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On the use of modular groups in quantum field theory

by

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ABSTRACT. – A survey of the use of the Tomita-Takesaki modular theory in the theory of local observables will be given. This contains, in particular, the characterization of chiral field theories in terms of the concept of modular inclusions. Moreover, it contains the reconstruction of the translation-group fulfilling spectrum condition out of the modular conjugations of the algebras associated with wedge-domains. Finally we will discuss the connection of modular transformations of the algebras of wedge-domains and Poincaré transformations in higher dimensions. This includes problems related to the wedge-duality.

RÉSUMÉ. – Nous présentons une revue de l'utilisation de la théorie modulaire de Tomita-Takesaki en théorie des observables locales. Elle contient, en particulier, la caractérisation des théories de champs chiraux par l'utilisation des inclusions modulaires. Elle contient aussi la reconstruction du groupe des translations satisfaisant à la condition spectrale à partir de la conjugaison modulaire des algèbres associées aux domaines en forme de coin. Enfin, nous examinons la relation entre la transformation modulaire des algèbres associées à ces domaines et les transformations de Poincaré en dimensions supérieures. Cette partie concerne aussi le problème de la dualité des coins.

Contents

1. HISTORICAL REMARKS

At the Baton Rouge conference 1967 Tomita [To] distributed a preprint containing his theory on the standard form of von Neumann algebras. At the same time Haag, Hugenholtz and Winnink [HHW] published their paper on the description of thermodynamic equilibrium states using the KMS-condition. Probably N. Hugenholtz and M. Winnink have been the first realizing the similarity between certain aspects of their approach and Tomita's theory and hence the importance of this new mathematical theory for theoretical physics. (*See* e.g. the thesis of M. Winnink [Win].) But Tomita's theory became general knowledge only by Takesaki's [Ta] treatment published in the Lecture Notes in Mathematics. Since then, this theory is usually called the Tomita-Takesaki theory.

A central role in this theory is played by faithful normal states of von Neumann algebras. As a consequence of the Reeh-Schlieder theorem [RS] we know that the vacuum-state has this property for every local algebra in quantum field theory. Therefore, several people hoped that the Tomita-Takesaki theory could be made a useful tool also for quantum field theory. It seems as if these wishes will become true only nowadays. The long delay is due to the fact that the modular group has only an abstract definition and therefore, its geometric meaning (if there is any) is not obvious.

The first application of the Tomita-Takesaki theory has been in thermodynamics. But we will not discuss this here. We also will not mention Connes' classification of factors, and the use of this by Driessler, Fredenhagen, Wollenberg, Borchers and Wollenberg, and others. Not mentioned shall be also the sporadic use of the Tomita-Takesaki theory by Eckmann and Osterwalder, Kraus, and many others. We are interested in the connection of the modular group with geometric transformation appearing in quantum field theory. In 1975 Bisognano and Wichmann [BW1,2] discovered that in a Lorenz-covariant Wightman theory the modular group of the algebra connected with a wedge domain coincides with the Lorentz boosts which leave this wedge invariant. This shows that in certain cases the modular group has a geometric meaning. Since in a massless field theory the interaction travels with the speed of light it follows that in such theory the algebras of two regions, which are timelike separated, commute with each other also. Therefore, in such theory the vacuum-vector is cyclic and separating for the algebra associated with the forward light cone. In 1977 Buchholtz [Bu] computed the modular group of this algebra. He found that in this situation the modular group acts as dilations. The conformal group is also the invariance group of the forward tube (Cartan group). Therefore, in a conformal covariant field theory one can obtain the algebra of a double cone by applying a certain conformal transformation to the algebra of a wedge. This transformation transforms also the corresponding modular groups. This computation has been done by Hislop and Longo [HL] and they found that also in this case the modular group acts as a geometric transformation. Knowing the geometric structure of the modular group usually leads to the possibility of answering questions of physical interest. In the case of Bisognano and Wichmann it was the wedge duality which could be proved by using the new techniques.

If the modular group acts geometrically on the algebras associated with the double cones then one can use the modular groups (after proper rescaling) as local dynamics. This means the following: We fix a double cone centered at the origin and look at the increasing family of double cones obtained by scaling. In this case the modular groups of the large double cones (after proper rescaling) approximate the time translation on the small double cone. Therefore, one hopes to be able to approximate the time translations also for general local nets and to obtain by this a "local dynamic".

If one has a local dynamic then it might be possible to define also local Gibbs states and to look at thermodynamical limits of such states. This would serve as the missing link between the abstractly defined KMS states and the local dynamics defined in the vacuum representation. This program has been carried through by Buchholz and Junglas [BJ] but with some extra condition and different local dynamics. The new input they need is the nuclearity condition which allows to define local dynamics in a different way. However, that the hamiltonian dynamics and the modular dynamics cannot be too different is implied by the coincidence of the nuclearity condition for the hamiltonian and the local modular operator. This has been discovered by Buchholz, D'Antoni and Longo [BDL1,2].

My interest in the modular group originates in the observation that the modular group Δ^{it}, applied to expressions of the form $A\Omega$, where A is an element of the algebra (or its commutant), permits some analytic continuation. This, together with the spectrum condition leads to matrix elements of products of operators analytic not only in the translations but also in the modular action. It is known from many experiences in field theory that such analyticity properties can give rise to drastic restrictions. The first result was the observation that looking at functions in space and modular variables one finds expressions which have some periodicity property in a complex direction, [Bch1] 1990. Mostly this is of little use since in general the domain of holomorphy does not contain any line in the direction of periodicity. If there is enough analyticity then interesting conclusions can be drawn. Such a case has been found by the author in 1992 [Bch2]. It implies that in two-dimensional quantum field theory the modular group of the wedge can be interpreted as a Lorentz transformation. This means the translations together with the modular group of the wedge give rise to a representation of the two-dimensional Poincaré group. In addition there is a localization which implies that a Poincaré covariant local net has been constructed. One can start from the right or the left wedge. In general the answer will be different. Only in a theory satisfying wedge-duality the two group representations coincide. If the two representations are different then also the localizations are different. This means if an algebra is localized in one scheme it is not localized in the other scheme.

The use of the modular theory for two-dimensional field theory has been further developped in 1992 by Wiesbrock [Wie1,2,3]. He observed that there are subalgebras of the wedge-algebra which are mapped into themselves by a part of the modular group of the wedge. Using this information he was able to reconstruct the translation group which necessarily fulfills the spectrum condition. Moreover, this information can be used to give an algebraic characterization of chiral field theories. Parts of these results are based on ideas of Schroer. Many of these results are based on the fact that one is dealing with conformal field theory and some of the results stay true also for higher dimensional conformal fields as it has been shown by Hislop and Longo [HL], Brunetti, Guido and Longo [BGL1], Gabbiani and Fröhlich [GF], and others.

Together with the modular structure there is a modular conjugation. It acts like a reflection but only in two dimensions. In higher dimensions we only know this for the Bisognano-Wichmann situation. On the other hand every Poincaré transformation can be composed out of reflections. Buchholz and Summers [BSu1] used this to reconstruct the translations out of these reflections. This can be done if certain additional requirements are fulfilled. These translations will fulfill the spectrum condition. The original requirements of Buchholz and Summers can be weakened as shown in a recent paper [Bch3]. Recently this method has been generalized in order to incorporate the whole Poincaré group [BSu2]. This program is not in a satisfying state since there appear some unsolved cohomological problems. A similar method is due to Brunetti, Guido and Longo [BGL2]. Instead of requiring geometric action of the modular conjugations of the wedge algebras they assumed geometric action of the modular groups (of the wedge algebras) themselves. In this situation the covering of the Poincaré group can be constructed. In both these cases one knows that the modular groups and modular conjugations of the wedges act geometrically in the two-plane defining the wedge. Therefore, the new requirement is about the behaviour of these transformations in the directions perpendicular to the defining two-plane.

Another problem which seems to be solvable with help of the modular group of the wedge algebra is the CPT-theorem in the theory of local observables. One knows from a result of Oksak and Todorov [OT] that for a Wightman field with an infinite number of components the CPT-theorem does not hold. Therefore, it is clear that not every Araki-Haag-Kastler theory is CPT-invariant. The covariance under Poincaré transformations is not sufficient to prove the CPT-theorem. From the investigation of Bisognano and Wichmann [BW1,2] we know that the theory must fulfill at least wedge duality. This means that the commutant of the algebra of a wedge W is the algebra of the opposite wedge W', *i.e.*

$$\mathcal{M}(W)' = \mathcal{M}(W').$$

Whether or not this condition suffices to show the CPT-theorem is not known. A necessary and sufficient condition for the wedge duality of a Poincaré covariant theory of local observables has been given by the author [Bch4].

2. PRELIMINARIES

In this section we want to collect the notations and the necessary background for understanding what is presented below. All results will be cited without proofs.

A. *Tomita-Takesaki theory*

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a von Neumann algebra acting on this space with commutant $\mathcal{M}'$. A vector Ω is cyclic and separating for $\mathcal{M}$ if $\mathcal{M}\,\Omega$ and $\mathcal{M}'\,\Omega$ are dense in $\mathcal{H}$. If these conditions are fulfilled then a modular operator Δ and a modular conjugation J are associated with the pair $(\mathcal{M},\ \Omega)$ such that

(i) Δ is self-adjoint, positive and invertible

$$\Delta\Omega = \Omega, \qquad J\,\Omega = \Omega.$$

(ii) The unitary group Δ^{it} defines a group of automorphisms of $\mathcal{M}$

$$\mathrm{ad}\,\Delta^{it}\,\mathcal{M} = \mathcal{M}, \qquad \forall\, t \in \mathbb{R}.$$

(iii) For every $A \in \mathcal{M}$ the vector $A\,\Omega$ belongs to the domain of $\Delta^{\frac{1}{2}}$.

(iv) The operator J is a conjugation, *i.e.* J is antilinear and $J^2 = 1$, where J commutes with Δ^{it}. This implies the relation

$$\mathrm{ad}\, J\,\Delta = \Delta^{-1}.$$

(v) J maps $\mathcal{M}$ onto its commutant

$$\mathrm{ad}\, J\,\mathcal{M} = \mathcal{M}'.$$

(vi) The operators $S := J\,\Delta^{\frac{1}{2}}$ and $S^* := J\,\Delta^{-\frac{1}{2}}$ have the property

$$S A\,\Omega = A^*\,\Omega, \qquad \forall\, A \in \mathcal{M},$$

$$S^* A'\,\Omega = A'^*\,\Omega, \qquad \forall\, A' \in \mathcal{M}'.$$

For the proof *see* Takesaki [Ta] or textbooks as Bratteli and Robinson [BR] or Kadison and Ringrose [KR].

B. *The theory of local observables*

In the theory of local observables one associates to every bounded open region O in Minkowski space $\mathbb{R}^d$ a C^*-algebra $\mathcal{A}(O)$. For any unbounded open set G the C^*-algebra $\mathcal{A}(G)$ is defined as the C^* inductive limit of the $\mathcal{A}(O)$ with $O \subset G$. These algebras are subject to the following conditions:

(1) They fulfill isotony, *i.e.* if $O_1 \subset O_2$ then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.

(2) They fulfill locality, *i.e.* if O_1 and O_2 are spacelike separated regions then the corresponding algebras commute, *i.e.*

$$A \in \mathcal{A}(O_1), \quad B \in \mathcal{A}(O_2) \text{ implies } [A,\ B] = 0.$$

(3) They fulfill translational covariance, *i.e.* the translation group of $\mathbb{R}^d$ acts as automorphisms on $\mathcal{A}(\mathbb{R}^d)$. For every $a \in \mathbb{R}^d$ there exists an automorphism $\alpha_a \in \mathrm{Aut}\,\mathcal{A}(\mathbb{R}^d)$ with

$$\alpha_a\,\mathcal{A}(O) = \mathcal{A}(O + a).$$

A representation π of $\mathcal{A}(\mathbb{R}^d)$ is called a particle representation if

(i) π a non-degenerate representation on a Hilbert space $\mathcal{H}$.

(ii) There exists a strongly continuous unitary representation of the translation group

$$a \mapsto U(a),$$

such that

(α) The spectrum of $U(a)$ is contained in the forward light cone.

(β) The representation $U(a)$ implements the automorphism α_a, which means that for every $A \in \mathcal{A}(\mathbb{R}^d)$ one has

$$\operatorname{ad} U(a)\, \pi(A) = \pi(\alpha_a A).$$

A representation π is called a vacuum representation if

(iii) π is a particle representation.

(iv) In $\mathcal{H}$, there exists a vector Ω such that:

$$U(a)\,\Omega = \Omega, \qquad \forall\, a \in \mathbb{R}^d.$$

In the following we will always deal with vacuum representations and we set

$$\mathcal{M}(O) = \pi(\mathcal{A}(O))''.$$

For more details about the theory of local observables see the book of R. Haag [Ha].

C. *The result of Bisognano and Wichmann*

The first calculation of a modular group in quantum field theory is due to Bisognano and Wichmann [BW1,2]. This is done in Wightman field theory where one makes the usual assumptions of this theory. In particular they assume that the fields have only a finite number of components, so that the CPT-theorem is valid. Moreover, they assume that they can pass from the algebra of unbounded operators to the algebra of bounded operators. (For details on the last problem *see* e.g. [BY].)

The domain which is of special importance is the *wedge*. Such a domain can be characterized in two ways:

(i) First characterization: Let t, s be two perpendicular vectors in $\mathbb{R}^d$. *i.e.* $(t,\ s) = 0$, such that $t^2 = 1$ and t belongs to the forward light cone and $s^2 = -1$ is spacelike. In this situation one defines

$$W(t,\ s) := \{a \in \mathbb{R}^d;\ |a,\ t)| < (a,\ s)\}.$$

If, for instance, t is the time direction and s is the 1-direction then this becomes $W_R = \{a;\ |a_0| < a_1\}$.

(ii) Second characterization: Every two-plane containing a timelike direction must cut the boundary of the forward light cone in two light rays. Let these light rays be described by the two lightlike vectors ℓ_1, ℓ_2 belonging to the forward light cone. These vectors are different. Now define:

$$W(\ell_1, \ell_2) := \{\lambda_1 \ell_1 - \lambda_2 \ell_2 + \tilde{a};\ \lambda_i > 0,\ (\tilde{a}, \ell_i) = 0,\ i = 1, 2\}.$$

It is easy to see that the two definitions results in the same set of wedges. The two definitions coincide if $\{t, s\}$ and $\{\ell_1, \ell_2\}$ span the same two-plane and if $s = \lambda_1 \ell_1 - \lambda_2 \ell_2$ with positive coefficients.

The opposite wedge of a wedge W is the negative of W and it is usually denoted by W'. It is obtained by replacing s by $-s$ in the first description and by interchanging the two lightlike vectors in the second description.

Given a wedge W there is exactly a one-parametric subgroup of the Lorentz boosts which maps this wedge onto itself. In the above example of the zero- and one-direction the Lorentz transformations are the boosts in the (0, 1)-plane. We will write these transformations (in case the wedge is the right wedge W_r in the (0, 1)-plane) as

$$\Lambda(t) = \begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t \\ -\sinh 2\pi t & \cosh 2\pi t \end{pmatrix}.$$

Bisognano and Wichmann showed the following results:

(i) One denotes by $\underline{f}$ the linear combination of products of test functions

$$\underline{f} = \sum_{n=0}^{N} \prod_{i=1}^{n} f_{n,i}(x_i)$$

and the field operator by

$$\Phi(\underline{f}) = \sum_{n=0}^{N} \prod_{i=1}^{n} \Phi(f_{n,i}(x_i)).$$

We say $\underline{f}$ has its support in G if every $f_{n,i}(x_i)$ has its support in G. If the support of $\underline{f}$ is contained in W_r then

$$U(\Lambda(t))\,\Phi(\underline{f})\,\Omega$$

allows in t an analytic continuation into the strip $-\frac{1}{2} < \Im m\,\tau < 0$.

(ii) Let R be the proper rotation such that RP is the reflection in the (0, 1)-plane where P is the reflection defined by the CPT-operator. Define $J = \Theta\, U(R)$ then

$$JU\left(\Lambda\left(-\frac{i}{2}\right)\right)\Phi(\underline{f})\,\Omega = \Phi(\underline{f})^*\,\Omega.$$

(iii) If the algebras generated by the field operators are affiliated with a local net of von Neumann algebras $\mathcal{M}(O)$ then one has wedge-duality which means

$$\mathcal{M}(W_r)' = \mathcal{M}(W_l).$$

We learn from these results that one can interprete $U(\Lambda(t))$ as the modular group of the wedge algebra and that J can be interpreted as its modular conjugation.

For the modular group of the forward lightcone or the double cone in conformal field theory consult the original papers ([Bu], [HL]).

D. *Remarks on the edge of the wedge problem*

The theory of several complex variables is an important tool in quantum field theory and we assume familiarity with these methods. The situation appearing here (and often in other physical problems) is the edge of the wedge problem. One deals with two analytic functions $f^+(z)$ and $f^-(z)$, $z \in \mathbb{C}^n$ defined in tubes T^+ and $T^- = -T^+$ respectively. The tube T^+ is based on a convex cone $C \subset \mathbb{R}^n$ with apex at the origin and defined by:

$$T^+ = \{z \in \mathbb{C}^n;\ z = x + iy,\ y \in C,\ x \in \mathbb{R}^n\}.$$

One assumes that $f^+(z)$ and $f^-(z)$ both have boundary values $f^+(x)$, $f^-(x)$ respectively (in the sense of distributions) and that these boundary values coincide on some open set $G \subset \mathbb{R}^n$. In this situation one knows from the edge of the wedge theorem [BÖT] that both functions are analytic continuations of each other and are analytic also in a complex neighbourhood of G. Therefore, the common function can be analytically continued into the envelope of holomorphy of $T^+ \cup T^- \cup \mathcal{N}(G)$, where $\mathcal{N}(G)$ denotes the complex neighbourhood of G obtained by the edge of the wedge theorem.

Here we need only the special case of two dimensions where T^+ is the tube based on the first quadrant. The coincidence domain in our case is also the first quadrant. In this case the envelope of holomorphy is easy to compute and the result can be found in [BEGS].

This calculation implies the following property about real lines: If one has a linear manifold which is real for real values of z_1, z_2 and if this real line intersects the first quadrant then all its non-real points belong to the envelope of holomorphy computed before.

If the coincidence domain is the half-space $ax_1 + bx_2 > 0$, $a, b \geq 0$ then the envelope of holomorphy consists of $\mathbb{C}^2$, except for a cut in the variable $az_1 + bz_2$ *i.e.* $\mathbb{C}^2 \backslash \{az_1 + bz_2 \in \mathbb{R}^-\}$.

3. THE FUNDAMENTAL RELATIONS

Large parts of modern investigations concerning modular groups are based on the following results:

THEOREM A. – *Let $\mathcal{M}$, $\mathcal{N}$ be two von Neumann algebras with the common cyclic and separating vector Ω and denote the modular operators and conjugations by $\Delta_{\mathcal{M}}$, $J_{\mathcal{M}}$ and $\Delta_{\mathcal{N}}$, $J_{\mathcal{N}}$, respectively. Let $V \in \mathcal{B}(\mathcal{H})$ be a unitary operator with*

(i) $V\Omega = \Omega$ *and*

(ii) $\mathrm{ad}\, V\mathcal{N} \subset \mathcal{M}$

then the function $V(t) := \Delta_{\mathcal{M}}^{-it} V \Delta_{\mathcal{N}}^{-it}$ has the properties

(*a*) *$V(t)$ is $*$-strong continuous in $t \in \mathbb{R}$.*

(*b*) *$V(t)$ possesses an analytic extension into the strip $S\left(0, \frac{1}{2}\right) = \left\{t \in \mathbb{C};\ 0 < \Im m\, t < \frac{1}{2}\right\}$ as holomorphic function with values in the normed space $\mathcal{B}(\mathcal{H})$*

(*c*) *In this strip we have the estimate*

$$\|V(\tau)\| \leq 1$$

(*d*) *$V(\tau)$ has boundary values at $\Im m\, \tau = 0$ and at $\Im m\, \tau = \frac{1}{2}$ in the $*$-strong topology.*

(*e*) *On the upper boundary the value is given by*

$$V\left(t + i\frac{1}{2}\right) = J_{\mathcal{M}} V(t) J_{\mathcal{N}},$$

hence by (*a*) *also this function is $*$-strong continuous in t.*

3.1 *Remark.* – The functions $V(t)$ fulfill the following chain rule: If $U\mathcal{P}U^* \subset \mathcal{N}$ and $V\mathcal{N}V^* \subset \mathcal{M}$ then $W = VU$ maps $\mathcal{P}$ into $\mathcal{M}$ and one finds

$$W(t) = V(t)\, U(t).$$

Moreover, with $\mathcal{N}' \supset V^* \mathcal{M}' V$ one obtains

$$V^*(t) = V(-t)^*.$$

Notice that the function $V(\bar{z})^*$ is again an analytic function holomorphic in $S\left(-\frac{1}{2}, 0\right) = \left\{t \in \mathbb{C};\ -\frac{1}{2} < \Im m\, t < 0\right\}$. Therefore, the last relation reads in the complex

$$V^*(z) = V(-\bar{z})^*.$$

THEOREM B. – *Let $\mathcal{M}$, $\mathcal{N}$ be two von Neumann algebras with the common cyclic and separating vector Ω. Let $W(s) \in \mathcal{B}(\mathcal{H})$ be an operator family fulfilling the following requirements with respect to the triple $(\mathcal{M}, \mathcal{N}, \Omega)$.*

(i) *For $(s) \in \mathbb{R}$ the operators $W(s)$ are unitary and strongly continuous and fulfill $W(s)\Omega = \Omega$.*

(ii) *The function $W(s)$ possesses an analytic continuation into the strip $S\left(0, \frac{1}{2}\right)$ and has continuous boundary values.*

(iii) *The operators $W\left(\frac{i}{2} + t\right)$ are again unitary.*

(iv) *The function $W(\sigma)$ is bounded, hence $\|W(\sigma)\| \leq 1$.*

(v) *For $t \in \mathbb{R}$ one has*

$$W(t)\,\mathcal{N}\,W(t)^* \subset \mathcal{M} \quad \text{and} \quad W\left(\frac{i}{2} + t\right)\mathcal{N}'\,W\left(\frac{i}{2} + t\right)^* \subset \mathcal{M}'.$$

In this situation the modular operator and the transformation $W(s)$ fulfill the following transformation rules:

$$\Delta_{\mathcal{M}}^{it}\,W(s)\,\Delta_{\mathcal{N}}^{-it} = W(s-t),$$
$$J_{\mathcal{M}}\,W(s)\,J_{\mathcal{N}} = W\left(\frac{i}{2} + s\right).$$

This result as well as that of Theorem A can easily be generalized to the situation where the boundary values are taken in the sense of distributions. We later use Theorem B in the case where the boundary value has eventually one discontinuity.

Special versions of Theorem A can be found in [Bch2] and in [BDL1] and of Theorem B in [Bch2] and [Wie2].

Proof of Theorem. A. – The continuity properties are shown by standard methods. The interesting parts are the analyticity properties. Take $A' \in \mathcal{N}'$. We consider the vector

$$\psi(t, s) := \Delta_{\mathcal{M}}^{-it}\,V\,\Delta_{\mathcal{N}}^{is}\,A'\,\Omega$$

and look at possible analytic continuations. We shall use the following abbreviations: $\sigma_{\mathcal{N}}^{s}(A) = \Delta_{\mathcal{N}}^{is}\,A\,\Delta_{\mathcal{N}}^{-is}$ and $j_{\mathcal{N}}(A) = J_{\mathcal{N}}\,A\,J_{\mathcal{N}}$. Since $\Delta_{\mathcal{N}}^{is}$ is the modular group for $\mathcal{N}$ we can analytically continue $\psi(t, s)$ in the variable s into the strip $S\left(0, \frac{1}{2}\right)$. This continuation has continuous boundary values and we find

$$\psi\left(t, s + i\frac{1}{2}\right) = \Delta_{\mathcal{M}}^{-it}\,V\,j_{\mathcal{N}}(\sigma_{\mathcal{N}}^{s}(A'^*))\,\Omega.$$

The operator $j_{\mathcal{N}}(\sigma^s_{\mathcal{N}}(A'^*))$ belongs to the algebra $\mathcal{N}$. Therefore, $V\, j_{\mathcal{N}}(\sigma^s_{\mathcal{N}}(A'^*))\, V^*$ belongs to $\mathcal{M}$. Now we can continue the vector $\psi\left(t,\, s + i\,\frac{1}{2}\right)$ in the variable t into the strip $S\left(0,\, \frac{1}{2}\right)$. Again we obtain continuous boundary values and find

$$\begin{aligned}\psi\left(t + i\,\frac{1}{2},\, s + i\,\frac{1}{2}\right) &= j_{\mathcal{M}}(\sigma^{-t}_{\mathcal{M}}(V\, j_{\mathcal{N}}(\sigma^s_{\mathcal{N}}(A'))\, V^*))\, \Omega \\ &= \Delta^{-it}_{\mathcal{M}}\, j_{\mathcal{M}}\, V\, j_{\mathcal{N}}\, \Delta^{is}_{\mathcal{N}}\, A'\, \Omega.\end{aligned}$$

Notice that the operator $\Delta^{-it}_{\mathcal{M}}\, J_{\mathcal{M}}\, V\, J_{\mathcal{N}}\, \Delta^{is}_{\mathcal{N}}$ is unitary, so that we obtain

$$\left\|\psi\left(t + i\,\frac{1}{2},\, s + i\,\frac{1}{2}\right)\right\| = \|A'\,\Omega\|.$$

Now using the Malgrange-Zerner-Theorem (*see* e.g. Epstein [Ep] we see that $\psi(t,\, s)$ has an analytic continuation into the tube based on the triangle defined by the points

$$\Im m\,(t,\, s) = \left\{(0,\, 0),\, \left(0,\, \frac{1}{2}\right),\, \left(\frac{1}{2},\, \frac{1}{2}\right)\right\}.$$

The strip $S\left(0,\, \frac{1}{2}\right)$ of the complex manifold $t = s$ belongs to the boundary of the above tube. Therefore, we have to prove that the function stays analytic in the remaining variable. To this end we define $\phi(x,\, y) = \psi(x + y,\, x - y)$. In these variables we have analyticity in the tube based on the triangle with the corners

$$\Im m\,(x,\, y) = \left\{(0,\, 0),\, \left(\frac{1}{4},\, \frac{1}{4}\right),\, \left(\frac{1}{2},\, 0\right)\right\}.$$

The functions $\phi(x,\, y)$ are for fixed y analytic in the strip $S\left(-\Im m\, y,\, \frac{1}{2} + \Im m\, y\right)$. We choose $\Re e\, y = 0$ and with $\Im m\, y$ we want to reach 0. For fixed $\eta < 0$ the function $\phi(x,\, i\,\eta)$ has continuous boundary values. Since the strip can be transformed bi-holomorphically onto the unit circle it follows that $\phi(x,\, i\,\eta)$ can be expressed with a transformed Cauchy kernel as an integral over the boundary values. This kernel is $\mathcal{L}^1$ on the boundary and depends real-analytically on η in a neighbourhood of zero. On the boundary we are only dealing with one modular group so that we know that the boundary values are continuous also for $\eta \to 0$. Hence the "Cauchy" integral converges to a function which is analytic in the variable t. The function $\psi(\tau,\, \tau)$ is analytic in the strip $S\left(0,\, \frac{1}{2}\right)$. On the lower and upper boundary we find $\|\psi(t,\, t)\| = \left\|\psi\left(t + i\,\frac{1}{2},\, t + i\,\frac{1}{2}\right)\right\| = \|A'\,\Omega\|$.

This implies by the maximum modulus theorem $||V(\tau) A' \Omega|| \leq ||A' \Omega||$. This shows that $V(\tau)$ is bounded in norm by 1 on $\mathcal{M}' \Omega$. Since this set of vectors is dense in $\mathcal{H}$ it can be continuously extended to a bounded operator with the same norm. The value of $V(\tau)$ at the upper boundary follows from the above remark. If τ is an interior point of $S\left(0, \frac{1}{2}\right)$ with distance d from the boundary and if we make a power series expansion around this point then we find that the n-th coefficient is a bounded operator with norm at most d^{-n}. Therefore, the power series is converging in the norm topology. □

The proof of Theorem B will be split into two parts. We start with the following result:

3.2. PROPOSITION. – *Let $W(s)$ fulfill the requirements listed in Theorem B. Then the operator function*

$$W(s, t) := \Delta_{\mathcal{M}}^{-it} W(s) \Delta_{\mathcal{N}}^{it}$$

has an analytic continuation into the tube domain based on the quadrangle with the four corners

$$\Im m(\sigma, \tau) = \left\{(0, 0), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right), \left(-\frac{1}{2}, \frac{1}{2}\right)\right\}.$$

In this domain we have the estimate

$$||W(\sigma, \tau)|| \leq 1.$$

At the four corners it takes the values

$$W(s, t) = \Delta_{\mathcal{M}}^{-it} W(s) \Delta_{\mathcal{N}}^{it}$$

$$W\left(s, \frac{i}{2} + t\right) = \Delta_{\mathcal{M}}^{-it} j_{\mathcal{M}} W(s) j_{\mathcal{N}} \Delta_{\mathcal{N}}^{it}$$

$$W\left(\frac{i}{2} + s, t\right) = \Delta_{\mathcal{M}}^{-it} W\left(\frac{i}{2} + s\right) \Delta_{\mathcal{N}}^{it}$$

$$W\left(-\frac{i}{2} + s, \frac{i}{2} + t\right) = \Delta_{\mathcal{M}}^{-it} j_{\mathcal{M}} W\left(\frac{i}{2} + s\right) j_{\mathcal{N}} \Delta_{\mathcal{N}}^{it}$$

Proof. – This result is a consequence of the last theorem together with the Malgrange-Zerner theorem. First we can continue for real s in the variable t into the strip $S\left(0, \frac{1}{2}\right)$ and for real t in the variable s into the strip $S\left(0, \frac{1}{2}\right)$. For $\tau = \frac{i}{2} + t$ we have the expression $W\left(s, \frac{i}{2} + t\right) = \Delta_{\mathcal{M}}^{-it} J_{\mathcal{M}} W(s) J_{\mathcal{N}} \Delta_{\mathcal{N}}^{it}$ which, because $J_{\mathcal{M}}$, $J_{\mathcal{N}}$ are antiunitary, has an analytic continuation in s into the strip $S\left(-\frac{1}{2}, 0\right)$. Since one has to take into consideration that J is an antilinear operator it follows that the function $J_{\mathcal{M}} W(\overline{\sigma}) J_{\mathcal{N}}$ is analytic. The values at the corners are obtained by simple computations. □

Proof of Theorem. B. – Since the function $W(s, t)$ is defined and bounded it suffices to look at its matrix-elements for studying its properties. We choose two operators $A \in \mathcal{N}$ and $B \in \mathcal{M}'$ and introduce two functions

$$F^{+}(s, t) := (\Omega, BW(s, t) A\Omega),$$
$$F^{-}(s, t) := (\Omega, AW(s, t)^{*} A\Omega).$$

It is clear that the operator function $W(s, t)^{*}$ is analytic in the conjugate complex of the domain of analyticity for $W(s, t)$. Now we look at the points where the two functions coincide. We have the following identities:

$$F^{+}(s, t) = F^{-}(s, t), \qquad s, t \in \mathbb{R},$$
$$F^{+}\left(-\frac{i}{2} + s, \frac{i}{2} + t\right) = F^{-}\left(\frac{i}{2} + s, -\frac{i}{2} + t\right), \qquad s, t \in \mathbb{R}.$$

Since $A \in \mathcal{N}$ one has $W(s, t) AW(s, t)^{*} \in \mathcal{M}$. This implies the first statement. Next the operator

$$\Delta_{\mathcal{M}}^{-it} J_{\mathcal{M}} W\left(\frac{i}{2} + s\right) J_{\mathcal{N}} \Delta_{\mathcal{N}}^{it} A \left\{\Delta_{\mathcal{N}}^{-it} J_{\mathcal{N}} W\left(\frac{i}{2} + s\right) J_{\mathcal{M}} \Delta_{\mathcal{M}}^{it}\right\}^{*}$$

belongs again to $\mathcal{M}$ since J interchanges the algebra and its commutant and $W\left(\frac{i}{2} + s\right)$ maps the commutant of $\mathcal{N}$ into the commutant of $\mathcal{M}$. From this follows the second statement. As a consequence of this result we see that the two functions are two different representations of one function $F(s, t)$ which is periodic, *i.e.*

$$F(s, t) = F(s + i, t - i).$$

Moreover, by the edge of the wedge theorem and the tube theorem we find that this function is analytic in the tube-domain

$$\left\{-\frac{1}{2} < \Im m\, \sigma + \Im m\, \tau < \frac{1}{2}\right\}.$$

Since the operator $W(s, t)$ is bounded in norm by 1 we see that the function $F(s, t)$ is bounded by $\max\{\|A\Omega\|\,\|B^{*}\Omega\|, \|A^{*}\Omega\|\,\|B\Omega\|\}$. Therefore, the function F is entire and bounded in the direction of periodicity which implies that the function $F(\sigma, \tau)$ depends only on one variable, *i.e.*

$$F(\sigma, \tau) = F(\sigma + z, \tau - z), \qquad z \in \mathbb{C}.$$

In the above equation for $F(\sigma, \tau)$ we choose real arguments and for z the value of t. Then we get $F(s, t) = F(s+t, 0)$. Inserting the expression for F we find $(\Omega, B\,\Delta_{\mathcal{M}}^{-it}\,W(s)\,\Delta_{\mathcal{N}}^{+it}\,A\,\Omega) = (\Omega\,, BW(s+t)\,A\,\Omega)$. This is the first statement in matrix elements. Since Ω is cyclic and separating the equation for the matrix elements becomes an equation for the operators. This yields the first relation. The second relation is obtained by choosing the value $\frac{i}{2}$ for τ. □

4. ONE-DIMENSIONAL SITUATION

In the two-dimensional theories the structure of quantum fields is much more transparent than in the general theory because of the product structure of the forward light cone. This simplification has its counterpart also in the structure of the modular groups. Therefore, we start with this subject. But we will simplify the situation even more by looking at theories depending only on one variable. Remember that in classical physics the solutions of the free wave equations in two dimensions split into the sum of two solutions depending only on one of the light cone coordinates $x_0 \pm x_1$. The same is true for certain quantum fields. First we shall look at theories depending only on one light cone variable. For the sake of definiteness we will deal with "right movers". First we want to show the following general result:

4.1. THEOREM. – *Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω and $\Delta_{\mathcal{M}}$, $J_{\mathcal{M}}$ be the modular operator and conjugation of the pair $(\mathcal{M}, \Omega)$. Then*

(*a*) *the following statements are equivalent:*

(i) *There exists a unitary group $U(s)$ with positive generator fulfilling*

$$U(s)\,\mathcal{M}\,U(-s) \subset \mathcal{M} \quad for\ s \geq 0, \qquad and \qquad U(s)\,\Omega = \Omega.$$

(ii) *There exists a proper subalgebra $\mathcal{N} \subset \mathcal{M}$ with Ω as cyclic vector, such that*

$$\Delta_{\mathcal{M}}^{it}\,\mathcal{N}\,\Delta_{\mathcal{M}}^{-it} \subset \mathcal{N} \qquad for \quad t \leq 0. \tag{*}$$

(*b*) *If the conditions* (*a*) *are fulfilled then the modular group of $\mathcal{M}$ and the translations $U(a)$ fulfill the relations*

$$\Delta_{\mathcal{M}}^{it}\,U(\lambda)\,\Delta_{\mathcal{M}}^{-it} = U(e^{-2\pi t}\,\lambda),$$
$$J_{\mathcal{M}}\,U(\lambda)\,J_{\mathcal{M}} = U(-\lambda).$$

If equation $(*)$ *is fulfilled then the group* $U(\lambda)$ *can be normalized such that the following equation holds:*

$$\mathcal{N} = U(1)\,\mathcal{M}\,U(-1).$$

(*c*) *If* (*a*) *or* (*b*) *is fulfilled and if the generator of* $U(a)$ *is bounded then* $U(a)$ *is identical to* $\mathbb{1}$.

The implication (a, i) $\rightarrow$ (b) and hence $\rightarrow$ (a, ii) can be found in [Bch2]. The implication (a, ii) $\rightarrow$ (a, i) is due to Wiesbrock [Wie2]. The situation described in (ii) of Theorem 4.1 is called *half-sided modular inclusion.* If the sign is of some importance we speak about $\pm$ half-sided modular inclusion. The second result deals with the uniqueness of the interpolating family of von Neumann algebras. It is generally believed and proved under additional assumptions that all local von Neumann algebras are the same. Therefore, the theory is not defined by one algebra only but one needs at least two of them. In this situation we have

4.2. THEOREM. – *Let* $\mathcal{M}_a$ *and* $\mathcal{N}_a$, $a \in \mathbb{R}$ *be two families of von Neumann algebras on the Hilbert spaces* $\mathcal{H}_m$, $\mathcal{H}_n$ *with the cyclic and separating vector* Ω_m, Ω_n, *respectively, which are introduced as follows. Assume there are continuous unitary one-parametric groups* $U_{\mathcal{M}}(a) U_{\mathcal{N}}(a)$ *both fulfilling spectrum condition and define*

$$\mathcal{M}_a = U_{\mathcal{M}}(a)\,\mathcal{M}_0\,U_{\mathcal{M}}(-a), \qquad \mathcal{N}_a = U_{\mathcal{N}}(a)\,\mathcal{N}_0 = U_{\mathcal{N}}(-a).$$

Assume that:

$$\mathcal{M}_a \subset \mathcal{M}_b, \qquad \mathcal{N}_a \subset \mathcal{N}_b \quad for\ a > b.$$

If we moreover assume that there exists a unitary map W *with* $W\,\mathcal{H}_n = \mathcal{H}_m$ *and* $W\,\Omega_n = \Omega_m$ *and in addition*

$$\mathcal{M}_0 = W\,\mathcal{N}_0\,W^*, \qquad and \qquad \mathcal{M}_1 = W\,\mathcal{N}_1\,W^*,$$

then it follows that:

$$\mathcal{M}_a = W\,\mathcal{N}_a\,W^*, \qquad \forall\, a \in \mathbb{R}$$
$$U_m(a) = W\,U_n(a)\,W^*.$$

The same is true if we require that $\mathcal{M}_0$ *and* $\mathcal{M}_1$ *as well as* $\mathcal{N}_0$ *and* $\mathcal{N}_1$ *both fulfill modular inclusion for negative arguments of the modular groups.*

The uniqueness of the interpolating families is taken from [Bch3].

Proof of Theorem. 4.1. – (a, i) $\rightarrow$ (b). If $U(a)$ fulfills the assumptions of (a, i) then it has an analytic continuation into the upper half plane. For positive arguments $U(a)$ maps $\mathcal{M}$ into itself by assumption and hence

$U(a)$ maps $\mathcal{M}'$ into itself for negative arguments. Hence we can apply Theorem B to the family $W(s) = U(e^{2\pi s})$ and obtain together with the analyticity of $U(a)$

$$\begin{aligned} \operatorname{ad} \Delta^{it} U(e^{2\pi s}) &= U(e^{2\pi (s-t)}), \\ \operatorname{ad} \Delta^{it} U(a) &= U(e^{-2\pi t} a), \\ \operatorname{ad} JU(a) &= U(-a). \end{aligned}$$

(b) → (a, ii). If we set $\mathcal{N} = U(1)\mathcal{M}U(-1)$ then we obtain $\Delta^{it}\mathcal{N}\Delta^{-it} = U(e^{-2\pi t})\mathcal{M}U(-e^{-2\pi t})$. This is contained in $\mathcal{N}$ for negative values of t.

(a, ii) → (a, i). Assume this to the true and assume $\mathcal{N} = U(1)\mathcal{M}U(-1)$ then one has $\Delta_{\mathcal{M}}^{-it}\Delta_{\mathcal{N}}^{it} = \Delta_{\mathcal{M}}^{-it}U(1)\Delta_{\mathcal{M}}^{it}U(-1) = U(e^{2\pi t}-1)$. Therefore, one has to show that the product $\Delta_{\mathcal{M}}^{-it}\Delta_{\mathcal{N}}^{it} =: D(t)$ commutes for different values of the arguments. For this one uses Theorem B again. In the situation $\mathcal{N} \subset \mathcal{M}$ one can apply Theorem A with $V = \mathbb{1}$ and will find that $D(t)$ has an analytic continuation into the strip $S\left(0, \frac{1}{2}\right)$. On both boundaries the expression is unitary. By assumption of the modular inclusion one obtains:

$$\begin{aligned} D(t)\mathcal{N}D(t)^* &\subset \mathcal{N}, && \text{for} \quad t \geq 0 \\ D(t)\mathcal{N}'D(t)^* &\subset \mathcal{N}', && \text{for} \quad t \leq 0 \\ D\left(\frac{i}{2}+t\right)\mathcal{N}'D\left(\frac{i}{2}+t\right)^* &\subset \mathcal{N}', && \text{for} \quad t \in \mathbb{R} \end{aligned}$$

The last statements follow from $D\left(\frac{i}{2}+t\right) = J_{\mathcal{M}}D(t)J_{\mathcal{N}}$. $J_{\mathcal{N}}$ maps $\mathcal{N}'$ onto $\mathcal{N}$, $D(t)$ maps this into $\mathcal{M}$ and finally $J_{\mathcal{M}}$ maps this into $\mathcal{M}' \subset \mathcal{N}'$. Consequently one can apply Theorem B to the expression

$$W(s) = D\left(\frac{1}{2\pi}\log(e^{2\pi s}+1)\right),$$

which leads to the relation

$$\Delta_{\mathcal{N}}^{it} D\left(\frac{1}{2\pi}\log(e^{2\pi s}+1)\right)\Delta_{\mathcal{N}}^{-it} = D\left(\frac{1}{2\pi}\log(e^{2\pi (s-t)}+1)\right).$$

Multiplying this equation from the left with $\Delta_{\mathcal{M}}^{-it}$ and from the right with $\Delta_{\mathcal{N}}^{it}$ then we get with $e^{2\pi x} = e^{2\pi s} + 1$

$$\begin{aligned}\Delta_{\mathcal{M}}^{-it}\,\Delta_{\mathcal{N}}^{it}\,\Delta_{\mathcal{M}}^{-ix}\,\Delta_{\mathcal{N}}^{ix} &= D\left(\frac{1}{2\pi}\log\left(e^{2\pi(s-t)}+1\right)+t\right)\\ &= D\left(\frac{1}{2\pi}\log\left(e^{2\pi s}+e^{2\pi t}\right)\right)\\ &= D\left(\frac{1}{2\pi}\log\left(e^{2\pi x}+e^{2\pi t}-1\right)\right).\end{aligned}$$

Since this expression is symmetric in x and t we obtain the commutativity of the operator family $D(t)$. If we set $U(e^{2\pi t}-1) = D(t)$ then the above equation reads

$$U(e^{2\pi x}-1)\,U(e^{2\pi t}-1) = U(e^{2\pi x}+e^{2\pi t}-2).$$

This shows that $U(a)$ is additive for positive arguments and by analytic continuation it follows that it is an additive unitary group with positive generator. It remains to show that $\mathcal{N}$ is of the form $U(1)\,\mathcal{M}\,U(-1)$. To this end we introduce:

4.3. Definition. – If the modular inclusion stated in Theorem 4.1 is fulfilled then we put

$$\begin{aligned}\mathcal{N}(e^{-2\pi t}) &= \Delta_{\mathcal{M}}^{it}\,\mathcal{N}\,\Delta_{\mathcal{M}}^{-it},\\ \mathcal{N}(-e^{-2\pi t}) &= \{\Delta_{\mathcal{M}}^{it}\,J_{\mathcal{M}}\,\mathcal{N}\,J_{\mathcal{M}}\,\Delta_{\mathcal{M}}^{-it}\}'.\\ \mathcal{N}(0) &= \{\bigcup_t \mathcal{N}(e^{-2\pi t})\}''.\end{aligned}$$

Next we will show that this is a good definition.

4.4. Lemma. – *The von Neumann algebras $\mathcal{N}(t)$, defined above, fulfill the following relations:*

$$\begin{aligned} t_1 < t_2 \quad &\textit{implies}\ \mathcal{N}(t_1) \supset \mathcal{N}(t_2),\\ &\mathcal{N}(0) = \mathcal{M},\\ t_1 < 0 \quad &\textit{implies}\ \mathcal{N}(t) \supset \mathcal{M},\\ t_1 > 0 \quad &\textit{implies}\ \mathcal{N}(t) \subset \mathcal{M}.\end{aligned}$$

Proof. – Because of modular inclusion we have $\mathcal{N}(t) \subset \mathcal{N}(1)$ for $t > 1$. Since unitary transformations preserve order we obtain the first statement for positive arguments. Moreover, $\mathcal{N} \subset \mathcal{M}$ implies $\mathcal{N}(t) \subset \mathcal{M}$ for positive t. For negative t we obtain the corresponding statements by the properties of $J_{\mathcal{M}}$. Finally the algebra $\mathcal{N}(0)$ is a subalgebra of $\mathcal{M}$ which is invariant under the modular group of $\mathcal{M}$ and hence coincides with $\mathcal{M}$. □

Proof of Theorem. 4.1, *continuation.* – From the observation that $U(a)$ is a continuous group it follows that the family $\mathcal{N}(t)$ is also continuous at zero. (In this context we understand that the relations $\mathcal{N}(0) = \{\bigcup_{t>0} \mathcal{N}(t)\}''$ and $\mathcal{N}(0) = \bigcap_{t<0} \mathcal{N}(t)$ hold.) Hence we obtain

$$\mathcal{M} = U(-1)\mathcal{N}U(1).$$

(c) If $U(a)$ has a bounded generator then $U(a)$ is entire analytic and hence the equation $[B, U(a)AU(-a)] = 0$ for $B \in \mathcal{M}'$ and $A \in \mathcal{M}$ is true for arbitrary a if it is true for positive a. This shows that $U(a)$ defines an automorphism of $\mathcal{M}$. Since the generator is bounded it defines an inner automorphism ([Ka], [Sak]) and since the state defined by Ω is left this automorphism is the identity. Hence also $U(a)$ is the identity.

This proves Theorem 4.1. □

It remains to show the statements concerning the uniqueness. Also the proof of Theorem 4.2 needs some preparation. We start with

4.5. LEMMA. – *Let $\mathcal{M}_a$ and $\mathcal{N}_a$ be two families with the same cyclic and separating vector Ω. Assume they fulfill the condition of Theorem* 4.1. *If we assume $\mathcal{M}_0 = \mathcal{N}_0$ and $\mathcal{M}_1 = \mathcal{N}_1$ then follows*

$$\mathcal{M}_i = \mathcal{N}_i, \quad for\ all\ i \in \mathbb{Z}.$$

Proof. – From Theorem 4.1 (*b*) we know that the modular conjugation of the algebra $\mathcal{M}_i$ maps the algebra $\mathcal{M}_{i+k}$ onto the commutant of the algebra $\mathcal{M}_{i-k}$. Therefore, if $\mathcal{M}_i = \mathcal{N}_i$ and $\mathcal{M}_{i+k} = \mathcal{N}_{i+k}$ then one has also $\mathcal{M}_{i-k} = \mathcal{N}_{i-k}$ by the identity of the modular conjugations. The smallest set containing 1 and 0, which is invariant under the above map, is the set of positive and negative integers. □

Proof of Theorem. 4.2. – From Lemma 4.5 we know that the two families of algebras coincide for all entire values of a. Let $U(a)$ be the unitary of the family $\mathcal{M}_a$ and $V(a)$ that of the family $\mathcal{N}_a$. Take $A \in \mathcal{M}_0$ and $B \in \mathcal{M}_0'$ and consider the two functions

$$F^+(a, b) = (\Omega, BU(a)V(b)A\Omega)$$
$$= (\Omega, BU(a)V(b)AV(-b)U(-a)\Omega)$$

$$F^-(a, b) = (\Omega, AV(-b)U(-a)B\Omega)$$
$$= (\Omega, U(a)V(b)AV(-b)U(-a)\Omega).$$

F^+ is the boundary value of an analytic function holomorphic in the tube

$$T^+ = \{z_1,\, z_2 \in \mathbb{C}^2;\ \Im m\, z_1 > 0,\ \Im m\, z_2 > 0\}.$$

F^- has an analytic extension into the tube $T^- = -T^+$. By the assumption of isotony it follows that the two functions coincide for $a > 0,\ b > 0$. However, since we know that $\mathcal{M}_i$ and $\mathcal{N}_i$ coincide for $i \in \pm\mathbb{N}$ we obtain a larger coincidence-domain namely, all points which are larger than a point $(i,\ -i),\ i \in \mathbb{Z}$ with respect to the order given by the first quadrant. The boundary of the coincidence domain is the sawtooth curve obtained by taking the boundary of the union of all the translated first quadrants. We want to enlarge this domain by computing the envelope of holomorphy. From the Remarks 2.D we know that the complex points of the line $a + b = C > 0$ belong to the envelope of holomorphy. Using these straight lines we obtain by the continuity theorem that the coincidence domain consists of all points $a + b > 0$. Again by the Remarks 2.D we obtain that the function is entire in the difference-variable $a - b$. Since it is bounded we obtain $U(a)\, V(-a) = 1$. Hence $U(a)$ and $V(a)$ coincide. Since $\mathcal{M}_0$ and $\mathcal{N}_0$ are the same it follows that the two families are the same. If the two families of von Neumann algebras are in different Hilbert spaces then we apply the above result to the families $\mathcal{M}_a$ and $W\mathcal{N}_a W^*$ and obtain the Theorem 4.2.

We end this section by looking at the case that we are dealing with a $+$ half-sided modular inclusion.

4.6. THEOREM. – *Let $\mathcal{M} \supset \mathcal{N}$ be two von Neumann algebras with cyclic and separating vector Ω, and Δ, J be the modular operator and conjugation of the pair $(\mathcal{M},\ \Omega)$. Assume, moreover,*

$$\Delta_{\mathcal{M}}^{it}\, \mathcal{N}\, \Delta_{\mathcal{M}}^{-it} \subset \mathcal{N} \qquad \textit{for} \quad t \geq 0,$$

then there exists a unitary group $U(s)$ with negative generator fulfilling

$$U(s)\, \mathcal{M}\, U(-s) \subset \mathcal{M} \quad \textit{for } s \geq 0, \qquad \textit{and} \qquad U(t)\, \Omega = \Omega.$$

Between the given two algebras one has the relation

$$\mathcal{N} = U(1)\, \mathcal{M}\, U(-1).$$

Proof. – Let us look at the operator function

$$D(t) = \Delta_{\mathcal{M}}^{-it}\, \Delta_{\mathcal{N}}^{it}.$$

This function has the following transformation properties:

$$\begin{aligned} \operatorname{ad} D(t)\mathcal{N} &\subset \mathcal{N}, & t<0,\\ \operatorname{ad} D(t)\mathcal{N}' &\subset \mathcal{N}', & t>0\\ \operatorname{ad} D\left(\frac{i}{2}+t\right)\mathcal{N}' &\subset \mathcal{N}', & \forall t\in\mathbb{R}. \end{aligned}$$

The first two equations follow by assumptions. The third equation is a consequence of the relation

$$\operatorname{ad}\{J_0\,\Delta_{\mathcal{M}}^{-it}\,\Delta_{\mathcal{N}}^{it}\,J_{\mathcal{N}}\}\,\mathcal{N}'\subset\mathcal{M}'\subset\mathcal{N}'.$$

This implies that for $t\leq 0$ $D(t)$ maps $\mathcal{N}$ into itself and for all other cases it maps $\mathcal{N}'$ into itself. As in Theorem 4.1 this implies that $D(t)$ is a commuting family of unitaries. Precisely speaking, the operator $W(s)$ of Theorem B is

$$W(s)=D\left(\frac{1}{2\pi}\log\frac{e^{2\pi s}}{e^{2\pi s}+1}\right)$$

which implies according to the same manipulation as in Theorem 4.1

$$D(t)\,D\left(\frac{1}{2\pi}\log\frac{e^{2\pi s}}{e^{2\pi s}+1}\right)=D\left(\left(\frac{1}{2\pi}\log\frac{e^{2\pi(s-t)}}{e^{2\pi(s-t)}+1}\right)+t\right).$$

Inserting $\dfrac{e^{2\pi s}}{e^{2\pi s}+1}=e^{2\pi x}$ which implies $e^{2\pi s}=\dfrac{e^{2\pi x}}{1-e^{2\pi x}}$ we find

$$D(t)\,D(x)=D\left(\frac{1}{2\pi}\log\frac{e^{2\pi(x+t)}}{e^{2\pi x}+e^{2\pi t}-e^{2\pi t}\,e^{2\pi x}}\right).$$

Defining $D(t)=:U(e^{-2\pi t}-1)$ then the same arguments as used in the proof of Theorem 4.1 show that $U(t)$ gives rise to an additive unitary but this time with a negative generator. Moreover one finds

$$\mathcal{N}=U(1)\,\mathcal{M}\,U(-1),\qquad \text{and}\qquad U(t)\,\mathcal{M}\,U(-t)\subset\mathcal{M}\quad\text{for}\quad t\geq 0.$$

Hence Theorem B can be applied to this algebra together with the group $U(t)$ and we obtain

$$\begin{aligned} \Delta_0^{it}\,U(s)\,\Delta_0^{-it}&=U(e^{2\pi t}\,s),\\ J_0\,U(s)\,J_0&=U(-s).\quad\square \end{aligned}$$

5. CHIRAL QUANTUM FIELDS

We start again with the assumption that we are dealing with a half sided modular inclusion, *i.e.* with a triple $(\mathcal{M} \supset \mathcal{N}, \Omega)$ such that $\Delta_{\mathcal{M}}^{it} \mathcal{N} \Delta_{\mathcal{M}}^{-it} \subset \mathcal{N}$ for $t \leq 0$. In addition we assume that the vector Ω is also cyclic for $\mathcal{M} \cap \mathcal{N}'$. First we have to introduce a notation.

5.1. Definition. – By a standard chiral field theory we understand an association of von Neumann algebras to the open intervals I of the unit-circle S^1 fulfilling:

(i) Isotony, *i.e.* $I_1 \subset I_2$ implies $\mathcal{M}(I_1) \subset \mathcal{M}(I_2)$.

(ii) Duality, *i.e.*, $M(S^1 \backslash I) = \mathcal{M}(I)'$, where in this formula the open complement is meant.

(iii) There exists a vector Ω cyclic and separating for every algebra $\mathcal{M}(I)$ if $I \neq S^1$ and $I \neq \emptyset$.

(iv) This family of algebras is covariant under the action of the Möbius group $\mathrm{Sl}(2, \mathbb{R})/\mathbb{Z}_2$, *i.e.* there exist automorphisms α_g implying

$$\alpha_g \mathcal{M}(I) = \mathcal{M}(I_g)$$

where I_g denotes the interval obtained by acting with the group element g on the interval I.

(v) There exists a unitary representation $W(g)$ of the Möbius group implementing the automorphisms α_g, *i.e.*

$$W(g) \mathcal{M}(I) W^*(g) = \alpha_g \mathcal{M}(I).$$

(vi) The vector Ω is left fixed by the group representation $W(g)$.

(vii) After transforming the circle onto the straight line the representation of the translations of this $W(a)$ has a positive generator.

The Möbius transformations are the maps of the unit-circle onto itself, which can be analytically continued into the interior of the circle. All of them are of the form:

$$w = e^{i\varphi} \frac{z - \alpha}{\overline{\alpha} z - 1}, \qquad |\alpha| < 1.$$

Typical examples are:

(i) Rigid rotations: $z \to e^{i\varphi} z$. These transformations appear for $\alpha = 0$.

If $\alpha \neq 0$ then the equation for fixed points is a quadratic equation. If the solutions are different we obtain:

(ii) Dilatations: If 1 and -1 are the fixed points then with $b > 0$ the transformation has the form:

$$w_b = \frac{w + \dfrac{1-b}{1+b}}{\dfrac{1-b}{1+b}\, w + 1}.$$

If the two fixed points become a double point then we speak of

(iii) Translations: If 1 is the fixed point then the transformation for $a \in \mathbb{R}$ is given by the formula:

$$w_a = -\frac{a-2i}{a+2i}\, \frac{w + \dfrac{a}{a-2i}}{\dfrac{a}{a+2i}\, w + 1}.$$

any three of the above mentioned subgroups containing a dilatation generate the whole Möbius group.

Using the triple $(\mathcal{M}, \mathcal{N}, \Omega)$ we can construct a dilatation $\Delta_{\mathcal{M}}^{it}$ and a translation $U(a)$, such that $\mathcal{N} = \mathrm{ad}\, U(1)\, \mathcal{M}$. With the help of these operators we introduce an algebra for every interval (a, b) by the

5.2. Definition. – For every pair $a, b \in \mathbb{R} \cup \infty$ and with $-\infty = \infty$ we define two algebras as follows:

(i) $\mathcal{M}_{a,\infty} = \mathrm{ad}\, U(a)\, \mathcal{M}$, $\mathcal{M}_{a,\infty} = \mathrm{ad}\, U(a)\, \mathcal{M}'$.

(ii) If $-\infty < a < b < \infty$ we put $\mathcal{M}_{a,b} = \mathcal{M}_{\infty,b} \cap \mathcal{M}_{a,\infty}$.

(iii) If $-\infty < b < a < \infty$ we set $\mathcal{M}_{a,b} = \mathcal{M}'_{b,a}$.

From the assumption that Ω is cyclic for $\mathcal{M} \cap \mathcal{N}' = \mathcal{M}_{0,1}$ it follows that Ω is cyclic and separating for every $\mathcal{M}_{a,b}$ with $a \neq b$.

By means of the triple $(\mathcal{M}, \mathcal{N}' \cap \mathcal{M}, \Omega)$ we obtain the dilatation $\Delta_{\mathcal{M}}^{it}$ and a second "translation" $V(t)$ such that one obtains $\mathcal{N}' \cap \mathcal{M} = \mathrm{ad}\, V(1)\, \mathcal{M}$. This transformation permits to give a second construction for the algebras of the intervals.

5.3. Definition. – With $1/\infty = 0$ we define two algebras for every pair $a, b \in \mathbb{R} \cup \infty$ as follows:

(i) $\tilde{\mathcal{M}}_{0,a} = \mathrm{ad}\, V\left(\dfrac{1}{a}\right) \mathcal{M}$, $\tilde{\mathcal{M}}_{a,0} = \mathrm{ad}\, V\left(\dfrac{1}{a}\right) \mathcal{M}'$, $a \neq 0$.

(ii) If $-\infty < \dfrac{1}{a} < \dfrac{1}{b} < \infty$ we put $\tilde{\mathcal{M}}_{a,b} = \tilde{\mathcal{M}}_{0,b} \cap \tilde{\mathcal{M}}'_{0,a}$.

(iii) If $-\infty < \frac{1}{b} < \frac{1}{a} < \infty$ we set $\tilde{\mathcal{M}}_{a,b} = \tilde{\mathcal{M}}'_{b,a}$.

With these notations we get the following characterization of standard chiral field theories.

5.4. PROPOSITION. – *Let $\mathcal{N} \subset \mathcal{M}$ be a pair of von Neumann algebras with the following restrictions:*

(i) *There exists a vector Ω which is cyclic and separating for $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{N}' \cap \mathcal{M}$.*

(ii) *The triple $(\mathcal{M}, \mathcal{N}, \Omega)$ fulfills the condition of – half sided modular inclusion.*

Then these data define a standard chiral field theory iff the algebras $\mathcal{M}_{a,\,b}$ and $\tilde{\mathcal{M}}_{a,b}$ coincide.

This result is essentially due to H. W. Wiesbrock [Wie4], although our formulation differs from his. Also our proof is not identical with his proof. Later we will give equivalent conditions implying the assumption of the theorem. This will coincide with the formulation of Wiesbrock.

Proof. – We transform the real line onto the circle by the transformation

$$w = \frac{i - x}{i + x}.$$

The inverse transformation is $(w = e^{i\varphi})$:

$$x = i\,\frac{1 - w}{1 + w} = \frac{\sin\,\varphi}{1 + \cos\,\varphi}.$$

By this map the algebra $\mathcal{M}$ becomes the algebra of the upper half circle, the algebra $\mathcal{N}$ is mapped onto the algebra of the second quadrant and $\mathcal{N}' \cap \mathcal{M}$ onto the algebra of the first quadrant.

The three groups, the modular automorphism of $\mathcal{M}$, the translations of the two triples $(\mathcal{M}, \mathcal{N},\text{-}\Omega)$ and $(\mathcal{M}, \mathcal{N}' \cap \mathcal{M}, \Omega)$ are transformed into geometric actions of the circle. The adjoint action of these groups operate by assumption in the correct geometric manner on the algebras of the intervals. In particular Δ^{it} and $U(a)$ transform the algebras $\mathcal{M}_{a,\,b}$ and Δ^{it} and $V(b)$ the algebras $\tilde{\mathcal{M}}_{a,b}$. Since $\mathcal{M}_{a,\,b}$ and $\tilde{\mathcal{M}}_{a,b}$ coincide and since the three groups generate the whole Möbius group we obtain a map from the Möbius group into the automorphisms of the family $\{\mathcal{M}(I)\}$. It remains to show that the three groups Δ^{it}_{μ}, $U(a)$, $V(b)$ generate a representation of the Möbius group (after the correct change of variables). Different representants can only differ by a local gauge because of the correct action on the set of algebras. Consequently one obtains a group Γ with a normal subgroup of local gauges such that Γ/N is the Möbius group. Hence one must show that N is trivial. This is due to the following: Take an interval $I_{a,\ b}$ of the circle. Then we can represent the modular operator of $\mathcal{M}(I_{a,\ b})$ in two different ways as function of $\Delta_{\mathcal{M}}$ namely using the algebras $\mathcal{M}_{c,\ d}$ or the algebras $\tilde{\mathcal{M}}_{c,b}$ for the calculation. From the fact that these algebras coincide we obtain relations between the groups $U(a)$ and $V(b)$. Together with the known relations between the modular group of $\mathcal{M}$ and $U(a)$ and $V(b)$ respectively one shows that different representations of elements of the Möbius group coincide. This implies that N is trivial and shows the theorem. □

It might be instructive to start only from the –half-sided modular inclusion $(\mathcal{M},\ \mathcal{N},\ \Omega)$ and to require that Ω is also cyclic for $\mathcal{M} \cap \mathcal{N}'$ and to look for different conditions in order to guarantee that we are dealing with a standard chiral field theory. Using Definition 5.2 we are able to construct every algebra $\mathcal{M}_{a,\ b}$. But instead of looking at the second translation $V(t)$ we look at the algebra $\mathcal{M}_{-1,\ 1}$ and in particular at its modular conjugation $J_{-1,\ 1}$.

5.5. PROPOSITION. – *Let Ω be cyclic and separating for $\mathcal{M}$, $\mathcal{N}$, $\mathcal{M} \cap \mathcal{N}'$ and assume that the triple $(\mathcal{M},\ \mathcal{N},\ \Omega)$ fulfills the condition of –half-sided modular inclusion. Define $\mathcal{M}_{a,\ b}$ as in Definition* 5.2. *Then this setting defines a standard chiral field theory iff the following two conditions are fulfilled:*

(i) $$\operatorname{ad} J_{-1,1}\,\mathcal{M} = \mathcal{M}$$

where $J_{-1,\ 1}$ is the modular conjugation of $\mathcal{M}_{-1,\ 1}$.

(ii) *The algebra $\mathcal{N}$ coincides with the relative commutant of $\mathcal{M} \cap \mathcal{N}'$ in $\mathcal{M}$,* i.e.:

$$\mathcal{N} = \mathcal{M} \cap \{\mathcal{M} \cap \mathcal{N}'\}'.$$

Proof. – From the relation ad $J_{-1,\ 1}\ \mathcal{M} = \mathcal{M}$ and the fact that $J_{-1,\ 1}$ is antiunitary we obtain the relations

$$J_{-1,1}\,J = J\,J_{-1,1}$$

$$J_{-1,1}\,\Delta^{it} = \Delta^{-it}\,J_{-1,1}$$

where J and Δ are the modular conjugation and modular operator of the algebra $\mathcal{M}$. This implies in particular ($\lambda \in \mathbb{R}$):

$$\operatorname{ad} J_{-1,1}\,\mathcal{M}_{-\lambda,\lambda} = \mathcal{M}_{1/\lambda,\,-1/\lambda}.$$

Taking the intersection with $\mathcal{M}$ or with $\mathcal{M}'$ we obtain:

$$\operatorname{ad} J_{-1,1}\,\mathcal{M}_{0,a} = \mathcal{M}_{1/a,\infty}, \qquad a \in \mathbb{R}.$$

This is the relation where it is necessary that $\mathcal{N}$ coincides with its second relative commutant in $\mathcal{M}$. Since $J_{-1,\,1}$ maps $\mathcal{M}_{0,\,1}$ onto $\mathcal{M}_{1,\,\infty}$ we also obtain a relation between the corresponding translations:

$$\mathrm{ad}\, J_{-1,1}\, V(a) = U(1/a).$$

Together with the invariance of $\mathcal{M}$ under ad $J_{-1,\,1}$ we find by this equation that the conditions of Proposition 5.4 are fulfilled. □

Sometimes the conditions for a standard chiral field theory are formulated in such a way that the difficulties are hidden. The result is the following:

5.6. THEOREM. – *Let $\mathcal{M}_1$, $\mathcal{M}_2$ be two von Neumann algebras and Ω be a vector, then these data define a standard chiral field theory iff the following conditions are fulfilled.*

(i) *Ω is cyclic for $\mathcal{M}_1 \cap \mathcal{M}_2$ and for $\mathcal{M}'_1 \cap \mathcal{M}'_2$.*

(ii) *Let J_i denote the modular conjugation of $\mathcal{M}_i$ then one has*

$$\mathrm{ad}\, J_1\, \mathcal{M}_2 = \mathcal{M}_2, \qquad \mathrm{ad}\, J_2\, \mathcal{M}_1 = \mathcal{M}_1.$$

(iii) *Let Δ_i denote the modular operator of $\mathcal{M}_i$ then*

$$\mathrm{ad}\, \Delta_1^{it}\, \mathcal{M}_2 \subset \mathcal{M}_2, \qquad \textit{for} \quad t \geq 0$$
$$\mathrm{ad}\, \Delta_2^{it}\, \mathcal{M}_1 \subset \mathcal{M}_1, \qquad \textit{for} \quad t \leq 0$$

Proof. – This setting allows many different interpretations namely $\mathcal{M}_1$ corresponds to the algebra of the upper half circle and $\mathcal{M}_2$ to the algebra of the left half circle. If we apply a rigid rotation of the circle to this scheme then we obtain the other possible interpretation. We will use the first interpretation. It is clear from the assumptions that the triple ($\mathcal{M}_1$, $\mathcal{M}_1 \cap \mathcal{M}_2$, Ω) fulfills the condition of + half sided modular inclusion. Therefore, the triple ($\mathcal{M}_1$, $\mathcal{M}_1 \cap \mathcal{M}'_2$, Ω) fulfills the condition of −half-sided modular inclusion. Since J_2 maps $\mathcal{M}_1$ onto itself if follows that $\mathcal{M}_1 \cap \mathcal{M}'_2$ coincides with its second relative commutant with respect to $\mathcal{M}_1$ and hence the conditions of Proposition 5.5 are fulfilled. □

There are other means of constructing standard chiral field theories from a local theory on a line. For details *see* Fredenhagen [Fre] or Buchholz and Schulz-Mirbach [BSM].

6. TWO-DIMENSIONAL THEORIES

In this section we deal with a two-dimensional quantum field theory. But, most of the results will be valid also for higher dimensional theories in the situation where we fix one time– and one space–coordinate and assume that all sets are cylindrical in the other directions. As distinguished set we use the right wedge. The algebra associated to this set will be denoted by $\mathcal{M}(W)$. It has the property that the translations in the direction of this wedge map the algebra into itself. Two of these directions lie in the boundary of the forward or in the backward lightcone respectively. The translations in the directions of the lightcone coordinates fulfill the spectrum condition. This yields the connection with the investigations of section 4.

6.1. THEOREM. – *Assume $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H}$ with cyclic and separating vector Ω. Assume $U(a)$ is a representation of the two-dimensional translation group which fulfills the spectrum condition and which has Ω as fixed point.*

If for every a in the closed right wedge one has

$$U(a)\,\mathcal{M}\,U(-a) \subset \mathcal{M}$$

then the following relations exist between the group representation and the modular group:

$$\Delta^{it}\,U(a)\,\Delta^{-it} = U(\Lambda(t)\,a),$$
$$JU(a)\,J = U(-a).$$

This means that we have a unitary representation of the two-dimensional Poincaré group. If we define

$$U(a)\,\mathcal{M}\,U(-a) = \mathcal{M}_a, \qquad U(a)\,\mathcal{M}'\,U(-a) = \mathcal{M}'_a$$
$$\mathcal{M}_{a,b} = \mathcal{M}_a \cap \mathcal{M}'_b,$$

provided one has $b - a \in W$, then this net transforms covariantly under the Poincaré group.

This result is taken from [Bch2].

Proof. – Introducing lightcone coordinates a^+, a^- we can use the results obtained in the last two sections. One should notice that the group $U(a^-)$ used here and the corresponding group used in the proof of Theorem 4.2 differ by the sign. This leads to

$$\Delta^{it}\,U(a^+)\,\Delta^{-it} = U(e^{-2\pi t}\,a^+),$$
$$\Delta^{it}\,U(a^-)\,\Delta^{-it} = U(e^{+2\pi t}\,a^+),$$
$$JU(a)\,J = U(-a).$$

This shows the first statement. The second statement follows from this by the definition and the commutation relations between the translations and the modular group. □

Also this situation can be charactized by using the modular group instead of the translations.

6.2. THEOREM. – *Let $\mathcal{N}_1 \subset \mathcal{M}$ and $\mathcal{N}_2 \subset \mathcal{M}$ be three von Neumann algebras with common cyclic and separating vector Ω and assume*

$$\sigma_{\mathcal{M}}(t)\mathcal{N}_1 \subset \mathcal{N}_1 \qquad for \quad t \le 0,$$
$$\sigma_{\mathcal{M}}(t)\mathcal{N}_2 \subset \mathcal{N}_2 \qquad for \quad t \ge 0.$$

Denote by $U_1(t)$, $U_2(t)$ the translations which exist according to Theorem 4.1 *and fulfill the spectrum condition and the relations*

$$\operatorname{ad} U_1(1)\mathcal{M} = \mathcal{N}_1,$$
$$\operatorname{ad} U_2(-1)\mathcal{M} = \mathcal{N}_2,$$

then the following conditions are equivalent:

(*a*) *$U_1(t)$ and $U_2(s)$ commute for arbitrary $t, s \in \mathbb{R}$.*

(*b*) ad $U_1(t)\mathcal{N}_2 \subset \mathcal{N}_2$ *for* $t \ge 0$.

(*c*) ad $U_2(t)\mathcal{N}_1 \subset \mathcal{N}_1$ *for* $t \le 0$.

(*d*) *One of the two products $J_1 J_{\mathcal{M}}$, $J_{\mathcal{M}} J_1$ commutes with one of the products $J_{\mathcal{M}} J_2$, $J_2 J_{\mathcal{M}}$. Here J_i stands for $J_{\mathcal{N}_i}$.*

If these equivalent conditions are fulfilled then there exists a two-parametric family $\mathcal{M}_a$, $a \in \mathbb{R}^2$ of von Neumann algebras. All of them have Ω as cyclic and separating vector. This family fulfills isotony, covariance under translations with a positive generator and hence also covariance under the Poincaré group. This family is connected to the given algebras by

$$\mathcal{M}_{(0,0)} = \mathcal{M}, \qquad \mathcal{M}_{(1,0)} = \mathcal{N}_1, \qquad \mathcal{M}_{(0,-1)} = \mathcal{N}_2.$$

Originally the algebras $\mathcal{M}_{a,b}$ have been constructed by the author. The construction of these algebras with help of the modular inclusion is due to H.-W. Wiesbrock [Wie3].

Proof. – If (*a*) is equivalent to (*b*) then it is also equivalent to (*c*) by symmetry. Assume (*a*) then one finds for $t \ge 0$:

$$\begin{aligned}\operatorname{ad} U_1(t)\mathcal{N}_2 &= \operatorname{ad} U_1(t) \operatorname{ad} U_2(-1)\mathcal{M} \\ &= \operatorname{ad} U_2(-1) \operatorname{ad} U_1(t)\mathcal{M} \subset \operatorname{ad} U_2(-1)\mathcal{M} = \mathcal{N}_2\end{aligned}$$

and hence (*b*) is fulfilled. Next assume (*b*) then $U_1(e^{2\pi s})$ fulfills the condition of Theorem B with respect to the sulbalgebra $\mathcal{N}_2$. Hence we obtain

$$\Delta_2^{it} U_1(a)\Delta_{\mathcal{M}}^{-it} = U_1(e^{-2\pi t} a) = \Delta_{\mathcal{M}}^{it} U_1(a)\Delta_{\mathcal{M}}^{-it}$$

and consequently

$$\Delta_{\mathcal{M}}^{-it}\,\Delta_2^{it}\,U_1(a) = U_1(a)\,\Delta_{\mathcal{M}}^{-it}\,\Delta_2^{it}.$$

From this follows (*a*) because of

$$\Delta_{\mathcal{M}}^{-it}\,\Delta_2^{it} = U_2(e^{2\pi t} - 1).$$

Since we have $J_1 = \mathrm{ad}\,U_1(1)\,J_{\mathcal{M}}$ it follows $J_1\,J_{\mathcal{M}} = U_1(2)$. We get $J_{\mathcal{M}}\,J_1 = U_1(-2)$ and $J_2\,J_{\mathcal{M}} = U_2(-2)$, $J_{\mathcal{M}}\,J_2 = U_2(2)$ accordingly. Hence (*a*) implies (*d*). It is sufficient to show the converse for one of the combinations. For the other combination the result follows in the same manner. So we choose the commutator of the first products. From $J_1\,J_{\mathcal{M}} = U_1(2)$ and $J_2\,J_{\mathcal{M}} = U_2(-2)$ we know that $U_1(2)$ and $U_2(-2)$ commute. Applying the modular automorphism of the algebra $\mathcal{M}$ to this commutator we obtain the commutation of $U_1(e^{-2\pi t}\,2)$ and $U_2(-e^{2\pi t}\,2)$. Since with two commuting operators also their powers commute we find that $U_1(e^{-2\pi t}\,2\,m)$ and $U_2(-e^{-2\pi t}\,2\,n)$ commute for $n, m \in \mathbb{Z}$. Let E_M^1 be the spectral projection of $U_1(a)$ for the interval $[0, M]$. Then the expression

$$E_M^1\,[U_1(z),\,U_2(-e^{-2\pi t}\,2\,n)]\,E_M^1$$

is in z an entire analytic function of order M and bounded on the reals with zeros at $e^{-2\pi t}\,2\,m$. But the density of zeros of a function of exponential type cannot be too high. (Carlsons Theorem, *see* Boas [Boa] 9.2.1) Hence this function vanishes if $e^{-2\pi t}\,2 < \{2\,\pi\,M\}^{-1}$. Now both unitary groups are boundary values of analytic functions, which implies that

$$E_M^1\,[U_1(z),\,U_2(y)]\,E_M^1$$

vanishes. Since the value M was arbitrary we ascertain that $U_1(x)$ commutes with $U_2(y)$. □

7. HIGHER DIMENSIONAL THEORIES

As we have seen in the last sections the modular groups give some insight into the structure of quantum field theory in one or two dimensions. In higher dimensions the situation is more complicated. This is mainly due to the structure of the light cone. However, there is one exception, *i.e.*, the theory covariant under the conformal group. This result is due to Brunetti, Guido, and Longo [BGL1]. Moreover, we will discuss the procedure of Buchholz and Summers [BuSu1] which gives a characterization of the vacuum state.

It is an important problem to find conditions in order that the modular groups of different wedge domains fit together and give a representation of the Lorentz group. Recently there have been two different attempts to solve this problem. One is due to Brunetti, Guido and Longo [BGL2] and the other is proposed by Buchholz and Summers [BuSu2]. The first case is solved by showing that the modular groups of different wedge domains fit together and give a representation of the covering group of the Poincaré group. In the second case the problem is solved modulo a cohomological question. Up to now it is not clear whether or not this leads to a real obstruction. At the end of this section we will indicate the starting points and discuss the appearing problems.

7.1. THEOREM. – *Let $\mathcal{K}$ be the family of sets consisting of double cones, wedges, and forward or backward translated light cones. Assume that for every $O \in \mathcal{K}$ we have a von Neumann algebra $\mathcal{M}(O)$ on a Hilbert space $\mathcal{H}$ fulfilling isotony and locality. Moreover, assume that there exists a vector $\Omega \in \mathcal{H}$ which is cyclic and separating for every $\mathcal{M}(O)$. Assume that this family is covariant under conformal transformations. Assume, in particular, that there exists a continuous unitary representation of the conformal group $U(g)$ with $U(g)\Omega = \Omega$ such that the translations fulfill the spectrum condition and $U(g)\mathcal{M}(O)U^*(g) = \mathcal{M}(O_g)$ whenever O and O_g belong to $\mathcal{K}$.*

If this is fulfilled then the modular group acts as geometric transformation i.e.:

as Lorentz boosts for wedge domains,

as dilatations for light cones,

as the transformations described by Hislop and Longo [HL] *for double cones.*

In particular one has

$$\Delta_O^{it} = U_O(t)$$

when the geometric transformations are properly defined, $U_O(t)$ denoting the image of the Lorentz transformations of the wedge under the map which sends the wedge onto O.

This is the result of Brunetti, Guido and Longo [BGL1]. Notice that we have absorbed the factor 2π which appears in the previous section when defining the geometric transformations. This simplifies the calculations.

Proof. – For the dimension equal to one we have seen in section 5 that the result is true, so we have to consider the case $d \geq 2$. First we want to show that for any $O \in \mathcal{K}$ the one-parametric group

$$W(t) = \Delta_O^{it} U_O(-t)$$

commutes with the representation of the conformal group and does not depend on the domain O. Since all these sets are the image of a wedge let us fix a wedge W and look at the equation

$$\Delta_W^{it} U(g) \Delta_W^{-it} = U_W(t) U(g) U_W(-t).$$

This is true for all transformations g which map the wedge onto itself as the Lorentz transformations of this wedge, the translations in the directions in the wedge, and the transformations

$$x \longrightarrow -\frac{x}{x^2}, \qquad \text{for} \quad d = 2$$

$$x \longrightarrow -\frac{\{x_0, x_1, -x_2, \ldots\}}{x^2}, \qquad \text{for} \quad d > 2.$$

These transformations belong to the connected component of the identity in the conformal group. Moreover, we known from the last section that this relation is also true for the translations in the two lightlike directions defining the wedge. But it is known [TMP] that these transformations together generate the connected component of the conformal group. Hence $W(t)$ commutes with the representation of the conformal group. Since $W(t)$ of one domain is mapped onto $W(t)$ of another domain by means of conformal transformations we get the independence of the domain. It remains to show that $W(t)$ is the identity. If $d > 2$ then we know that there is a rotation which maps the wedge onto the opposite wedge. The adjoint action of this rotation maps Lorentz boosts and the modular group onto their inverse. Hence we obtain

$$\begin{aligned} W(t) &= U(R) W(t) U(R) \\ &= (U(R) \Delta_W U(R))^{it} U_W(t) = W(-t). \end{aligned}$$

In two dimensions we have to use the modular conjugaison J_W and obtain the same result. □

Next we turn to the result of Buchholz and Summers [BuSu1] concerning a characterization of the vacuum state. The main idea is the following: It is well known that every translation can be decomposed into reflections. In the vacuum sector of local quantum field theory the modular conjugations of the algebras belonging to wedges are reflections in the two-plane, spanned by the two light rays defining the wedge. Therefore, if these conjugations act locally on every local algebra it should be possible to construct a representation of the translation group. If this representation of the translation group is continuous then it will have the correct properties in configuration space and therefore, by a result of Wiesbrock [Wie1], this representation must necessarily fulfill the spectrum condition. So we obtain:

7.2. THEOREM. – *Assume that we are dealing with a representation of the theory of local observables on a Hilbert space $\mathcal{H}$ such that there is a vector $\Omega \in \mathcal{H}$ which is cyclic and separating for all wedge algebras $\mathcal{M}(W(\ell_1, \ell_2 a))$. (All our lightlike vectors belong to the closed forward lightcone.) Denote by $J(\ell_1, \ell_2, a)$ the corresponding modular conjugations and assume that these induce a geometric action on all wedge algebras* i.e. *like a reflection in the two-plane spanned by ℓ_1 and ℓ_2 with fixed point a. Suppose in addition that for every wedge $W(\ell_1, \ell_2 0)$ the modular group of this wedge satisfies the condition of $\mp$ modular inclusion for the wedges obtained by translation in the direction ℓ_1 and $-\ell_2$ respectively. Then there exists a continuous unitary representation $U(x)$ of the translation group fulfilling*

(*a*) $U(x)\mathcal{M}(W(\ell_1, \ell_2, a))U(-x) = \mathcal{M}(W(\ell_1, \ell_2, a) + x)$,

(*b*) $U(x)\Omega = \Omega$,

(*c*) $U(x)$ *fulfills the spectrum condition.*

This action of the group gives the correct action on algebras of the double cones provided the algebas of the double cones are identical with the intersections of the algebra of all the wedges which contain this double cone.

Before proving this theorem we need some explanations:

Given the triple (ℓ_1, ℓ_2, a) we can decompose the Minkowski space into the two-plane $E(\ell_1, \ell_2)$ spanned by the two lightlike vectors and the complement $E(\ell_1, \ell_2)^\perp$ where the orthogonal complement is computed with respect to the Minkowski scalar product. Now the vector a has a unique decomposition $a = a_1 + a^\perp$ where a_1 belongs to $E(\ell_1, \ell_2)$. If x is an arbitrary vector we can also decompose it: $x = x_1 + x^\perp$, where x_1 belongs again to $E(\ell_1, \ell_2)$. Now we define the reflection $r(\ell_1, \ell_2, a)$ by the formula

$$r(\ell_1, \ell_2, a)x = -x_1 + 2a_1 + x^\perp.$$

The correct reflection, mentioned in the theorem, is given by the formula

$$J(\ell_1, \ell_2, a)\mathcal{M}(W)J(\ell_1, \ell_2, a) = \mathcal{M}(r(\ell_1, \ell_2, a)W).$$

By the $\mp$ modular inclusion we mean the following: Let $\Delta(\ell_1, \ell_2)$ be the modular operator of the algebra $\mathcal{M}(\ell_1, \ell_2, 0)$ and $p > 0$. Then we require

$$\Delta^{it}(\ell_1, \ell_2)\,\mathcal{M}(W(\ell_1, \ell_2, p\,\ell_1))\,\Delta^{-it}(\ell_1, \ell_2) \subset \mathcal{M}(W(\ell_1, \ell_2, p\,\ell_1)) \quad \text{for} \quad t \leq 0,$$

$$\Delta^{it}(\ell_1, \ell_2)\,\mathcal{M}(W(\ell_1, \ell_2, -p\,\ell_2))\,\Delta^{-it}(\ell_1, \ell_2) \subset \mathcal{M}(W(\ell_1, \ell_2, -p\,\ell_2)) \quad \text{for} \quad t \geq 0,$$

The original work of Buchholz and Summers contains much more assumptions. Here we follow the ideas of Borchers [Bch3] reducing the assumptions to the essential ones. The strategy consists in looking first in a fixed lightlike direction and constructing the translation group in this direction. Afterwards one has to show that all these one-dimensional groups fit together.

7.3. PROPOSITION. – *With the assumptions of Theorem* 7.2 *let* ℓ_1, ℓ_2 *be two fixed lightlike vectors in the boundary of the forward lightcone. Then there exists a one parametric continuous unitary group* $U(a\,\ell_1)$ *fulfilling the assumptions of the theorem. This group is given by the formula*

$$U(2\,a\,\ell_1) = J(\ell_1, \ell_2, 0)\,J(\ell_1, \ell_2, -a\,\ell_1).$$

Proof. – During this proof we denote the algebra $\mathcal{M}(W(\ell_1, \ell_2, a\,\ell_1))$ by $\mathcal{M}_a$ and its commutant by $\mathcal{M}'_a$. The corresponding modular conjugation will simply be denoted by J_a. The requirement concerning the action of the modular conjugations becomes

$$J_a\,\mathcal{M}_b\,J_a = \mathcal{M}'_{(2a-b)}.$$

Since the algebra and its commutant have the same modular conjugation we obtain from this relation the equation

$$J_a\,J_b\,J_a = J_{(2a-b)}. \qquad (*)$$

As a consequence of this relation we show that the products $J_0\,J_{-\frac{1}{2}a}$ form a group for rational values of a.

7.4. LEMMA. – *Assume equation* $(*)$ *and let* $c \in \mathbb{R}$ *be fixed. If* $a, b \in \mathbb{Q}\,c$ *then*

1) *the products* $J_a\,J_b$ *depend only on the difference* $b - a$,

2) *the unitary operators* $V_c(a) := J_0\,J_a$ *define a unitary representation of the additive group of* $\mathbb{Q}$ i.e.

$$V_c(a)\,V_c(b) = V_c(a + b).$$

Proof. – Equation $(*)$ leads to the relation $J_0\,J_a = J_{-a}\,J_0$. First we show that equation $(*)$ implies

$$(J_0\,J_a)^n = J_{-ka}\,J_{(n-k)\,a}, \qquad k = 0, \ldots, n. \qquad (**)$$

We prove this relation by induction with respect to n. The statement is obviously correct for $n = 1$. Assume we know the statement for $i = 1 \ldots n-1$. Then we want to show it for n. Let $0 < k < n$ and write $(J_0\,J_a)^n = (J_0\,J_a)^k\,(J_0\,J_a)^{n-k}$. Inserting the induction hypothesis we obtain

$$(J_0\,J_a)^n = J_{-la}\,J_{(k-l)\,a}\,J_{-ma}\,J_{(n-k-l)\,a}.$$

In order to use equation $(*)$ we must choose $l = m$. We obtain $J_{-(l+k)\,a}\,J_{(n-l-k)\,a}$. This is the stated formula. We can also choose $(k-l) = (n-k-m)$. This again gives the representation $J_{-la}\,J_{(n-l)\,a}$. As restriction we have $l = 0, \ldots, k$ and $m = 0, \ldots, n-k$. Since k does not take the values 0 and n only both constructions together cover the whole of formula $(**)$. Now note that the equation $J_0\,J_{-a}\,J_0\,J_a = J_a\,J_0\,J_0\,J_a = 1$, implies $J_0\,J_{ma}\,J_0\,J_{na} = J_0\,J_{(m+n)\,a}$ for $n, m \in \mathbb{N}$. Let now q_1, q_2 be two rational numbers. Then we obtain $J_0\,J_{q_1\,a}\,J_0\,J_{q_2\,a} = J_0\,J_{q_2\,a}\,J_0\,J_{q_1\,a} = J_0\,J_{a\,(q_1+q_2)\,a}$ for arbitrary $q_1, q_2 \in \mathbb{Q}$ by writing $q_1 = \frac{m}{r}$, $q_2 = \frac{n}{r}$. □

If we would know the continuity of the expression $J_0\,J_a$ in the variable a then we would obtain a continuous unitary representation of the additive group of the real line. In order to obtain the continuity we start from the requirement that the algebra $\mathcal{M}_1$ fulfills the condition of –modular inclusion with respect to the algebra $\mathcal{M}_0$. Hence by Theorem 4.1 there is a unitary group $U_1(a)$ of the real line fulfilling the spectrum condition, leaving the vector Ω fixed and mapping $\mathcal{M}_0$ onto $\mathcal{M}_1$ *i.e.*

$$U_1(1)\,\mathcal{M}_0\,U_1(-1) = \mathrm{ad}\,J_0\,J_1\,\mathcal{M}_0 = \mathcal{M}_1.$$

We could have started with another subalgebras as e.g. $\mathcal{M}_{\frac{1}{2}}$. Then we would have obtained another group $U_{\frac{1}{2}}(a)$ and, after proper rescaling, the relation

$$U_{\frac{1}{2}}(1)\,\mathcal{M}_0\,U_{\frac{1}{2}}(-1) = \mathrm{ad}\,(J_0\,J_{\frac{1}{2}})^2\,\mathcal{M}_0 = \mathcal{M}_1.$$

We know from Theorem 4.2 that these are the same groups.

For every $a > 0$ we can construct a unitary group $U_a(t)$ fulfilling spectrum condition and the equation

$$U_a(a)\,\mathcal{M}_0\,U_a^*(a) = \mathcal{M}_a$$

by using Theorem 4.1. Moreover, the family

$$\mathcal{M}_t^a = U_a(t)\, \mathcal{M}_0\, U_a(-t)$$

fulfills the condition of isotony. Moreover, we have $\mathcal{M}^a(2a) = \mathcal{M}^{2a}(2a)$. It follows from Theorem 4.2 that $U_a(t)$ and $U_{2a}(t)$ coincide. Repeating the argument for other values with a common multiple we find that $U_q(t) = U(t)$ does not depend on q for rational values of q. The isotony implies that the family $\mathcal{M}_a$ is continuous because for every irrational a $\mathcal{M}^a$ can be approximated from the inside and the outside by algebras with rational values of the index. On the algebras we have the action of $U(t)$ which is a continuous group. Hence we obtain continuity for $\mathcal{M}_a$. This implies the representation $\mathcal{M}_a = U(a)\,\mathcal{M}_0\,U(-a)$ with $U(a)$ fulfilling the spectrum condition. Moreover, from Theorem 4.1 we obtain

$$J_0\, J_{-\frac{1}{2}a} = J_0\, U\left(-\frac{1}{2}a\right) J_0\, U\left(\frac{1}{2}a\right) = U(a).$$

This shows the statements of Proposition 7.2. □

Next we look at a fixed wedge $W(\ell_1,\, \ell_2,\, 0)$ and we find two unitary groups $U(a^+)$ and $U(a^-)$ translating in the directions ℓ_1 and ℓ_2 both fulfilling spectrum condition. Now we have to show that these groups commute. For this we need a preparation.

7.5. Lemma. – *Let $\mathcal{M}_a$ be a one-parametric family of von Neumann algebras with a common cyclic and separating vector Ω. Assume that this family has the properties of isotony and covariance with respect to a continuous unitary group of translations fulfilling the spectrum condition. Let V be a unitary operator which has Ω as fixed point and maps every $\mathcal{M}_a$ into itself. Then V commutes with the translations.*

Proof. – Let $U(a)$ be the unitary group defining the covariance and fulfilling the spectrum condition. Define a family of unitary operators $T(a) = U(a)\,VU(-a)$. Since V maps every $\mathcal{M}_b$ into itself, it follows from the definition that the same is true for $T(a)$. Now choose $A \in \mathcal{M}_0$ and $B \in \mathcal{M}_0'$. Then $U(b)\,BU(-b)$ and $T(a)\,AT^*(a)$ commute for $b \leq 0$. Therefore, the two functions

$$\begin{aligned} F^+(b,\, a) &= (\Omega,\, BU(-b)\,T(a)\,A\,\Omega) \\ &= (\Omega,\, U(b)\,BU(-b)\,T(a)\,AT^*(a)\,\Omega), \\ F^+(b,\, a) &= (\Omega,\, AT^*(a)\,U(b)\,B\,\Omega) \\ &= (\Omega,\, T(a)\,AT^*(a)\,U(b)\,BU(-b)\;\Omega) \end{aligned}$$

coincide for negative values of b. Because of $U(-b)T(a) = U(a-b) VU(-a)$ it follows that $F^+(b, a)$ is the boundary-value of a bounded analytic function holomorphic in the tube $\Im m(a-b) > 0$, $\Im m\, a < 0$. Since we get $T^*(a)U(b) = U(a)V^*U(b-a)$ if follows that $F^-(b, a)$ has an analytic extension into the opposite tube. Hence we have to deal with an edge of the wedge problem. Let us denote the common analytic extension of both functions by $F(b, a)$. For solving this edge of the wedge problem let us introduce the variables $z_1 = a$, $z_2 = a - b$. Then we obtain the two tubes based on the second and on the fourth quadrants. The coincidence domain becomes $x_1 - x_2 > 0$. According to II.D $F(b, a)$ is holomorphic except for the cut in the b-variable along the negative axis. In particular F is entire analytic in the variable a. Since F is also bounded, it does not depend on this variable. Because the vector Ω is cyclic, as well as for $\mathcal{M}_0$ as for $\mathcal{M}'_0$, $T(a)$ does not depend on a. The equation $T(a) = T(0)$ is equivalent to the statement of the lemma. □

Next we show the commuting of the two groups. Let a^+ be the positive lightcone coordinate. Then $U(a^+) = J_0 J_{-\frac{1}{2}a^+}$ fulfills spectrum condition. Now the operator $J_0 J_{-\frac{1}{2}a^-}$ fulfills the relation:

$$J_0 J_{-\frac{1}{2}a^-} \mathcal{M}_{a^+} J_{-\frac{1}{2}a^-} J_0 = J_0 \mathcal{M}'_{-a^- -a^+} J_0 = \mathcal{M}_{a^+ + a^-}.$$

From the isotony follows that for negative values of a^- the product $J_0 J_{-\frac{1}{2}a^-}$ maps every $\mathcal{M}_{a^+}$ into itself. Hence by 7.5 it commutes with $U(a^+)$. Since we have modular inclusion for both lightlike directions also the group $U(a^-) = J_0 J_{-\frac{1}{2}a^-}$ fulfills the spectrum condition. Hence by analytic continuation $U(a^+)$ and $U(a^-)$ commute for all values of their arguments.

Proof of Theorem. – We start with two wedges which have a common lightlike vector ℓ *i.e.* the two wedges $W(\ell, \ell_1, 0)$ and $W(\ell, \ell_2, 0)$. We can apply the two-dimensional situation to every of these wedges and obtain two representations of the translations along the direction ℓ. Both of these representations fulfill the spectrum condition. We have to show that these representations coincide. Define $U(\ell, \ell_1)(a^+) = J(\ell, \ell_2, 0)\, J\left(\ell, \ell_1, -\frac{a^+}{2}\ell\right)$, then the reflection symmetry yields:

$$U(\ell, \ell_1)(a^+)\, \mathcal{M}\, (\ell, \ell_2, b^+)\, U^*(\ell, \ell_1)(a^+)$$
$$= \mathcal{M}\left(\ell, \ell_2, r(\ell, \ell_1, 0)\, r\left(\ell, \ell_1, \frac{a^+}{2}\right) b^+\right).$$

This implies that $U(\ell, \ell_1)(a^+)$ maps the family of wedge algebras $M(\ell, \ell_2)(b^+)$ onto itself. Hence, by Lemma 7.5, the two groups

$U(\ell, \ell_1)(a^+)$ and $U(\ell, \ell_2)(a^+)$ coincide. Therefore, the groups in every two different lightlike directions commute. Since every vector can be decomposed into two lightlike vectors we see that the system is covariant and that it fulfills the spectrum condition, because it is true for every lightlike direction. □

Now we are coming to the construction of the whole Poincaré group mentioned in the beginning of this section. We start from a theory of local observables on a Hilbert space $\mathcal{H}$ defined by algebras of all wedge regions $M(W)$ (including translated wedges). One assumes that there exists a vector $\Omega \in \mathcal{H}$ cyclic and separating for all these algebras. The algebras of double cones are defined by the intersection of all algebras belonging to the wedges containing the given double cone.

There are different startingpoints for the two approaches.

7.6. REQUIREMENT OF BRUNETI, GUIDO AND LONGO. – We assume that Ω is also cyclic and separating for all $M(O)$ where (O) runs through the set of double cones and wedges. To every pair $(M(W), \Omega)$ exists a modular group Δ_W^{it} which acts as a Lorentz boost $\Lambda_W(t)$. It is now required that all these modular groups act locally:

$$\Delta_W^{it}\,\mathcal{M}(O)\,\Delta_W^{-it} = \mathcal{M}(\Lambda_W(t)\,O).$$

Starting from these assumptions then for every of these wedges one finds ±modular inclusions in the two timelike directions defining the wedge. Hence one obtains, as shown in section 4, translations in these timelike directions. Using all these transformations one might construct, for every element in the Poincaré group, several representants. But from the assumption of local action one sees that these different representants can only differ by a local gauge, *i.e.* by a transformation which keeps every $M(O)$ fixed. Hence we obtain a large group γ and normal subgroup N of local gauges such that Γ/N is isomorphic to the Poincaré group. These authors succeed to show that in their situation the normal subgroup can consist at most of the elements $(1, -1)$ so that they obtain a representation of the covering group. By the arguments of section 4 this representation must necessarily fulfill the spectrum condition.

7.7. THEOREM. – *Let the theory of local observables fulfill the requirements 7.6 then there exists a continuous unitary representation of the covering group of the Poincaré group $U(g)$ fulfilling spectrum condition and having Ω as fixed point with*

$$U(\pm\Lambda_W(t)) = \Delta_W^{it}.$$

(The equality holds for one of the signs.)

The approach of Buchholz and Summers is an extension of the method described in Theorem 7.2, *i.e.* they only use the modular conjugations of the wedge algebras for constructing the Poincaré group.

7.8. Requirement of Buchholz and Summers. – The modular conjugation of every wedge J_W maps the algebra of every wedge onto the algebra of another wedge:

$$J_W \mathcal{M}(W_1) J_W = \mathcal{M}(W_2).$$

From this setting one obtains first a map from the set of wedges to the set of wedges. Since J_W is its own inverse this map is onto and one to one. Since the wedges generate a topology on the Minkowski space these maps can be identified with bijections of the Minkowski space. Since every light ray is uniquely characterized by the family of wedges containing half of the light ray in the boundary one sees that these maps send light rays onto light rays. Therefore, by Zeeman's [Ze] theorem the maps are Poincaré transformations followed by dilatations. Again we obtain a large group Γ containing a normal subgroup N of local gauges such that Γ/N coincides with that part of the conformal group which consists of bijections of the Minkowski space. The investigation of the group N seems in the moment not to be in a completely satisfactory state and one has to wait until a preprint is accessible.

8. LORENTZ GROUP AND WEDGE DUALITY, EXAMPLES

In the last section we have discussed the construction of the Poincaré group from the modular groups of the local algebras. Now we will assume that we deal with a local theory covariant under the whole Poincaré group and we will look for the CPT-theorem. Jost's proof of this Theorem [Jo] in the Wightman frame needed the assumption that one is dealing with a finite number of fields. As known from examples of Streater [Str] and of Oksak and Todorov [OT] the CPT-theorem can fail if one is dealing with a field having an infinity of components. Therefore, the CPT-theorem cannot hold in general in a Poincaré covariant theory of local observables and one needs conditions replacing the assumptions of finiteness of the components in Wightman theory. Unfortunately this problem is still open but there is a first step which shall be discussed here. A necessary condition for the CPT-theorem is that the wedge duality holds which is the condition that the commutant of the von Neumann algebra of the right wedge coincides with the von Neumann algebra of the left wedge. The first attempt to this problem is due to Brunetti, Guido and Longo [BGL1]. They used the nuclearity condition of Buchholz and Wichmann [BuWi] and showed that in a Lorentz covariant theory the given Lorentz boosts coincide with the Lorentz boosts constructed with help of the modular theory. In Wightman theory there are fields with an infinite number of components which fulfill the CPT-theorem but do not enjoy the nuclearity condition. Therefore, this result does not give the full answer to the duality problem. The general problem of the wedge duality has recently been solved [Bch4] with help of the modular theory. We will give a discussion of the problem and its solution without going through all the proofs. For details I must refer to the original paper.

For a long time it was an unsolved problem whether or not the wedge duality is a consequence of the axioms of the theory of local observables (not including Lorentz symmetry). This has been solved by Yngvason [Yng] by constructing counterexamples. He also gave examples showing that the modular group of the wedge does in general not act locally in directions perpendicular to the two-plane defining the wedge. These examples will be discussed at the end of this section.

The result about the wedge duality we are presenting here is essentially a two-dimensional statement and hence easier to derive than the CPT-theorem. In the proof we can also think of sets which are cylindrical in all directions perpendicular to the two directions defining the two wedges. Hence all the expressions depend only on two variables. In this situation we have two wedges which we call the right wedge W^r and the left wedge W^l. The wedges obtained by applying a shift by a will be denoted by W_a^r and W_a^l respectively. If we denote the double-cones by K then this can be characterized by the intersection of two wedges.

$$K_{a,b} = W_a^r \cap W_b^l, \qquad b - a \in W^r.$$

Let $\mathcal{M}_{a,b}^0$ be the given von Neumann algebra associated with $K_{a,b}$ fulfilling the standard assumption. Starting from this we obtain for the wedges the algebras:

$$\mathcal{M}_a^r = \{ \bigcup_{K \subset W_a^r} \mathcal{M}_K^0 \}'',$$

$$\mathcal{M}_a^l = \{ \bigcup_{K \subset W_a^l} \mathcal{M}_K^0 \}''.$$

Without loss of generality we can construct a net which might be slightly larger:

$$\mathcal{M}_{a,b} := \mathcal{M}(K_{a,b}) = \mathcal{M}_a^r \cap \mathcal{M}_b^l.$$

This net fulfills again all requirements listed in the beginning. Moreover, the wedge-algebra constructed with $\mathcal{M}(K)$ coincides with the wedge-algebra constructed with the $\mathcal{M}^0(K)$. In what follows we will only work with the algebras $\mathcal{M}(K)$.

In Wightman field theory one is dealing with quantities $\Phi_n(x)$ localized at a point. If x belongs to the right wedge one can analytically continue the expresion $U(\Lambda(t))\,\Phi_n(x)\,\Omega$ into the strip $S\left(-\frac{1}{2},\,0\right)$. This is due to the fact that the representation of the Lorentz group in the index-space is defined for complex Lorentz transformations. The result which one obtains is an element belonging to the left wedge namely $U(\Lambda(t))\,\Phi_n(-x)\,\Omega$ (for entire spin). There are two problems if one wants to generalize this:

First our objects are not localized at a point but in bounded domains. Here we will find a natural generalization of the description.

The second problem consists of understanding the exchange of the left and the right wedge by the complex Lorentz transformations because of the following

8.1. *Remark*. – If we are dealing with a von Neumann algebra $\mathcal{M}$ and a strongly continuous, one parametric group of automorphisms α_t then one can define analytic elements $\mathcal{M}^{\text{anal}}$ for which $\alpha_t\,A$ has an entire analytic extension. The set $\mathcal{M}^{\text{anal}}$ is a $*$-strong dense subalgebra of $\mathcal{M}$ and the elements $\alpha_z\,A$, $A \in \mathcal{M}^{\text{anal}}$ also belong to $\mathcal{M}$.

Therefore, it is not easy to understand why for element A localized in the right wedge, the expression $U\left(\Lambda\left(-\frac{i}{2}\right)\right)A\,\Omega$ can be written as $\hat{A}\,\Omega$ with an element $\hat{A}$ localized in the left wedge.

We must start from the assumption that wedge duality is not present. Therefore, to every wedge there are associated two algebras defined above and the commutant of the algebra belonging to the opposite wedge. This leads to three different local algebras. We set

$$\mathcal{M}'^{r}_{a} = \{\mathcal{M}^{r}_{a}\}', \qquad \mathcal{M}'^{l}_{a} = \{\mathcal{M}^{l}_{a}\}'.$$

With help of these wedge-algebras we can construct two other local nets

$$\mathcal{M}^{r}_{a,\,b} = \mathcal{M}^{r}_{a} \cap \mathcal{M}'^{r}_{b}, \qquad \mathcal{M}^{l}_{a,\,b} = \mathcal{M}'^{l}_{a} \cap \mathcal{M}^{l}_{b}.$$

The algebras $\mathcal{M}^{l}_{a,\,b}$ and $\mathcal{M}^{r}_{c,\,d}$ are not relatively local to each other, however, the algebras commute if $c - b \in W^r$.

The modular operator of the algebra $\mathcal{M}^{r}_{0}$ will be denoted by Δ_r and that of $\mathcal{M}'^{l}_{0}$ by Δ_l. The modular operators of $\mathcal{M}'^{r}_{0}$ and $\mathcal{M}^{l}_{0}$ are the inverse of Δ_r and Δ_l respectively. The corresponding modular conjugations will be denoted by J_r and J_l. Due to the result described in Theorem 6.1 these operators fulfill the following commutation relations with the representation $T(a)$ of the translations:

$$\Delta_r^{it} T(a) \Delta_r^{-it} = T(\Lambda(t) a), \qquad J_r T(a) J_r = T(-a)$$
$$\Delta_l^{it} T(a) \Delta_l^{-it} = T(\Lambda(t) a), \qquad J_l T(a) J_l = T(-a).$$

In this formula $\Lambda(t)$ denotes the Lorentz transformation appearing in the modular theory of the wedge algebra. This implies the following transformation laws for the local algebras (*see* [Bch2]):

$$\Delta_r^{it} \mathcal{M}_{a,b}^r \Delta_r^{-it} = \mathcal{M}_{\Lambda(t) a, \Lambda(t) b}^r, \qquad \Delta_l^{it} \mathcal{M}_{a,b}^l \Delta_l^{-it} = \mathcal{M}_{\Lambda(t) a, \Lambda(t) b}^l,$$
$$J_r \mathcal{M}_{a,b}^r J_r = \mathcal{M}_{-b,-a}^r, \qquad J_l \mathcal{M}_{a,b}^l J_l = \mathcal{M}_{-b,-a}^l.$$

We deal with a Lorentz covariant theory, *i.e.,* we have a continuous representation of the Lorentz group satisfying

$$U(\Lambda) T(a) U(\Lambda)^* = T(\Lambda a),$$

and

$$U(\Lambda) \mathcal{M}_{a,b} U(\Lambda)^* = \mathcal{M}_{(\Lambda a, \Lambda b)}.$$

Using this and remembering the construction of the algebras $\mathcal{M}_{a,b}$ and $\mathcal{M}_{a,b}^{r,l}$ we see that $U(\Lambda)$ transforms these algebras in the same manner. These equations permit to compare the Lorentz transformations with the two modular groups. First notice that $U(\Lambda)$ maps the four algebras of the two wedges into themselves and hence $U(\Lambda)$ commutes with the modular groups and the modular conjugations (*see* e.g. [BrRo]). Therefore, we obtain the following representations of the Lorentz group:

$$R(t) \Delta_r^{it} = U(\Lambda(t))$$
$$L(t) \Delta_l^{it} = U(\Lambda(t)).$$

Since $U(\Lambda)$ commutes with the modular groups and acts on the translations in the same manner as the modular groups we obtain the following commutation relations:

$$[R(s), \Delta_r^{it}] = [R(s), U(\Lambda)] = [R(s), T(a)] = [R(s), J_r] = 0,$$
$$[L(s), \Delta_l^{it}] = [L(s), U(\Lambda)] = [L(s), T(a)] = [L(s), J_l] = 0.$$

Notice that the algebra $\mathcal{M}_0^r$ is contained in $\mathcal{M}_0'^l$. Therefore, we can apply Theorem A to the expression

$$D(t) := \Delta_l^{-it}\,\Delta_r^{it} = L(t)\,R(-t)$$

and get:

8.2. LEMMA. – *As a consequence of the definition of* $R(t)$ *and* $L(t)$ *we obtain:*

(*a*) *If* $A \in \mathcal{B}(\mathcal{H})$ *(the set of bounded linear operators on* $\mathcal{H}$*) and if* $L(t)\,A\,\Omega$ *has a bounded analytic continuation into the strip* $S\left(-\frac{1}{2},\,0\right)$ *then the same is true for* $R(t)\,A\,\Omega$. *If* $A \in \mathcal{B}(\mathcal{H})$ *is such that* $R(t)\,A\,\Omega$ *has a bounded analytic extension into the strip* $S\left(0,\,\frac{1}{2}\right)$ *then the same holds for* $L(t)\,A\,\Omega$.

(*b*) *Moreover, we obtain the following identities:*

$$J_l\,L\left(-\frac{i}{2}\right) = J_r\,R\left(-\frac{i}{2}\right) \quad on\ \mathcal{D}\left(L\left(-\frac{i}{2}\right)\right),$$

$$J_l\,L\left(\frac{i}{2}\right) = J_r\,R\left(\frac{i}{2}\right) \quad on\ \mathcal{D}\left(R\left(\frac{i}{2}\right)\right),$$

where $\mathcal{D}(X)$ *denotes the domain of definition of the operator* X.

As in the proof of the CPT-theorem and in the investigation of Bisognano and Wichmann one has to look at analytic continuation on the Lorentz group. But there is a problem. If $A(K)$ is an operator localized in the double cone $K \subset W^r$ and such that $U(\Lambda(t))\,A(K)\,\Omega$ can in t be analytically continued into the strip $S\left(-\frac{1}{2},\,0\right)$ then we expect that we can write $U\left(\Lambda\left(-\frac{i}{2}\right)\right)A(K)\,\Omega$ in the form $\hat{A}(-K)\,\Omega$. This operator should be localized in $-K \subset W^l$. There is, however, one problem: at the beginning we do not know wedge-duality. Hence we cannot conclude that there exist elements $A(K)$ such that the corresponding operator $\hat{A}(-K)$ is bounded. Therefore, we must include unbounded operators in our investigation.

We write $X(K)$ for unbounded operators which shall imply that this operator is closable and affiliated with the algebra $\mathcal{M}(K)$. Without further mentioning, the domain of definition of $X(K)$ and of its adjoint shall contain $\mathcal{M}'(K)\,\Omega$. We always identify $X(K)$ with the restriction of X to the domain $\mathcal{M}'(K)\,\Omega$. This has the advantage that we have the transformation

$$T(y)\,U(\Lambda)\,X(K)\,U(\Lambda^{-1})\,T(-y) = X(K_\Lambda + y).$$

The restriction of the adjoint of $X(K)$ to this domain will be denoted by $X^\dagger(K)$. This definition implies that $X(K)\,\Omega$ belongs for $K \subset W^r$ to the domain of the Tomita conjugation S_K of the algebra $\mathcal{M}(K)$ which leads to the relation

$$S_K\, X(K)\,\Omega = X^\dagger(K)\,\Omega.$$

Using the relations between the translations $T(a)$ and the Lorentz transformations $U(\Lambda)$ and the local gauges $R(t)$, $L(t)$ and using methods of the theory of analytic functions in two variables one finds:

8.3. LEMMA. – (1) *Let $X(K)$ be such that $K \subset W^r$ and the vector function*

$$U(\Lambda(t))\,X(K)\,\Omega$$

has a bounded analytic continuation into the strip $S\left(-\frac{1}{2},\, 0\right)$ and continuous boundary-values at $\Im m\, t = -\frac{1}{2}$. Then

$$L(t)\,J_l\,X^\dagger(K+x)\,\Omega \quad \textit{and} \quad R(t)\,J_r\,X^\dagger(K+x)\,\Omega, \qquad x \in \mathbb{R}^2$$

have bounded analytic continuations into the strip $S\left(-\frac{1}{2},\, 0\right)$ and continuous boundary values at $\Im m\, t = -\frac{1}{2}$.

(2) *If $K \subset W^l$ and if the vector function*

$$U(\Lambda(t))\,X(K)\,\Omega$$

has a bounded analytic continuation into the strip $S\left(0,\, \frac{1}{2}\right)$ and continuous boundary-values at $\Im m\, t = \frac{1}{2}$ then

$$L(t)\,J_l\,X^\dagger(K+y)\,\Omega \quad \textit{and} \quad R(t)\,J_r\,X^\dagger(K+y)\,\Omega, \qquad y \in \mathbb{R}^2$$

have bounded analytic continuations into the strip $S\left(0,\, \frac{1}{2}\right)$ and continuous boundary values at $\Im m\, t = \frac{1}{2}$.

The result that $U(\Lambda(t))\,X(K)\,\Omega$ has an analytic extension into the strip $S\left(-\frac{1}{2},\, 0\right)$ is independant of the situation of K in W^r. This is the consequence of Lemma 8.3. The same statement holds also for the left wedge. Collecting the results obtained so far one gets:

8.4. THEOREM. – (i) *For every $X(K)$ with $K \subset W^r$ with the property that $U(\Lambda(t))\,X(K)\,\Omega$ has a bounded analytic extension into* *the strip* $S\left(-\frac{1}{2},\,0\right)$ *and continuous boundary-values at* $\Im m\, t = -\frac{1}{2}$ *and* $U(\Lambda(t))\, X^\dagger(K-x)\,\Omega$, *with* $K-x \subset W^l$ *has a bounded analytic extension into the strip* $S\left(0,\,\frac{1}{2}\right)$ *and continuous boundary-values at* $\Im m\, t = \frac{1}{2}$, *there exists an element* $\hat{X}(-K)$ *affiliated with* $\mathcal{M}(-K)$ *fulfilling*

$$U\left(\Lambda\left(-\frac{i}{2}\right)\right) X(K)\,\Omega = \hat{X}(-K)\,\Omega.$$

A corresponding result holds if we interchange W^r *with* W^l.

Since this is the crucial result we will indicate the demonstration:

Indication of the proof. – Let K be the double-cone $K_{a,\,b}$ with $a \in W^r$, and $b - a \in W^r$. Choose an element $B'_r \in \mathcal{M}'^r_{-b}$ such that ad $R(t)\, B'_r$ is analytic and an element $B'_l \in \mathcal{M}'^l_{-a}$ such that ad $L(t)\, B'_l$ is analytic. These elements are $*$-strongly dense in the respective algebras since $R(t)$ and $L(t)$ are gauges of the respective algebras. We look at the expression

$$\left(B'_r\, B'_l\,\Omega,\, U\left(\Lambda\left(-\frac{i}{2}\right)\right) X(K)\,\Omega\right)$$
$$= \left(B'_r\, B'_l\,\Omega,\, R\left(-\frac{i}{2}\right) J_r\, X^\dagger(K)\,\Omega\right).$$

By Lemma 8.2 $R(t)\; B'_l\,\Omega$ is analytic in $S\left(-\frac{1}{2},\,0\right)$. Hence we obtain

$$= \left(\left\{\mathrm{ad}\, R\left(\frac{i}{2}\right) B'_r\right\} R\left(-\frac{i}{2}\right) B'_l\,\Omega,\; J_r\, X^\dagger(K)\, J_r\,\Omega\right).$$

The operator $J_r\, X^\dagger(K)\, J_r$ is affiliated with $\mathcal{M}^r(-K)$. Together with the Remark 8.1 this implies

$$= \left(R\left(-\frac{i}{2}\right) B'_l\,\Omega,\; J_r\, X^\dagger(K)\, J_r\; R\left(\frac{i}{2}\right) B'^*_r\,\Omega\right)$$
$$= \left(X^\dagger(K)\, J_r\; R\left(\frac{i}{2}\right) B'^*_r\,\Omega,\; J_r\, R\left(-\frac{i}{2}\right) B'_l\,\Omega\right).$$

The vector $B'^*_r\,\Omega$ belongs to the domain of $R\left(\frac{i}{2}\right)$ by choice of B'_r. Hence by Lemma 8.2 this vector belongs also to the domain of $L\left(\frac{i}{2}\right)$. The other vector belongs by choice of B'_l to the domain of $L\left(-\frac{i}{2}\right)$. Hence Lemma 8.2 applies and we obtain

$$= \left(X^\dagger(K)\,J_l\,L\left(\frac{i}{2}\right)B'^*_r\Omega,\ J_l\,L\left(-\frac{i}{2}\right)B'_l\,\Omega\right)$$
$$= \left(\operatorname{ad} L\left(-\frac{i}{2}\right)B'_l\Omega,\ J_l\,X^\dagger(K)\,J_l\,L\left(\frac{i}{2}\right)B'^*_r\Omega\right).$$

By the Remark 8.1 we find

$$= \left(\Omega,\ J_l\,X^\dagger(K)\,J_l\,L\left(\frac{i}{2}\right)B'^*_l\,B'^*_r\,\Omega\right)$$
$$= \left(J_l\,X(K)\,\Omega,\ L\left(\frac{i}{2}\right)B'^*_l\,B'^*_r\,\Omega\right).$$

Since the translations commute with $L(t)$ it follows by Lemma 8.3 and by the properties of $X(K)$ that the vector $J_l\,X(K)\,\Omega$ belongs to the domain of definition of $L\left(\frac{i}{2}\right)$ and hence the expression becomes

$$= \left(L\left(\frac{i}{2}\right)J_l\,X(K)\,\Omega,\ B'^*_l\,B'^*_r\,\Omega\right)$$
$$= \left(J_l\,L\left(-\frac{i}{2}\right)X(K)\,\Omega,\ B'^*_l\,B'^*_r\,\Omega\right).$$

Using that sums of the products of the two B are $*$-strong dense in $\mathcal{M}(-K)'$ we conclude that the two vectors $U\left(\Lambda\left(-\frac{i}{2}\right)\right)X(K)\,\Omega$ and $J_l\,L\left(-\frac{i}{2}\right)(K)\,\Omega$ belong to the domain of definition of the Tomita conjugation S_{-K} and satisfy

$$S_{-K}\,U\left(\Lambda\left(-\frac{i}{2}\right)\right)X(K)\,\Omega = J_1\,L\left(-\frac{i}{2}\right)(K)\,\Omega.$$

Hence there exists an operator $\hat{X}(-K)$ affiliated with $\mathcal{M}(-K)$ such that (*see* e.g. [BrRo] Prop. 2.5.9)

$$U\left(\Lambda\left(-\frac{i}{2}\right)\right)X(K)\,\Omega = \hat{X}(-K)\,\Omega$$

holds. Now it is easy to show that operator $\hat{X}(-K)$ has the analytic properties required for elements belonging to the left wedge. □

By this result one has established a map from a family of operators affiliated with $\mathcal{M}(W^r)$ to the corresponding family affiliated with $\mathcal{M}(W^l)$ and conversely. Unfortunately one cannot conclude directly that the modular operators of $\mathcal{M}(W^r)$ and $\mathcal{M}(W^l)'$ must coincide. This is due to the fact that we do not known whether or not there are elements in this family which are analytic with respect to the modular groups. Therefore one has to dualize Theorem 8.4 and show the same result for a large family affiliated with the algebras $\mathcal{M}(W^l)'$ and $\mathcal{M}(W^r)'$ respectively. If these families are now large enough that one can act on them with the corresponding modular groups and prove the identity of the modular operators of $\mathcal{M}(W^r)$ and $\mathcal{M}(W^l)'$, this will imply the wedge duality. Large enough means also that these families applied to the vacuum generate a core for the repective modular operators. This leads to the following requirement:

8.5. CONDITION. – *Let $\mathcal{A}_r$ be the set of operators $A(K)$ with the properties:*

(i) *the operator $A(K)$ with $K \subset W^r$ is such that $U(\Lambda(t))\, A(K)\, \Omega$ has a bounded analytic continuation into the strip $S\left(-\frac{1}{2},\, 0\right)$ with continuous boundary-values and*

(ii) *$A^*(K-x)$ with $K-x \subset W^l$ is such that $U(\Lambda(t))\, A^*(K-x)\, \Omega$ has a bounded analytic continuation into the strip $S\left(0,\, \frac{1}{2}\right)$ with continuous boundary-values.*

The set obtained by interchanging the right and the left wedge will be denoted by $\mathcal{A}_l$. It has the corresponding property with respect to $\mathcal{M}_0^l$.

We require that the sets $\mathcal{A}_r$ and $\mathcal{A}_l$ are $*$-strong dense in $\mathcal{M}_0^r$ and $\mathcal{M}_0^l$ respectively.

The result which one now obtains is the following.

8.6. THEOREM. – *We consider a Lorentz covariant theory of local observables in the vacuum-sector. This theory fulfills wedge-duality exactly if Condition* 8.5 *is fulfilled.*

So far the discussion might give the impression that the wedge duality is only a question of the number of fields. This is not the case. We want to present some examples constructed from one free field showing that duality and wedge-duality do not result from the usual axioms without Lorentz covariance. From these models we construct examples in higher dimensions showing that the modular group does not always act locally in the direction of invariance of the wedge. These results are due to J. Yngvason [Yng].

Suppose Φ is a hermitian Wightman field that transforms covariantly under space-time translations, but not necessarily under Lorentz transformations, and depends only on one light cone coordinate, say x^+. Locality implies that the commutator $[\Phi(x^+),\, \Phi(y^+)]$ has support only for $x^+ = y^+$. Moreover, from the spectrum condition it follows that the generator for translations of Φ in the x^+-direction, $P^0 - P^1$, is positive semidefinite. This implies that the Fourier transform of the two point function, $\mathcal{W}_2$, defined by $(\Omega, \Phi(x^+)\Phi(y^+)\Omega) = (1/2\pi) \int \exp[ip(x^+ - y^+)]\hat{\mathcal{W}}_2(p)\,dp$ has the form

$$\hat{\mathcal{W}}_2(p) = \theta(p)\,p\,Q(p^2) + c\,\delta(p).$$

In this formula Ω is the vacuum vector, $Q(p^2)$ is a positive, even polynomial in $p \in \mathbb{R}$ and $\theta(s) = 1$ for $s \geq 0$ and zero else, and $c = (\Omega, \Phi(x^+)\Omega)^2 \geq 0$ is a constant. Subtracting $c^{1/2}$ from Φ if necessary, we may drop the $\delta(p)$-term. For simplicity of notation from now on we write x, y instead of x^+, y^+.

The models we consider are generalized free fields with the two point function given above (without the δ-term). They are characterized by the commutation relations

$$[\Phi(x), \Phi(y)] = Q\,Q(D^2)\delta(x - y)\,\mathbb{1},$$

where we have for convenience denoted id/dx by D. Let $\mathcal{H}_{Q,1}$ be the Hilbert space of functions $f(p)$ such that $\int_0^\infty |f(p)|\,p\,Q(p)\,dp < \infty$. Define for $f \in \mathcal{H}_{Q,1}$ the unitary Weyl operators as usual by

$$W(f) = e^{i\Phi(f)}.$$

The Weyl relations are

$$W(f)\,W(g) = e^{-K(f,g)/2}\,W(f+g)$$

with

$$K(f, g) = (\Omega, [\Phi(f), \Phi(g)]\,\Omega) = \int_{-\infty}^{\infty} p\,Q(p^2)\,\tilde{f}(-p)\,\tilde{g}(p)\,dp.$$

It follows that $W(f)$ commutes with $W(g)$ if and only if $K(f, g) = 0$, in particular if f and g have disjoint supports. The Weyl operators are defined on the Fock space $\mathcal{H}_Q$. For our future investigations we can restrict our attention to the one-particle Hilbert space $\mathcal{H}_{Q,1}$.

We know that the modular group of the half line acts as a dilatation by the factor $e^{-2\pi t}$. This amounts in momentum space to a dilatation by the factor $\lambda = e^{2\pi t}$. If we denote the restriction of the modular group of the positive half line Δ_+^{it} to the one-particle Hilbert space $\mathcal{H}_{Q,1}$ by $V_+(\lambda)$ we must get

$$(V_+(\lambda)\,\psi)(p) = \lambda\sqrt{\frac{Q(\lambda p)}{Q(p)}}\,e^{i\Phi(p)}\,\mu(\lambda p),$$

where the phasefactor $e^{2\Phi(p)}$ has to be determined. If $\psi(p)$ is analytic in the upper half plane then the same must be true for $(V_+(\lambda)\psi(p)$. This condition can be solved by remembering the structure of $Q(p)$ which permits us to write

$$Q(p) = L(p)\,L(-p), \qquad \text{with} \quad L(-p) = L(p)^*.$$

The polynomial $L(p)$ is fixed up to a sign by the requirement that its zeros lie in the closed upper half plane. Hence we find:

$$(V_+(\lambda)\,\psi)(p) = \lambda\,\frac{L(-\lambda\,p)}{L(-p)}\,\psi(\lambda\,p).$$

That this is the correct expression for the modular group can be checked by showing that the KMS-condition is fulfilled. For this one uses the analyticity property as well in p as in λ.

In the same manner we obtain for the left half-line

$$(V_-(\lambda)\,\psi)(p) = \lambda\,\frac{L(\lambda\,p)}{L(p)}\,\psi(\lambda\,p).$$

Since the algebra and its commutant have the same modular group we see that wedge duality is fulfilled iff $L(p)$ has only real zeros.

The duality conditions for bounded intervals is a little more difficult. Yngvason has shown that it is violated if $L(p)$ and hence $Q(p)$ is not a constant.

Finally we consider fields in n-dimensional Minkowski space. Guided by the low-dimensional examples considered above we shall compute the modular groups of the wedge algebras for generalized free fields on $\mathbb{R}^n$. We treat the special case when the two-point function has in Fourier space the form

$$\mathcal{W}_2(p) = M(p)\,d\mu(p),$$

where $d\mu$ is a positive Lorentz-invariant measure with support in the forward light cone and M is a polynomial that is positive on the support of $d\mu$. The polynomial M allows a factorization,

$$M(p) = F(p)\,F(-p),$$

where $F(p)$ is a function (in general not a polynomial) with certain analyticity properties to be specified below.

To describe the properties of F we use the light cone coordinates $x^\pm = x^0 \pm x^1$ for $x = (x^0, \ldots, x^{(n-1)}) \in \mathbb{R}^n$ and denote $(x^2, \ldots, x^n)$ by $\hat{x}$. The Minkowski scalar product is

$$\langle x, y\rangle = \frac{1}{2}(x^+ y^- + x^- y^+) - \hat{x}\cdot\hat{y}.$$

The right wedge, W_r, is characterized by $x^+ > 0$, $x^- < 0$; hence the Fourier transform, $\tilde{f}(p) = \int \exp(-i\langle p, x\rangle) f(x)\, d^n x$ of a test function f with support in W_r has for fixed $\hat{p} \in \mathbb{R}^{n-2}$ an analytic continuation in p^+ and p^- into the half planes Im $p^+ > 0$, Im $p^- < 0$. We require for F that $F(\pm p)$ is analytic and that $F(-p)$ is *without zeros* in this domain, with $F(-p) = F(p)^*$ for $p \in \mathbb{R}^n$. There is no lack of polynomials M allowing such a factorization; one example is

$$M(p) = (p^1)^2 + \cdots + (p^n)^2 + m^2$$

with

$$F(p) = \sqrt{\hat{p} \cdot \hat{p} + m^2} + ip^1 = \sqrt{\hat{p} \cdot \hat{p} + m^2} + \frac{i}{2}(p^+ - p^-).$$

If $d\mu(p) = \theta(p^0)\,\delta(\langle p, p\rangle - m^2)$ we can replace the polynomial by $(p^0)^2$. Hence the corresponding generalized free field is nothing but the time derivative $(d/dx^0)\,\Phi_m(x)$, where Φ_m is the free field of mass m.

In analogy with the first example we define for $\lambda > 0$ the unitary operators $V_R(\lambda)$ on the Fock space $\mathcal{H}$ over the one-particle space $\mathcal{H}_1 = L^2(\mathbb{R}^n, M(p)\, d\mu(p))$ by

$$V_R(\lambda)\,\varphi(p) = \frac{F(-\lambda p^+, -\lambda^{-1} p^-, -\hat{p})}{F(-p^+, -p^-, -\hat{p})}\,\varphi(\lambda p^+, \lambda^{-1} p^-, \hat{p}) \qquad (*)$$

for $\varphi \in \mathcal{H}_1$ and canonical extension to $\mathcal{H}$. Then we define by means of $V_R(\lambda)$ a one parameter group of automorphisms of the Neumann algebra $\mathcal{M}(W_r)$ on $\mathcal{H}$ generated by the Weyl operators $W(f)$ with supp $f \subset W_r$. By essentially the same computation that verified the example of the half line one shows that $(*)$ satisfies the KMS condition and that it is therefore, the modular group defined by the vacuum state on $\mathcal{M}(W_r)$.

For the left wedge $W_l = \{x | x^+ < 0,\ x^- > 0\}$ the corresponding operators are

$$V_L(\lambda)\,\varphi(p) = \frac{F(\lambda p^+, \lambda^{-1} p^-, \hat{p})}{F(p^+, p^-, \hat{p})}\,\varphi(\lambda p^+, \lambda^{-1} p^-, \hat{p}).$$

By comparing the two modular groups we see that the field satisfies the wedge duality condition $\mathcal{M}(W_r)' = \mathcal{M}(W_l)$ if and only if $F(p) = F(-p)$ on the support of $d\mu$. This condition is, e.g., violated in the above mentioned example.

This example demonstrates also that the modular group of $\mathcal{M}(W_r)$ may act nonlocally in the $\hat{x}$-directions. In fact, let f be a test function with compact support in W_r. Under the transformation (7.7) the Fourier transform $\hat{f}$ is mapped into

$$\tilde{f}_\lambda(p) = \frac{\sqrt{\hat{p}\cdot\hat{p}+m^2} - \frac{i}{2}(\lambda p^+ - \lambda^{-1}p^-)}{\sqrt{\hat{p}\cdot\hat{p}+m^2} - \frac{i}{2}(p^+ - p^-)} \tilde{f}(\lambda p^+, \lambda^{-1}p^-, \hat{p}).$$

This is no longer the Fourier transform of a function of compact support in the $\hat{x}$-directions, because it is not analytic in $\hat{p}$. From this lack of analyticity it is not difficult to deduce that $W(f_\lambda)$ does not belong to any wedge algebra generated by the field unless the wedge is a translate of W_r or W_l, but we refrain from presenting a formal proof. The operator $W(f_\lambda)$ is still localized in the x^0, x^1-directions in the sense that it is contained in $\mathcal{M}(W_r + a) \cap \mathcal{M}(W_r + b)'$ for some $a, b \in W_r$.

REFERENCES

[BW1] J. Bisognano and E. H. Wichmann, On the duality condition for a Hermitian scalar field, *J. Math. Phys.*, Vol. **16**, 1975, pp. 985-1007.

[BW2] J. Bisognano and E. H. Wichmann, On the duality condition for quantitum fields, *J. Math. Phys.*, Vol. **17**, 1976, pp. 303-321.

[Boa] R. P. Boas, *Entire Functions*, Academic Press, New York, 1954.

[Bch1] H.-J. Borchers, Translation Group and Modular Automorphisms for Local Regions, *Commun. Math. Phys.*, Vol. **132**, 1990, pp. 189-199.

[Bch2] H.-J. Borchers, The CPT-Theorem in Two-dimensional Theories of Local Observables, *Commun. Math. Phys.*, Vol. **143**, 1992, pp. 315-332.

[Bch3] H.-J. Borchers, On Modular Inclusion and Spectrum Condition, *Lett. Math. Phys.*, Vol. **27**, 1993, pp. 311-324.

[Bch4] H.-J. Borchers, When does Lorentz Invariance imply Wedge-Duality?, *Lett. Math. Phys.*, Vol. **35**, 1995, pp. 39-60.

[BY] H.-J. Borchers and J. Yngvason, From Quantum Field to local von Neumann Algebras, *Rev. Math. Phys.*, Special Issue, 1992, pp. 15-47.

[BR] O. Bratteli and D. W. Robinson, Operator Algebras and Quantum Statistical Mechanics I, Springer Verlag, New York, Heidelberg, Berlin, 1979.

[BÖT] H. J. Bremermann, R. Oehme and J. G. Taylor, Proof of dispersion relation in quantized field theories, *Phys. Rev.,* Vol. **109**, 1958, pp. 2178-2190.

[BEGS] J. Bros, H. Epstein, V. Glaser and R. Stora, Quelques aspects globaux des problèmes d'Edge-of-the-Wedge, In: Hyperfunctions and Theoretical Physics (Nide Rencontre 1973), *Lecture Notes in Mathematics,* Vol. **449**, Springer-Verlag, Berlin, Heidelberg, New York, 1975.

[BGL1] R. Brunetti, D. Guido and R. Longo, Modular structure and duality in conformal quantum field theory, *Commun. Math. Phys.*, Vol. **156**, 1993, pp. 201-219.

[BGL2] R. Brunetti, D. Guido and R. Longo, Group cohomology, modular theory and space-time symmetrics, to appear in *Rev. Math. Phys.*

[Bu] D. Buchholz, On the Structure of Local Quantum Fields with Non-trivial Interactions, In: *Proceedings of the International Conference on Operator Algebras,* Ideals and their Applications in Thepretical Physics, Leipzig, 1977, Teubner-Texte zur Mathematik, 1978, pp. 146-153.

[BDL1,2] D. BUCHHOLZ, C. D'ANTONI and R. LONGO, Nuclear Maps and Modular Structures, I. General Properties, *Jour. Func. Analysis*, Vol. **88**, 1990, pp. 233-250. II. Application to Quantum Field Theory, *Commun. Math. Phys.*, Vol. **129**, 1990, pp. 115-138.

[BJ] D. BUCHHOLZ and P. JUNGLAS, On the existence of equilibrium states in local quantum field theory, *Commun. Math. Phys.*, Vol. **121**, 1989, pp. 255-270.

[BSM] D. BUCHHOLZ and H. SCHULZ-MIRBACH, Haag-duality in conformal quantum field theory, *Rev. Math. Phys.*, Vol. **2**, 1990, p. 105.

[BuSu1] D. BUCHHOLZ and S. J. SUMMERS, An Algebraic Characterization of Vacuum States in Minkowsky Space, *Commun. Math. Phys.*, Vol. **155**, 1993, pp. 442-458.

[BuSu2] D. BUCHHOLZ and S. J. SUMMERS, Geometric modular action and representations of the Poincaré group, in preparation.

[BuWi] D. BUCHHOLZ and E. H. WICHMANN, Causal independence and the energy-level density of states in local quantum field theory, *Commun. Math. Phys.*, Vol. **106**, 1986, pp. 321-344.

[Ep] H. EPSTEIN, Some Analytic Properties of Scattering Amplitudes in Quantum Field Theory, in 1965 Brandeis Summer Institute, Gordon and Breach, New York, London, Paris, 1966.

[Fre] K. FREDENHAGEN, Generalization of the Theory of Superselection Sectors, In: The Algebraic Theory of Superselection Sectors. Introduction and Recent Results, World Scientifique 1990, p. 379.

[GF] F. GABBIANI and J. FRÖHLICH, Operator algebras and conformal field theory, *Commun. Math. Phys.*, Vol. **155**, 1993, pp. 569-640.

[Ha] R. HAAG, Local Quantum Physics, Springer verlag, Berlin, Heidelberg, New York, 1992.

[HHW] R. HAAG, N. HUGENHOLTZ and M. WINNINK, On the equilibrium state in quantum statistical mechanics, *Commun. Math. Phys.*, Vol. **5**, 1967, pp. 215-236.

[HL] P. D. HISLOP and R. LONGO, Modular structure of the local algebra associated with a free massless scalar field theory, *Commun. Math. Phys.*, Vol. **84**, 1982, pp. 71-85.

[Jo] R. JOST, Eine Bemerkung zum CTP Theorem, *Helv. Phys. Acta*, Vol. **30**, 1957, pp. 409-416.

[Ka] R. V. KADISON, Derivations of operator algebras, *Ann. of Math.*, Vol. **83**, 1966, pp. 280-293.

[KR] R. V. KADISON and J. R. RINGROSE, Fundamentals of the Theory of Operator Algebras II, New York: Academic press, 1986.

[Lo] R. LONGO, Algebraic and modular structure of von Neumann algebras in physics, *Proc. Symp. Pure Math.*, Vol. **38**, 1982, pp. 551-566.

[OT] A. I. OKSAK and I. T. TODOROV, Invalidity of the TCP-Theorem for Infinite-Component Fields, *Commun. Math. Phys.*, Vol. **11**, 1968, p. 125.

[RS] H. REEH and S. SCHLIEDER, Eine Bemerkung zur Unitäräquivalenz von Lorentzinvarianten Feldern, *Nuovo Cimento*, Vol. **22**, 1961, p. 1051.

[Sak] S. SAKAI, Derivations of W^*-algebras, *Ann. of Math.*, Vol. **83**, 1966, pp. 273-279.

[Str] R. STREATER, Local Fields with the Wrong Connection Between Spin and Statistics, *Commun. Math. Phys.*, Vol. **5**, 1967, pp. 88-96.

[Ta] M. TAKESAKI, Tomita's Theory of Modular Hilbert Algebras and its applications, Lecture Notes in Mathematics, Vol. **128**, Springer verlag, Berlin, Heidelberg, New York, 1970.

[TMP] I. T. TODOROV, M. C. MINTCHEV and V. B. PETKOVA, Conformal invariance in quantum field theory, *Publ. Scuola Normale Superiore, Pisa* 1978.

[To] M. TOMITA, Quasi-standard von Neumann algebras, Preprint, 1967.

[Wie1] H.-W. WIESBROCK, A comment on a recent work of Borchers, *Lett. Math. Phys.*, Vol. **25**, 1992, pp. 157-159.

[Wie2] H.-W. WIESBROCK, Half-Sided Modular Inclusions of von Neuman Algebras, Preprint, FU Berlin, 1992.

[Wie3] H.-W. WIESBROCK, Symmetries and Half-Sided Modular Inclusions of von Neumann Algebras, *Lett. Math. Phys.*, Vol. **28**, 1993, pp. 107-114.

[Wie4] H.-W. WIESBROCK, Conformal Quantum Field Theory and Half-Sided Modular Inclusions of von Neumann Algebras, *Commun. Math. Phys.*, Vol. **158**, 1993, pp. 537-543.

[Win] M. WINNINK, An Application of C^*-Algebras to Quantum Statistical Mechanics of Systems in Equilibrium, Thesis groningen, 1968.

[Yng] J. YNGVASON, A Note on Essential Duality, *Lett. Math. Phys.,* Vol. **31**, 1994, pp. 127-141.

[Ze] E. C. ZEEMAN, Causality Implies the Lorentz Group, *J. Math. Phys.*, Vol. **5**, 1964, pp. 490-493.

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