{(\vec{r}\dot{\vec{x}}_1)}{r} - \kappa\lambda\frac{(\vec{r}\dot{\vec{x}}_2))}{r}\right) x_a{}^i. \tag{5.15}$$

Here $d_T = \partial/\partial t + D_1 + D_2$ is the total time derivative and $r = |\vec{x}_2(t) - \vec{x}_1(t)|$. Thus, the Lagrangian function (5.9), and corresponding equations of motion, are defined on the $J^\infty\varrho$. This Lagrangian can be expanded into a Taylor series. It differs from Kerner's Lagrangian [2] in terms which are the total time derivatives.

Having used the results of Section 3 (see eq. (3.6)), infinite-order Euler-Lagrange equations

$$\sum_{s=0}^{\infty} \left(-\frac{d}{dt}\right)^s \frac{\partial L}{\partial \overset{s}{x}_a{}^i} = 0 \tag{5.16}$$

can be written in the following form

$$T_{abi}^{(+)j}\left(\frac{\partial L^{(+)}}{\partial y_b{}^{j(+)}} - \frac{d}{dt}\frac{\partial L^{(+)}}{\partial \dot{y}_b{}^{j(+)}}\right) + T_{abi}^{(-)j}\left(\frac{\partial L^{(-)}}{\partial y_b{}^{j(-)}} - \frac{d}{dt}\frac{\partial L^{(-)}}{\partial \dot{y}_b{}^{j(-)}}\right) = 0. \tag{5.17}$$

Here

$$T_{abi}^{(\kappa)j} = \sum_{s=0}^{\infty} \left(-\frac{d}{dt}\right)^s \frac{\partial f_{bj}^{(\kappa)}}{\partial \overset{s}{x}_a{}^i} \tag{5.18}$$

are the components of the differential operators $\widehat{T}^{(+)}$ and $\widehat{T}^{(-)}$.

To prove the time-symmetry of the Lagrangian function (5.9), we establish the transformational properties of the functions $y_a{}^{i(\kappa)}(t)$ with respect to the time inversion operator $\mathcal{T}$. The functions $x_a{}^i(t)$ which satisfy equations of motion (5.16), are invariant with respect to time inversion. For the higher derivatives we have the rule $\mathcal{T}\,\overset{s}{x}_a{}^i(t) = (-1)^s\,\overset{s}{x}_a{}^i(t)$. From (5.15) we obtain the transformation laws of the functions $y_a{}^{i(\kappa)}(t)$ under the influence of the time inversion operator $\mathcal{T}$ (i.e. substitution $t \to -t$):

$$\mathcal{T} y_a{}^{i(+)}(t) = y_a{}^{i(-)}(t), \qquad \mathcal{T} y_a{}^{i(-)}(t) = y_a{}^{i(+)}(t). \tag{5.19}$$

It is easy to see, $\mathcal{T}L^{(+)} = L^{(-)}$, $\mathcal{T}\widehat{T}^{(+)} = \widehat{T}^{(-)}$ and vice versa. Therefore the Lagrangian function (5.9) and equations of motion (5.17) are time-symmetric.

The space inversion operator $\mathcal{P}$ acts on the functions ${x_a}^i(t)$ and their time derivatives according to the rule $\mathcal{P}\overset{s}{x}_a{}^i(t) = -\overset{s}{x}_a{}^i(t)$. Using eqs. (5.15) we obtain the transformation law $\mathcal{P}{y_a}^{i(\kappa)}(t) = -{y_a}^{i(\kappa)}(t)$. Using it in eqs. (5.10) we see that both function $L^{(+)}$ and function $L^{(-)}$ are invariant with respect to space inversion.

Let us prove that both operator $\widehat{T}^{(+)}$ and operator $\widehat{T}^{(-)}$ have a trivial kernel. Taking into account the Theorem given in Section 3 it is sufficient to find the relations inverted to the expression (5.15). We act on the function ${y_1}^{i(\kappa)}(t) = {x_1}^i[t - \kappa(1-\lambda)y^{(\kappa)}(t)]$ by the integral operator $\int_{\mathbb{R}} d\xi\delta(\xi - \zeta)$ where the parameter $\xi = t - \kappa(1-\lambda)y^{(\kappa)}(t)$, and on the function ${y_2}^{i(\kappa)}(t) = {x_2}^i[t + \kappa\lambda y^{(\kappa)}(t)]$ by the same operator where $\xi = t + \kappa\lambda y^{(\kappa)}(t)$. On the left sides of transformed equalities we develop the δ-functions into Taylor series in the neighbourhood of the point $t - \zeta$, integrate over t and redefine $\zeta \to t$. Thence we construct the following expansions

$$
{x_1}^i = {y_1}^{i(\kappa)} + \sum_{s=1}^{\infty} \frac{[\kappa(1-\lambda)]^s}{s!} {d_T}^{s-1}\left(y^{(\kappa)}\right)^s {\dot{y}_1}^{i(\kappa)}, \tag{5.20a}
$$

$$
{x_2}^i = {y_2}^{i(\kappa)} + \sum_{s=1}^{\infty} \frac{[-\kappa\lambda]^s}{s!} {d_T}^{s-1}\left(y^{(\kappa)}\right)^s {\dot{y}_2}^{i(\kappa)}, \tag{5.20b}
$$

which are inverted to the expressions (5.15). Q.E.D.

In ref. [27] the time-asymmetric Wheeler-Feynman electrodynamics for two point charges was investigated. In this model the first particle moves in the retarded (advanced) Lienard-Wiechert potential of the second particle, while the second particle moves in the advanced (retarded) Lienard-Wiechert potential of the first particle. Four-dimensional "coordinates on the cone" were used as the Lagrangian variables. In order to get three-dimensional formalism of the instant form of dynamics [26] it is necessary to restrict the gauge reparametrization group by imposing the constraint $x^0 = t$. As a result, single-time Lagrangian function coincides with the expression (5.10) where the value $\kappa = +1$ corresponds to Fokker-type action with advanced Green's function of the d'Alembert equation in ref. [27], and $\kappa = -1$ – for retarded Green's function in ref. [27]. Hence, in case of time-asymmetric model, the set $\Gamma(s)$ of motions $s(t) = (t, {x_a}^i(t))$ which are solutions of the infinite-order Euler-Lagrange equations, is isomorphic to the set $\Gamma(s')$ which consists of the motions $s'(t) = (t, {y_a}^{i(+)}(t))$ or $s'(t) = (t, {y_a}^{i(-)}(t))$ satisfying the equations of motion corresponding to first-order Lagrangian $L^{(+)}$ or $L^{(-)}$, respectively. If we change the time direction, the trajectory ${y_a}^{i(+)}(t)$ becomes ${y_a}^{i(-)}(t)$ and vice versa (see eqs. (5.19)). An analogous result is obtained in ref. [28] where Wheeler-Feynman N-body electrodynamics in two-dimensional Minkowski space is examined. On the Hamiltonian level the constrained Hamiltonian formalism is expected for the time-asymmetric Lagrangian function defined on the bundle $J^\infty \varrho$. This Hamiltonian theory has to contain an infinite set of constraints including the primary constraints of the type (4.13).

Damour and Shäfer [8] elaborate an iterative procedure which permits to reduce such a higher-order perturbative Lagrangian, so that a lower-derivative function is obtained finally. Thus the higher-derivative degrees of freedom are rejected as well as corresponding additional solutions of motion equations. The reason is that eliminated motions do not satisfy the requirement of analyticity with respect to the expansion parameter. Does it mean, that exact solutions, which could have such neglected approximations, do not exist? In case of time-asymmetric electrodynamics the answer is affirmative. Indeed, any finite-order approximation of the contact transformations (5.15) or (5.20) is not invertible. The claim of approximative invertibility of them is the mathematical realization of the requirement of analyticity mentioned above. A wide set of constraints of type (4.13) arises due to an invertibility of the *exact* contact transformation. Such a constrained dynamical system does not posses redundant degrees of freedom, which are caused by the higher derivatives.

As far as the time-symmetric Lagrangian (5.9) is concerned, it would be interesting to consider the functions ${y_a}^{i(+)}(t)$ and ${y_a}^{i(-)}(t)$ as Lagrangian variables. They are not independent quantities because the position variables ${x_a}^i$ can be written in the form of expansion $g_a^{i(+)}(y_a^{i(+)}, \ldots, \overset{s}{y}_a{}^{i(+)}, \ldots)$ as well as $g_a^{i(-)}(y_a^{i(-)}, \ldots, \overset{s}{y}_a{}^{i(-)}, \ldots)$ (see eqs. (5.20) for $\kappa = +1$ and $\kappa = -1$ respectively). The variational problem leading to Euler-Lagrange equations (5.17) is equivalent to the variational problem on the extremum of action $S = \int_{\mathbb{R}} dtL'$, associated with Lagrangian

$$L' = \frac{1}{2}(L^{(+)} + L^{(-)}) + \mu_{ai}(g_a^{i(+)} - g_a^{i(-)}). \tag{5.21}$$

Here symbols μ_{ai} denote Lagrange multipliers which can be assumed as independent Lagrangian variables. Thus we obtain an infinite-order Lagrangian function (5.21) instead of the first-order one (5.9), supplemented with infinite-derivative conditions $g_a^{i(+)} - g_a^{i(-)} = 0$ on the Lagrangian variables. One way of continuing the present work would be the investigation of its approximations.

APPENDIX A

Euler-Lagrange equations and generalized momenta

Expression of the original Euler-Lagrange equations (3.3) can be written as

$$\left[\frac{\delta S}{\delta x_a}\right] = \frac{\partial L}{\partial x_a} - d_T \, \hat{p}_{a,0}, \tag{A1}$$

where $\hat{p}_{a,0}$ is a zeroth-order Jacobi-Ostrogradski momentum. Generalized Jacobi-Ostrogradski momenta

$$\hat{p}_{a,s} = \sum_{l=s+1}^{k} (-d_T)^{l-s-1} \frac{\partial L}{\partial {x_a}^l}, \quad s = \overline{0, k-1}, \tag{A2}$$

are linked together by the recurrent relationships

$$\hat{p}_{a,s} = \frac{\partial L}{\partial {x_a}^{s+1}} - d_T \, \hat{p}_{a,s+1}, \quad s = \overline{0, k-2}. \tag{A3}$$

In this Appendix we establish the relations between the expression of new Euler-Lagrange equations (3.5) written as

$$\left[\frac{\delta \widetilde{S}}{\delta y_b}\right] = \frac{\partial \widetilde{L}}{\partial y_b} - d_T \, \hat{\pi}_{b,0}, \tag{A4}$$

new generalized momenta

$$\hat{\pi}_{b,r} = \sum_{l=r+1}^{k+n} (-d_T)^{l-r-1} \frac{\partial \widetilde{L}}{\partial {y_b}^l}, \quad r = \overline{0, k+n-1}, \tag{A5}$$

and original equations of motion (A1) and momenta (A2). We have the simplest connection between the maximal order momenta:

$$\hat{\pi}_{b,k+n-1} = \hat{p}_{a,k-1} \frac{\partial f_a}{\partial {y_b}^n}. \tag{A6}$$

Here and below in this Appendix we think that $\hat{p}_{a,s} = \hat{p}_{a,s}|_{\mathcal{F}}$ (contact transformation $\mathcal{F}$ is defined by eqs. (2.7)).

Having used recurrent relations analogous to (A3)

$$\hat{\pi}_{b,r} = \frac{\partial \widetilde{L}}{\partial {y_b}^{r+1}} - d_T\, \hat{\pi}_{b,r+1}, \quad s = \overline{0, k+n-2}, \tag{A7}$$

and the operator equality

$$\frac{\partial}{\partial {y_b}^i} {d_T}^m = \sum_{l=0}^{\alpha} \mathrm{C}_m^l\, {d_T}^{m-l} \frac{\partial}{\partial {y_b}^{i-l}}, \quad \alpha = \begin{cases} m, m < i \\ i, m \geq i \end{cases}, \tag{A8}$$

we obtain momenta $\hat{\pi}_{b,k+n-2}$ and $\hat{\pi}_{b,k+n-3}$. Usage of the method of mathematical induction gives the general laws:

$$\hat{\pi}_{b,n+i} = \sum_{r=i}^{k-1} \hat{p}_{a,r} \sum_{l=i}^{\beta} \mathrm{C}_r^l\, {d_T}^{r-l} \frac{\partial f_a}{\partial {y_b}^{n+i-l}}, \quad \beta = \begin{cases} r, r < n+i \\ n+i, r \geq n+i \end{cases}; \tag{A9a}$$

$$\begin{aligned} \hat{\pi}_{b,j} = &\sum_{r=0}^{k-1} \hat{p}_{a,r} \sum_{l=0}^{\gamma} \mathrm{C}_r^l\, {d_T}^{r-l} \frac{\partial f_a}{\partial {y_b}^{j-l}} + \\ &+ \sum_{l=j}^{n-1} (-d_T)^{l-j} \frac{\partial f_a}{\partial {y_b}^{l+1}} \left[\frac{\delta S}{\delta x_a}\right]_{\mathcal{F}}, \quad \gamma = \begin{cases} r, r < j \\ j, r \geq j \end{cases}, \end{aligned} \tag{A9b}$$

where index i runs from 0 to $k-1$ and index j runs from 0 to $n-1$. In particular zeroth-order momentum $\hat{\pi}_{b,0}$ is

$$\hat{\pi}_{b,0} = \sum_{r=0}^{k-1} \hat{p}_{a,r}\, {d_T}^r \left(\frac{\partial f_a}{\partial y_b}\right) + \sum_{l=0}^{n-1} (-d_T)^l \frac{\partial f_a}{\partial {y_b}^{l+1}} \left[\frac{\delta S}{\delta x_a}\right]_{\mathcal{F}}. \tag{A10}$$

Thanks to the commutation $[\partial/\partial y_b, d_T] = 0$ one easily proves that

$$\frac{\partial \widetilde{L}}{\partial y_b} = \sum_{l=0}^{k} \left.\frac{\partial L}{\partial {x_a}^l}\right|_{\mathcal{F}} {d_T}^l \left(\frac{\partial f_a}{\partial y_b}\right). \tag{A11}$$

Finally using (A10) and (A11) in (A4), after some calculations we arrive at

$$\left[\frac{\delta \widetilde{S}}{\delta y_b}\right] = \sum_{l=0}^{n} (-d_T)^l \frac{\partial f_a}{\partial {y_b}^l} \left[\frac{\delta S}{\delta x_a}\right]_{\mathcal{F}}. \tag{A12}$$

This formula explains the connection between the original and new Euler-Lagrange equations.

APPENDIX B

Gauge transformation as an example of invertible contact transformation

We examine a many-body system based on Lagrangian function L which is defined on the bundle $J^k\pi$ (e.g. (3.1)). Let us consider that $\mathcal{R}$ Lagrangian variables μ_p are not contained in L explicitly. Corresponding variational derivatives $\delta S/\delta\mu_p$ are identically equal to zero. Hence time-dependent functions $x_a(t)$ which are solutions of eqs. (3.2) together with arbitrary time-dependent functions $\mu_p(t)$ are coordinates of motion $s(t)$. Having carried out the invertible contact transformation, say $\mathcal{A}$

$$\begin{aligned} &x_a = f_a(t, y_b; \nu_p, {\nu_p}^1, \ldots, {\nu_p}^m), \quad \mu_q = \nu_q; \\ &{x_a}^s = d_T^s f_a, \quad {\mu_q}^s = {\nu_q}^s, \end{aligned} \tag{B1}$$

where indices p, q run from 1 to $\mathcal{R}$, we obtain the Lagrangian

$$\begin{aligned} &\widetilde{L}(t, y_b, \dot{y}_b, \ldots, \overset{k}{y}_b; \nu_p, \dot{\nu}_p, \ldots, \overset{k+m}{\nu_p}) = \\ &= L(t, f_a(t, y_b; \nu_p, \ldots, \overset{m}{\nu}_p), \ldots, \frac{d^k}{dt^k} f_a(t, y_b; \nu_p, \ldots, \overset{m}{\nu}_p)) \end{aligned} \tag{B2}$$

which is determined on the bundle $J^{k+m}\pi$ of $(k+m)$-jets $j_t^{k+m}\sigma'$ of sections $\sigma' = (t, y_b(t), \nu_p(t))$. Using eqs. (3.6) we write the relations between the original equations of motion (3.3) and the new ones in the form

$$\frac{\delta\widetilde{S}}{\delta y_b} = \frac{\partial f_a}{\partial y_b} \left.\frac{\delta S}{\delta x_a}\right|_{\mathcal{A}}, \tag{B3}$$

$$\frac{\delta\widetilde{S}}{\delta\nu_p} = \sum_{l=0}^{m} \left(-\frac{d}{dt}\right)^l \frac{\partial {f_a}^l}{\partial\, \nu_p} \left\{ \left.\frac{\delta S}{\delta x_a}\right|_{\mathcal{A}} \right\}. \tag{B4}$$

It is obvious that $M_I(s') \cong M_I(s)$.

If the new Lagrangian $\widetilde{L}$ given by (B2) does not depend on variables ν_p, then contact transformation (B1) becomes the gauge transformation [16]. In this case the left-hand side in (B4) vanishes identically. Whence we obtain $\mathcal{R}$ relationships including the original Euler-Lagrange equations which are transformed by the substitution (B1). Taking the $\nu_p \to 0$ limits we find correlations between "pure" original equations of motion. In order to write them in the form given in ref. [16]:

$$\sum_{l=0}^{m} \int_{\mathbb{R}} dt' \delta_{t'}^{(l)}(t'-t) \left.\frac{\partial f_a}{\partial \nu_p}^{l}\right|_{\nu=0} \frac{\delta S}{\delta x_a(t')} = 0, \tag{B5}$$

we have to use integral representation for the operator components T_{ap} such as

$$T_{ap} = \sum_{l=0}^{m} \int_{\mathbb{R}} dt' \delta_{t'}^{(l)}(t'-t) \frac{\partial f_a}{\partial \nu_p}^{l} . \tag{B6}$$

Here $\delta_{t'}^{(l)}(t'-t)$ is the l-th derivative of δ-function $\delta(t'-t)$ with respect to t'.

APPENDIX C

Effect of an irreversible contact transformation on the dynamics of a simple harmonic oscillator

In the Lagrangian of a simple harmonic oscillator in one dimension

$$L(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 \tag{C1}$$

we change of variables

$$x = y + \alpha\dot{y}, \qquad \dot{x} = \dot{y} + \alpha\ddot{y} \tag{C2}$$

where α is an arbitrary real parameter. We obtain a Lagrangian function determined on the 2-nd order tangent bundle T^2Q of one-dimensional configuration space Q (cf. eq. (10) in ref. [4]):

$$\widetilde{L}(y, \dot{y}, \ddot{y}) = \frac{1}{2}\left(1 - \alpha^2\omega^2\right)\dot{y}^2 - \frac{1}{2}\omega^2 y^2 + \frac{1}{2}\alpha^2\ddot{y}^2. \tag{C3}$$

The Euler-Lagrange equation is

$$\frac{\delta\widetilde{S}}{\delta y} \equiv \alpha^2 \overset{4}{y} - \left(1 - \alpha^2\omega^2\right)\ddot{y} - \omega^2 y = 0. \tag{C4}$$

We find its solution with the help of the Laplace transform:

$$\begin{aligned} y(t) =& \frac{y_0 - \alpha^2 \ddot{y}_0}{1+\alpha^2\omega^2}\cos\omega t + \frac{\dot{y}_0 - \alpha^2 {y_0}^3}{1+\alpha^2\omega^2}\omega^{-1}\sin\omega t + \\ &+ \frac{y_0\omega^2 + \ddot{y}_0}{1+\alpha^2\omega^2}\alpha^2\cosh(\alpha^{-1}t) + \frac{\dot{y}_0\omega^2 + {y_0}^3}{1+\alpha^2\omega^2}\alpha^3\sinh(\alpha^{-1}t). \end{aligned} \tag{C5}$$

Integration constants ${y_0}^k$ with $k = \overline{0,3}$ are coordinates of the point ${q_0}^3 = (0, y_0, \dot{y}_0, \ddot{y}_0, {y_0}^3)$ belonging to $\mathbb{R} \times T^3Q$. These constants correspond to initial-values of the higher derivatives. Trivial redefinition of the initial data allows to identify this solution with that given in ref. [4, eq. (15a)].

If we take eqs. (3.7) into account, it leads to

$$\left(1 - \alpha\frac{d}{dt}\right)\chi = 0 \quad \text{(a)}, \qquad \left.\frac{\delta S}{\delta x}\right|_{(C2)} \equiv \ddot{y} + \alpha \overset{3}{y} + \omega^2(y + \alpha\dot{y}) = \chi \quad \text{(b)} \tag{C6}$$

instead of the motion equation (C4). Kernel of the differential operator $\widehat{T} = 1 - \alpha d/dt$ is one-dimensional vector space spanned by the only basic function $\exp(\alpha^{-1}t)$. The following system of differential equations:

$$\ddot{x} + \omega^2 x = A\exp(\alpha^{-1}t), \tag{C7}$$

$$y + \alpha\dot{y} = x. \tag{C8}$$

is the fibre $\tau^{-1}(\xi)$ over $\xi = A\exp(\alpha^{-1}t)$. Here A is a real constant. Having applied the Laplace transform we arrive at

$$\begin{aligned} y(t) =& \frac{x_0 - \alpha\dot{x}_0}{1+\alpha^2\omega^2}\cos\omega t + \frac{\alpha\omega^2 x_0 + \dot{x}_0}{1+\alpha^2\omega^2}\omega^{-1}\sin\omega t \\ &+ \left(y_0' - \frac{x_0 - \alpha\dot{x}_0}{1+\alpha^2\omega^2}\right)\exp(-\alpha^{-1}t) \\ &+ \frac{\alpha C}{1+\alpha^2\omega^2}(\alpha\sinh(\alpha^{-1}t) - \omega^{-1}\sin\omega t). \end{aligned} \tag{C9}$$

Putting

$$\begin{aligned} x_0 &= y_0 + \alpha\dot{y}_0, & A &= \omega^2(y_0 + \alpha\dot{y}_0) + \ddot{y}_0 + \alpha {y_0}^3, \\ \dot{x}_0 &= \dot{y}_0 + \alpha\ddot{y}_0, & y_0' &= y_0, \end{aligned} \tag{C10}$$

we see that solutions (C9) and (C5) are identical. Original simple harmonic oscillation correlates with the one-parameter family of curves $M_{\mathbb{R}}^{0}(s')$ consisting of motions (C9) with $A = 0$. If $A \neq 0$ we have the set $M_{\mathbb{R}}^{A}(s')$ which corresponds to the oscillation in presence of an exponential external force (see eq. (C7)). This solution has not an original prototype.

In formulae (C3)-(C5) we substitute the imaginary constant $-i\varepsilon$ for the real parameter α. It allows to analyse additional aspects of the dynamics of the higher-derivative system, described in ref. [4]. Transformed expressions are the same as those in ref. [4] (see eqs. (10), (12) and (15)). Thus the contact transformation such as (C2) with imaginary parameter α is mathematically incorrect in the sense that it is outside the method developed in Sections 2 and 3. Nevertheless, ignoring this fact and neglecting the imaginary parts of the above relations, we present the equation of motion

$$\varepsilon^2 \overset{4}{y} + \left(1 + \varepsilon^2\omega^2\right)\ddot{y} + \omega^2 y = 0 \tag{C11}$$

in the following form:

$$\ddot{y} + \omega^2 y = A\cos(\varepsilon^{-1}t + \phi). \tag{C12}$$

Comparing of the solutions of these equations leads to the relations

$$A\cos\phi = y_0\omega^2 + \ddot{y}_0, \qquad A\sin\phi = -\varepsilon(\dot{y}_0\omega^2 + {y_0}^3). \tag{C13}$$

We are sure that the higher-derivative system described in ref. [4] (a simple harmonic oscillator with the mass slightly modified, and an acceleration-squared term) has a dynamically identical lower-derivative counterpart (a simple harmonic oscillator in presence of a sinusoidal external force). Note that amplitude A and phase ϕ of this force depend on the initial data of the higher-derivative initial-value problem. Thus, if $y_0\omega^2 + \ddot{y}_0 = 0$ and $\dot{y}_0\omega^2 + {y_0}^3 = 0$, the external force vanishes and the dynamical system reduces to the original simple harmonic oscillator.

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REFERENCES

[1] J. A. WHEELER and R. P. FEYNMAN, Classical electrodynamics in terms of direct interparticle action, *Rev. Mod. Phys.*, Vol. **21**, 3, 1949, pp. 425-433.

[2] E. H. KERNER, Hamiltonian formulation of action-at-a-distanse in electrodynamics, *J. Math. Phys.*, Vol. **3**, 1, 1962, pp. 35-42.

[3] C. G. DARWIN, The dynamical motions of charged particles, *Philos. Mag.*, Vol. **39**, 233, 1920, pp. 537-551.

[4] J. Z. SIMON, Higher-derivative Lagrangians, nonlocality, problems, and solutions, *Phys. Rev. D*, vol.**41**, 12, 1990, pp. 3720-3733.

[5] R. P. GAIDA, Yu. B. KLUCHKOVSKY, and V. I. TRETYAK, Three-Dimensional Lagrangian Approach to the Classical Relativistic Dynamics of Directly Interacting Particles, *Constraint's Theory and Relativistic Dynamics*, Arcetri, Firenze (Italy), May 28-30, 1986. Ed. by G. Longhi and L. Lusanna, *World Scientific, Singapore*, 1987, pp. 210-241.

[6] X. JAÉN, J. LLOSA, and A. MOLINA, A reduction of order two for infinite-order Lagrangians, *Phys. Rev. D.*, Vol. **34**, 8, 1986, pp. 2302-2311.

[7] D. A. ELIEZER and R. P. WOODARD, The problem of nonlocality in string theory, *Nucl. Phys. B.*, Vol. **325**, 1989, pp. 389-469.

[8] T. DAMOUR and G. SHÄFER, Redefinition of position variables and the reduction of higher-order Lagrangians, *J. Math. Phys.*, Vol. **32**, 1, 1991, pp. 127-134.

[9] R. P. GAIDA, Non-Point Transformations in Classical Mechanics, *Prepr. ICMP-94-5E*, Lviv (Ukraine), 1994.

[10] D. J. SAUNDERS, *The Geometry of Jet Bundles*, London Math. Soc., Lecture Notes Series **142**, Cambridge Univ. Press, 1989.

[11] W. M. TULCZYJEW, Sur la différentiellee de Lagrange, *C. R. Acad. Sci. Paris*, T. **280**, Sér. A, 1975, pp. 1295-1298.

[12] O. KRUPKOVÁ, A geometrical setting for higher order Dirac-Bergmann theory of constraints, *J. Math. Phys.*, Vol. **35**, 12, 1994, pp. 6557-6576.

[13] I. BUCUR and A. DELEANU, *Introduction to the Theory of Categories and Functors*, John Wiley & Sons, London·New York·Sydney, 1968.

[14] N. H. IBRAGIMOV and R. L. ANDERSON, Lie-Bäcklund tangent transformations, *J. Math. Anal. Appl.*, Vol. **59**, 1, 1977, pp. 145-162.

[15] N. H. IBRAGIMOV, Group theoretical nature of conservation theorem, *Letters in Math. Phys.*, Vol. **1**, 5, 1977, pp. 423-428.

[16] D. M. GITMAN and I. V. TYUTIN, *Canonical Quantization of Fields with Constraints*, Nauka, Moscow, 1986 *(in Russian)*.

[17] P. W. HEBDA, Treatment of higher-order Lagrangians via the construction of dynamically equivalent first-order Lagrangians, *J. Math. Phys.*, Vol. **31**, 9, 1990, pp. 2116-2125.

[18] M. OSTROGRADSKI, Mémoire sur les équations différentielles relatives aux problèmes des isopérimètres, *Mem. Acad. St.-Pétersbourg*, Vol. **VI**, 1850, pp. 385-517.

[19] M. DE LEÓN and P. R. RODRIGUES, *Generalized Classical Mechanics and Field Theory*, North-Holland Math. Studies, Ser. **112**, Amsterdam, 1985.

[20] X. GRÀCIA, J. M. PONS and N. ROMÁN-ROY, Higher-order Lagrangian systems Geometric structures, dynamics, and constraints, *J. Math. Phys.*, Vol. **32**, 10, 1991, pp. 2744-2763.

[21] M. DE LEÓN and D. M. DE DIEGO, Symmetries and constants of the motion for higher-order Lagrangian systems, *J. Math. Phys.*, Vol. **36**, 8, 1995, pp. 4138-4161.

[22] M. DE LEÓN and P. R. RODRIGUES, *Methods of Differential Geometry in Analytical Mechanics*, North-Holland Math. Studies, Ser. **158**, Amsterdam, 1989.

[23] M. J. GOTAY, J. M. NESTER and G. HINDS, Presymplectic manifolds and the Dirac-Bergmann theory of constraints, *J. Math. Phys.*, Vol. **19**, 11, 1978, pp. 2388-2399.

[24] A. D. FOKKER, Ein invarianter Variationsatz für die Bewegung mehrerer elektrischer Massenteilchen, *Z. Phys.*, Vol. **28**, 5-6, 1929, pp. 386-393.

[25] R. P. GAIDA and V. I. TRETYAK, Single-time form of the Fokker-type relativistic dynamics.I, *Acta Phys. Pol.*, Vol. **B11**, 7, 1980, pp. 509-522.

[26] R. P. GAIDA, Yu. B. KLUCHKOVSKY, and V. I. TRETYAK, Lagrangian classical relativistic mechanics of a system of directly interacting particles.I, *Theor. Math. Phys.*, Vol. **44**, 2, 1980, pp. 687-697.
[27] A. STARUSZKIEWICZ, Canonical theory of the two-body problem in the classical relativistic electrodynamics, *Ann. Inst. H. Poincaré*, Vol. **14**, 1, 1971, pp. 69-77.
[28] A. STARUSZKIEWICZ, An example of a consistent relativistic mechanics of point particles, *Ann. der Physik*, Vol. **25**, 4, 1970, pp. 362-367.

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