ANNALES DE L'I. H. P., SECTION A

L. PASTUR

Eigenvalue distribution of random matrices : some recent results

Annales de l'I. H. P., section A, tome 64, nº 3 (1996), p. 325-337
<http://www.numdam.org/item?id=AIHPA_1996__64_3_325_0>

Ann. Inst. Henri Poincaré,

Vol. 64, n° 3, 1996, p. 325-337. *Physique théorique*

Eigenvalue distribution of random matrices: Some recent results

by

L. PASTUR*

UFR de Mathématiques, Université de Paris-VII
2, place Jussieu, 75251 Paris Cedex 05
Mathematical Division
B. Verkin Institute for Low Temperature Physics
310164 Kharkiv, Ukraine
E-mail: pastur@mathp7.jussieu.fr

ABSTRACT. – We review recent results obtained for the eigenvalue statistics of $n \times n$ Hermitian (real symmetric) random matrices with nonunitary (nonorthogonal) invariant probability distributions. Most of the paper is devoted to the normalized counting measure of eigenvalues (NCM). We describe formulae for the nonrandom limit form (known as the integrated density of states (IDS)) of this random measure corresponding to a variety of the random matrix ensembles and obtained by an unique method, based on the study of the Stieltjes transform of the NCM. We mention also results on the $1/n$ corrections to the IDS and to more complex statistical characteristics of the eigenvalue distribution obtained by the same method.

RÉSUMÉ. – Dans cet article, nous présentons une revue des résultats obtenus pour la distribution statistique des valeurs propres de matrices aléatoires $n \times n$ hermitiennes (réelles symétriques) ayant une distribution de probabilité non nécessairement invariante par les transformations unitaires (orthogonales). La plupart de ce travail est consacrée à la mesure de comptage normalisée des valeurs propres (NCM). Nous en donnons les formules pour la limite non aléatoire $n \to \infty$, connue sous le nom de

* The research described in this publication was made possible in part by Grant U2S00 from the International Science Foundation and by Grant 1/3/132 from the Fund of Fundamental Research of the DKNT of Ukraine.

densité d'état intégrée (IDS), pour une classe de matrices aléatoires sur la base de la même méthode utilisant la transformée de Stieljes de la NCM. Nous mentionnons aussi quelques résultats concernant les corrections d'ordre $1/n$ de l'IDS ainsi que certaines caractéristiques plus fines de la distribution statistique des valeurs propres, qu'il est possible d'obtenir par cette méthode.

1. INTRODUCTION

In recent decades there has been a rising interest in the theory of random matrices. Initiated mainly by statisticans (*see* e.g. book [1]), random matrix theory has been extensively and successfully developed and applied in various branches of theoretical and mathematical physics: statistical nuclear physics ([2], [3]), quantum chaology ([4], [5]), quantum field theory [6], condensed matter theory [7], string theory and two dimensional gravity ([8], [9]) (we mention here only recent reviews and books, containing numerous references and historical remarks).

The internal logic of the random matrix theory (RMT) and nature of its numerous applications suggest three classes of matrices to be studied: real symmetric, Hermitian and quaternion real [2]. In this paper we consider mainly the technically simplest ensembles of $n \times n$ Hermitian matrices, however many of results mentioned below are valid for two other classes of ensembles as well.

A considerable amount of studies in the RMT and its applications deals with the large n behaviour of the normalized counting measure (NCM)

$$N_n(\Delta) = \#\{\lambda_i^{(n)} \in \Delta\}n^{-1}, \qquad \Delta = (a, b) \subset \mathbf{R} \tag{1.1}$$

of eigenvalues $\lambda_1^{(n)} \leq ... \leq \lambda_n^{(n)}$ of random matrices.

Simplest but rather impotant probabilitic characteristics of this random measure are

(a) the expectation value

$$\mathbf{E}\{N_n(\Delta)\}, \tag{1.2}$$

(b) the variance

$$\Delta_n\{N_n^2(\Delta)\} = \mathbf{E}\{N_n^2(\Delta)\} - \mathbf{E}^2\{N_n(\Delta)\}, \tag{1.3}$$

and, more generaly, the correlation function

$$K_n(\Delta_1, \Delta_2) = \mathbf{E}\{N_n(\Delta_1)N_n(\Delta_2)\} - \mathbf{E}\{N_n(\Delta_1)\}\,\mathbf{E}\{N_n(\Delta_2)\} \quad (1.4)$$

The goal of the RMT is to present asymptotic formulae for these and other spectral characteristics of respective random matrix ensembles provided that the probability distribution of matrix elements is given. To make the goal more clear we discuss shortly the archetype case of the Gaussian ensemble introduced by E. Wigner in the early fifties and studied quite thoroughly afterwards (*see* e.g. book [2]). In this case the ensemble probability distribution (the joint distribution of matrix elements) is

$$p_n(M)\mathrm{d}M = Z_n^{-1}\exp\{-1/2\mathrm{Tr}M^2\}\mathrm{d}M, \quad (1.5)$$

where

$$\mathrm{d}M \equiv \prod_{j=1}^{n}\mathrm{d}M_{jj}\prod_{j<k}\mathrm{dRe}M_{jk}\mathrm{dIm}\ M_{jk} \quad (1.6)$$

as the "Lebesgue" measure for Hermitian matrices. The ensemble (1.5), (1.6) is known as the Gaussian unitary ensemble (GUE), because its density is invariant with respect to an arbitrary unitary transformation $M \to UMU^*$, $U \in \mathbf{U}(n)$.

It is well known that [2]:

$$\lim_{n\to\infty}\mathbf{E}\{N_n(\Delta)\} = \int_{\Delta}\rho(\lambda)\mathrm{d}\lambda, \quad (1.7)$$

$$\rho(\lambda) = \pi_+^{-1}\sqrt{4-\lambda^2},\ {}_+\sqrt{x} = [\max(0,x)]^{1/2}; \quad (1.8)$$

$$K_n(\Delta_1,\Delta_2) = \begin{cases} c_1/n(1+O(1/n)), & \Delta_1\cap\Delta_2 = \emptyset, \quad (1.9a) \\ c_2/n^2(1+O(1/n)), & \Delta_1\cap\Delta_2 \neq \emptyset, \quad (1.9b) \end{cases}$$

if $\Delta_{1,2} = (a_{1,2}, b_{1,2})$ are n-independent intervals, and

$$\lim_{n\to\infty} K_n(\Delta_1,\Delta_2) = \rho^2(\lambda_0)\int_{\delta_1\times\delta_2}\mathrm{d}\xi\mathrm{d}\eta\frac{\sin^2\pi(\xi-\eta)}{\pi^2(\xi-\eta)^2} \quad (1.10)$$

if

$$\Delta_{1,2} = \left(\lambda_0 + \frac{\alpha_{1,2}}{n\rho(\lambda_0)}\ ,\ \lambda_0 + \frac{\beta_{1,2}}{n\rho(\lambda_0)}\right) \quad (1.11)$$

$$\rho(\lambda_0) \neq 0, \qquad \delta_{1,2} = (\alpha_{1,2}, \beta_{1,2}). \tag{1.12}$$

Formulae (1.9) for $\Delta_1 = \Delta_2 = \Delta$ imply that the NCM (1.1) converges in probability to the nonrandom limit given by (1.7) and (1.8). More careful analysis snows that if a $n \times n$ Hermitian matrix $M = \{M_{jk}\}_{j,k=1}^n$ is written as

$$M_{jk} = \frac{1}{\sqrt{n}} W_{jk} \tag{1.13}$$

and an infinite collection $\{W_{jk}\}_{j,k=1}^\infty$ of complex Gaussian random variables $W_{jk} = X_{jk} + iY_{jk}$

$$\mathbf{E} = \{W_{jk}\} = 0, \tag{1.14}$$

$$\mathbf{E}\{X_{jk}X_{lm}\} = \delta_{jl}\delta_{km}+\delta_{jm}\delta_{kl}, \quad \mathbf{E}\{Y_{jk}Y_{lm}\} = \delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl} \tag{1.15}$$

is defined on the same probability space, then the NCM (1.1) converges to (1.7) and (1.8) with probability 1 as $n \to \infty$.

The limiting eigenvalue distribution defined in (1.7) and (1.8) is known as the Wigner semicircle law.

Formulae (1.9) and (1.10) demonstrate the existence of two asymptotic regimes for the correlation function (1.4). The first one is the case when we deal with n-independent intervals $\Delta_{1,2}$. According to (1.7) and (1.9a) for $\Delta_1 = \Delta_2 = \Delta$ such intervals contain $O(n)$ eigenvalues as $n \to \infty$. We will call this regime global [2]. In this regime the statistical fluctuations of the NCM vanish (in particular, all moments of $N_n(\Delta)$ factorize in the products of the first moments as $n \to \infty$ up to the order $O(1/n)$ at least). The second asymptotic regime corresponds to intervals whose length is of the order $1/n$ (of the order of mean distanse $(n\rho(\lambda))^{-1}$ between eigenvalues). These intervals contains finitely many eigenvalues as $n \to \infty$. We will call this regime (or limit) local or scaling. According to (1.10) in this regime there exists rather strong correlation between eigenvalues.

The GUE (1.5), (1.6) possesses two properties:

(i) all functionally independent matrix elements $M_{ij}, i \leq j$, are statistically independent;

(ii) the probability distribution is unitary invariant.

It is easy to prove that the converse statement is also true [2]. This result makes natural two classes of the random matrix ensembles generalizing the GUE. The first class consisits of ensembles with statistically independent matrix elements

$$P_n(\mathrm{d}M) = \prod_{i \leq j} P_n^{(ij)}(\mathrm{d}M_{ij}), \tag{1.16}$$

i.e. possesses property (i) of the GUE. The second class consists of ensembles whose probability distribution is unitary invariant. In this paper we resrict ourselves mainly the first class of ensembles and to some their generalizations. As for the second class, we only mention here that a certain subclass of these ensembles defined by the density (*cf.* (1.5))

$$Z_n^{-1} \exp\{-n\mathrm{Tr}V(M)\} \tag{1.17}$$

with a bounded from below polynomial V(t) has been actively studied in the quantum field theory since 80's. We refer the reader to reviews ([6], [8], [9]) devoted to the quantum field theory aspects of these studies and to [25] for some probabilistic and spectral aspects.

2. RESULTS

1. Deformed semicircle law

Consider Hermitian $n \times n$ random matrices of the form

$$H_n = h_n + n^{-1/2} W_n. \tag{2.1}$$

Here h_n is a nonradom matrix such that its NCM $N_n^{(0)}(\mathrm{d}\lambda)$ converges weakly to a limiting measure $N^{(0)}(\mathrm{d}\lambda)$ and W_n is a random matrix whose matrix elements W_{ij} are independent for $i \leq j$ random variables satisfying conditions (1.14), (1.15) and an à la Lindeberg condition

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} \int_{|w|\geq\tau\sqrt{n}} |w|^2 P\{W_{ij} \in \mathrm{d}w\} = 0 \tag{2.2}$$

for all $\tau > 0$. Then the NCM of ensemble (2.1) converges in probability to the nonrandom limiting measure $N(\mathrm{d}\lambda)$

$$N_n(\mathrm{d}\lambda) \xrightarrow{P} N(\mathrm{d}\lambda), \tag{2.3}$$

whose Stieltjes transform

$$f(z) = \int \frac{N(\mathrm{d}\lambda)}{\lambda - z}, \quad \text{for } z \neq 0 \tag{2.4}$$

is the unique solution of the functional equation

$$f(z) = f_0(z + f(z)) \tag{2.5}$$

in the class of functions that are analytic for nonreal z and satisfy the condition

$$\sup_{\eta\geq 1} \eta|f(i\eta)| = 1, \qquad \mathrm{Im}\ f(z)\mathrm{Im}\ z > 0 \tag{2.6}$$

In (2.5) $f_0(z)$ is the Stieltjes transform of the "unpertrubed" limiting measure $N^{(0)}(\mathrm{d}\lambda)$. We will call $N(\lambda)$ the integrated density of states (IDS). This result was proved in [10] for diagonal matrices h_n and under stronger version of the condition (2.2). The case of $h_n = 0$ was thoroughly studied in [13] under condition (2.2). In particular the neccessity of this condition and convergence with probability 1 was proved. The general case (2.1) (including various properties of the solution of functional equation (2.5)) was considered in recent paper [13]. The limiting eigenvalue distribution defined by (2.5) is known as the deformed semicircle law. We see that this limiting measure is independent of a particular form of the probability distribution (1.15) of matrix elements of the matrix W_n provided that condition (2.2) is satisfied. Condition (2.2) is a natural matrix analogue of the well known Lindeberg condition which is necessary and sufficient condition of the validity of the central limit theorem.

The proof of (2.5) given in [13] is based on the study of certain identities for moments of matrix elements of the resolvent $(H_n - z)^{-1}$. This method is described in [12], [14], [16] and can be applied to the study of a wide variety of global spectral characteristics of random matrices and random operators.

We mention here two more results obtained by this method.

2. Band matrices

Consider random matrices of the form

$$b^{-1/2}\phi(\frac{i-j}{b})W_{ij}, \tag{2.7}$$

where $n = 2m + 1$, $|i|, |j| \leq m$, $\phi(t), t \in \mathbf{R}$, is a real-valued, piecewise continuous and even function having a compact support and satisfying the conditions

$$\sup_t |\phi(t)| < \infty, \qquad \int \phi^2(t)\mathrm{d}t = 1,$$

and W_{ij} are as in (2.1). The case where $\phi(t)$ is the characteristic function of the interval $(-1/2, 1/2)$ corresponds to the band matrices [17], whose matrix elements are equal to zero outside of the band of width b centered at the principal diagonal. Assume that $n \to \infty, b \to \infty$ and

$$\frac{n}{b} \to 2\gamma \geq 1. \tag{2.8}$$

Then, according to [16] the NCM of (2.7) satisfies (2.3) in which the limiting distribution is the semicircle law ($N'(\lambda)$ is given by (1.7)) if $\gamma = \infty$ and is different from the semicircle law if $\gamma < \infty$. The Stieltjes transform of the latter limiting measure can be found as the unique solution of a certain nonlinear integral equation. Similar results were obtained in [18], [19], [20] by different methods. Concerning other rather interesting properties of the band matrices see physical papers ([21], [22]).

3. Random matrices with statistically dependent matrix elements

The GUE probability distribution is uniquely determined by conditions (1.14) and (1.15). Consider now the random matrices of the form (2.1) in which

$$(a) \qquad (h_n)_{jk} = h_{j-k} \tag{2.9}$$

(*b*) W_{jk} are complex Gaussian random variables satisfying (1.14) and (*cf.* (1.15))

$$\mathbf{E}\{W_{ij}W_{kl}\} = B_{i-k,j-l}, \qquad \mathbf{E}\{W_{ij}W_{kl}^*\} = B_{i-l,j-k} \tag{2.10}$$

where $B_{jk} = B_{kj} = B_{-jk}$ is a positive definite function and

$$\sum_{i,j\in\mathbf{Z}} B_{ij} = 1.$$

Then according to [13] for the NCM of this ensemble the relation (2.3) is valid and the Stieltjes transform of the IDS $N(\lambda)$ can be found as

$$f(z) = \int_0^1 f(z,p)\mathrm{d}p \tag{2.11}$$

where $f(z,p)$ is a solution of the nonlinear integral equation

$$f(z,p) = \left(\tilde{h} - \int_0^1 \tilde{B}(p,q)f(z,q)\mathrm{d}q\right)^{-1} \tag{2.12}$$

in which

$$\tilde{h}(p) = \sum_{k\in\mathbf{Z}} e^{ikp}h_k, \tag{2.13}$$

$$\tilde{B}(p,q) = \sum_{k,l\in\mathbf{Z}} e^{ikp+ilq}B_{kl}.$$

The equation (2.12) is uniquely solvable in the class of functions $f(z,p)$ bounded in $p \in [0,1]$ for each nonreal z, analytic in z for Im $z \neq 0$ for each $p \in [0,1]$ and such that Im $g(z,p)$Im $z > 0$, Im $z \neq 0$.

It is easy to check that for $B_{ij} = \delta_{ij}$ (2.11) and (2.12) reduce to (2.5), because for difference matrices (2.9) the IDS is [26]

$$N_0(\mathrm{d}\lambda) = mes\{p \in [0,1], \tilde{h}(p) \in \mathrm{d}\lambda\}.$$

Consider now somewhat different random matrices with statistically dependent matrix elements:

$$H_{n,p} = h_n + \frac{1}{n}\sum_{\mu=1}^{p} \tau_\mu(\cdot, \xi_\mu)\xi_\mu, \tag{2.14}$$

where h_n is as in (2.1), τ_μ are independent identically distributed random variables with the common probability distribution $T(\mathrm{d}\tau)$ and $\xi_\mu = \{\xi_{\mu i}\}_{i=1}^{n}$ are independent identically distributed random vectors such that

$$\mathbf{E}\{\xi_{\mu i}\} = 0, \qquad \mathbf{E}\{\xi_{\mu i}^2\} = \frac{1}{n}, \qquad \mathbf{E}\{\xi_{\mu i}^4\} < \infty \tag{2.15}$$

and $\xi_{\mu i}$ are either independent for different i's or weakly dependent random variables. For instance, one can assume that ξ_μ are uniformly distributed over the unit spere in $\mathbf{R}^n$.

Assume that $n \to \infty$, $p \to \infty$ and

$$\frac{p}{n} \to c < \infty. \tag{2.16}$$

Then the limiting relation (2.3) is valid and the Stieltjes transform of the IDS $N(\lambda)$ is a unique solution of the functional equation

$$f(z) = f_0\left(z - c\int \frac{\tau T(\mathrm{d}\tau)}{1 + \tau f(z)}\right) \tag{2.17}$$

in the class (2.6).

It is obvious that for $c < \infty$ matrix elements of (2.14) are statistically dependent. Thus, no wonder that the limiting counting measure is different from the deformed semicircle law (2.5) for all $c < \infty$ and coincides with the latter only in the limit $c \to \infty, \tau \to 0, c\tau^2 = 1$ if we replace the spectral parameter λ by $\lambda - c\tau$.

Formula (2.17) was obtained long time ago [27]. Its proof in that paper was rather long. Simpler proof and under weaker than (2.15) conditions were given in [11] and [14]. In particular, the latter paper contains simple proof of (2.16) for $f_0 = -1/z$, *i.e.* for $h_n = 0$ in (2.14). The advantage of this proof is that it can be used in the cases of rather strong dependence of vectors ξ_μ in (2.14). We mention here the random matrices of the form

$$M_{jk} = \frac{1}{n}\sum_{\mu=1}^{p} g(x_{\mu j} - x_{\mu k}) \tag{2.18}$$

where $g(t), t \in \mathbf{R}$, is a 2π-periodic and continuous at zero function with zero mean

$$\int_0^{2\pi} g(t)\mathrm{d}t = 1,$$

and $\xi_{\mu i}$ are independent random variables uniformly distributed on $[0, 2\pi]$. These matrices (as well as (2.14)) arise in the statistical mechanics of disordered systems (*see* e.g. [28]).

According to [23] if $n \to \infty, p \to \infty$ and $n^{-1}p \to c < \infty$, then the Stieltjes trasform of the IDS satisfies the equation

$$zf(z) = 1 + cg(0)f(z) + cf^2(z)\sum_{l=1}^{\infty}\frac{\tau_l}{1+\tau_l f(z)} \tag{2.19}$$

where τ_l are the Fourier coefficients (real and even) of the function $g(t)$ in (2.18).

The proof of (2.19) in [23] uses some special method. However let us rewrite ensemble (2.18) as follows

$$M_{jk} = \frac{1}{n}\sum_{\mu=1}^{p}\sum_{l\in\mathbf{Z}}\tau_l e^{ilx_{j\mu}}e^{-ilx_{k\mu}},$$

$$M = \sum_{\mu=1}^{p}\sum_{l\in\mathbf{Z}}\tau_l(\cdot, \xi_{l\mu})\xi_{l\mu} \tag{2.20}$$

where $\xi_{l\mu} = n^{-1/2}\{e^{ilx_{j\mu}}\}_{j=1}^{n}$ are n-dimensional vectors. This form of (2.17) is obviously a generalization of (2.14) for the case of two-parameter collection of $\xi's$. Thus by analogy with (2.16) we can write the equation for the Stieltjes transform of the IDS as follows

$$f(z) = -\left(z - c\sum_{l=1}^{\infty}\frac{\tau_l}{1+\tau_l f(z)}\right)^{-1}.$$

It is easy to check that this equation is equivalent to (2.19).

However, unlike (2.14) where all ξ_μ are statisticaly independent, in (2.20) $\xi_{l\mu}$ are independent for different μ and only orthogonal ("weakly" independent) for different l. Nevertheless, it can be shown that the method proposed in [14] to prove (2.17) for ensemble (2.14) with independent ξ's can be extended to the case (2.20) in which vectors $\xi_{l\mu}$ are independent for different μ and orthogonal for different l [34].

4. $1/n$ corrections

We have discussed above the form of the IDS defined by (1.1) and (2.3). The IDS is a simple but rather important example of global quantities. Now we present certain results on the large n behaviour of the correlation function (1.4) of the NCM or, in other words, on corrections to the limiting relation (2.3). Common wisdom of the spectral theory tells us that the optimal strategy to study next terms of the eigenvalue counting measure asymptotics is to pass on to some smothed form of this measure. We will use the Stieltjes transform, *i.e.* we will study corrections to

$$f(z) = \lim_{n\to\infty} f_n(z), \qquad f_n(z) = n^{-1}\mathrm{Tr}(M-z)^{-1}. \tag{2.21}$$

Consider technically simplest case of Hermitean random matrices. More precisely we assume that (*cf.* (1.13))

$$M_{jl} = n^{-1/2}(X_{jl}+Y_{jl}), \qquad j,l = 1...n$$

where $X_{jl} = X_{lj}, Y_{jl} = Y_{lj}$ are independent random variables such that

$$\mathbf{E}\{X_{jl}\} = \mathbf{E}\{Y_{jl}\} = 0;\ \mathbf{E}\{X_{jl}^2\} = (1+\delta_{jl})/2,\ \mathbf{E}\{Y_{jl}^2\} = (1-\delta_{jl})/2;$$

$$\mathbf{E}\{X_{jl}^4\} - 3\,\mathbf{E}^2\{X_{jl}^2\} = \sigma_x,\ \mathbf{E}\{Y_{jl}^4\} - 3\,\mathbf{E}^2\{Y_{jl}^2\} = \sigma_y,\ \sigma_x+\sigma_y = \sigma, j<l;$$

$$\mathbf{E}\{X_{jj}^4\} - 3\,\mathbf{E}^2\{X_{jj}^2\} = \sigma, \quad \sup_{j\le l} \mathbf{E}\{|X_{jl}|^5 + |Y_{jl}|^5\} \le C < \infty$$

where σ and C are n-independent quantities.

According to (2.3) the correlation function

$$K_n(z_1,z_2) = \mathbf{E}\{f_n(z_1)f_n(z_2)\} - \mathbf{E}\{f_n(z_1)\}\,\mathbf{E}\{f_n(z_2)\} \tag{2.22}$$

tend to zero as $n\to\infty$ for Im $z_{1,2}\neq 0$ uniformly in n. It turns out [24] that

$$K_n(z_1,z_2) = n^{-2}F(z_1,z_2) + O(n^{-5/2}), \qquad n\to\infty \tag{2.23}$$

where Im $z_{1,2} \ge 2$ and

$$F(z_1,z_2) = \frac{2(f_1-f_2)^2}{(z_1-z_2)^2(1-f_1^2)(1-f_2^2)} + \frac{2\sigma f_1^3 f_2^3}{(1-f_1^2)(1-f_2^2)} \tag{2.24}$$

Since in (2.23), (2.24) $z_{1,2}$ are n-independent variables, this asymptotic formula corresponds to the global regime and depends on the form of the matrix elements probability distribution (via excess σ) in the full agreement with the general rule of the RMT mentioned at the end of Introduction.

Formulae (2.23), (2.24) are obtained by using the extention of the method which was proposed in [12-16] in order to compute the limiting form of a wide variety of global spectral characteristics of random matrix ensembles, the IDS, in particular. We have seen above a number of results for the IDS, obtained by this rather general method, based on derivation and analysis of identities for certain collection of moments of the Stieltjes transforms of respective quantities. By iterating these identities we can construct corrections (and even expansions) for respective quantities, correlation function (2.23) in particular.

Formulae (2.23), (2.24) are rigorous for $|\text{Im } z_{1,2}| \geq 2$. Now we use these formulae to draw certain nonrigorous conclusions on the large-n form of correlation function $S_n(\lambda_1, \lambda_2)$ of the formal density $\rho_n(\lambda) = n^{-1} \sum_{j=1}^{n} \delta(\lambda - \lambda_j^{(n)})$ of the NCM (1.1), *i.e.* the density of the correlation function (1.4). Since $\rho_n(\lambda) = \pi^{-1}\text{Im } f_n(\lambda + i0) \equiv I_\lambda[f_n(z)]$, where $f_n(z)$ is defined in (2.21), we see, that to obtain $S_n(\lambda_1, \lambda_2)$ we have to use (2.23) and (2.24) outside of the strip $|\text{Im } z_{1,2}| \geq 2$. Nevertheless, since the function $F(z_1, z_2)$ in (2.24) can be obviously continued up to the real axis with respect to both variables $z_{1,2}$, we can apply to this expression the operations I_{λ_1} and $I_{\lambda_2}, \lambda_1 \neq \lambda_2$. We obtain for the leading term:

$$-\rho(\lambda_1)\rho(\lambda_2)\left[\frac{2}{n^2(\lambda_1 - \lambda_2)^2}(4 - \lambda_1\lambda_2) - \frac{4\sigma}{n^2}(2 - \lambda_1^2)(2 - \lambda_2^2)\right] \quad (2.25)$$

This result in the case $\sigma = 0$ (Gaussian distributed matrix elements) was obtained in physical papers ([29], [30]).

Let us consider now the so-called scaling limit, when $\lambda_{1.2} \to \lambda_0$, $n(\lambda_1 - \lambda_2) \to s/\rho(\lambda_0)$ (*cf.* (1.10)). We obtain remarkably simple expression

$$\frac{-1}{2\pi^2 s^2} \quad (2.26)$$

which does not contain the excess σ. On the other hand, according to (1.9) the exact form of this quantity for the Gaussian case is

$$-\frac{\sin^2 \pi s}{\pi^2 s^2}.$$

Comparing these expressions we see that our procedure of computing the density-density correlation function yields for the general case the leading order expression coinciding with the Gaussian correlation function smoothed over energy intervals $\Delta s \gg 1$. This fact can be regarded as a support of the universality conjecture of the RMT mentioned at the end of the Introduction.

Let us mention three more supports of that conjecture. The first one concerns so-called sparse (or diluted) random matrices whose matrix elements are zeroes with probability p/n. According to the physical paper [31] in this case we have the form (2.26) of the density-density correlation function if p is large enough. The second one was obtained in [32] for the ensemble (2.14). In this case the analogues of (2.23) and (2.24) are obtained and it is shown that the scaling limit of these expression coincides with (2.26). The third one was obtained in [33] for the unitary ivariant ensemble (1.16) with an even polynomial $V(\lambda)$. Authors established a number of interesting results concerning the eigenvalue statistics of these ensembles, in particular the relation (2.24) for $\sigma = 0$, *i.e.* the universal form of the correlation function.

ACKNOWLEDGMENTS

I am grateful to the Erwin Schrodinger International Institute for Mathematical Physics where this work was completed.

REFERENCES

[1] T. ANDERSON, An Introduction to Multivariate Statistical Analysis, Wiley, N. Y., 1984.
[2] M. L. MEHTA, Random Matrices. AP, N. Y., 1990.
[3] T. A. BRODY, J. FLORES, J. FRENCH, P. A. MELLO, A. PANDY and S. WONG, *Rev. Mod. Phys.*, Vol. **53**, 1981, pp. 185-480.
[4] O. BOHIGAS, In: "Chaos and Quantum Physics" M.-J. Giannoni, A. Voros, J. Zinn-Justin (Eds.) North-Holland, Amsterdam, 1991.
[5] F. HAAKE, Quantum Signature of Chaos. Springer, Heidelberg, 1991.
[6] D. BESSIS, C. ITZYKSON and J. ZUBER, *Adv. Appl. Math.*, Vol. **1**, 1980, pp. 109-157.
[7] J.-L. PICHARD, In: "Quantum Coherence in Mesoscopic Systems." B.Kramer (Ed.), Plenum, N. Y., 1991, pp. 369-400.
[8] K. DEMETREFI, *Inter. J. Mod. Physics*, Vol. **A8**, 1993, pp. 1185-1244.
[9] A. MIRONOV, *Inter. J. Mod. Phys.*, Vol. **A9**, 1994, pp. 4355-4401.
[10] L. PASTUR, *Teor. Math. Phys.*, Vol. **10**, 1972, pp. 102-112.
[11] V. GIRKO, Spectral Theory of Random Matrices, Moscow: Nauka, 1988, 376 p. (in Russian).
[12] A. KHORUNZHY and L. PASTUR, Preprint of *Inst. for Low Temp. Phys.*, Kharkov, 1989, pp. 3-89 (in Russian).
[13] A. KHORUNZHY and L. PASTUR, Advances in *Soviet Math.*, Vol. **19**, 1994, pp. 97-127.
[14] A. KHORUNZHY, B. KHORUNZHENKO, L. PASTUR and M. SHCHERBINA, Phase Transitions and Critical Phenomena, Vol. **15**, 1992, pp. 73-201.
[15] A. KHORUNZHY, S. MOLCHANOV and L. PASTUR, *Teor. Math. Phys.*, Vol. **92**, 1992.
[16] A. KHORUNZHY and L. PASTUR, *Commun. Math. Phys.*, Vol. **153**, 1993, pp. 605-646.
[17] G. CASATI, F. IZRAILEV and L. MOLINARI, *Phys. Rev. Lett.*, Vol. **64**, 1990, pp. 1851-1855.

[18] G. Casati and V. Girko, Random Oper. and Stochastic Eq. 1, 1993, pp. 15-21.
[19] L. Bogachev and S. Molchanov, *L. Pastur. Matem. Zametki*, Vol. **50**, 1991, pp. 31-42.
[20] M. Kus, M. Lewenstein and F. Haake, *Phys. Rev.*, Vol. **A44**, 1991, pp. 2500-2508.
[21] Y. Fyodorov and A. Mirlin, *Phys. Rev. Lett.*, Vol. **67**, 1991, pp. 2405-2409.
[22] Y. Fyodorov and A. Mirlin, *Phys. Rev. Lett.*, Vol. **69**, 1992, pp. 1093-1096.
[23] L. Pastur and M. Shcherbina, *Math. Phys., Analysis and Geometry*, Vol. **1**, 1994, pp. 131-138.
[24] A. Khorunzhy, B. Khoruzhenko and L. Pastur, Preprint BiBos, 627/3/94, Bielefeld, 1994.
[25] L. Pastur, Preprint IMA, Minneapolis, 1995.
[26] U. Grenander and G. Szegö, Toeplitz Forms and their Applications. UC Press, LA, 1958.
[27] V. Marchenko and L. Pastur, *Mat. Sbornik*, Vol. **72**, 1967, pp. 507-536.
[28] M. Mezard, G. Parisi and M. Virasoro, Spin Glasses and Beyond. World Sci., Singapore, 1986.
[29] J. B. French, P. A. Mello and A. Pandy, *Ann. Phys.*, (N.Y.), Vol. **113**, 1978, pp. 277-290.
[30] A. Pandy, *Ann. Phys.*, (N.Y.), Vol. **134**, 1981, pp. 110-129.
[31] A. Mirlin and Y. Fyodorov, *J. Phys.*, Vol. **A24**, 1991, pp. 227-239.
[32] A. Khorunzhy, L. Pastur and V. Vasilchuk, Prerint of the *Inst. for Low Temp. Phys.*, Kharkov, 1994.
[33] E. Brezin and A. Zee, *Nuclear Physics*, Vol. **B402**, 1993, pp. 613-615.
[34] A. Boutet de Monvel, M. Shcherbina and D. Shepelsky, Preprint BiBoS 671/11/94, Bielefeld, 1994.

Notes added to the proofs: One more evidence of the universality of the semicircle law is the fact that it can also be obtained in the apparently different context of the operator algebras and it is strongly related to the new important notion of free random variables (see e.g. D. Voiculescu, K. Dykema, A. Nica, Free Random Variables. AMS, Providence (1992) and references therein). The random matrix content of this finding is discussed in the recent paper P. New, R. Speicher, *J. Stat. Phys.*, Vol. **80**, 1995, pp. 1287-1308.

(Manuscript received February 4, 1995;
Revised version received December 4, 1995.)