ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 65, nº 1 (1996), p. 57-79

<http://www.numdam.org/item?id=AIHPA_1996__65_1_57_0>

Ann. Inst. Henri Poincaré,

Vol. 65, n° 1, 1996, p. 57-79. *Physique théorique*

On the ground states of vector nonlinear Schrödinger equations

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ABSTRACT. – We consider a class of vector *nonlinear Schrödinger equations*, which arise in the infinite ion acoustic speed limit of the *Zakharov equations*. We define a ground state as a minimizer of an appropriate energy functional. Ground states satisfy a nonlinear elliptic system of partial differential equations. We show, in certain parameter regimes, that a ground state cannot be a radial vector field (the gradient of a function which depends only on the distance to some fixed origin of coordinates). This was conjectured and supported by numerical observations of Zakharov *et al.* ([26], [6]). In a special case, corresponding to a Ginzburg Landau energy functional, we prove that the ground state is a vector field whose components are constant multiples of the ground state of the analogous scalar *variational problem*. It follows, in this case, that the ground state is essentially unique. This gives a characterization of ground states (or minimum action solutions) constructed by Brezis and Lieb [1].

RÉSUMÉ. – On considère une classe d'équations de Schrödinger non linéaires qui interviennent en physique des plasmas comme la limite des équations de Zakharov lorsque la vitesse acoustique ionique tend vers l'infini. On définit un état fondamental comme étant un minimiseur d'une fonctionnelle appropriée. Les états fondamentaux vérifient un système elliptique non linéaire d'équations aux dérivées partielles. On montre, pour une plage de paramètres, qu'un état fondamental ne peut être un champ de vecteur radial (*i.e.* le gradient d'une fonction ne dépendant que de la distance à une origine de coordonnées). Cela était conjecturé et des simulations numériques de Zakharov *et al.* ([26], [6]) le mettaient en évidence. Dans un cas particulier, correspondant à une fonctionnelle d'énergie de type Ginzburg Landau, on montre que l'état fondamental est un champs de vecteurs dont toutes les composantes sont des multiples de l'état fondamental du problème variationnel scalaire associé. Cela donne une caractérisation des états fondamentaux (ou solutions d'action minimale) construits par Brezis et Lieb [1].

1. INTRODUCTION

1.1. Vector nonlinear Schrödinger systems

This paper concerns the structure of ground state solutions of a class of vector nonlinear Schrödinger equations:

$$i\,\frac{\partial E}{\partial t} + \nabla\,(\nabla\cdot E) - \alpha^2\,\nabla\times\nabla\times E + |\,E\,|^{2\,\sigma}\,E = 0. \tag{1}$$

Here, $E = E\,(x,\,t) : \mathbf{R}^{\mathbf{n}}\times\mathbf{R}^{\mathbf{1}} \to \mathbf{C}^{\mathbf{n}}$ is a complex vector field and α is a real non-zero parameter. We shall also consider the case where the vector field, E, is irrotational, *i.e.* $E = \nabla\,\psi$. In this case, (1) becomes the equation

$$i\,\frac{\partial\nabla\,\psi}{\partial t} + \Delta\nabla\,\psi + \Delta^{-1}\,\nabla\nabla\cdot(\,|\,\nabla\,\psi\,|^{2\,\sigma}\,\nabla\,\psi) = 0. \tag{2}$$

Here $\Delta^{-1}\,\nabla\nabla\cdot\vec{F}$ is the projection of $\vec{F}$ onto its curl free part.

In the special case, $\alpha^2 = 1$, as a consequence of the identity: $\nabla\times\nabla\times A = \nabla\,(\nabla\cdot A) - \Delta A$, equation (1) reduces to the standard nonlinear Schrödinger equation:

$$i\,E_t = -\Delta E - |\,E\,|^{2\,\sigma}\,E. \tag{3}$$

The consideration of the more general α-dependent equation (1) above is motivated by the mathematical description of nonlinear (Langmuir) waves in a collisionless plasma. A short discussion of the physical problem is presented below in Section 1.3.

Though a vector equation, the known techniques on nonlinear Schrödinger equations apply to the study of its well-posedness, *see* [11], [13] for example. The Cauchy problem for (2) has been investigated in [4].

Remark on terminology. – If E denotes a vector field on $\mathbf{R}^n$, by $E \in H^1$ we mean that each component of E lies in H^1. A *radial function* refers to a function which depends only on $r = |x|$. Finally, a *radial vector field*, E is a vector field of the form $E(x) = \frac{x}{|x|} f(r)$.

1.2. Bound states

Of interest for these evolution equations are nonlinear bound state solutions. These are finite energy localized solutions $(E \in H^1)$. It is known for scalar nonlinear Schrödinger type equations that *ground states*, least energy bound states, participate in the structure of general solutions in, for example, global in time solutions, as stable states ([5], [3], [8], [24]) or as a universal profile in collapsing or self-focusing solutions in the *critical case* ([27], [15], [23]).

By a bound state solution of (1) or (2) we mean a solution of the form $e^{i\omega t} E(x)$ or $e^{i\omega t} \nabla \psi(x)$, $(\omega \in \mathbf{R})$, where $E(x) \in L^2$ and $\nabla \psi(x) \in L^2$ do not depend on the time t. The vector field E satisfies the system of elliptic partial differential equations

$$-\omega E + \nabla(\nabla \cdot E) - \alpha^2 \nabla \times \nabla \times E + |E|^{2\sigma} E = 0. \tag{4}$$

In the irrotational case, we have from (2):

$$-\omega \nabla \psi + \Delta \nabla \psi + \Delta^{-1} \nabla\nabla \cdot (|\nabla \psi|^{2\sigma} \nabla \psi) = 0, \tag{5}$$

where $\sigma > 0$.

Taking the divergence, gives the equivalent fourth order elliptic partial differential equation for the scalar potential, ψ:

$$\Delta(-\omega\psi + \Delta\psi) = \nabla \cdot (|\nabla \psi|^{2\sigma} \nabla \psi). \tag{6}$$

In the special case $\alpha^2 = 1$, E satisfies the equation

$$-\Delta E - |E|^{2\sigma} E = -\omega E, \qquad x \in \mathbf{R}^n. \tag{7}$$

The scalar variant of (7) is the equation of a bound state $u : \mathbf{R}^n \to \mathbf{C}$ of the scalar nonlinear Schrödinger equation:

$$-\Delta u - |u|^{2\sigma} u = -\omega u, \qquad x \in \mathbf{R}^n \tag{8}$$

has been studied in depth. The existence of H^1 solutions to (8), under the restriction $0 < \sigma < 2/(n-2)$ for $n > 2$ and $0 < \sigma$, for $n = 1, 2$ is a special case of results in [20] and [16]. In these works, the existence problem for a *ground state* is formulated and solved as a constrained minimization problem for a functional, whose Euler-Lagrange equation is (8). A constrained minimizer is constructed which is a function of $r = |x|$ and is monotonically decreasing to zero as $r \to \infty$. Now, any positive solution is radial with respect to one point [10]. The uniqueness problem is then reduced to the study of a nonlinear ordinary differential equation and was resolved for $0 < \sigma < 2/(n-2)$ in [14]. We denote the ground state profile associated with (8) by u_g.

In the remainder of this section we define the notion of *ground state*, give the statements of our main results, and outline the remainder of the paper.

Our approach to the construction of solutions to (4) and (6) is also by a variational methods. Our characterization of ground states is in terms of a natural *unconstrained* minimization problem. Let

$$J_\alpha^{\sigma,n}(E) \equiv \frac{\left(\int |\nabla \cdot E|^2 + \alpha^2 \int |\nabla \times E|^2\right)^{\sigma n/2} \left(\int |E|^2\right)^{1+\sigma(2-n)/2}}{\int |E|^{2\sigma+2}}. \tag{9}$$

Formally, a critical point of $J_\alpha^{\sigma,n}$ is a solution of (4). Since

$$\int |\nabla \cdot E|^2 + \int |\nabla \times E|^2 = \int |\nabla E|^2 \equiv \sum_i \int |\nabla E_i|^2,$$

for $\alpha^2 = 1$ we have

$$J_1^{\sigma,n}(E) = \frac{\left(\int |\nabla E|^2\right)^{\sigma n/2} \left(\int |E|^2\right)^{1+\sigma(2-n)/2}}{\int |E|^{2\sigma+2}}.$$

In the case of irrotational vector fields, we have $E = \nabla \psi$, and $J_\alpha^{\sigma,n}$ reduces to the functional:

$$I^{\sigma,n}(\nabla \psi) \equiv \frac{\left(\int |\Delta \psi|^2\right)^{\sigma n/2} \left(\int |\nabla \psi|^2\right)^{1+\sigma(2-n)/2}}{\int |\nabla \psi|^{2\sigma+2}}.$$

Let

$$J_0^\alpha \equiv \inf \{J_\alpha^{\sigma, n}(E) : E \in H^1(\mathbf{R^n}),\ E \neq 0\}, \tag{10}$$

$$J_1 \equiv \inf \{J^{\sigma, n}(E) = I^{\sigma, n}(\nabla \psi) : E = \nabla \psi \in H^1(\mathbf{R^n}),\ E \neq 0\}, \tag{11}$$

$$\begin{aligned} J_2 \equiv \inf \{J^{\sigma, n}(E) = I^{\sigma, n}(\nabla \psi(\,|x|\,)) :& \\ E = \nabla \psi(\,|x|\,) \in H^1(\mathbf{R^n}),\ E \neq 0\}.& \end{aligned} \tag{12}$$

Clearly, $J_0^\alpha \leq J_1 \leq J_2$. The main results of this paper are:

(i) Theorem 1: For $\alpha^2 = 1$, $J_0^\alpha < J_1$ and J_0^1 is attained at a vector field all of whose components are constant multiples of the ground state, $u_g(\,|x|\,)$, of a scalar nonlinear Schrödinger equation (8). This completely characterizes the ground states (minimum action) solutions constructed by Brezis and Lieb [1].

(ii) Theorem 2: If $\alpha^2 \neq 1$ and $\alpha^2 < \alpha_*(\sigma, n)$, then $J_0^\alpha < J_2$. In this case, the components of a minimizer are not all radial functions.

(iii) Theorem 3: As $\alpha \to \infty$, $J_0^\alpha \to J_1$. Moreover, if $\alpha_j \to \infty$, then a subsequence α_{j_k} can be found, such that, up to translations in space and phase, the corresponding minimizers of $J_{\alpha_{j_k}}^{\sigma, n}$, $E^{\alpha_{j_k}}$ converge strongly to a minimizer among gradient vector fields [see (11)].

CONJECTURE. – We conjecture, in general, that for finite α^2 that $J_0^\alpha < J_1 < J_2$, *i.e.* that the ground state among irrotational fields is not a radial vector field.

Why minimize the functional $J_\alpha^{\sigma, n}$? Our motivation comes from existence and dynamical stability considerations in the scalar case. In analogy with the scalar case, the system (3) has conserved integrals

$$\mathcal{H}[E] = \int |\nabla E|^2 - (\sigma + 1)^{-1} |E|^{2\sigma+2}\, dx,$$

and

$$\mathcal{N}[E] = \int |E|^2\, dx.$$

The functional $\mathcal{H}$ is a Hamiltonian for the system and $\mathcal{N}$ is a momentum or impulse functional. At the heart of the proof of dynamical stability of the ground state is its characterization as a minimizer of $\mathcal{H}[E]$ subject to fixed $\mathcal{N}[E]$:

$$I_\lambda = \inf \{\mathcal{H}[u] : \mathcal{N}[u] = \lambda\}. \tag{13}$$

This characterization and stability hold if $\sigma < 2/n$ ([3], [8], [5], [24]).

Now the variational principle J_0^α, can be seen as a generalization of the problem (13) as follows. Let $E_\eta(x) = \eta^{n/2} E(\eta x)$ for any $\eta > 0$. Note that the map $E \mapsto E_\eta$ leaves the L^2 norm unchanged. Now fix E. If $\sigma < 2/n$, then

$$\mathcal{H}[E] \geq \min_{\eta>0} \mathcal{H}[E_\eta]$$

$$\geq \left(\frac{\sigma n}{2} - 1\right) (2\sigma + 2)^{n\sigma/(2-n\sigma)} \left(\frac{\left(\int |E|^{2\sigma+2} dx\right)^2}{\left(\int |\nabla E|^2 dx\right)^{\sigma n}} \right)^{1/(2-n\sigma)}.$$

For $\sigma < 2/n$, we replace E by $\lambda E/\| E \|_2$, to obtain:

$$I_\lambda = \left(\frac{\sigma n}{2} - 1\right) (2\sigma + 2)^{\frac{n\sigma}{(2-n\sigma)}} \lambda^{\frac{2\sigma+2-n\sigma}{2-n\sigma}} \inf J_1^{\sigma, n}. \tag{14}$$

On the other hand, if $\sigma > 2/n$, then

$$I_\lambda = -\infty.$$

Although, $I_\lambda = -\infty$ for $\sigma > 2/n$, the minimization problem appearing on the right hand side of (14) is well posed for *any* σ satisfying $0 < \sigma < 2/(n-2)$ for $n \geq 3$ and $0 < \sigma < \infty$ if $n = 2$. Thus it is natural to view a ground state, more generally, as being a minimizer of $J_\alpha^{\sigma, n}$. An analogous variational characterization for the ground state of the scalar variant of (7) was used in [22].

1.3. Physical background

A collisionless plasma can be described by the coupled evolution the electron plasma wave envelope, $E(x, t)$, governing Langmuir waves, and ion density fluctuations about its equilibrium value, $\delta n(x, t)$. The evolution equations are known as the Zakharov equations ([25], [9], [7]).

$$i \frac{\partial E}{\partial t} = \alpha^2 \nabla \times \nabla \times E - \nabla(\nabla \cdot E) + n E, \tag{15}$$

$$\frac{1}{c_s^2} \frac{\partial^2 n}{\partial t^2} = \Delta(n + |E|^2). \tag{16}$$

Here E, the slowly varying envelope of the electric field, $\mathcal{E}$, is related to $\mathcal{E}$ by

$$\mathcal{E} = \frac{1}{2} (E e^{-i\omega_p t} + E^* e^{i\omega_p t}).$$

The equations are in dimensionless variables. The parameter α^2, is given by

$$\alpha^2 = \frac{c^2}{3\, v_{T_e}^2}$$

where c is the speed of light, and v_{T_e} is the thermal velocity of the electrons, and ω_p is the plasma frequency (*see* [7]). In the limit $c_s \to \infty$, the system (15), (16) reduces to the vector nonlinear Schrödinger equation

$$i \frac{\partial E}{\partial t} + \nabla (\nabla \cdot E) - \alpha^2 \nabla \times \nabla \times E + |E|^2 E = 0. \tag{17}$$

In the case where $E = \nabla \psi$ we have

$$\Delta \left(i \frac{\partial \psi}{\partial t} + \Delta \psi \right) = \nabla \cdot (| \nabla \psi^2 | \nabla \psi). \tag{18}$$

These equations are special cases of the equations with more general nonlinearities introduced in Section 1.

1.4. Outline of the paper

The paper is structured as follows.

In Section 2 we prove the existence of ground states by proving that a minimizer is attained in our variational formulations.

In Section 3 we prove our results concerning the structure of ground states (Theorems 1 and 2). We use a technique of Lopes [17] for proving the symmetry of energy minimizers. It follows that the minimizer is not a gradient field and, in particular, not a radial vector field.

In Section 4 we give a summary of our results, discuss some open problems and state a theorem (Theorem 4) on nonlinear orbital dynamical stability of the ground states.

2. EXISTENCE OF SOLUTIONS AND MINIMIZATION PROBLEMS

Throughout this paper we shall assume

$$0 < \sigma < \frac{2}{n-2} \qquad \text{for} \quad n \geq 3,$$

and

$$0 < \sigma < \infty \qquad \text{for} \quad n = 2.$$

The first result is the following.

PROPOSITION 1. – (i) *The functional* $J_\alpha^{\sigma, n}$ *attains its minimum at a vector field*

$$E_* \in H^2 (\mathbf{R}^n).$$

(ii) *Equivalently,* E_* *is a minimizer of*

$$- \int | E |^{2\sigma+2}$$

subject to the constraints

$$\int | E |^2 = \lambda \qquad \textit{and} \qquad \int | \nabla \cdot E |^2 + \alpha^2 \int | \nabla \times E |^2 = \mu,$$

for appropriate choice of λ *and* μ.

(iii) *There exists an* $\omega > 0$, E_* *is a solution of* (1). *We therefore also write* $E_* = E_\omega$.

Proof. – This result can be proved by using the concentration compactness approach [16]. See also the approach taken in [1]. ■

For equation (6) or equivalently (5), the situation is a little bit more complicated. We have

PROPOSITION 2. – *For* $\omega > 0$, *there exists a solution* $E_\omega = \nabla \psi_\omega$ *to* (5), *with* $\nabla \psi_\omega \in H^2 (\mathbf{R}^n)$. *Moreover* $\psi_\omega \in L^{2n/(n-2)} (\mathbf{R}^n)$ *for* $n > 2$ *and* $\psi_\omega \in BMO (\mathbf{R}^2)$ *for* $n = 2$.

$\nabla \psi_\omega$ *minimizes the functional*

$$- \int | \nabla \psi |^{2\sigma+2} \tag{19}$$

subject to the constraints

$$\int | \nabla \psi |^2 = \lambda \qquad \textit{and} \qquad \int | \Delta \psi |^2 = \mu$$

for an appropriate choice of λ *and* μ.

Equivalently, $\nabla \psi_\omega$ *is an unconstrainted minimizer of*

$$I^{\sigma, n} (\nabla \psi) = \frac{\left(\int | \Delta \psi |^2 \right)^{\sigma n/2} \left(\int | \nabla \psi |^2 \right)^{1+\sigma (2-n)/2}}{\int | \nabla \psi |^{2\sigma+2}}.$$

Proof. – These results have been proved, for the case $n = 3$, in [5] using the concentration-compactness approach. For technical reasons, this proof appears not to work in the spatial dimension 2. Here, we present a simpler proof valid for all space dimensions. Let

$$I_0(\lambda,\,\mu) = \inf_{\nabla\psi\in H^1(\mathbf{R}^n)} \left\{ -\int_{\mathbf{R}^n} |\nabla\psi|^{2\sigma+2} \,\middle|\, \int_{\mathbf{R}^n} |\nabla\psi|^2 = \lambda, \right.$$
$$\left. \int_{\mathbf{R}^n} |\Delta\psi|^2 = \mu \right\},$$

for λ and μ in $\mathbf{R}^{+*}$ and

$$I_\varepsilon(\lambda,\,\mu) = \inf_{E\in H^1(\mathbf{R}^n)} \left\{ -\int_{\mathbf{R}^n} |E|^{2\sigma+2} \,\middle|\, \int_{\mathbf{R}^n} |E|^2 = \lambda, \right.$$
$$\left. \int_{\mathbf{R}^n} |\nabla E|^2 = \mu,\ \int_{\mathbf{R}^n} |\nabla\times E|^2 = \varepsilon \right\}.$$

Note that by scaling we have

$$I_0(\lambda,\,\mu) = I_0(1,\,1)\,\lambda^{\sigma(1-n/2)+1}\,\mu^{n\sigma/2}, \tag{20}$$

and therefore that $I_0(\lambda,\,\mu)$ is continuous. We first have:

PROPOSITION 3

$$\lim_{\varepsilon\to 0} I_\varepsilon(\lambda,\,\mu) = I_0(\lambda,\,\mu).$$

Proof. – Consider a minimizing sequence of functions, denoted by φ^ε with $\varepsilon\to 0$ and such that

$$|\,I_0(\lambda-\varepsilon,\,\mu-\varepsilon) + \int |\nabla\varphi^\varepsilon|^{2\sigma+2}\,| \le \varepsilon$$

and

$$\int |\nabla\varphi^\varepsilon|^2 = \lambda-\varepsilon, \qquad \int |\Delta\varphi^\varepsilon|^2 = \mu-\varepsilon.$$

Since $I_0(\lambda,\,\mu)$ is continuous we have

$$-\int |\nabla\varphi^\varepsilon|^{2\sigma+2} \to I_0(\lambda,\,\mu)$$

as $\varepsilon\to 0$. We introduce $f_\varepsilon = \nabla\varphi_\varepsilon + \sqrt{\varepsilon}\,\nabla\times\psi$, where ψ is a regular vector field such that $\int |\nabla\times\psi|^2 = 1$ and $\int |\nabla\times\nabla\times\psi|^2 = 1$. In dimension 2, one has to replace $\nabla\times\psi$ by $\nabla^\perp$ and $\nabla\times\nabla\times\psi$ by $\Delta\psi$.

We therefore obtain $\int |f_\varepsilon|^2 = \lambda$, $\int |\nabla f_\varepsilon|^2 = \mu$ and $\int |\nabla \times f_\varepsilon|^2 = \varepsilon$.

It follows that $-\int |f_\varepsilon|^{2\sigma+2} \geq I_\varepsilon(\lambda, \mu)$. Letting $\varepsilon \to 0$ leads to

$$I_0(\lambda, \mu) \geq \limsup I_\varepsilon(\lambda, \mu). \tag{21}$$

We now have to prove the converse inequality. Let E_ε be such that

$$\left| I_\varepsilon(\lambda, \mu) + \int |E_\varepsilon|^{2\sigma+2} \right| \leq \varepsilon$$

and satisfying $\int |E_\varepsilon|^2 = \lambda$, $\int |\nabla E_\varepsilon|^2 = \mu$ and $\int |\nabla \times E_\varepsilon|^2 = \varepsilon$.
Then,

$$\liminf_{\varepsilon \to 0} I_\varepsilon(\lambda, \mu) = \liminf_{\varepsilon \to 0} - \int |E_\varepsilon|^{2\sigma+2}.$$

We express E_ε in the form

$$E_\varepsilon = \nabla \varphi_\varepsilon + \nabla \times \psi_\varepsilon, \tag{22}$$

(resp. $\nabla \varphi_\varepsilon + \nabla^\perp \psi_\varepsilon$ in space dimension 2) with $\int |\nabla \times \nabla \times \psi_\varepsilon|^2 = \varepsilon$, $\int |\Delta \varphi_\varepsilon|^2 = \mu - \varepsilon$, and $\int |\nabla \varphi_\varepsilon|^2 + \int |\nabla \times \psi_\varepsilon|^2 = \lambda$. We then have $\int |\nabla \varphi_\varepsilon|^2 = \lambda - \int |\nabla \times \psi_\varepsilon|^2 \equiv \lambda_\varepsilon \to \lambda_1$ with $0 \leqq \lambda_1 \leqq \lambda$.

Moreover $|\nabla \times \psi_\varepsilon|_{L^{2\sigma+2}} \to 0$. Indeed, by the Gagliardo-Nirenberg inequality

$$\begin{aligned} |\nabla \times \psi_\varepsilon|_{L^{2\sigma+2}}^{2\sigma+2} &\leq C |\nabla \times \nabla \times \psi_\varepsilon|_{L^2}^{2+\sigma(2-n)} |\nabla \times \psi_\varepsilon|_{L^2}^{\sigma n} \\ &\leq C \lambda^{1+\sigma(2-n)/2} \varepsilon^{n\sigma/2}. \end{aligned} \tag{23}$$

Therefore (22) and (23) imply

$$\left| \int |\nabla \varphi_\varepsilon|^{2\sigma+2} - \int |E_\varepsilon|^{2\sigma+2} \right| \to 0.$$

It follows that

$$I_0(\lambda_\varepsilon, \mu - \varepsilon) \leq - \int |\nabla \varphi_\varepsilon|^{2\sigma+2}, \tag{24}$$

and passing to the limit in (24) leads to

$$I_0(\lambda_1, \mu) \leq \liminf_{\varepsilon \to 0} I^\varepsilon(\lambda, \mu). \tag{25}$$

On the other hand, by (20), since $\lambda_1 \leq \lambda$, we have $I_0(\lambda_1, \mu) \geq I_0(\lambda, \mu)$. Hence (25) gives

$$I_0(\lambda, \mu) \leq \liminf_{\varepsilon \to 0} I^{\varepsilon}(\lambda, \mu). \tag{26}$$

Equation (21) and (26) prove the Proposition 3. ■

In order to prove the Proposition 2, we take a sequence E_ε such that

$$\left| I^{\varepsilon}(\lambda, \mu) + \int |E_\varepsilon|^{2\sigma+2} \right| \leq \varepsilon$$

and $\int |E_\varepsilon|^2 = \lambda$, $\int |\nabla E_\varepsilon|^2 = \mu$ and $\int |\nabla \times E_\varepsilon|^2 = \varepsilon$. We now apply the concentration-compactness principle [16] to the sequence E_ε. We briefly sketch the proof. The idea is to rule out the occurance of (i) *vanishing* and (ii) *dichotomy*.

(i) To show that (i) does not occur, one can show the existence of a ball of radius $R > 0$ such that

$$\limsup_{\varepsilon \to 0} \int_{y+B_R} |E_\varepsilon|^{2\sigma+2} > 0.$$

This follows since the infimum of (19) in Proposition 2 is strictly negative. The covering techniques in [16] or [1] can be used to then to preclude vanishing.

(ii) If *dichotomy* occurs, then $E_\varepsilon = E_1^\varepsilon + E_2^\varepsilon + \eta$ with $|\eta|_{H^1} \ll 1$ and $\operatorname{supp}(E_1^\varepsilon) \cap \operatorname{supp}(E_2^\varepsilon) = \emptyset$. Then

$$-\int |E_\varepsilon|^{2\sigma+2} = -\int |E_1^\varepsilon|^{2\sigma+2} - \int |E_2^\varepsilon|^{2\sigma+2} + f(\eta)$$

with $f(\eta) \to 0$ when $\eta \to 0$. This implies

$$-\int |E_\varepsilon|^{2\sigma+2} \geq I_{\varepsilon_1}(\beta, \mu_1) + I_{\varepsilon_2}(\lambda - \beta, \mu_2) + f(\eta) \tag{27}$$

with $0 \leq \mu_1 + \mu_2 \leq \mu$, $0 < \beta < \lambda$, $0 \leq \varepsilon_1 + \varepsilon_2 \leqq \varepsilon$. We now let ε tend to zero in (27) using lemma 1:

$$I_0(\lambda, \mu) \geq I_0(\beta, \mu_1) + I_0(\lambda - \beta, \mu_2) + f(\eta), \tag{28}$$

and let $\eta \to 0$ in (28):

$$I_0(\lambda, \mu) \geq I_0(\beta, \mu_1) + I_0(\lambda - \beta, \mu_2). \tag{29}$$

This leads to a contradiction as can be seen from the explicit expression for I_0 in (20).

(iii) It follows that we have compactness, *i.e.* $E_\varepsilon \to E_*$ in $H^1(\mathbf{R}^n)$ (along a subsequence and modulo translations in space and phase) strongly and $\nabla \times E_* = 0$, which implies $E_* = \nabla \psi_*$ and $\nabla \psi_*$ is a minimizer for $I_0(\lambda, \mu)$. The fact that $\psi_* \in L^{2n/(n-2)}$ for $n > 2$ follows from the Sobolev imbedding and $\psi_* \in BMO(\mathbf{R}^2)$ for $n = 2$ follows from the Poincaré inequality. ■

3. COMPARISON BETWEEN THE MINIMA OF $I^{\sigma, n}(\nabla \psi)$ AND $J_\alpha^{\sigma, n}(E)$

Here we shall compare the vector fields that achieve the minima of $I^{\sigma, n}$ and $J_\alpha^{\sigma, n}$ for the cases $\alpha^2 = 1$ and $\alpha^2 \neq 1$.

3.1. The case $\alpha^2 = 1$

In all this section, we work with $\alpha^2 = 1$. The main result is

THEOREM 1. – (a) $J_0^\alpha < J_1$ *for* $\alpha^2 = 1$.

(b) *Let E_* be a vector field that minimize $J_\alpha^{\sigma, n}$ for $\alpha = 1$. Then, up to a change in the origin of coordinates, we have*

$$E_* = E_*(r) = \frac{\gamma}{\eta_*} R(\omega^{1/2} r), \tag{30}$$

where $R(\rho)$ is the unique positive and decaying solution of

$$R''(\rho) + \frac{n-1}{\rho} R'(\rho) - R(\rho) + R(\rho)^{2\sigma+1} = 0, \tag{31}$$

with $R'(0) = 0$ and $R(0) = \eta_*$. Moreover, we have the constraint on γ

$$\gamma \cdot \gamma = \omega^{1/\sigma} \eta_*^2.$$

Finally, the value of $J_0^1 = J_1^{\sigma, n}(R)$ is given by

$$J_0^1 = \frac{\left(\int |\nabla R|^2\right)^{\sigma n/2} \left(\int |R|^2\right)^{1+\frac{\sigma}{2}(2-n)}}{\int |R|^{2\sigma+2}}.$$

Remark. – It follows that $C = 1/J_0^1$ is the best constant in the Gagliardo-Nirenberg estimate:

$$\int |E|^{2\sigma+2} \leq C \Big(\int |\nabla E|^2 \Big)^{\sigma n/2} \Big(\int |E|^2 \Big)^{1+\frac{\sigma}{2}(2-n)}.$$

See also [22]. The proof of the theorem will be carried out in two steps. The first one is:

PROPOSITION 4. – *Let* $\alpha^2 = 1$.

(a) *An alternative characterization of a ground state,* E_* *is as a minimizer of the functional*

$$\frac{\omega}{2} \int |E|^2 - \frac{1}{2\sigma+2} \int |E|^{2\sigma+2}$$

subject to the constraint

$$\int |\nabla E|^2 = \int |\nabla E_*|^2.$$

(b) *There exists* $x_0 \in \mathbf{R}^n$ *such that each component of* E_* *is radial with respect to* x_0.

Proof of Proposition 4. – Part (a) is obtain by the same methods as in [2] or [5]. We first prove that this last minimization problem has a solution and then compare both minimizers by a scaling analysis.

(b) We use the method of O. Lopes [17]. Let E be a minimizer; for any hyperplane Π in $\mathbf{R}^n$, there exists a (unique) hyperplane $\tilde{\Pi}$ parallel to Π such that E is symmetric with respect to $\tilde{\Pi}$. Indeed, let $\tilde{\Pi}$ be an hyperplane parallel to Π such that

$$\int_{\tilde{\Pi}_+} |\nabla E|^2 = \int_{\tilde{\Pi}_-} |\nabla E|^2 = \frac{1}{2} \int_{\mathbf{R}^n} |\nabla E_*|^2,$$

where $\tilde{\Pi}_+$ and $\tilde{\Pi}_-$ denote the half spaces defined by $\tilde{\Pi}$. Let V the vector field in $H^1(\mathbf{R}^n)$ which is symmetric with respect to $\tilde{\Pi}$ and $V = E$ on $\tilde{\Pi}_+$. Then V is a minimizer also.

This implies that V satisfies (7):

$$-\omega V + \Delta V = -|V|^{2\sigma} V.$$

The vector field $W \equiv E - V$ therefore satisfies

$$-\omega W + \Delta W = MW,$$

where

$$M = \int_0^1 (|u|^{2\sigma} u)' (t V + (1 - t) E) \, dt.$$

Now since $W \equiv 0$ on $\tilde{\Pi}_+$, the unique continuation principle, which applies to such weakly coupled systems, implies $W \equiv 0$, *see* [21] Theorem 2.1 p. 45 and E is symmetric with respect to $\tilde{\Pi}$. We now follow the idea of Lopes. Applying the above argument to the coordinate hyperplanes and after a possible change of origin of coordinates we can arrange for E to be symmetric with respect to all coordinates hyperplanes. In particular, E is symmetric with respect to the origin. We still must show that E is symmetric with respect to any hyperplane passing through the origin. Suppose not. Then, there is a hyperplane, Π, passing through the origin such that E is not symmetric with respect to Π. As above, we construct a hyperplane $\tilde{\Pi}$ parallel to Π such that E is symmetric with respect to $\tilde{\Pi}$. Let T_0 denote the simplex delimited by $\tilde{\Pi}$ and the coordinates hyperplanes. Let T_1 be the simplex obtained from T_0 by reflection about $\tilde{\Pi}$, and T_2 that obtained from T_1 by reflection about the origin and so on. In this way we obtain a sequence of disjoint simplicies, T_i, such that $\int_{T_i} |E|^2 = \int_{T_i} |E|^2$ for all i. Since $E \in L^2$, this implies that $E \equiv 0$ on T_1. The unique continuation principle implies that $E \equiv 0$, which is a contradiction. ■

Proof of Theorem 1. – a) Note that this result enables us to prove that $J_0^\alpha < J_1$. Indeed suppose that $E_* = \nabla \psi_*$, then for all i, j one has

$$\frac{\partial E_j^*}{\partial x_i} = \frac{\partial E_i^*}{\partial x_j}$$

and this implies

$$E_j^{*\prime}(r) \frac{x_i}{r} = E_i^{*\prime}(r) \frac{x_j}{r}.$$

For $i \neq j$, taking $x_i = 0$ and $x_j \neq 0$ yields $E_i^{*\prime}(r) = 0$ which is a contradiction.

b) We first remark that i-th component of E_* satisfies the *linear* ordinary differential equation:

$$y'' + \frac{n-1}{r} y' - \omega y = -|E_*|^{2\sigma} y, \tag{32}$$

with $y'(0) = 0$ and $y(0) = E_{*i}$. This ordinary differential equation is equivalent to the following integral equation:

$$\left.\begin{aligned} y(r) &= y(0) + \int_0^r s\,(\omega - |E_*|^{2\sigma})\, y(s) \ln(r/s)\, ds, \\ &\qquad \text{for} \quad n = 2, \\ y(r) &= y(0) + \int_0^r s^{n-1}(\omega - |E_*|^{2\sigma})\, y(s) \frac{1}{2-n} \\ &\quad \times \left(\frac{1}{r^{n-2}} - \frac{1}{s^{n-2}}\right) ds \qquad \text{for} \quad n \geq 3. \end{aligned}\right\} \tag{33}$$

If $f_*(r)$ denotes the solution to (32) with initial conditions $f_*(0) = 1$ and $f_*'(0) = 0$, then, by uniqueness, we have $E_{*i} = \gamma_i\, f_*$, where $\gamma_i = E_{*i}(0)$, $0 \leq i \leq n$, which can be chosen to be positive.

If f_* vanishes at some point r_0, we take

$$\tilde{E} = (\gamma_1 \,|\, f_* \,|\, (r), \ldots, \gamma_n \,|\, f_* \,|\, (r));$$

$\tilde{E}$ is also a minimizer hence $\tilde{E} \in (C^1(\mathbf{R}^n))^n$, it follows that $f_*'(r_0) = 0$.

If $r_0 \neq 0$, Cauchy-Lipschitz uniqueness theorem implies $f_* \equiv 0$ which is a contradiction. If $r_0 = 0$, then we use the integral equation (33) with $y(0) = 0$ and Gronwall's lemma implies $f_* \equiv 0$, hence f_* can not vanish.

The function $f_*(r)$ satisfies the equation

$$f_*'' + \frac{n-1}{r} f_*' - \omega\, f_* + (\gamma \cdot \gamma)^\sigma\, f_*^{2\sigma+1} = 0.$$

If we set

$$f_*(r) = \frac{\omega^{1/2\sigma}}{(\gamma \cdot \gamma)^{1/2}} R(\omega^{1/2}\, r),$$

we then find that $R(\rho)$ satisfies the equation (31). The existence of a unique η_* for which the solution to (31) with initial data $R(0) = \eta_*$ and $R'(0) = 0$ is positive and decreasing was settled in [14]. ■

3.2. The case $\alpha^2 \neq 1$

In this section, we restrict ourselves to dimensions 2 and 3. The first result is:

PROPOSITION 5. – *A minimizer E of $J_\alpha^{\sigma, n}$ can not be of the form $E = (E_i(r))_{i=1 \ldots n}$.*

Proof. – We carry out the computation in the case $n = 3$. The equations satisfied by the components, E_i, are:

$$\begin{aligned} & -\omega\, E_1 + E_{1\,xx} + E_{2\,xy} + E_{3\,xz} \\ & \quad - \alpha^2\,(E_{2\,xy} - E_{1\,yy} - E_{1\,zz} + E_{3\,xz}) = -\,|\,E\,|^{2\,\sigma}\,E_1, \end{aligned} \tag{34}$$

$$\begin{aligned} & -\omega\, E_2 + E_{1\,xy} + E_{2\,yy} + E_{3\,yz} \\ & \quad - \alpha^2\,(E_{1\,xy} - E_{2\,xx} - E_{2\,zz} + E_{3\,yz}) = -\,|\,E\,|^{2\,\sigma}\,E_2, \end{aligned} \tag{35}$$

$$\begin{aligned} & -\omega\, E_3 + E_{1\,zx} + E_{2\,zy} + E_{3\,zz} \\ & \quad - \alpha^2\,(E_{1\,xz} + E_{2\,yz} - E_{3\,yy} - E_{3\,xx}) = -\,|\,E\,|^{2\,\sigma}\,E_3, \end{aligned} \tag{36}$$

Suppose E is a minimizer with $E_i = E_i\,(r)$. Equation (34) then reduces to

$$\begin{aligned} & -\omega\,E_1 + \frac{x^2 + \alpha^2\,(y^2 + z^2)}{r^2}\,E_1'' + \frac{1 + 2\,\alpha^2}{r}\,E_1' - \frac{x^2 + \alpha^2\,(y^2 + z^2)}{r^3}\,E_1' \\ & + (1 - \alpha^2)\left(\frac{xy}{r^2}\,E_2'' - \frac{xy}{r^3}\,E_2'\right) + (1 - \alpha^2)\left(\frac{xz}{r^2}\,E_3'' - \frac{xz}{r^3}\,E_3'\right) \\ & = -\,|\,E\,|^{2\,\sigma}\,E_1. \end{aligned} \tag{37}$$

Letting $x = 0$ in (37) leads to

$$-\omega\,E_1 + \alpha^2\,E_1'' + \frac{1 + 2\,\alpha^2}{r}\,E_1' - \frac{\alpha^2}{r}\,E_1' = -\,|\,E\,|^{2\,\sigma}\,E_1. \tag{38}$$

Letting now $y = z = 0$ in (37) leads to

$$-\omega\,E_1 + E_1'' + \frac{1 + 2\,\alpha^2}{r}\,E_1' - \frac{1}{r}\,E_1' = -\,|\,E\,|^{2\,\sigma}\,E_1. \tag{39}$$

subtracting (39) from (38) gives

$$(\alpha^2 - 1)\left(E_1'' - \frac{1}{r}\,E_1'\right) = 0,$$

so that if $\alpha^2 \neq 1$

$$E_1'' - \frac{1}{r}\,E_1' = 0.$$

Since $E_1 \in L^2$ we conclude $E_1 \equiv 0$. A similar computation can be carried out for the other components of E. ■

THEOREM 2. – *Let us suppose that* $\alpha^2 < \dfrac{\sigma+1}{3} + 1$ *for* $n = 3$ *and* $\alpha^2 < 2\sigma + 3$ *for* $n = 2$, *then* $J_0^\alpha < J_2$.

Remarks. – (i) That $J_0^\alpha < J_2$ was argued on physical grounds in ([26], [6]).

(ii) We expect that the above restrictions on α are not optimal. Numerical simulations of [26] indicate that the result still holds for larger values of α^2 (they take $\alpha^2 = 6$ for $\sigma = 1$ and $n = 2$). The case $\sigma = 1$ and $n = 2$ is a *critical case.* Here, solutions to the initial value problem develop singularities in finite time. These numerical simulations track solutions which are on their way to becoming singular. In [6], the formation of the singularity in solutions to the equation (18) is reported to be roughly self-similar, with a dipole-like profile. In analogy with the formation of singularities in the critical case for scalar nonlinear Schrödinger equations (*see* for example ([27], [15], [23]), one expects that near the singularity the profile is well-approximated by the ground state.

Proof of Theorem 2. – For convenience, we take $\omega = 1$ and let E_* be the ground state which minimize $J_\alpha^{\sigma,n}$. We now calculate the second variation of $J_\alpha^{\sigma,n}$ at the critical point E_*. Since E_* is a minimizer, the following quadratic form must be non-negative:

$$
\begin{aligned}
B(F, F) = & \int |\nabla \cdot F|^2 + \alpha^2 |\nabla \times F|^2 + \int |F|^2 - \int |E_*|^{2\sigma} F^2 \\
& - 2\sigma \int |E_*|^{2\sigma-2} (E_* \cdot F)^2 + \frac{2\sigma}{n \int |E_*|^2} \left(\int E_* \cdot F \right) \\
& \times \left(\int (\nabla \cdot E_*)(\nabla \cdot F) + \alpha^2 (\nabla \times E_*) \cdot (\nabla \times F) \right) \\
& + \frac{1}{n\mu} \left(\int (\nabla \cdot E_*)(\nabla \cdot F) + \alpha^2 (\nabla \times E_*) \cdot (\nabla \times F) \right)^2, \quad (40)
\end{aligned}
$$

where $\mu = \displaystyle\int |\nabla \cdot E_*|^2 + \alpha^2 |\nabla \times E_*|^2$. To see for what values of α and σ we have $J_0^\alpha < J_2$, we assume that $E_* = \nabla g(r)$, and construct admissible trial vector fields F which make the quadratic form attain strictly negative values. The cases of spatial dimension $n = 2$ and $n = 3$ are treated separately.

Note that if $E_* = \nabla g(r) = \dfrac{1}{r} f(r)(x, y, z)$, then $f(r)$ satisfies the ordinary differential equation:

$$f'' + \frac{n-1}{r} f' - f - \frac{n-1}{r^2} f + |f|^{2\sigma} f = 0. \tag{41}$$

(i) Dimension $n = 2$:

From (41), one can derive the Pohozaev type identities (*see* [2], p. 161 for example):

$$\int \left(|f'|^2 + \frac{1}{r^2} |f|^2 \right) r dr = \int |f|^{2\sigma+2} r dr - \int |f|^2 r dr \tag{42}$$

and

$$\int |f|^2 r dr = \frac{1}{\sigma+1} \int |f|^{2\sigma+2} r dr. \tag{43}$$

Suppose $F = \dfrac{f(r)}{r}(y, x)$. Then we find

$$\frac{1}{2\pi} B(F, F) = \frac{1+\alpha^2}{2} \int_0^\infty ((f')^2 + r^{-2} f^2) r dr + \int_0^\infty f^2 r dr \\ - \int_0^\infty |f|^{2\sigma+2} r dr - \sigma \int_0^\infty |f|^{2\sigma+2} r dr.$$

Using (42) and (43) we find

$$B(F, F) = 2\pi \left(\frac{1+\alpha^2}{2} \frac{\sigma}{\sigma+1} + \frac{1}{\sigma+1} - 1 - \sigma \right) \int_0^\infty |f|^{2\sigma+2} r dr.$$

This last expression being negative provided

$$\alpha^2 < 2\sigma + 3.$$

(ii) Dimension $n = 3$:

Suppose $E_* = \dfrac{1}{r} f(r)(x, y, z)$ and take $F = \dfrac{1}{r} f(r)(y, z, x)$. We compute each term:

First we obtain

$$\nabla \cdot F = \left(\frac{f'(r)}{r^2} - \frac{f(r)}{r^3} \right)(xy + yz + zx). \tag{44}$$

On the other hand the components of $\nabla \times F$ are given respectively by:

$$f'(r)\,\frac{xy - z^2}{r^2} + f(r)\left(\frac{z^2 - xy}{r^3} - \frac{1}{r}\right) \tag{45}$$

$$f'(r)\,\frac{yz - x^2}{r^2} + f(r)\left(\frac{x^2 - yz}{r^3} - \frac{1}{r}\right) \tag{46}$$

$$f'(r)\,\frac{xz - y^2}{r^2} + f(r)\left(\frac{y^2 - xz}{r^3} - \frac{1}{r}\right). \tag{47}$$

Moreover we need to compute some integrals on the sphere:

$$\int_{S^2} (xy + yz + xz)^2\, dS^2 = \frac{4\,\pi}{5},$$

$$\int_{S^2} (xy - z^2)^2\, dS^2 = \frac{16\,\pi}{15},$$

$$\int_{S^2} (z^2 - xy - 1)^2\, dS^2 = \frac{12\,\pi}{5},$$

$$\int_{S^2} (xy + yz + xz)^2\,(z^2 - xy - 1)\, dS^2 = \frac{4\,\pi}{15}.$$

Hence we get:

$$\begin{aligned}\int |\nabla \times F|^2 &= 3\int_0^\infty (f')^2\, r^2\, dr\,\frac{16\,\pi}{15} + 3\int_0^\infty f^2\, dr\,\frac{12\,\pi}{5}\\ &\quad + 6\int_0^\infty f\, f'\, r dr\,\frac{4\,\pi}{15},\\ &= \frac{16\,\pi}{5}\left(\int_0^\infty (f')^2\, r^2\, dr + 2\int_0^\infty f^2\, dr\right)\end{aligned}$$

and

$$\int |\nabla . F|^2 = \frac{4\,\pi}{5}\left(\int_0^\infty (f')^2\, r^2\, dr + 2\int_0^\infty f^2\, dr\right).$$

We also remark that

$$\int_{S^2} (xy + yz + xz)\, dS^2 = 0.$$

We finally obtain

$$\begin{aligned} B(F, F) = & \int_0^\infty ((f')^2 + 2r^{-2} f^2)\, r^2\, dr \left(\frac{4\pi}{5} + \alpha^2 \frac{16\pi}{5}\right) \\ & + 4\pi \int_0^\infty f^2 r^2\, dr - 4\pi \int_0^\infty |f|^{2\sigma+2} r^2\, dr \\ & - \frac{4\pi}{5} 2\sigma \int_0^\infty |f|^{2\sigma+2} r^2\, dr. \end{aligned} \tag{48}$$

In the same way as in the two dimension case, this expression reduces to

$$B(F, F) = \frac{8\pi\sigma}{5} \left((\alpha^2 - 1)\frac{3}{\sigma+1} - 1\right) \int_0^\infty |f|^{2\sigma+2} r^2\, dr.$$

This last expression is negative as long as

$$\alpha^2 < \frac{\sigma+1}{3} + 1. \quad \blacksquare$$

Finally, regarding the convergence of J_0^α to J_1, as $\alpha^2 \to \infty$ we have:

THEOREM 3. – *Let E_α be a minimizer of $J_\alpha^{\sigma,n}$ over all vector fields. Then $J_0^\alpha \to J_1$ as $\alpha^2 \to \infty$ and E_α converges to some $\nabla \psi$ which is a minimizer of $J^{\sigma,n}$ over all irrotational vector fields* $(\nabla \times E = 0)$.

The proof of this result is very similar to those of Propositions 2 and 3.

4. DISCUSSION AND SUMMARY

We have a complete description of the ground state of the vector nonlinear Schrödinger equation (1) when $\alpha^2 = 1$. For general α^2, we have shown in dimensions $n = 2$ and $n = 3$ that there is a number $\alpha_*(n, \sigma)$ such that if $\alpha^2 < \alpha_*(n, \sigma)$, then the ground state is not the gradient of a radial function. This was argued on physical grounds and supported numerically in ([26], [6]). (There it was argued that if the minimizer were a smooth radial vector field, then necessarily the corresponding electric field would vanish at the origin, contrary physical expectations.)

Our characterization of the ground state for the case $\alpha^2 = 1$ yields a dynamical stability theorem.

THEOREM 4. – *Suppose that $\alpha^2 = 1$ and $\sigma < 2/n$. Let E_* denote the ground state solution of* (7). *We denote by $P(\theta_1, \ldots, \theta_n)$ the operator that* *consists in multiplying the j-th component of a vector field by* $\exp(i\,\theta_j)$. *Then* E_* *is stable under the flow of* (1) *for* $\alpha = 1$ *in the following sense: For any* $\varepsilon > 0$ *there is a* δ *such that if*

$$E_0 \in H^1, \quad \| E_0 - E_* \|_{H^1} < \delta$$

then

$$\inf_{x_0 \in \mathbf{R}^n,\, L \in SO(\mathbf{R}^n),\, (\theta_j)_{j=1\ldots n} \in \mathbf{R}^n} \| E(\cdot, t) - L(P(\theta_1, \ldots, \theta_n)\, E_*(\cdot - x_0)) \|_{H^1} \leq \varepsilon.$$

The result follows from our characterization of minimizers and from (13) and can be obtained using the technique of [3].

Of physical interest, are the level sets of the function E_*, where E_* is a ground state. These correspond to the level sets of the electric field intensity. For the case of arbitrary H^1 vector fields, a consequence of Theorem 1 is that the minimizer associated with J_0^1 has rotationally symmetric level sets. In the case of irrotational vector fields, it remains an open problem to establish whether one has $J_1 < J_2$. If $J_1 < J_2$, it may be that the level sets of $|E_*|$ or the associated potential φ_*, for which $E_* = \nabla \varphi_*$ corresponding to J_1, are not rotationally symmetric. Numerical simulations ([26], [6]) suggests a dipole structure of level curves associated with the ground state. This would imply a kind of symmetry breaking, as the functionals are invariant under orthogonal transformations in x.

Finally, note that if $E = \nabla f(r)$ is a radial vector field, then f satisfies (41). The techniques of [14] do not apply in this case, so the question of uniqueness of the ground state among radial vector fields associated with (12), is open.

ACKNOWLEDGMENTS

M. I. W. was supported in part by a grant from the U.S. National Science Foundation. He wishes to thank Harvey A. Rose for introducing him to the problems studied in this paper, and the Mittag-Leffler Institute in Djursholm, Sweden, where part of this manuscript was prepared.

Part of this work was done while T. C. was a visitor of the University of Michigan, Ann Arbor. T. C. wishes to thank the Mathematics Department for its hospitality.

REFERENCES

[1] H. BREZIS and E. H. LIEB, Minimum Action Solutions of Some Vector Field Equations, *Commun. Math. Phys.*, Vol. **96**, 1984, pp. 97-113.

[2] T. CAZENAVE, An Introduction to Nonlinear Schrödinger Equations, *Textos de Métodos Matemáticos 26*, Instituto de Matemática-UFRJ Rio de Janeiro, 1993.

[3] T. CAZENAVE and P.-L. LIONS, Orbital Stability of Standing Waves for Some Nonlinear Schrödinger equations, *Comm. Math. Phys.*, Vol. **85**, 1982, pp. 549-561.

[4] T. COLIN, On the Cauchy Problem for a Nonlocal, Nonlinear Schrödinger Equation Occuring in Plasma Physics, *Differential and Integral Equations*, Vol. **6**, n° 6, Nov. 1994, pp. 1431-1450.

[5] T. COLIN, On the Standing Waves Solutions to a Nonlocal, Nonlinear Schrödinger Equation Occuring in Plasma Physics, *Physica D*, **64**, 1993, pp. 215-236.

[6] L. M. DEGTYAREV and V. E. ZAKHAROV, Dipole Character of the Collapse of Langmuir Waves, *JETP Lett.*, **20**, 1974, pp. 164-165.

[7] R. O. DENDY, *Plasma Dynamics*, Oxford University press, 1990.

[8] M. GRILLAKIS, J. SHATAH and W. A. STRAUSS, Stability Theory of Solitary Waves in the Presence of Symmetry I, *J. Func. Anal.*, **74**, 1987, pp. 263-272.

[9] J. GIBBONS, S. G. THORNHILL, M. J. WARDROP and D. TER HARR, On the Theory of Langmuir Solitons, *J. Plasma Phys.*, **17**, 1977, pp. 153-170.

[10] B. GIDAS, W. M. NI and L. NIRENBERG, Symmetry of Positive Solutions of Nonlinear Elliptic Equations in $\mathbf{R}^n$, *Mathematical analysis and applications, Part A, Advances in mathematics supplementary studies*, **7A**, 1981, pp. 369-402.

[11] J. GINIBRE and G. VELO, On a Class of Nonlinear Schrödinger Equations, I the Cauchy Problem, General Case, *J. Func. Anal.*, **32**, 1979, pp. 1-32.

[12] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators*, Vol. III, Springer Verlag.

[13] T. KATO, On Nonlinear Schrödinger Equations, *Ann. Inst. Henri Poincaré, Physique théorique*, Vol. **46**, n° 1, 1987, pp. 113-129.

[14] M. K. KWONG, Uniqueness of Positive Solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$, *Arch. Rat. Mech. Anal.*, **105**, 1991, pp. 583-599.

[15] G. LE MESURIER, G. C. PAPANICOLAOU, C. SULEM and P.-L. SULEM, The Focusing Singularity of the Nonlinear Schrödinger Equation, in Directions in Partial Differential Equations, edited by M. G. CRANDALL, P. H. RABINOWITZ and R. E. TURNER, Academic, New York, 1987, pp. 159-201.

[16] P.-L. LIONS, The Concentration Compactness Principle in the Calculus of Variations. The Locally Compact Case, Part I, *Ann. Inst. Henri Poincaré, Analyse non linéaire*, **1**, n° 2, 1984, pp. 109-145, and Part II, *Ann. Inst. Henri Poincaré, Analyse non linéaire*, **1**, n° 4, 1984, pp. 223-283.

[17] O. LOPES, *Radial Symmetry of Minimizer for some Translation and Rotation Invariant Functionals*, Preprint UNICAMP, Campinas, Brazil, to appear in *Journal of Differential Equations*.

[18] T. OZAWA and Y. TSUTSUMI, Existence and Smoothing Effect of Solutions for the Zakharov Equations, *Publ. RIMS, Kyoto Univ.*, **28**, 1992, pp. 329-361.

[19] S. H. SCHOCHET and M. I. WEINSTEIN, The Nonlinear Schrödinger Limit of the Zakharov Equations Governing Langmuir Turbulence, *Comm. Math. Phys.*, **106**, 1986, pp. 569-580.

[20] W. A. STRAUSS, Existence of Solitary Waves in Higher Dimensions, *Comm. Math. Phys.*, **55**, 1977, pp. 149-162.

[21] F. TRÈVES, *Linear Partial Differential Equations*, Gordon and Breach, 1970.

[22] M. I. WEINSTEIN, Nonlinear Schrödinger Equation and Sharp Interpolation Estimates, *Comm. Math. Phys.*, **87**, 1983, pp. 567-576.

[23] M. I. WEINSTEIN, On the Structure and Formation of Singularities in Solutions of Nonlinear Dispersive Evolution Equations, *Comm. in Partial Diff. Eqns*, **11**, 1986, pp. 545-565.

[24] M. I. WEINSTEIN, Lyapunov Stability of Ground States of Nonlinear Dispersive Evolution Equations, *Commun. Pure Appl. Math.*, **39**, 1986, pp. 51-68.

[25] V. E. ZAKHAROV, S. L. MUSHER and A. M. RUBENCHIK, Hamiltonian Approach to the Description of Nonlinear Plasma Phenomena, *Physics Reports*, **129**, n° 5, 1985, pp. 285-366.

[26] V. E. ZAKHAROV, A. F. MASTRYUKOV and V. S. SYNAKH, Two-Dimensional Collapse of Langmuir Waves, *JETP Lett.*, **20**, n° 1, July 1974.

[27] V. E. ZAKHAROV and V. S. SYNAKH, The Nature of the Self-Focusing Singularity, *Sov. Phys. JETP*, **41**, 1976, pp. 465-468.

(Manuscript received April 25, 1995;
Revised version received July 4, 1995.)