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QUANTUM SPACETIME [1]

by

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ABSTRACT. – We review some recent result and work in progress on the quantum structure of Spacetime at scales comparable with the Planck length; the models discussed here are operationally motivated by the limitations in the accuracy of localization of events in spacetime imposed by the interplay between Quantum Mechanics and classical general relativity.

RÉSUMÉ. – Nous exposons de façon synthétique quelques résultats récents ainsi que des travaux en cours sur la structure quantique de l'Espace-Temps à des échelles comparables à la longueur de Planck; les modèles discutés ici sont motivés d'une façon opérationnelle par la limitation sur la précision de la localisation d'un événement dans l'espace-temps imposée par l'effet conjoint de la Mécanique Quantique et de la Relativité Générale classique.

1. THE BASIC MODEL

According to the Heisenberg principle, if we measure the spacetime coordinates of an event with high precision, we necessarily transfer to the system some energy. According to Einstein's classical theory of gravity,

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this energy generates a gravitational field which, in the case of an extremely precise measurement, would prevent any signal from reaching a far distant observer. Thus probing spacetime at very short distances would cause instability of spacetime itself. The critical length in this connection is Planck length, $\lambda_p^2 = \frac{G\hbar}{c^3}$.

It is therefore *operationally impossible* to localize an event up to a scale smaller than λ_p, otherwise we would put the space region in question out of reach of any observation.

If, however, at least one space coordinate is measured with a poorer accuracy, say with an uncertainty a, while the errors τ, b in time and in the other space coordinates are small, the energy transmitted might spread over the distance a so that, if a is large enough, the associated gravitational field is nowhere too strong. This heuristic argument can be carried through and leads to the relations

$$a\,\tau \gtrsim \lambda_p^2; \qquad ab \gtrsim \lambda_p^2, \tag{1}$$

which motivate the (weaker) Spacetime Uncertainty Relations (STUR)

$$\begin{aligned} \sum_{1\le j<k\le 3} \Delta x_j \cdot \Delta x_k &\gtrsim \lambda_p^2, \\ \Delta x_0 \cdot \sum_{j=1}^{3} \Delta x_j &\gtrsim \lambda_p^2. \end{aligned} \tag{2}$$

These relations have been deduced in a joint research with Klaus Fredenhagen and John E. Roberts where, on this basis, we proposed to view the coordinates of events as (unbounded) *operators* q_μ in (or affiliated to) a *non commutative* C^*-algebra $\mathcal{E}$, which replaces $\mathcal{C}_0(\mathbb{R}^4)$ (the continuous functions vanishing at infinity) and describes Quantum Spacetime (QST) [1, 2, 3].

The basic model of QST studied in [1, 2] is defined by the following *Quantum Conditions* (in units $\hbar = c = G = 1$ so that $\lambda_p = 1$)

$$\begin{aligned} \left(\frac{1}{8}\,\varepsilon^{\mu\nu\lambda\rho}\,[q_\mu, q_\nu]\,[q_\lambda, q_\rho]\right)^2 &= I, \\ [q_\mu, q_\nu] \cdot [q^\mu, q^\nu] &= 0, \\ [q_\mu, [q_\lambda, q_\rho]] &= 0. \end{aligned} \tag{3}$$

These relations appear as the simplest way to implement the STUR (2) in a way which is covariant under the full Poincaré group and gives back the ordinary spacetime $\mathbb{R}^4$ in the large scale limit (*cf.* [1]).

By these relations the closures $Q_{\mu\nu}$ of the $-i\,[q_\mu,\, q_\nu]$ (assumed to be self-adjoint) are *central* operators with joint spectrum lying in the manifold Σ of real antisymmetric 2-tensors σ s.t. (*cf.* [3])

$$\begin{aligned} \sigma \cdot \sigma &= 0, \\ \frac{1}{4}\,\sigma \cdot *\sigma &= \pm 1, \end{aligned} \tag{4}$$

where the dot indicates contraction and * the Hodge dual. If we specify the antisymmetric tensor σ by its "electric" and "magnetic" components, $\sigma = (\vec{e},\, \vec{m})\,(e_j = \sigma_{0j},\, m_j = \sigma_{kl},\, (j\,k\,l)$ a cyclic permutation of $(123))$, we have equivalently

$$\vec{e}^{\,2} = \vec{m}^2, \qquad \vec{e} \cdot \vec{m} = \pm 1. \tag{5}$$

Accordingly, Σ is a two sheeted manifold $\Sigma = \Sigma_+ \cup \Sigma_-$, where $\Sigma_+ \sim \Sigma_- \sim SL\,(2,\, \mathbb{C})/\text{diagonal} \sim TS^2$, the tangent manifold of the unit sphere in $\mathbb{R}^3$ (*cf.* [1]).

If ω denotes a state in the domain of the commutators, by a well-known consequence of Schwarz inequality

$$\Delta_\omega\, q_\mu \cdot \Delta_\omega\, q_\nu \geq \frac{1}{2}\,|\omega\,(Q_{\mu\nu})|$$

and if ω is pure on the centre $\omega\,(Q_{\mu\nu}) = \sigma_{\mu\nu},\ \sigma \in \Sigma$, so that

$$(\Delta_\omega\, q_0)^2 \sum_{j=1}^{3} (\Delta_\omega\, q_j)^2 \geq \frac{1}{4}\,\vec{e}^{\,2},$$

$$\sum_{1 \leq j < k \leq 3} (\Delta_\omega\, q_j)^2\,(\Delta_\omega\, q_k)^2 \geq \frac{1}{4}\,\vec{m}^2,$$

where $\sigma = (\vec{e},\, \vec{m})$. Since by (5) $\vec{e}^{\,2} = \vec{m}^2 \geq 1$, we get the relations (2) for states which are definite on the $Q_{\mu\nu}$'s. By reduction theory, the STUR then hold for each state [1, 2].

While (3) do imply (2), they do *not* imply (1): a representation of (3) where $Q_{02} = -Q_{20} = Q_{31} = -Q_{13} = I$ and where the other components vanish is provided by setting

$$\begin{cases} q_0 = I \otimes q, & q_2 = I \otimes p, \\ q_1 = p \otimes I, & q_3 = q \otimes I, \end{cases} \tag{6}$$

where q, p are the Schroedinger operators for a particle in one dimension.

Accordingly for each $\varepsilon > 0$ there is a state ω in this representation s.t.

$$\Delta_\omega\, q_1 = \Delta_\omega\, q_3 = \frac{1}{\sqrt{2}}, \qquad \Delta_\omega\, q_2 < \varepsilon. \tag{7}$$

We will discuss in the next section a deformation of the basic model where such states are no longer admissible.

The state ω_0 in the representation (6) s.t. $\omega_0(q_\mu) = 0$ and where $\Delta q_\mu = \frac{1}{\sqrt{2}}$, $\mu = 0, ..., 3$, minimizes the quantity $\Sigma_{\mu=0}^{(3)} (\Delta q_\mu)^2$. Any state with this property can be shown to be, up to a translation, a mixture of the transforms of ω_0 under space rotations. Thus states with *optimal localization* are concentrated on the unit ball $\Sigma^{(1)}$ of Σ, $\Sigma^{(1)} = \{(\vec{e}, \vec{m}) \in \Sigma / \vec{e}^{\,2} = \vec{m}^2 = 1\}$; the sign of $\vec{e} \cdot \vec{m} = \pm 1$ (*i.e.* $\vec{e} = \pm \vec{m}$ for $(\vec{e}, \vec{m}) \in \Sigma^{(1)}$) splits $\Sigma^{(1)}$ into $\Sigma_+^{(1)} \cup \Sigma_-^{(1)}$, $\Sigma_\pm^{(1)} \sim S^2$, the base space of $TS^2 \sim \Sigma_\pm$.

As the phase space of a Schroendinger particle, spacetime thus subdivides into cells of volume $(2\pi)^2 \lambda_p^4$ (in a different context, *cf.* also [10]).

By definition, the C^**-algebra* $\mathcal{E}$ is associated to the *regular* representations of (3) for which the "Weyl relations" hold:

$$e^{i\alpha q} e^{i\beta p} = e^{-\frac{1}{2} i\alpha Q\beta} e^{i(\alpha+\beta) q}, \qquad \alpha, \beta \in \mathbb{R}^4. \tag{8}$$

This C^*-algebra is the completion of the $*$-subalgebra spanned by

$$\int f(Q, \alpha) e^{i\alpha q} d^4 \alpha, \qquad f \in \mathcal{C}_0(\Sigma) \otimes L^1(\mathbb{R}^4);$$

It can be shown that $\mathcal{E}$ is isomorphic to $\mathcal{C}_0(\Sigma) \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of all compact operators on a fixed infinite dimensional separable Hilbert space [1]. It can be viewed as a strict deformation quantization in the sense of Rieffel [12].

In the large scale limit $(\lambda_p \to 0)$, $\mathcal{E}$ deforms to $\mathcal{C}_0(\Sigma) \otimes \mathcal{C}_0(\mathbb{R}^4)$, *i.e.* the QST deforms to the product $\mathbb{R}^4 \times \{\pm\} \times \Sigma_+$.

If we limit our attention to *optimally localized states,* only is relevant the submanifold

$$\mathbb{R}^4 \times \{\pm\} \times S^2 \tag{9}$$

where, in generic units, we must think of S^2 as the sphere of radius λ_p^2 in $\mathbb{R}^3$.

The classical limit (9) shows an unexpected relation to Alain Connes' theory of the Standard Model [4, 5], described by the non commutative geometry of a product manifold of $\mathbb{R}^4$ and a finite discrete space.

The discrete space as a factor in the classical limit of QST is not a peculiarity of our basic model; it appears rather as a generic feature of spacetime quantization. For, if we consider general relations

$$[q_\mu, q_\nu] = i\, Q_{\mu\nu} \tag{10}$$

restricted only by covariance under the *full* Poincaré group, the $Q_{\mu\nu}$ will be invariant under the transformations

$$q_\mu \rightarrow \pm q_\mu + a_\mu \cdot I \tag{11}$$

hence will be acted upon by the two component quotient of the full Lorentz group modulo total reflection. The same group acts on the spectrum Ω of the centre of the algebra generated by the $Q_{\mu\nu}$'s and Ω will be two-sheeted in general; if the basic algebra is a field of Lie algebras over Ω, Ω will survive the classical limit appearing as a factor.

The full reflection $q_\mu \rightarrow -q_\mu$ might fail to leave the basic commutation relations invariant if their form a Lie algebra (with central operators as structure coefficients) with other generators besides the q_μ's, including e.g. a Lorentz invariant R (*cf.* next section). In this case the discrete factor in the classical limit would be $\mathbb{Z}_4$.

The root of the discrete factor in (7) is therefore the invariance under the *full* Poincaré group $\mathcal{P}$.

Poincaré transformations are clearly symmetries of (3) and act as automorphisms τ of $\mathcal{E}$ s.t.

$$\tau_{(a,\Lambda)^{-1}}(q) = \Lambda q + a \cdot I.$$

We may thus say that spacetime is quantized but its group of global motions is still the *classical* group $\mathcal{P}$. This must be the case since the global motions ought to act in the same way at small as well as at large scales, and reduce to the classical group in the last case, where the spacetime looks again classical.

A similar situation occurs in nonrelativistic Quantum Mechanics, where the quantum variables $\vec{q}$, $\vec{p}$ are acted upon by the *classical* Galilei group

$$\vec{q} \rightarrow \vec{q} + \vec{a} \cdot I, \qquad \vec{p} \rightarrow \vec{p} + m\vec{v} \cdot I.$$

In the formulation of QFT over QST we may then retain Wigner's notion of elementary particles. Another ingredient is calculus on QST. Functions of the quantum variables q_μ can be defined à la von Neumann-Wigner-Moyal by setting

$$f(q) = \int \check{f}(\alpha)\, e^{i\alpha q}\, d^4\alpha \tag{12}$$

where $\check{f}$ is the inverse Fourier transform of f. Differentiation and integration on $\mathcal{E}$ can be defined in a consistent way (*cf.* [1]) and allow us to take the first steps into perturbative Quantum Field Theory on QST. The result is a *non ad hoc regularization*: QST replaces a local interaction Hamiltonian by a *non local effective interaction* on ordinary spacetime. One such procedure is described in [1], but it is not unique; other, equally standard procedures lead to more drastic regularizations [6].

We close this section with a comment on our STUR (2). In the case of a single particle, neglecting its rest mass, we may take the Heisenberg time-energy uncertainty relation to give the order of Δq_0:

$$\Delta q_0 \sim \frac{1}{\Delta p}; \tag{13}$$

hence by (2) and (13)

$$\sum_{j=1}^{3} \Delta q_j \gtrsim \Delta p;$$

combining with Heisenberg's relation $\Delta q_j \gtrsim \frac{1}{\Delta p_j}$ we deduce that a particle on QST must obey

$$\sum_{j=1}^{3} \Delta q_j \gtrsim \sum_{j=1}^{3} \frac{1}{\Delta p_j} + \Delta p. \tag{14}$$

A relation of this kind is known to follow from Mead's analysis of the Heisenberg microscope experiment if one takes into account the (classical) gravitational force between photon and electron (*cf.* [7]); it also follows from Quantum black hole physics [8], or from String Theory [9].

Such relations have also been taken as a basis to study deformations of the Heisenberg algebra obeyed by the p, q which do imply them ([8]; *cf.* also [13]). These deformations are related to Quantum deformations of the Poincaré algebra [8, 11]. Our approach differs markedly: it is spacetime itself, rather than the Heisenberg algebra of position and momentum of a particle, to be quantized, while the Poincaré group, as argued earlier on, is not quantized.

Limitations to the accuracy of localization result also from Ashtekar's approach to quantum gravity [14].

2. DEFORMATIONS OF QUANTUM SPACETIME

The material of this section is based on joint work with K. Fredenhagen [6].

The basic model of QST has been singled out by some criteria:

(i) the quantum structure should imply the STUR (2);

(ii) invariance under the full Poincaré group;

(iii) ordinary Minkowski space should appear in the large scale limit of QST;

(iv) simplicity: the "degree of non commutativity" was minimal.

If we release the last requirement (iv), there is room for other models where the $Q_{\mu,\nu} = -i\,[q_\mu,\, q_\nu]$ are no longer central. We are interested in other models where the stronger STUR (1) might be implemented as well.

We wish to retain the possibility of a reasonable functional calculus on the quantum variables q_μ given by (8). As in the basic model, we look for a field of Lie algebras over some base space Ω, which, by (ii) and a minimality requirement, will carry a transitive action of the full Poincaré group.

Factorial representations (where the centre is mapped to $\mathbb{C}$) will each select a point in Ω and will be called *regular* if they are integrable to a unitary representation of the associated simply connected Lie group (*Generalized Weyl relations*). General *regular representations* are then defined by reduction theory. By an argument in [1, 2], if the STUR are fulfilled by each factorial representation, they will be fulfilled in general.

In the basic model the base space was Σ and the Lie algebra at $\sigma \in \Sigma$ was the Heisenberg algebra

$$[q_\mu,\, q_\nu] = i\,\sigma_{\mu\nu}\,\cdot\,I. \tag{15}$$

We consider the following deformation of (15):

$$[q_\mu,\, q_\nu] = i\,(\sigma_{\mu\nu}\,\cdot\,I + \tau_{\mu\nu}\,R) \tag{16.1}$$

$$[q_\mu,\, R] = ic_\mu\, S \tag{16.2}$$

$$[q_\mu,\, S] = -ic_\mu\, R \tag{16.3}$$

$$[R,\, S] = 0 \tag{16.4}$$

$$I \text{ is central} \tag{16.5}$$

where $\sigma \in \Sigma$ and by Jacoby identities

$$c_\mu\,\tau_{\nu\lambda} + c_\nu\,\tau_{\lambda\mu} + c_\lambda\,\tau_{\mu\nu} = 0. \tag{17}$$

(Of course, (16.1) could be closed to a Lie algebra differently: e.g., replacing (16.2-4) by $[q_\mu,\, R] = ic_\mu\,\cdot\,I$).

The generators R, S are taken to be scalars under the full Poincaré group while σ, τ will be antisymmetric 2–tensors (invariant under (11)) and c a vector. The manifold Ω is here the Lorentz orbit of the triple $(\sigma,\, \tau,\, c)$.

But when are the STUR (2) implemented? The argument in section 1 shows that the following *general quantum condition* suffices: if

$$[q_\mu, q_\nu] = i\,Q_{\mu\nu}$$

for each factorial state ω we must have

$$\{\omega\,(Q_{\mu\nu});\ \mu,\ \nu = 0, \ldots, 3\} \in \Sigma. \tag{18}$$

In the case of (16), the left hand side of (18) is $\sigma + \omega\,(R)\,\tau$, hence the general quantum condition is fulfilled if

$$\sigma + \lambda\tau \in \Sigma, \qquad \lambda \in \mathbb{R}. \tag{19}$$

For any antisymmetric two tensors τ_1, τ_2, $\tau_1 \perp \tau_2$ will mean $\tau_{1\,\mu\nu}\,\tau_2^{\mu\nu} = \tau_{1\,\mu\nu}\,(*\tau_2)^{\mu\nu} = 0$.

Let Σ_0 denote the manifold of real antisymmetric 2-tensors τ s.t. $\tau \perp \tau$ [1]. Then (σ, τ) fulfill (19) if and only if

$$\sigma \in \Sigma; \qquad \tau \in \Sigma_0; \qquad \sigma \perp \tau. \tag{20}$$

The Jacobi identity (17) is fulfilled if, for some vector d,

$$\tau_{\mu\nu} = c_\mu\, d_\nu - c_\nu\, d_\mu. \tag{21}$$

To find solutions of (20), (21) for a given $\sigma \in \Sigma$ choose first a Lorentz frame where $\sigma = (\vec{u}, \vec{u})$, so that $\sigma \in \Sigma_+^{(1)}$. If $\tau = (\vec{v}, \vec{w})$, (20) says that $\vec{v}^2 = \vec{w}^2$ and the triple $(\vec{u}, \vec{v}, \vec{w})$ is orthogonal. Thus any solution must be a Lorentz transform of a pair

$$\sigma = (\vec{e}_2, \vec{e}_2), \qquad \tau = a\,(\vec{e}_3, \vec{e}_1), \tag{22}$$

with some $a \in \mathbb{R}$. The choice

$$\begin{aligned} c &\equiv (c_0, c_1, c_2, c_3) = (c_0, 0, c_0, 1) \\ d &= \lambda\,(0, 0, 0, 1) \end{aligned} \tag{23}$$

gives τ as in (22) with $a = \lambda\, c_0$. Thus assigning the pair (σ, τ) amounts to assigning a "vierbein".

As in the case of Heisenberg relations, we are interested only in representations which are *regular*, *i.e.* obey generalized Weyl relations (≡integrable to a unitary representation of the associated simply connected Lie group) and *unital*, *i.e.* they assign the identity to the generator I (in the Heisenberg case, this means fixing the value of $\hbar$).

There is another parameter, similarly associated to irreducible representations of (16). The operator $R^2 + S^2$ is central, hence in irreducible representations

$$R^2 + S^2 = z^2 \cdot I \tag{24}$$

where $z \in [0, +\infty)$ is the "weight" of the representation. An irreducible representation with weight $z \neq 0$ of (16) at the values (λ, c_0) of the parameters in (23) determines also a representation *with weight one* of (16) at the values (λ, zc_0). If the weight is zero of course we have a representation of (15). For this reason we may restrict attention to representation *with weight one.*

The STUR (2) are obeyed by construction in the model defined by (16), (21), (22), (23). In the basic model, at the spot $\sigma \in \Sigma$, $\sigma = (\vec{e}_2, \vec{e}_2)$, we had

$$[q_1, q_3] = -i I \tag{25}$$

while q_2 commutes with q_1, q_3; this allowed us to find states localized in a lens region with thickness as small as we want in the 2-direction and radius ~ 1 (*cf.* (7)). In the present model (25) holds too but q_2 does not commute with q_1, q_3; if we choose a state ω, induced by a unit vector ξ in a Hilbert space representation, s.t.

$$(\xi, q_j \xi) = \omega(q_j) = 0, \qquad j = 1, 2, 3,$$

we then have

$$\sum_{j=1}^{3} (\Delta_\omega q_j)^2 = 1 + \|(q_1 - iq_3)\xi\|^2 + \|q_2 \xi\|^2 \tag{26}$$

and we may again choose ω so that $(q_1 - iq_3)\xi = 0$, *i.e.*

$$\Delta_\omega q_1 = \Delta_\omega q_3 = \frac{1}{\sqrt{2}} \tag{27}$$

and *optimal localization in space* amounts to letting $\Delta_\omega q_2$ approach its infimum under the constraint (27) for ω. Our main result can be then formulated as the

THEOREM [6] (i). – *The regular unital representations with weight one of* (16) (*with the specifications* (21), (22), (23)), *such that the operators* q_μ, $\mu = 0, \ldots, 3$ *form an irreducible family, fulfill* $R - iS = be^{-i\sigma_{\mu\nu} q^\mu c^\nu}$, *with* $b \in \mathbb{T}$. *Up to unitary equivalence, there is exactly one such representation for each* $b \in \mathbb{T}$. *Translation invariance hold in these representations only for translations a s.t.* $a^\mu \sigma_{\mu\nu} c^\nu \in 2\pi\mathbb{Z}$, i.e. *it is spontaneously broken along the direction* $\sigma_{\mu\nu} c^\nu$ *at Planck scale.*

(ii) *States in these representations fulfilling* (27) *satisfy the bound*

$$(\Delta q_2)^2 \geq c_0^2 \left(\mu \frac{\sqrt{e} - 1}{2} + \varepsilon_0(\mu) \right) \tag{28}$$

where $\mu = \lambda^2 \frac{1}{\sqrt{e}} \left(1 - \frac{1}{\sqrt{e}} \right)$ *and* $\varepsilon_0(\mu)$ *denotes the ground energy level of the one dimensional Schroedinger operator* $P^2 + \mu^2 \cos^2 Q$. *The bound* (28) *is optimal and as* Δq_2 *approaches that* inf, Δq_0 *tends to infinity.*

The two free parameters λ, c_0 in the model could in principle be fixed by requiring that

$$\inf_{\omega} \{ (\inf_{j=1,2,3} \Delta_\omega q_j) \cdot (\sup_{j=1,2,3} \Delta_\omega q_j) \} = \frac{1}{2}, \tag{29}$$

$$\inf_{\omega} \sum_{\mu=0}^{3} (\Delta_\omega q_\mu)^2 \text{ is minimal}, \tag{30}$$

where the second quantity is always bounded below by 2.

3. CONCLUSIONS

The Quantum Conditions in their general form (18) do imply the STUR (2) but allow different models of QST besides the basic model, where (1) are fulfilled as well. The merits of these models should be judged by developing first the *classical* theory of gravity over QST. A fully consistent model should have gravitational *stability* against optimal localization of events. This would provide a better underlying geometry to develop, if possible, Quantum Gravity as a QFT over QST.

At a less ambitions level, the quantum structure of Spacetime acts as a non *ad hoc* regularization of any QFT [1, 6].

REFERENCES

[1] S. DOPLICHER, K. FREDENHAGEN, and J. E. ROBERTS, The quantum structure of spacetime at the Planck scale and quantum fields, *Commun. Math. Phys.*, **172**, 1995, pp. 187-220.
[2] S. DOPLICHER, K. FREDENHAGEN, J. E. ROBERTS, Spacetime quantization induced by classical gravity, *Phys. Lett.*, **B331**, 1994, pp. 39-44.
[3] S. DOPLICHER, *Quantum physics, classical gravity, and non-commutative spacetime*, Proceedings of the ICMP, Paris, July 18-23, 1994; International Press, 1995.
[4] A. CONNES, *Non-commutative geometry*, Academic Press, 1994.
[5] A. CONNES, Non-commutative geometry and reality, *Journal of Math. Phys.*, **36**, n° 11, 1995.
[6] S. DOPLICHER, K. FREDENHAGEN (in preparation).
[7] L. GARAY, Quantum gravity and minimum length, *Int. Journ. Mod. Phys.*, **A10**, 1995, pp. 145-165.
[8] M. MAGGIORE, Quantum groups, gravity, and the generalized uncertainty principle, *Phys. Rev.*, **D49**, 1994, pp. 5182-5187.
[9] G. VENEZIANO, *Europhys. Lett.*, **2**, 1986, p. 199; D. AMATI, M. CIAFALONI and G. VENEZIANO, *Nucl. Phys.*, **B347**, 1990, p. 551.
[10] J. MADORE, Fuzzy physics, *Ann. Phys.*, **219**, 1992, pp. 187-198.
[11] J. LUKIERSKI, A. NOWICKI and H. RUEGG, New quantum Poincaré algebra and k-deformed field theory, *Phys. Lett.*, **B293**, 1992, pp. 344-352.
[12] M. RIEFFEL, Quantization and C^*-algebras, *Contemporary Math.*, **167**, 1994, pp. 67-96.

[13] A. Kempf, G. Mangano and R. B. Mann, Hilbert space representations of the minimal uncertainty relations (preprint).

[14] A. Ashtekar, Quantum gravity, a mathematical physics perspective, in Mathematical Physics towards the 21st Century, R. Sen and A. Gersten eds, Ben Gourion University of Neghev Press, 1994.

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