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Axiomatic analyticity properties and representations of particles in thermal quantum field theory

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ABSTRACT. – We provide an axiomatic framework for Quantum Field Theory at finite temperature which implies the existence of general analyticity properties of the n-point functions; the latter parallel the properties derived from the usual Wightman axioms in the vacuum representation of Quantum Field Theory. Complete results are given for the propagators, including a generalization of the Källén-Lehmann representation. Some known examples of "hard-thermal-loop calculations" and the representation of "quasiparticles" are discussed in this general framework.

RÉSUMÉ. – Nous présentons un cadre axiomatique pour la Théorie Quantique des Champs à température finie qui implique l'existence de propriétés générales d'analyticité des fonctions à n points; celles-ci forment un parallèle avec les propriétés découlant des axiomes de Wightman habituels pour la représentation du vide de la Théorie Quantique des Champs. Nous donnons des résultats complets pour les propagateurs, incluant une généralisation de la représentation de Källén-Lehmann, et nous discutons quelques exemples de « calculs de boucles à haute température » ainsi que la représentation des « quasiparticules » dans ce cadre général.

1. INTRODUCTION

In the past fifteen years, there has been an increasing interest for the physics of media at very high temperature; in the latter, the basic quantum fields of matter are supposed to manifest themselves through new properties which highly deserve to be investigated from a theoretical viewpoint. In particular, the so-called "quark-gluon plasma" has been the object of a number of theoretical studies and computations [1-3]. These are generally based on an adaptation of the standard methods of quantum statistical mechanics to the relativistic system consisting of the basic fields of quantum chromodynamics. The fields are supposed to be in an equilibrium state of infinite volume at temperature $T = 1/\beta$ in a certain privileged Lorentz frame; the latter fixes the time and space variables $(x_0, \vec{x})$ and the corresponding Fourier conjugate variables, namely the energy and momentum $(\omega, \vec{p})$.

The use of the Matsubara *imaginary-time formalism* (ITF) (*see* e.g. [4] and references therein) has resulted in computations involving discrete imaginary-energy summations for obtaining perturbative approximations of the retarded propagators of the fields in the space of (complex) energy and (real) momentum variables. In this formalism, the *quasiparticles* are associated with "modes" obeying a (real or complex) "dispersion law" $\omega = f(\vec{p})$, which appear as poles of the form $Z(\vec{p})/[\omega - f(\vec{p})]$ in the retarded propagators of the fields. Such a structure has been displayed in the so-called "hard-thermal-loop calculations" which aim to exhibit the very high temperature behaviour of a quark-gluon plasma; summing the one-loop self-energy contributions in a leading approximation at large T yields the following result: the gluon propagator exhibits two typical poles at $\omega = f_t(\vec{p})$ and $\omega = f_l(\vec{p})$, interpreted respectively as "transverse and longitudinal plasmon modes"[5,6].

The independent elaboration of a formalism which only involves *real-time quantities* (RTF) has resulted in a double-field matrix formulation, now called *thermo-field dynamics* (*see* e.g. [7] and references therein). This approach introduces as preferable basic functions the time-ordered and anti-time-ordered expectation values.

In the past years, there has been a long debate about the consistency of the ITF and RTF approaches, the latter being both expressed in terms of appropriate versions of the path-integral formalism; the controversy has generally been set in this framework, where genuine subtleties appear in handling "time-ordered paths" in the complex plane of the time-variable (*see* e.g. [8]). Although the situation now seems to have been clarified in favour of the consistency [9], one feels a real need for a general structural study of the thermal Green functions going beyond the time-ordered path technique.

In the present work, we adopt such a general (non-perturbative) viewpoint on Thermal Quantum Field Theory: we suggest that the validity and consistency of the ITF and RTF formalisms are by-products of model-independent structural properties of the thermal n-point functions which follow rigorously from an appropriate set of general physical principles. As a matter of fact, we will show that one can define an axiomatic program for studying the analytic and algebraic properties of the n-point functions of fields in a thermal equilibrium state. This program is completely similar to the one which has been developped in the sixties for Quantum Field Theory in the vacuum state within the Wightman axiomatic framework and which has led to such important structural insights as the PCT and Spin-Statistics theorems, collision theory, dispersion relations and the Osterwalder-Schrader theorem (*see* [10-13] and references therein).

The main discrepancy of our axiomatic starting point with respect to the familiar Wightman axioms will consist in releasing the Lorentz covariance properties and replacing the *spectral condition* by an appropriate formulation of the *KMS-condition* : this idea relies on the basic analysis of [14], completed by the results of [15,16] which display the KMS-property as being a general criterion for systems in equilibrium states. Moreover, in spite of the breaking of Lorentz covariance due to the thermal bath, the basic role played by the causality cone $V^+ = \{x = (x_0, \vec{x}); x_0 > |\vec{x}|\}$ for quantum field systems implies that a (stronger) *relativistic form of the KMS-condition* can be justified [17], as a remnant of the *relativistic spectral condition.*

In this preliminary work, we shall present substantial results for the structure of the two-point functions; only partial results will be given as far as the general program of n-point functions is concerned. However, the case of the thermal two-point functions being already of physical interest, we give a special importance to a Källén-Lehmann-type representation (already presented in [18, 19]) which we are able to derive from the general principles; among other advantages, this representation opens a new possibility for the characterization of particles in Thermal Quantum Field Theory.

After having presented our axiomatic framework for the representations of Quantum Field Theory in thermal equilibrium states in Sec. 2, we devote Sec. 3 and 4 to the pure implications of locality: this part of the program reproduces in the setting of Thermal Quantum Field Theory properties of general field theory in the vacuum state. It concerns:

a) in Sec. 3, the basic analyticity properties of n-point Green functions in the space of complex energy and momentum variables,

b) in Sec. 4, the derivation of Källén-Lehmann-type integral representations for the two-point commutator and retarded functions which are valid *in the absence of Lorentz covariance and of spectral condition.*

The implications of the KMS-condition will be studied in Sec. 5. In particular, we shall exhibit the resulting *double analytic structure* of the two-point functions in the time and energy variables and the relations between the Fourier transforms of the retarded and time-ordered functions which replace the usual "coincidence relations" implied by the spectral condition. The derivation of similar properties for the n-point functions will only be initiated there and mentioned as a further important part of our program to be implemented.

In Sec. 6, the following complements on the structure of two-point functions will be presented:

a) incorporating the KMS-condition into the Källén-Lehmann-type formula of Sec. 4 in order to provide a corresponding representation for the two-point correlation function itself,

b) completing the results of Sec. 5 by analyticity properties in the *complex space and time variables* which result from our relativistic form of the KMS-condition,

c) studying the basic Feynman-type operations on two-point functions and illustrating all the previous results on the perturbative examples mentioned at the beginning of this introduction,

d) giving a comparative discussion of two different characterizations of the notion of particle in Thermal Quantum Field Theory.

2. THE AXIOMATIC FRAMEWORK

The adaptation of the Wightman axiomatic framework to the case of quantum fields in a thermal equilibrium state Ω_β of temperature $T = 1/\beta$ relies on the following well-known ideas. For simplicity, we consider the case of a single hermitian field $\phi(x)$, namely a system of "observables" $\phi(f) = \int \phi(x) f(x) dx$ depending continuously on test-functions f on Minkowski space (taken for convenience in the Schwartz space $\mathcal{S}(\mathbb{R}^4)$). These observables form an algebra which satisfies the general axiom of "locality" or "local commutativity", namely: $[\phi(f), \phi(g)] = 0$ for all pairs of test-functions (f, g) whose supports are mutually space-like separated in $\mathbb{R}^4$.

This axiom, which expresses the principle of Einstein causality, is independent of the *representation* in which the system is described; it therefore holds in the Hilbert space $\mathcal{H}_\beta$ of thermal states in which the field observables $\phi(f)$ are supposed to act, as operators defined on a suitable dense domain generated by the state vector Ω_β. It is understood that Ω_β determines a fixed Lorentz frame, *i.e.* a distinguished set of time and space variables; the corresponding unitary representation U_t of the time-translation group in $\mathcal{H}_\beta$ (*i.e.* the evolution operator group) leaves Ω_β invariant, since an equilibrium state is stationary. We shall only consider here the case when Ω_β is also invariant under the (unitary) representation of space-translations $U_{\vec{a}}$ in $\mathcal{H}_\beta$. The action of the space and time-translations on the field is, as usual: $\phi(x_0 + t, \vec{x} + \vec{a}) = U_t U_{\vec{a}} \phi(x_0, \vec{x}) U_{\vec{a}}^{-1} U_t^{-1}$. The previous axioms do not differ from the corresponding Wightman axioms in the Hilbert space $\mathcal{H}_{vac}$ generated by the vacuum state Ω_{vac}; however, there are no unitary operators which implement Lorentz transformations in $\mathcal{H}_\beta$.

In order to complete our axiomatic framework in $\mathcal{H}_\beta$, we shall introduce the "correlation functions" or "Wightman functions" $\mathcal{W}_n^{(\beta)}(x_1, \ldots, x_n) = < \Omega_\beta, \phi(x_1) \cdots \phi(x_n) \Omega_\beta >$ and adopt the viewpoint of the reconstruction theorem [10]. A Thermal Quantum Field Theory is thus entirely specified by the knowledge of the corresponding set of tempered distributions $\{\mathcal{W}_n^{(\beta)}; n \in \mathbb{N}\}$ which have to satisfy the usual *positivity conditions* of Wightman functions [10]. (It can equivalently be presented as *the representation associated with a certain positive functional* on the Borchers-Uhlmann algebra [20,21] of terminating sequences of test-functions $\{f_0, f_1(x_1), \ldots, f_n(x_1, \ldots, x_n), \ldots, 0, \ldots\}$.)

Apart from the properties which express the previous axioms, namely:

i) Locality:

$$\mathcal{W}_n^{(\beta)}(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = \mathcal{W}_n^{(\beta)}(x_1, \ldots, x_{i+1}, x_i \ldots, x_n)$$

for $x_i - x_{i+1} \notin \overline{V^+} \cup \overline{V^-}$ with $V^- = -V^+$,

ii) Translation Invariance:

$$\mathcal{W}_n^{(\beta)}(x_1, x_2, \ldots, x_n) = \mathcal{W}_n^{(\beta)}(x_1 + a, x_2 + a, \ldots, x_n + a)$$

for all a in $\mathbb{R}^4$, the thermal Wightman functions should satisfy the following analyticity properties which replace those implied by the spectral condition in the vacuum case. These properties are the mathematical expression of the fact that the state Ω_β is in thermal equilibrium [14-16].

iii) KMS-condition:

Let e_0 be the unit vector of the time-axis. For every $n \in \mathbb{N}$ and every pair (I, J) of *ordered sets* $I = \{1, \ldots, m\}, J = \{m+1, \ldots, n\}$, there exists a tempered distribution $F_{IJ}(x_1, \ldots, x_n; t)$ which is holomorphic with respect to the complex variable t in the strip $-\beta < Im t < 0$ and admits distribution boundary values on $\{t \in R\}$ and $\{t \in R - i\beta\}$ (still denoted by F_{IJ}) satisfying the following conditions:

$$F_{IJ}(x_1, \ldots, x_n; t) = \mathcal{W}_n^{(\beta)}(\{x_i + te_0\}_{i \in I}, \{x_j\}_{j \in J}), \tag{1}$$

$$F_{IJ}(x_1, \ldots, x_n; t - i\beta) = \mathcal{W}_n^{(\beta)}(\{x_j\}_{j \in J}, \{x_i + te_0\}_{i \in I}). \tag{2}$$

Let us introduce the Fourier transforms of the Wightman functions, which (in view of ii)) can be written as $\hat{\mathcal{W}}_n^{(\beta)}(p_1, \ldots, p_n)\ \delta(p_1 + \cdots + p_n)$, with $p_l = (\omega_l, \vec{p}_l), 1 \leq l \leq n$. Then the distributions $\hat{\mathcal{W}}_n^{(\beta)}$ on the linear manifold $p_1 + \cdots + p_n = 0$ satisfy the following property which is equivalent to iii):

iii)' KMS-condition in the energy variable:

For each pair (I, J), the following identity holds:

$$\hat{\mathcal{W}}_n^{(\beta)}(J, I) = e^{-\beta\omega_I} \hat{\mathcal{W}}_n^{(\beta)}(I, J), \tag{3}$$

where $\omega_I = \sum_{i \in I} \omega_i$ and $\hat{\mathcal{W}}_n^{(\beta)}(I, J)$ stands for $\hat{\mathcal{W}}_n^{(\beta)}(\{p_i\}_{i \in I}, \{p_j\}_{j \in J})$.

Remark. – If we introduce the commutator functions

$$\hat{\mathcal{C}}_n^{(\beta)}(I, J) = \hat{\mathcal{W}}_n^{(\beta)}(I, J) - \hat{\mathcal{W}}_n^{(\beta)}(J, I) \tag{4}$$

it follows from Eq. (3) that $\hat{\mathcal{W}}_n^{(\beta)}$ is "essentially" determined from $\hat{\mathcal{C}}_n^{(\beta)}$ (*i.e.* determined up to a part which factorizes $\delta(\omega_I)$) by the formula:

$$(1 - e^{-\beta\omega_I})\hat{\mathcal{W}}_n^{(\beta)}(I, J) = \hat{\mathcal{C}}_n^{(\beta)}(I, J). \tag{5}$$

We notice that this formula replaces the usual one expressing the positivity of the energy (or *spectral condition*) it in the vacuum case:

$$\hat{\mathcal{W}}_n^{(vac)}(I, J) = \theta(\omega_I)\hat{\mathcal{C}}_n^{(vac)}(I, J) \tag{6}$$

(θ denoting the Heaviside step-function).

In the limit of zero temperature ($\beta \to \infty$), Eq. (5) tends to Eq. (6); the occurrence of the "inverse Bose-Einstein factor" $(1 - e^{-\beta\omega_I})$ in Eq. (5)

implies (since $\hat{\mathcal{W}}_n^{(\beta)}$ is a tempered distribution) that $\hat{\mathcal{W}}_n^{(\beta)}(I,J)$ has an *exponentially decreasing behaviour* of the form $e^{-\beta|\omega_I|}$ for ω_I tending to $-\infty$, instead of being equal to zero for $\omega_I < 0$ as $\hat{\mathcal{W}}_n^{(vac)}(I,J)$. This occurrence of negative energy contributions in $\hat{\mathcal{W}}_n^{(\beta)}$ corresponds to the possibility of "extracting energy from the ambient thermal bath", but their exponential suppression can be seen as *a remnant of the spectral condition* of the vacuum case. As a matter of fact, the Lorentz invariance of the latter implies a more stringent support condition for $\hat{\mathcal{W}}_n^{(vac)}(I,J)$ than the condition $\omega_I > 0$ exhibited by Eq. (6), namely:

$$\text{supp}\ \hat{\mathcal{W}}_n^{(vac)}(I,J) \subset \{(p_1,\ldots,p_n); p_I = -p_J \in \overline{V^+}\}\ , \tag{7}$$

where $p_I = \sum_{i\in I} p_i$ and p_J is defined analogously.

Correspondingly, in Thermal Quantum Field Theory, there is *a remnant of this relativistic spectral condition* which is a relativistic form of the KMS-condition. In fact, the analysis of [17] based on the general principles of "Local Quantum Physics" [22] gives a firm background to the following properties (stronger than iii), iii)') of the Wightman functions $\mathcal{W}_n^{(\beta)}(I,J)$.

iv) Relativistic KMS-condition:

For each pair (I,J), the distribution F_{IJ} of condition iii) admits an analytic continuation (which we still call) $F_{IJ}(x_1,\ldots,x_n;z)$ with respect to the complex four-vector variable $z=(z_0,\vec{z})$ in the following tube-domain:

$$T_\beta = \{z\in\mathbb{C}^4; \text{Im}z\in V^-, \text{Im}z+\beta e_0\in V^+\}.$$

The boundary value equations (1) and (2) can then be replaced respectively by:

$$\lim_{\epsilon\to 0,\epsilon\in V^+} F_{IJ}(x_1,\ldots,x_n;x-i\epsilon) = \mathcal{W}_n^{(\beta)}(\{x_i+x\}_{i\in I};\{x_j\}_{j\in J}), \tag{8}$$

$$\lim_{\epsilon\to 0,\epsilon\in V^+} F_{IJ}(x_1,\ldots,x_n;x-i\beta+i\epsilon) = \mathcal{W}_n^{(\beta)}(\{x_j\}_{j\in J};\{x_i+x\}_{i\in I}). \tag{9}$$

In the space of energy and momentum variables, condition iv) can be equivalently expressed by iii)' supplemented by the following condition:

iv)' Essential support conditions in p-space:

For each pair (I,J), the distribution $\hat{\mathcal{W}}_n^{(\beta)}(I,J)$ admits the cone $\{(p_1,\ldots,p_n); p_I = -p_J \in \overline{V^+}\}$ as a majorant of its "essential support in the sense of exponential decrease". A more precise formulation of this condition is that the products of the tempered distribution $\hat{\mathcal{W}}_n^{(\beta)}(I,J)$ by the (smooth) functions $e^{\frac{\beta}{2}[(1+\vec{p}_I^2)^{\frac{1}{2}}-\omega_I]}$ and $e^{-\beta\omega_I}$ are required to be also tempered.

For completeness, we should also mention an additional postulate whose role is to restrict our framework to the case of equilibrium states Ω_β which are *pure phases.*

v) Time-clustering postulate:

For each pair (I, J), one has in the limit $|t| \to \infty$ (with the notation used in Eq. (1)):

$$F_{IJ}(x_1, \ldots, x_n; t) = \mathcal{W}_m^{(\beta)}(\{x_i\}_{i\in I})\ \mathcal{W}_{n-m}^{(\beta)}(\{x_j\}_{j\in J}) + o(1). \qquad (10)$$

3. ANALYTICITY PROPERTIES IN THE ENERGY AND MOMENTUM VARIABLES

Recently, a procedure for studying the usual n-point retarded and advanced functions and possibly introducing generalized retarded functions characterized by various analyticity properties in the energy variables has been presented in the traditional imaginary-time formalism of Thermal Quantum Field Theory [23]. Here, we would like to emphasize that a general understanding of the algebraic and analytic properties of these Green functions follows from the Wightman axiomatic approach of Quantum Field Theory, and that the latter allows one to control exactly which results are identical to those of the vacuum case and which ones are different.

We devote this section to the *introduction of n-point thermal Green functions enjoying analyticity properties in the energy and momentum variables.* Their definitions and algebraic study in the previous axiomatic framework and the (correlated) derivation of their analyticity properties as *pure consequences of the axiom of locality* are completely identical to those which have been given for the vacuum representation of Quantum Field Theory. This structure has been the object of a number of works in the past [24-28].

The basic idea is that for each n, one can define certain privileged combinations r_α of permuted n-point Wightman functions multiplied by appropriate products of step-functions of the (differences of) time-variables. Each distribution r_α enjoys remarkable *support properties* which are *implied by locality,* namely the support of r_α is contained in a *(salient) convex Lorentz invariant cone* Γ_α of the space of vector differences $x_i - x_j$. Then, in view of a basic result of complex analysis (*see* e.g. [10, 29]), the Fourier transform of r_α is the boundary value on the reals of a holomorphic function $\tilde{r}_\alpha$ in a tube-domain T_α of the *complex energy-momentum space* which is of the following form: $\{k = (k_1, \ldots, k_n); k_1 + \cdots + k_n = 0; Imk \in \tilde{\Gamma}_\alpha\}$, where $\tilde{\Gamma}_\alpha$ (called the *basis of the tube* T_α) is the dual cone of Γ_α (i.e. the set of vectors q such that $qx = q_1x_1 + \cdots + q_nx_n$ is positive for all x in Γ_α). Each of these distributions r_α (or holomorphic functions $\tilde{r}_\alpha$) will be called a *generalized retarded function.* It is the expectation value $< \Omega_\beta, R_\alpha\Omega_\beta >$ of a corresponding *generalized retarded operator* $R_\alpha(x_1, \ldots, x_n)$. The generalized retarded operators (resp. functions) satisfy two sets of basic algebraic relations, namely:

i) discontinuity relations between the various R_α (resp. r_α); in this connection, *generalized absorptive parts* are introduced;

ii) relations between each R_α and the *(anti) time-ordered operator products.*

The two-point function:

Let $\mathcal{C}(x)$ be the commutator function, *i.e.*, $\mathcal{C}(x_1 - x_2) = \mathcal{W}_2^{(\beta)}(x_1, x_2) - \mathcal{W}_2^{(\beta)}(x_2, x_1)$. Locality implies that supp $\mathcal{C} \subset \overline{V^+} \cup \overline{V^-}$. There are two distributions r_α in this case, namely the "retarded and advanced functions", defined formally as $r(x) = i\theta(x_0)\mathcal{C}(x)$ and $a(x) = -i\theta(-x_0)\mathcal{C}(x)$.

The relation $r - a = i\mathcal{C}$ corresponds to the splitting of the support of $\mathcal{C}$ into its two convex components, since supp $r \subset \overline{V^+}$ and supp $a \subset \overline{V^-}$. (Note that this splitting is defined up to a distribution with support at the origin).

This splitting is in turn equivalent to the following one for the Fourier-transformed quantities: $i\tilde{\mathcal{C}}(p) = \tilde{r}(p) - \tilde{a}(p)$, where $\tilde{\mathcal{C}}$ is usually called *spectral function* [1-8] and $\tilde{r}$ and $\tilde{a}$ are the boundary values of holomorphic functions (denoted similarly by) $\tilde{r}(k)$ and $\tilde{a}(k)$ $(k = p + iq)$ in the respective tube-domains $T^+ = \{k = p + iq; q \in V^+\}$ and $T^- = -T^+$ of $\mathbb{C}^4$. Conversely, given a holomorphic function in $T^+ \cup T^-$ (with at most a tempered behaviour at infinity and near the reals), the spectral function obtained by taking the difference of the corresponding boundary values defines a commutator function which does satisfy locality. According to the common use, the holomorphic function $\tilde{r}(k)$ (or $\tilde{a}(k)$) is then called the *propagator* whose associated spectral function is $\tilde{\mathcal{C}}(p)$. (In view of the Hermitian character of the field, $\tilde{\mathcal{C}}(p)$ is also the imaginary part of $\tilde{r}(p)$ and is commonly described as such, although its characterization as *the discontinuity of the holomorphic function* $\tilde{r}(k)$ *across the reals* is more substantial).

Let $\tau(x)$, $x = x_1 - x_2$ (resp. $\overline{\tau}(x)$) be the two-point time-ordered (resp. anti-time-ordered) function. The following relations hold:

$$r = i(\tau - \mathcal{W}') = -i(\overline{\tau} - \mathcal{W}), \qquad a = i(\tau - \mathcal{W}) = -i(\overline{\tau} - \mathcal{W}'), \quad (11)$$

where $\mathcal{W}(x_1 - x_2) = \mathcal{W}_2^{(\beta)}(x_1, x_2)$ and $\mathcal{W}'(x_1 - x_2) = \mathcal{W}_2^{(\beta)}(x_2, x_1)$.

Examples:

1) *The free field propagators:*

For any free scalar field of mass m, the commutator function $\mathcal{C}^{(m)}$ and therefore the associated retarded and advanced functions $r^{(m)}$ and $a^{(m)}$ are independent of the representation generated by the thermal state Ω_β. In fact, in this case $\mathcal{C}^{(m)}(x)$ is a structural function of the field algebra determined by the c-number commutation relations of the field. Therefore, for every thermal representation with temperature β^{-1}, one has (as in the vacuum representation): $\mathcal{C}^{(m)}(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} e^{-ipx}\, \tilde{\mathcal{C}}^{(m)}(p)\, dp$, with

$$\tilde{\mathcal{C}}^{(m)}(p) = \frac{1}{2\pi} \epsilon(\omega) \delta(\omega^2 - \vec{p}^2 - m^2) \quad (12)$$

and

$$\tilde{r}^{(m)}(k) \text{ (resp. } \tilde{a}^{(m)}(k)) = -\frac{1}{4\pi^2} (k_0^2 - \vec{k}^2 - m^2)^{-1} \quad (13)$$

for $k = (k_0, \vec{k}) = p + iq \in T^+$(resp. $k \in T^-$).

2) *The Weldon-Pisarski (WP) propagators:*

The following expressions of the high-temperature transverse and longitudinal gluon propagators have been obtained by the resummation of dominant contributions at small β from the one-loop self-energy diagrams:

$$\Delta_t(k) = \left[k_0^2 - \vec{k}^2 - \frac{M^2}{2\vec{k}^2} \left(k_0^2 + \frac{k_0(\vec{k}^2 - k_0^2)}{2\sqrt{\vec{k}^2}} \log \left(\frac{k_0 + \sqrt{\vec{k}^2}}{k_0 - \sqrt{\vec{k}^2}} \right) \right) \right]^{-1}, \quad (14)$$

$$\Delta_l(k) = \frac{\vec{k}^2}{k_0^2 - \vec{k}^2} \left[\vec{k}^2 + M^2 \left(1 - \frac{k_0}{2\sqrt{\vec{k}^2}} \log \left(\frac{k_0 + \sqrt{\vec{k}^2}}{k_0 - \sqrt{\vec{k}^2}} \right) \right) \right]^{-1}. \quad (15)$$

In these expressions, M denotes the "Debye screening mass" which is directly related to the plasma frequency. The propagators Δ_t and Δ_l can be checked to be holomorphic in the tubes T^+ and T^- in the privileged sheet of the logarithm which defines the *physical sheet* of these functions. Therefore the associated two-point functions satisfy locality (in this connection, *see* [30] for a detailed numerical analysis of these examples and further comments on the construction of causal propagators).

The n-point functions:

For each n, the distributions r_α are labelled by a "cell-function" α which represents a geometrical cell γ_α (namely a polyhedral cone) in the space $H^{(n)} = \{h = (h_1, \dots, h_n) \in \mathbb{R}^n; h_1 + \dots + h_n = 0\}$ deprived from all the hyperplanes $H^{(n)}_{I,J}$ with equation $h_I = -h_J = 0$. Here, (I, J) denotes an *arbitrary* partition of $\{1, \dots, n\}$ and we use the notation $h_I = \sum_{i \in I} h_i$. A cell α is defined as the "sign-valued function" $I \to \alpha(I) = +$ or $-$ such that $\gamma_\alpha = \{h \in H^{(n)}; \forall I \subset \{1, 2, \dots, n\}, \alpha(I) h_I > 0\}$. Two cells α_1 and α_2 are called "adjacent cells along the face $H^{(n)}_{I_0,J_0}$" if they only differ on the complementary sets I_0 and J_0, namely if $\alpha_1(I_0) = -\alpha_1(J_0) = -\alpha_2(I_0) = \alpha_2(J_0)$. The corresponding generalized retarded operators or functions will also be said to be adjacent.

Each generalized retarded function $\tilde{r}_\alpha$ is holomorphic in the tube T_α whose conical basis $\tilde{\Gamma}_\alpha$ in $\{q = Im k = (q_1, \dots, q_n); q_1 + \dots + q_n = 0\}$ is defined by the set of conditions $\alpha(I) q_I \in V^+$ (for all proper subsets I of $\{1, 2, \dots, n\}$) [27, 28]. Each ordinary retarded function $r^{(i)}$ corresponds to the cell α such that $\alpha(\{j\}) = +$ for all $j \in \{1, 2, \dots, n\}$ except for $j = i$.

Discontinuity relations:

They are generated by the following set of relations which connect all the pairs of adjacent generalized retarded operators [25, 26]: for every partition (I, J) of $\{1, \dots, n\}$ and for every pair of cells (α_1, α_2) which are adjacent along the face $H^{(n)}_{I,J}$, there holds:

$$R_{\alpha_1} - R_{\alpha_2} = i[R_{\alpha^{(I)}}, R_{\alpha^{(J)}}], \tag{16}$$

where $\alpha^{(I)}, \alpha^{(J)}$ denote the common restrictions of the cell-functions α_1, α_2 respectively to the proper subsets of I and of J, and $R_{\alpha^{(I)}}, R_{\alpha^{(J)}}$ are the corresponding generalized retarded operators in the space of the variables labelled by the elements of I and J. Correspondingly, there holds the following set of relations, which generalize the relation $r - a = iC$ of the two-point function:

$$r_{\alpha_1} - r_{\alpha_2} = i(\langle \Omega_\beta, R_{\alpha^{(I)}} R_{\alpha^{(J)}} \Omega_\beta \rangle - \langle \Omega_\beta, R_{\alpha^{(J)}} R_{\alpha^{(I)}} \Omega_\beta \rangle). \tag{17}$$

In the latter, the terms at the r.h.s. are interpreted as *generalized absorptive parts* in the respective channels (I, J) and (J, I).

Steinmann relations [24]:

The generalized retarded operators R_α are *not linearly independent. In fact, if two adjacent pairs* (α_1, α_2) *and* (α'_1, α'_2) *along the same face* $H^{(n)}_{I,J}$ *admit the same restrictions* $\alpha^{(I)}, \alpha^{(J)}$, *the following corresponding relation holds:* $R_{\alpha_1} - R_{\alpha_2} = R_{\alpha'_1} - R_{\alpha'_2}$.

Relations with the time-ordered operator products:

Let $T(I)$ denote the time-ordered product of the n_I field-operators $\phi(x_i)$ for $i \in I \subset \{1, 2, \ldots, n\}$. For each cell α the following expression of R_α is valid [28]:

$$\begin{aligned} (-i)^{n-1}\, R_\alpha(x_1, \ldots, x_n) &= T(\{1, 2, \ldots, n\}) \\ &+ \sum\nolimits^{(\alpha)} (-1)^{r-1}\, T(I_1) \ldots T(I_r), \end{aligned} \tag{18}$$

where the sum $\sum^{(\alpha)}$ runs over all *ordered* partitions $(I_1, \ldots, I_r), 2 \leq r \leq n$, of $\{1, 2, \ldots, n\}$ such that $\alpha(I_1) = \alpha(I_1 \cup I_2) = \alpha(I_1 \cup \ldots \cup I_{r-1}) = -$. Similar relations involving the anti-time-ordered operator products can also be written. Eq. (18) generalizes the relations (11) of the two-point function.

4. INTEGRAL REPRESENTATION OF THE TWO-POINT SPECTRAL FUNCTION

We present here an integral representation of the thermal two-point commutator and spectral functions $\mathcal{C}(x)$ and $\tilde{\mathcal{C}}(p)$ which is *a pure consequence of locality;* from a technical viewpoint, it also relies on the fact that $\tilde{\mathcal{C}}(p)/\omega$ has to be a (positive and even) *measure* as a by-product of the positivity and KMS conditions (*see* Sec. 5). Our representation is characterized by a certain *"weight-function"* which plays the same role as that of the Källén-Lehmann representation of the vacuum two-point function. This weight-function can also be reconstructed from $\mathcal{C}(x)$ by a simple inversion formula.

PROPOSITION

i) *The following integral representation holds:*

$$\mathcal{C}(x) = \int_0^\infty dm\ D(\vec{x}; m)\ \mathcal{C}^{(m)}(x)\ ; \tag{19}$$

in the latter, the weight-function $D(\vec{x};m)$ is a tempered distribution with support in $\mathbb{R}^3 \times \overline{\mathbb{R}^+}$ which is uniquely defined and computable in terms of $\mathcal{C}$ by the following inversion formula

$$D(\vec{x};m) = 2i\pi \frac{\partial}{\partial m}\left[\theta(m) \int_{-\infty}^{+\infty} dx_0 x_0 J_0(m\sqrt{x^2})\ \mathcal{C}(x)\right], \qquad (20)$$

where J_0 is the zeroth-order Bessel function of the first kind.

ii) *The following corresponding representation applies to the spectral function:*

$$\tilde{\mathcal{C}}(\omega,\vec{p}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\epsilon(\omega) \int_{\mathbb{R}^3} d\vec{u} \int_0^{\infty} ds\ \delta(\omega^2 - (\vec{p}-\vec{u})^2 - s)\ \tilde{\rho}(\vec{u},s), \quad (21)$$

where

$$\tilde{\rho}(\vec{u},s) = \frac{1}{2\ (2\pi)^{\frac{5}{2}}\sqrt{s}} \int_{\mathbb{R}^3} d\vec{x}\ e^{-i\vec{u}.\vec{x}} D(\vec{x};\sqrt{s}) \qquad (22)$$

belongs to $\mathcal{S}'(\mathbb{R}^4)$, **with supp** $\tilde{\rho} \subset \mathbb{R}^3 \times \overline{\mathbb{R}^+}$.

Similar results were first stated by Gervais and Yndurain in some unpublished paper [31]; Eq. (21) can be interpreted as a Jost-Lehmann-Dyson representation (*see* [12] and references therein) in the absence of spectral condition. The proof which we present here is self-contained; the missing technical details of distribution theory will be given elsewhere [32].

a) *Gauss-type transforms in the time and energy-variables:* we first associate with the (4d-) Fourier pair $(\mathcal{C}, \tilde{\mathcal{C}})$ the following (3d-) Fourier pair $(\Psi(\vec{x};\lambda), \tilde{\Psi}(\vec{p};\lambda))$

$$\Psi(\vec{x};\lambda) = i \int e^{\frac{-x_0^2}{4\lambda}} x_0\ \mathcal{C}(x)\ dx_0, \qquad (23)$$

$$\tilde{\Psi}(\vec{p};\lambda) = (2\lambda)^{\frac{3}{2}} \int e^{-\lambda\omega^2} \omega\ \tilde{\mathcal{C}}(\omega,\vec{p})\ d\omega\ . \qquad (24)$$

It can be checked that Ψ and $\tilde{\Psi}$ are tempered distributions (resp. in $\vec{x}$ and $\vec{p}$) which are (slowly-increasing) holomorphic functions of λ in the complex half-plane $\mathbb{C}^+ = \{\lambda; Re\lambda > 0\}$. Moreover, the mapping $\tilde{\mathcal{C}} \to \tilde{\Psi}$ and (thereby) $\mathcal{C} \to \Psi$ are one-to-one since the integral transformation (24) can be considered as a Laplace-transformation in the variable ω^2 and therefore inverted (here, one makes use of the fact that $\frac{\tilde{\mathcal{C}}(\omega,\vec{p})}{\omega}$ is an even measure in ω).

b) *The transforms Φ and $\tilde{\Phi}$:* We define

$$\Phi(\vec{x};\lambda) = e^{\frac{\vec{x}^2}{4\lambda}}\ \Psi(\vec{x};\lambda)\ . \qquad (25)$$

Locality implies that the distribution-valued holomorphic function $\Phi(\vec{x}, \lambda)$ is *tempered* like $\Psi(\vec{x}, \lambda)$. By inverting Eq. (25), we obtain the following relation between the Fourier transforms $\tilde{\Phi}(\vec{p}; \lambda)$ and $\tilde{\Psi}(\vec{p}; \lambda)$ of Φ and Ψ:

$$\tilde{\Psi}(\vec{p}; \lambda) = \left(\frac{\lambda}{\pi}\right)^{\frac{3}{2}} \int_{\mathbb{R}^3} e^{-\lambda(\vec{p}-\vec{u})^2} \tilde{\Phi}(\vec{u}; \lambda)\, d\vec{u}\,, \tag{26}$$

the latter being valid for all λ in $\mathbb{C}^+$.

c) *The "weight-functions" ρ and $\tilde{\rho}$:* we call respectively $\rho(\vec{x}; s)$ and $\tilde{\rho}(\vec{p}; s)$ the inverse Laplace-transforms of Φ and $\tilde{\Phi}$ with respect to the variable λ; $(\rho, \tilde{\rho})$ is a (3d-) Fourier pair of tempered distributions on $\mathbb{R}^4$ with support contained in $\mathbb{R}^3 \times \overline{\mathbb{R}^+}$. For our needs, we write explicitly the mappings $\Phi \to \rho$ and $\tilde{\rho} \to \tilde{\Phi}$:

$$\rho(\vec{x}; s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\nu s} \Phi(\vec{x}; i\nu)\, d\nu\,, \tag{27}$$

$$\tilde{\Phi}(\vec{p}; \lambda) = \int_0^{+\infty} e^{-\lambda s} \tilde{\rho}(\vec{p}; s)\, ds\,. \tag{28}$$

d) *The integral representations* (19), (21): by plugging Eq. (28) into Eq. (26), we obtain

$$\begin{aligned} \tilde{\Psi}(\vec{p}; \lambda) &= \left(\frac{\lambda}{\pi}\right)^{\frac{3}{2}} \int_{\mathbb{R}^3 \times \mathbb{R}^+} e^{-\lambda[(\vec{p}-\vec{u})^2+s]} \rho(\vec{u}; s)\, d\vec{u}\, ds \\ &= \left(\frac{\lambda}{\pi}\right)^{\frac{3}{2}} \int e^{-\lambda\omega^2} d\omega^2 \\ &\quad \times \left[\int_{\mathbb{R}^3 \times \mathbb{R}^+} \delta(\omega^2 - (\vec{p}-\vec{u})^2 - s) \rho(\vec{u}; s)\, d\vec{u} ds\right]. \end{aligned} \tag{29}$$

By comparing Eqs. (24) and (29), we see that $(2\pi)^{\frac{3}{2}} \theta(\omega) \tilde{C}(\omega, \vec{p})$ and the bracket in the r.h.s. of Eq. (29) admit the same Laplace-transform with respect to the variable ω^2 and are therefore equal in view of a). Since $\tilde{C}(\omega, \vec{p})$ is odd in the variable ω, Eq. (21) is therefore established and Eq. (19) immediately follows by Fourier transformation.

e) *The inversion formula* (20): by plugging Eq. (23) into Eq. (25) and the latter into Eq. (27), we readily obtain (after an admissible interchange of integration, the following formula being understood in the sense of distributions in the variables $\vec{x}$ and s):

$$\rho(\vec{x}; s) = \int_{-\infty}^{+\infty} x_0 C(x) dx_0 \left[\frac{i}{2\pi} \int_{-\infty}^{+\infty} e^{i\nu s - \frac{x^2}{4i\nu}} d\nu\right]. \tag{30}$$

Since the bracket in the r.h.s. of Eq. (30) is an integral representation of $i\frac{\partial}{\partial s}[\theta(s)J_0(\sqrt{sx^2})]$(*see* [33], page 357, formula (9)), Eq. (30) takes the form (20) if one replaces ρ by the distribution $D(\vec{x};m) = 4\pi m\ \rho(\vec{x};m^2)$ (linked to $\tilde{\rho}$ by Eq. (22)).

From the previous proposition, one easily deduces the following

COROLLARY

The (Fourier pair of) retarded thermal propagators $r(x)$ and $\tilde{r}(k)$ satisfy the following general integral representations

$$r(x) = \int_0^\infty dm D(\vec{x};m)\ r^{(m)}(x) \tag{31}$$

$$\tilde{r}(k) = \frac{-1}{(2\pi)^{\frac{5}{2}}} \int_{\mathbb{R}^3} d\vec{u} \int_0^\infty ds\ \frac{1}{[k_0^2 - (\vec{k}-\vec{u})^2 - s]}\ \tilde{\rho}(\vec{u};s), \tag{32}$$

where k varies in T^+. The same representation holds for $\tilde{a}(k)$ with k varying in T^-.

Remark. – The Källén-Lehmann representation, which is valid for a general propagator in the vacuum state in the axiomatic Wightman framework (*see* [12] and references therein), appears as a special case of the representation (31); it is obtained when the weight-function D is of the form $D(\vec{x};m) = 2m\rho_0(m^2)$ (i.e. $\tilde{\rho}(\vec{u};s) = (2\pi)^{\frac{1}{2}}\ \rho_0(s)\ \delta(\vec{u})$), ρ being a tempered (positive) measure with support contained in $\overline{\mathbb{R}^+}$.

5. CONSEQUENCES OF THE KMS-CONDITION

The absence of the spectral condition has a drastic effect on the structure of the n-point Green functions in the space of energy and momentum variables: as we shall explain it below, the various generalized retarded functions in complex energy- momentum space do not admit in general a common analytic continuation; in contrast with the case of the vacuum representation, *there do not exist analytic n-point Green functions in (connected) primitive domains for the thermal representations of quantum field theory.* It also appears as a related fact that *the (anti-) time-ordered products and (generalized) retarded products have in general no mutual coincidence regions in real energy-momentum space,* which plagues the usual connection between the approach in real time and energy variables (RTF) (in terms of time-ordered products) and the use of the Green functions in complex energy variables. As a matter of fact, for the case of representations of field theory at finite temperature $T = \beta^{-1}$, the KMS-condition provides substitutes to the usual coincidence relations due to spectrum. These substitutes are relations which contain the Bose-Einstein factor $(1 - e^{-\beta\omega})^{-1}$ and thereby imply that coincidence relations can be only "essentially restored, up to exponential tails" in the limit of very high energies.

Another consequence of the KMS-condition concerns the justification of the approach in purely imaginary time and energy variables. It is a standard practice of field-theorists to use the so-called "Euclidean n-point functions", namely the Wightman functions at purely imaginary times (or Schwinger functions) and correspondingly the n-point Green functions at purely imaginary energies. In the vacuum representation of Quantum Field Theory, the justification of this practice is a by-product of the double analytic structure of the n-point functions which results from the interplay of spectral condition and locality. In particular, it is known that the Green functions at imaginary energies are obtained as the Fourier transforms of the corresponding Wightman functions at imaginary times. In the case of thermal representations of Quantum Field Theory, a similar double analytic structure of the n-point functions also results from the interplay of KMS-condition and locality. However, in a representation at temperature $T = \beta^{-1}$, the n-point Wightman functions of the complex time-variables acquire *periodicity conditions with period* $i\beta$ as a consequence of the KMS-condition. Correspondingly, the Green functions of the complex energy-variables (namely the Fourier-Laplace transforms of the generalized retarded functions) are expected to be completely determined from their values on a certain lattice of *discrete purely imaginary energies* of the form $\omega = \frac{2i\pi l}{\beta}$, with l integer, these values being the Fourier coefficients of the corresponding ($i\beta$-periodic) Schwinger functions.

These consequences of the KMS-condition have been understood to a large extent in the time-ordered path approach (*see* e.g. [4, 7, 9, 23]). We shall exhibit them here in our general axiomatic setting and give complete results for the case of the two-point function, but only indicate how their generalization to the n-point functions could be worked out.

The two-point function

Substitutes to the coincidence relations in energy-momentum space:

Let $\tilde{\mathcal{W}}(p)$ be the Fourier transform of the two-point function $\mathcal{W}(x)$. The KMS-condition in the energy variable is expressed as a special case of formulas (3), (5) (*see* Sec 1 iii)'), namely:

$$\tilde{\mathcal{W}}(p) = e^{\beta\omega}\tilde{\mathcal{W}}(-p) = \frac{\tilde{\mathcal{C}}(p)}{1 - e^{-\beta\omega}}. \tag{33}$$

This way of writing Eq. (5) for the case of the two-point function (with the Bose-Einstein factor at the r.h.s. of this equation) is justified and unambiguous, *as a relation between positive measures,* in view of the additional postulate v), namely, the time-clustering property for $\mathcal{W}$. The latter implies that (after the addition of a suitable constant to the field) $\mathcal{W}(x)$ tends to zero when the time-variable x_0 tends to infinity (at fixed $\vec{x}$) and correspondingly that the measure $\tilde{\mathcal{W}}(p)$ contains no term proportional to $\delta(\omega)$. Therefore, Eq. (33) determines uniquely the splitting of the spectral function $\tilde{\mathcal{C}}(p) = \tilde{\mathcal{W}}(p) - \tilde{\mathcal{W}}(-p)$ and replaces the usual splitting $\tilde{\mathcal{W}}^{(vac)}(p) = \theta(\omega)\tilde{\mathcal{C}}^{(vac)}(p)$ which results from the spectral condition in the vacuum representation. Eq. (33) implies that $\tilde{\mathcal{C}}(p)$ and $\tilde{\mathcal{W}}(p)$ "coincide up to an exponential tail" at very high energies, while $\tilde{\mathcal{W}}(p)$ " vanishes up to an exponential tail" at negative energies of very high absolute value.

The fact that $\tilde{\mathcal{C}}(p)$ does *not* vanish in general on any open subset of energy-momentum space (the examples given in Sec 2 are exceptions which will be commented in Sec. 6c) below) implies that $\tilde{r}(p)$ and $\tilde{a}(p)$ do not coincide on any open set and therefore that the corresponding holomorphic functions in the tubes T^+ and T^- are *not* the analytic continuation of each other.

Let $\tilde{\tau}(p)$ be the Fourier transform of the time-ordered function $\tau(x)$. In view of Eqs (10), (11) and (33), the following relations hold:

$$\tilde{\tau}(p) = -i\tilde{a}(p) + \tilde{\mathcal{W}}(p) = \frac{-i\tilde{r}(p) + i\tilde{a}(p)e^{-\beta\omega}}{1 - e^{-\beta\omega}}, \tag{34}$$

which show that $\tilde{\tau}$ and $-i\tilde{r}$ only "coincide up to an exponential tail" at very high energies.

The double analytic structure:

The KMS-condition (*see* Sec. 2 iii)) implies that there exists a holomorphic function $W(z_0, \vec{x})$ of z_0 in the strip $-\beta < Im z_0 < 0$ whose boundary values on the edges of this strip are respectively: $W(x_0, \vec{x}) = \mathcal{W}(x)$ and $W(x_0 - i\beta, \vec{x}) = \mathcal{W}'(x)$. Locality then implies that the holomorphic function W can be analytically continued as an $i\beta$-periodic function of z_0 in the whole complex plane minus the cuts $\{z_0 = x_0 + iy_0; |x_0| \geq |\vec{x}|, y_0 = l\beta, l \in \mathbb{Z}\}$. Moreover, the jump ΔW of

W across the right-hand cut $x_0 \geq |\vec{x}|$ is (as in the case of the vacuum representation):

$$\Delta W(x) = \theta(x_0)\ [\mathcal{W}(x) - \mathcal{W}'(x)] = -ir(x). \tag{35}$$

If one now considers conjointly

i) the Fourier-Laplace transform

$$\tilde{r}(k_0, \vec{p}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^+ \times \mathbb{R}^3} e^{i(k_0 x_0 - \vec{p}\cdot\vec{x})}\ r(x_0, \vec{x})\ dx_0\ d\vec{x} \tag{36}$$

of r in its holomorphy domain $Imk_0 > 0$, and

ii) the Fourier coefficients of the (periodic) Schwinger function $W(iy_0, \vec{x})$, namely

$$\tilde{w}_l(\vec{p}) = \frac{1}{(2\pi)^2} \int_{\frac{-\beta}{2}}^{\frac{\beta}{2}} e^{\frac{-2i\pi l}{\beta} y_0} dy_0 \int_{\mathbb{R}^3} e^{-i\vec{p}\cdot\vec{x}} W(iy_0, \vec{x})\ d\vec{x}, \tag{37}$$

for l non-negative integer,

one can show the following basic relations:

$$\tilde{w}_l(\vec{p}) = \tilde{r}\left(\frac{2i\pi l}{\beta}, \vec{p}\right), \quad l = 0,\ 1,\ 2, \ldots \tag{38}$$

by means of a simple contour distortion argument. In fact, one checks that the integral of the product $-(2\pi)^{-2} i e^{i(k_0 z_0 - \vec{p}.\vec{x})} \times W(z_0, \vec{x})$ over the complex cycle $\{z_0 \in [\infty - i\frac{\beta}{2}, -i\frac{\beta}{2}] \cup [-i\frac{\beta}{2}, i\frac{\beta}{2}] \cup [i\frac{\beta}{2}, \infty + i\frac{\beta}{2}]\} \times \{\vec{x} \in \mathbb{R}^3\}$ reduces to Eq (36) by shrinking the integration cycle onto the real set $\mathbb{R}^+ \times \mathbb{R}^3$, while it reduces to Eq (37) *for the special values* $k_0 = \frac{2i\pi l}{\beta}$, l integer, $l \geq 0$ due to the $i\beta$-periodicity of $W(z_0, \vec{x})$.

Since $\tilde{r}(k_0, \vec{p})$ is of moderate growth in the complex half-plane $Imk_0 > 0$ (for each $\vec{p}$), it can be characterized as the *(unique) Carlsonian interpolation* (*see* e.g. [34] p. 153) of the sequence of its values given by Eq. (38) at the set of discrete imaginary energies $\frac{2i\pi l}{\beta}$, l integer, $l \geq 0$; a corresponding result holds for the advanced propagator $\tilde{a}$ which satisfies in the lower half-plane equations similar to Eq. (38) with $l \leq 0$. This result together with formula (34) make completely clear the connection between the imaginary and real-time formalisms for the two-point function and their equivalence in the axiomatic framework.

The n-point functions

The derivation of various formulas which are substitutes to coincidence relations in energy-momentum space relies on the following fact. The relations (5) implied by the KMS-condition can be extended to a more general set of similar formulas, namely

$$(1 - e^{-\beta\omega_I}) \langle \Omega_\beta, \tilde{\Pi}(I)\tilde{\Pi}(J)\Omega_\beta \rangle = \langle \Omega_\beta, [\tilde{\Pi}(I), \tilde{\Pi}(J)]\Omega_\beta \rangle, \tag{39}$$

where $\tilde{\Pi}(I)$ denotes the Fourier transform of any (product of) generalized retarded or (anti-) time-ordered products of field operators depending on the energy-momentum variables $p_i = (\omega_i, \vec{p}_i); i \in I$. Since all the distributions involved are tempered, it follows from Eq. (39) that all the expressions of the form $\langle \Omega_\beta, \tilde{\Pi}(I)\tilde{\Pi}(J)\Omega_\beta \rangle$ decrease exponentially as $e^{-\beta|\omega_I|}$ for ω_I tending to $-\infty$.

By taking the latter property into account for all the terms of the sum at the r.h.s. of Eq. (18) (after applying a Fourier transformation to both sides of this equality), one easily checks the following statement:

Consider for each cell α the region $\hat{\gamma}_\alpha$ of (real) energy- momentum space which is defined by the condition $\omega = (\omega_1, \ldots, \omega_n) \in \gamma_\alpha$; then, in this region $\hat{\gamma}_\alpha$, the corresponding distribution $\tilde{r}_\alpha(p)$ "essentially coincides" with the Fourier transform $\tilde{\tau}(p)$ of the time-ordered expectation value $\langle \Omega_\beta, T(\{1, 2, \ldots, n\})\Omega_\beta \rangle$ up to terms which are exponentially decreasing with respect to the various energy variables inside $\hat{\gamma}_\alpha$.

Formula (39) also applies to the r.h.s. of the following relations which result from Eq. (17) by a Fourier transformation:

$$\tilde{r}_{\alpha_1} - \tilde{r}_{\alpha_2} = \langle \Omega_\beta, [\tilde{R}_{\alpha(I)}, \tilde{R}_{\alpha(J)}]\Omega_\beta \rangle. \tag{40}$$

However, in contrast with what happens for the vacuum representation (in view of the spectral condition), formula (39) does not imply any support property for the commutator function at the r.h.s. of Eq. (40). Therefore, *the holomorphic functions $\tilde{r}_{\alpha_1}(k)$ and $\tilde{r}_{\alpha_2}(k)$ whose boundary values on the reals are linked by Eq. (40) do not admit mutual analytic continuation via the edge-of-the-wedge property* as it is the case for the Green functions of the vacuum representation.

The analytic structure of the n-point functions in the complex time-variables results from the KMS-conditions (*see* Eqs (1), (2)) which one writes for *all the permuted Wightman distributions* $\mathcal{W}_n^{(\beta)}(I, J)$, (I, J) denoting any pair of ordered sets forming an ordered partition of $\{1, 2, \ldots, n\}$. As in the case of the vacuum representation, there exists (for each n) a distribution $W_n^{(\beta)}((z_{1,0},\vec{x}_1),\ldots,(z_{n,0},\vec{x}_n))$ which is holomorphic in a union of permuted tubes in the space of complex time-variables $z_{i,0},\ \ 1\leq i\leq n$, together with regions of mutual crossover provided by locality. However, what is different and characteristic of the representations at temperature β^{-1} is the fact that the tubes have *bounded* polyhedral bases which form a multiperiodic paving of the space of purely imaginary time-variables $y_0=(y_{1,0},\ldots,y_{n,0})$. In particular, the holomorphy domain of the $W_n^{(\beta)}$ contains all the Euclidean configurations $z_j=(iy_{j,0},\vec{x}_j),\ 1\leq j\leq n$, except those which satisfy conditions of the form $z_j=z_k+il\beta e_0,\ l\in\mathbb{Z}$, for all possible pairs (j,k). One therefore also expects a generalization to take place concerning *the double analytic structure in the time and energy variables* which we have exhibited above for the two-point function. More precisely, one expects the derivation of formulas of the form (38) to be feasible in the general case; such formulas would relate the values of the generalized retarded n-point functions $\tilde{r}_\alpha$ on appropriate lattices of purely imaginary energies to the Fourier coefficients of the Schwinger n-point functions which should be computed (for each r_α) on a corresponding periodicity pattern of y_0-space. This is an algebraic problem which can be solved rather easily for n=3 and is expected to be solvable for general n with some amount of work [35].

6. COMPLEMENTS AND DISCUSSION

a) Integral representation of the two-point correlation function

It follows from Eqs (12) and (33) that the Fourier transform of the two-point correlation function of the free field at temperature β^{-1} is

$$\tilde{\mathcal{W}}_\beta^{(m)}(\omega,\vec{p})=\frac{1}{2\pi}\frac{\epsilon(\omega)\delta(\omega^2-\vec{p}^{\,2}-m^2)}{1-e^{-\beta\omega}}. \tag{41}$$

By multiplying both sides of Eq. (21) by $(1-e^{-\beta\omega})^{-1}$ and taking Eq. (33) into account, one then obtains the following general representation for the Fourier transform of the two-point correlation function of any local scalar field at temperature β^{-1}:

$$\tilde{\mathcal{W}}(\omega,\vec{p})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}d\vec{u}\int_0^\infty ds\ \tilde{\rho}(\vec{u},s)\ \tilde{\mathcal{W}}_\beta^{(\sqrt{s})}(\omega,\vec{p}-\vec{u}). \tag{42}$$

By coming back to the space-time variables, we obtain correspondingly the following integral representation for the correlation function itself, which is comparable to formula (19):

$$\mathcal{W}(x) = \int_0^\infty dm\ D(\vec{x}; m)\ \mathcal{W}_\beta^{(m)}(x). \tag{43}$$

Remark. – The consideration of "generalized free thermal fields" completely defined by specifying their two-point function (and imposing the prescription that all the truncated n-point functions, with $n > 2$, should vanish) is now possible. We can say that all the information which is necessary to construct such a field satisfying all the general principles of Thermal Quantum Field Theory defined above (*see* Sec. 2) is encoded in the "weight-function" $D(\vec{x}; m)$ (or $\tilde{\rho}(\vec{u}, s)$) of the integral representation (43) (or (42)) of its two-point function; the only restriction which the tempered distribution $\tilde{\rho}(\vec{u}, s)$ has to comply with is the following one: the convolution expressed (formally) by the double integral at the r.h.s. of Eq. (21) must give a positive measure. This is clearly the case if $\tilde{\rho}(\vec{u}, s)$ happens to be positive.

b) Consequences of the relativistic KMS-condition

The implications of the relativistic KMS-condition (presented above in Sec. 2, conditions iv), iv)') concern the existence of an *analytic structure of the n-point functions in the complex space-time vector variables* and the exploitation of corresponding *exponential decay properties in the space of (real) energy-momentum variables.* The latter are produced not only at very large negative energies but also at very large momenta; they in fact apply to energy-momentum regions and to distributions for which the relativistic spectral condition would imply vanishing properties (in the vacuum representation of field theory).

For the case of the two-point function, a detailed study will be given in [32] with the following type of results. Even under the weakest form of relativistic KMS-condition (see [17]), the function $W(z_0, \vec{x})$ introduced in Sec. 5 admits an analytic continuation $W(z)$ as a holomorphic function of the complex four-vector $z = (z_0, \vec{z})$ in a domain which contains the $i\beta$-periodic "flat cut-domain" of W in $(z_0, \vec{x})$-space. This shows (as a by-product) the regular (i.e. $\mathcal{C}^\infty$) character of the distribution $\mathcal{W}(x_0, \vec{x})$ with respect to the spatial coordinates $\vec{x}$. Moreover, the distribution $D(\vec{x}, m)$ appearing as a "weight-function" in the representations (19) and (43) is also shown to have a $\mathcal{C}^\infty$-dependence in $\vec{x}$. Under the strongest form of relativistic KMS-condition (namely, the one presented as condition iv) in Sec. 2), $W(z)$ is holomorphic in the union of the tube $\mathcal{T}_\beta$ together with all those which are obtained from the latter by the translations $il\beta e_0$, $l \in \mathbb{Z}$. All these tubes are connected together by complex neighbourhoods of the regions $\{z;\ y_0 = l\beta,\ x_0^2 - \vec{x}^2 < 0,\ l \in \mathbb{Z}\}$ given by locality. The weight-function D is also proved to admit an analytic continuation $D(\vec{z}, m)$ in a corresponding tube with imaginary basis (containing the origin) in the space of complex coordinates $\vec{z}$. In the energy-momentum variables $(\omega, \vec{p})$, the product of the spectral function $\tilde{C}(p)$ by the function $e^{\frac{\beta}{2}[(1+\vec{p}^2)^{\frac{1}{2}} - (1+\omega^2)^{\frac{1}{2}}]}$ must be a tempered distribution and the "exponential tail" of $\tilde{\tau} + i\tilde{r}$ extends to the whole complement of the forward cone in $\mathbb{R}^4$; correspondingly, the "weight-function" $\tilde{\rho}(\vec{u}, s)$ of the representation (20) is exponentially decreasing with respect to $|\vec{u}|$. Similar properties could be derived for the n-point functions.

c) Feynman-type operations on two-point functions and discussion of examples

The double analytic structure of thermal two-point functions which we have described (*see* Sec. 5) can be shown to be preserved under the two basic Feynman-type operations, namely

i) "N-line Wick-contraction" (represented by the diagram with two vertices z_1 and z_2 connected by N lines):

With such a diagram Γ is associated the product $W_{[\Gamma]}(z_1 - z_2)$ of N two-point functions taken at the same complex point $z_1 - z_2$ (each factor $W_{(\lambda)}(z_1 - z_2)$ being associated with a line λ). It is clear that such a product of holomorphic functions still satisfies the same properties ($i\beta$-periodicity, analyticity domain expressing the relativistic KMS-condition, boundary values on the reals in the sense of distributions, locality) as each individual factor. These properties characterize $W_{[\Gamma]}$ as being a thermal two-point function whose associated propagator $\tilde{r}_{[\Gamma]}(k)$ or $\tilde{a}_{[\Gamma]}(k)$ can be computed on the discrete sequence of energies $\omega_l = i\frac{2\pi l}{\beta}$, $l \in \mathbb{Z}$, (*see* Eq. (38)) as a (discrete) convolution of the N propagators $\tilde{r}_{(\lambda)}$ or $\tilde{a}_{(\lambda)}$ taken on the same sequence of imaginary energies. Moreover, the Fourier transform of the correlation function $\tilde{\mathcal{W}}_{[\Gamma]}(p)$ is also the convolution product on the (real) energy-momentum space of the Fourier transforms of the N correlation functions $\tilde{\mathcal{W}}_{(\lambda)}$.

ii) "Vertex convolution":

The vertex convolution of the thermal two-point functions $W_{(1)}$ and $W_{(2)}$ is the following convolution on Euclidean space-time: $W(z_1 - z_2) = \big(W_{(1)} * W_{(2)}\big)(z_1 - z_2) = \int_{-i\frac{\beta}{2}}^{i\frac{\beta}{2}} dz_0 \int_{\mathbb{R}^3} d\vec{z}\ W_{(1)}(z_1 - z)\ W_{(2)}(z - z_2)$. The resulting holomorphic function W still satisfies all the structural properties of thermal two-point functions and the associated propagator $\tilde{r}(k)$ in complex energy-momentum space is the ordinary product of the propagators $\tilde{r}_1(k)$ and $\tilde{r}_2(k)$ associated with W_1 and W_2.

In particular, if one starts from the free thermal two-point functions (*see* Sec. 3, Example 1, and Sec. 6a)), one can construct as an application of the operation i) appropriate perturbative two-point functions associated with "self-energy bubble diagrams" of the thermal interacting field. Then, the operation ii) defines in a rigorous way the two-point function obtained by "resummation of the corresponding iterated self-energy diagrams". If the self-energy contribution is denoted (in complex energy-momentum space) by $\Sigma(k)$, the associated "complete" propagator is given, as usual, in the complex domains (T^+ and T^-) by the formula $\Delta(k) = \frac{1}{k_0^2 - \vec{k}^2 - m^2 - \Sigma(k)}$.

All these structural properties could of course be derived similarly for non-scalar fields such as those of QCD. One can thus understand in this general framework such computations of gluon propagators as those given by Weldon and Pisarski [5, 6] (*see* Sec. 3, Example 2). In the latter, however, a high-temperature approximation has been taken which makes the expressions (14) and (15) of the propagators Δ_l and Δ_t somewhat peculiar, as far as the support and decrease properties of the associated spectral functions are concerned.

We first notice that, apart from those of the free field and of the two-lined bubble diagram, the spectral functions of all the perturbative two-point functions generated by the previous operations i) and ii) have *no support restrictions;* however, they all enjoy a property of exponential decrease at large momenta which expresses the fact that the relativistic KMS-condition is satisfied. These two properties are violated by the spectral functions of the WP-propagators, since:

1) the latter only satisfy a condition of power decrease at large momenta: but this "hard thermal loop approximation" is generally used only in the low momentum region;

2) in the usual relativistic spectral region $|\omega| > |\vec{p}|$, their support is restricted to a pair of hypersurfaces of the form $\omega = \pm\omega_{t,l}(\vec{p})$ which correspond to sharp terms containing the factors $\delta(\omega \mp \omega_{t,l}(\vec{p}))$; the existence of such real dispersion laws, interpreted as "plasmon modes", calls for further comments which open our last topic.

We conclude these remarks by noting that a general perturbative approach to the construction of thermal correlation functions, based on the systematic exploitation of locality and KMS-condition, has been proposed by Steinmann [36].

d) Representations of particles

The concept of *"dispersion law"* $\omega = \omega_{part}(\vec{p})$ for representing a "particle" (or "mode") in Thermal Quantum Field Theory may seem the most natural one to be inherited from the familiar formalism of quantum field theory in the vacuum representation, having in mind that the hypersurface defined by this law in the space of the energy-momentum variables might not be (in general) a relativistic hyperboloid shell. Such a hypersurface should appear as the singular set of a *real pole* in the thermal propagators of the theory, or equivalently as the support of a *δ-term* in the corresponding spectral functions. However, this attractive picture turns out to be *false, except precisely in the trivial case of a relativistic free particle moving across the thermal bath without any interaction:* so is the content of the Narnhofer-Requardt-Thirring theorem [37] which has been proved by these authors in the general framework of "Local Quantum Physics" [22]. This frustrating result holds true, even if one allows the support of the δ-term to be imbedded in a region where the spectral function has a non-zero continuous background. As a matter of fact, this is not so surprising since a δ-term in $\tilde{C}(p)$ would imply that the correlation function $\mathcal{W}(t, \vec{x})$ has the "normal" slow decay property (as $|t|^{-\frac{3}{2}}$) along any world-line $\vec{x} = \vec{v}t$, and therefore that the particle is not submitted to dissipative effects due to the interactions with the thermal bath. For example, the WP-gluon-propagators considered above represent an approximation which does not provide a realistic description in terms of particles in a thermal equilibrium state.

A reasonable way out of this deadlock, which has been proposed in particular by Landsman [38], consists in adopting the same viewpoint as for the representation of unstable particles in the vacuum state of field theory. This amounts to assume that the retarded propagator (or a distinguished part of it) admits an analytic continuation from the upper k_0-plane (or the tube T^+) into the lower k_0-plane, across (part of) the reals, and that a *complex pole* $k_0 = \omega_{part}(\vec{p}) - i\gamma_{part}(\vec{p})$ is present in this *second-sheet* domain. Of course, this *complex dispersion law* should present some characteristics which would distinguish the type of decay (due to statistical dissipative effects) of a particle in a thermal bath from the one (due to intrinsic unstability) of a resonance. In particular, one would expect the law to be such that the dissipative effects are barely felt by the particle *at rest* , namely that $\mathcal{W}(t, \vec{v}t)$ keeps the $|t|^{-\frac{3}{2}}$ behaviour for $\vec{v} = 0$, while being exponentially decreasing in t for $\vec{v} \neq 0$. This is certainly not the case for the simplest ways of choosing the "width" $\gamma_{part}(\vec{p})$ (i.e. constant or proportional to $\omega_{part}(\vec{p})$).

At this point, we would like to advocate (as in [18.19]) that our general integral representations (21) and (43) of the thermal spectral functions and correlation functions provide a somewhat more natural prescription for representing particles: let us assume that, as in the Källén-Lehmann formula for the spectral function in the vacuum state, a particle is associated with *a δ-term in the weight-function $\tilde{\rho}(\vec{u}, s)$ of the representation (21),* namely, with a term of the form $\tilde{\rho}_{part}(\vec{u})\ \delta(s - m_0^2)$. The representation (43) then contains the distinguished term $D_{part}(\vec{x})\ \mathcal{W}_\beta^{(m_0)}(x)$, where $D_{part}(\vec{x})$ is (up to a factor) the Fourier transform of $\tilde{\rho}_{part}(\vec{u})$. It is now clear that, if we interpret $D_{part}(\vec{x})$ as a *"dissipative for damping factor"* which should decrease, for instance, exponentially at large $\vec{x}$, this distinguished term is a good candidate for representing a particle behaviour. (Note that it would satisfy the exponential decrease property along all world-lines $\vec{x} = \vec{v}t$, but would behave as $|t|^{-\frac{3}{2}}$ at rest, like the free correlation function $\mathcal{W}_\beta^{(m_0)}$.) Since in this case, the function $\tilde{\rho}_{part}$ would be holomorphic in a tube domain of the complexified variables $\vec{u}$, one can prove that the corresponding propagator represented by formula (32) would then have analytic continuation properties in the lower half-plane of the variable k_0. This indicates (and a further general study confirms it as well as the special example presented in [18,19]) that this *δ-representation of particles,* far from being contradictory with the previous one based on "second sheet complex poles" of the propagators, *selects inside the latter class* candidates which might be the most appropriate representatives of the notion of particle in Relativistic Thermal Quantum Field Theory.

Finally, one can also show that the integral representations (19) and (21) are a useful tool for investigating the manifestations of "thermal Goldstone particles" in the case of "spontaneous symmetry breaking"; this will be the object of a further work [39].

In conclusion, we have presented a general framework "a la Wightman" for the concepts of Thermal Quantum Field Theory and a set of (preliminary) results, mainly expressed in terms of analytic structural properties of the n-point functions of the fields. These results are already rather complete for the case $n = 2$. They are useful for clarifying some structural aspects of perturbative results, obtained in the previous years by various authors. Moreover, the general integral representation of thermal two-point functions that we have obtained is interesting under several respects: providing a complete knowledge of "generalized free thermal fields", it also suggests a rather promising approach to the concept of particle, integrating the previously known ideas and consistent with the general principles of relativistic thermal field theory. Many open problems remain, as far as the properties and the use of n-point functions are concerned (for $n > 2$);

among the important ones, let us mention in particular the problem of the possible definition of several-particle states as a manifestation of the field interactions in a spirit comparable to the collision theory in the vacuum state.

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