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Localization for random Schrödinger operators with Poisson potential

by

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ABSTRACT. – Localization at all energies is proved for the one-dimensional random Schrödinger operator with Poisson potential $\sum_j f(x - X_j(\omega))$. The single site potential f is assumed to be non-negative and compactly supported. The result holds for arbitrary density of the Poisson process. Eigenfunctions decay exponentially at the rate of the Lyapunov exponent. Crucial to the proof is a new result on spectral averaging.

RÉSUMÉ. – On montre la localisation à toute énergie pour le Hamiltonien de Schrödinger aléatoire à 1 dimension avec un potential de Poisson $\sum_j f(x - X_j(\omega))$. Le potentiel de site f est supposé non négatif et à support compact. Ce résultat est valable pour une densité arbitraire du processus de Poisson. Les fonctions propres décroissent exponentiellement avec un taux égal à l'exponent de Lyapunov. Un élément crucial de la démonstration est fourni par un résultat nouveau sur les moyennes spectrales.

1. INTRODUCTION

The Poisson model is a random Schrödinger operator, which describes random media with extreme structural disorder. For the one-dimensional

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case it is known that this model does not exhibit absolutely continuous spectrum, *cf.* [14]. But in contrast to other random operators, in particular Anderson-type models describing random alloys, no proof of localization could be given so far. The main problem for the Poisson model is the apparent lack of "monotonicity" in the random parameter, a property which was successfully used in proofs of localization for Anderson-type models, e.g. [27, 20, 4].

For the one-dimensional Poisson model we will be able to overcome this difficulty by providing a new result on *spectral averaging*, which allows to give a version of "Kotani's trick" adapted to the Poisson model and can be seen as a consequence of some kind of monotonicity of the Poisson model in the distance of neighboring Poisson points. Once we have established Kotani's trick we can use positivity of the Lyapunov exponent and the strategy used previously for Anderson models [20] to prove localization for the one-dimensional Poisson model.

To define the one-dimensional Poisson model (for general information *see* [22] and [2]) we start with the Poisson random measure μ, which depends on a parameter $\alpha > 0$ called the concentration (density, intensity). This is a weakly measurable random variable μ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values μ_ω in the space $\mathcal{M}_+(\mathbb{R})$ of non-negative Borel measures on $\mathbb{R}$. It has the property that for every Borel set $B \subset \mathbb{R}$ the real random variable $\mu_\omega(B)$ is Poisson distributed with parameter $\alpha|B|$, *i.e.* $\mathbb{P}(\mu_\omega(B) = n) = (\alpha|B|)^n e^{-\alpha|B|}/n!$, and that $\mu_\omega(B_1), \ldots, \mu_\omega(B_n)$ are independent if $B_1, \ldots, B_n$ are mutually disjoint. The distribution of μ makes $\mathcal{M}_+(\mathbb{R})$ a probability space, which henceforth is identified with $(\Omega, \mathcal{F}, \mathbb{P})$.

The process x_t defined on $\mathcal{M}_+(\mathbb{R})$ by $x_t(\mu) = \mu((0, t])$ for $t > 0$ and $x_t(\mu) = -\mu((0, t])$ for $t \leq 0$ is strictly stationary in the sense that the multivariate distributions of $x_{t_2+t} - x_{t_1+t}, \ldots, x_{t_n+t} - x_{t_{n-1}+t}$ do not depend on t. Furthermore it has independent increments $x_{t_2} - x_{t_1}, \ldots, x_{t_n} - x_{t_{n-1}}$. Thus the shifts T_t, $t \in \mathbb{R}$ defined by $(T_t\rho)(A) = \rho(A + \{t\})$, $\rho \in \mathcal{M}_+(\mathbb{R})$, are a metrically transitive family of one-to-one measure preserving transformations on $\mathcal{M}_+(\mathbb{R})$ (*cf.* [7, Ch. XI.1]).

With probability one μ_ω is a counting measure for a locally finite set of points $X_i(\omega) \in \mathbb{R}$, *i.e.* $\mu_w(B) = \#\{i;\, X_i(\omega) \in B\}$ [12]. The X_i can be labeled by $i \in \mathbb{Z}\backslash\{0\}$ in a measurable way such that

$$\ldots < X_{-1}(\omega) < 0 < X_1(\omega) < X_2(\omega) < \ldots$$

The concentration α is the average number of points $X_i(\omega)$ per unit interval. The random variables $Y_{\pm 1} := X_{\pm 1}$, $Y_{n:} = X_n - X_{n-1}$ and

$Y_{-n:} := X_{-(n-1)} - X_{-n}$ $(n \geq 2)$ are independent and each has absolutely continuous distribution with density $\alpha e^{-\alpha t}$, *cf.* [13, Ch. 4]. Note here that the distribution of $X_1 - X_{-1}$ is different from the distributions of Y_n, $|n| \geq 2$, an example of the *waiting time paradox* [13].

We take a *single site potential*

$$f \in L^2(\mathbb{R}), \quad \text{real valued and compactly supported,} \tag{1}$$

and define the *Poisson potential* by

$$V_\omega(\omega) = F(T_x \mu_\omega), \tag{2}$$

where

$$F(\mu) = \int f(-y)\, d\mu(y).$$

This means that

$$V_\omega(x) = \int f(x-y)\, d\mu_\omega(y) = \sum_i f(x - X_i(\omega)).$$

By Campbell's Theorem ([13], Sect. 3.2]) we have

$$\begin{aligned} \mathbb{E}(V(0)^2) &= \operatorname{Var}(V(0)) + \mathbb{E}(V(0))^2 \\ &= \alpha^2 \int f(-x)^2\, dx + \alpha^2 \left(\int f(-x)\, dx \right)^2 < \infty. \end{aligned} \tag{3}$$

Thus

$$H_\omega = -\frac{d^2}{dx^2} + V_\omega \tag{4}$$

is almost surely essentially selfadjoint on $C_0^\infty(\mathbb{R})$ [14, App. 2]. By (2) the family H_ω is metrically transitive, thus (*cf.* [22]) there exist susbsets Σ_{ac}, Σ_{sc} and Σ_{pp} of $\mathbb{R}$ such that almost surely

$$\sigma_{ac}(H_\omega) = \Sigma_{ac}, \qquad \sigma_{sc}(H_\omega) = \Sigma_{sc}, \qquad \sigma_{pp}(H_\omega) = \Sigma_{pp},$$

with σ_{ac}, σ_{sc} and σ_{pp} denoting the absolutely continuous, singular continuous and point spectrum.

We will now assume that in addition

$$f \geq 0 \quad \text{and not identically } 0. \tag{5}$$

Our main result is

THEOREM 1. – *For the one-dimensional Poisson model H_ω defined by* (1), (2), (4) *and* (5) *one has*

$$\Sigma_{ac} = \Sigma_{sc} = \emptyset,$$

i.e. *H_ω almost surely has pure point spectrum. The eigenfunctions decay exponentially at the rate of the Lyapunov exponent.*

From $f \geq 0$ it easily follows that $\sigma(H_\omega) = [0, \infty)$ almost surely, *i.e.* Theorem 1 actually proves *dense* pure point spectrum for H_ω.

To prove Theorem 1 we will rely on Kotani's general results [18] on one-dimensional random Schrödinger operators. Kotani assumed boundedness of the potential, which does not hold for the Poisson potential. (In the case of bounded f there almost surely is a logarithmic bound for V_ω [10]). But is was shown in [14, App. 2] that the results to be used below extend to general metrically transitive $H_\omega = -d^2/dx^2 + V_\omega$ as long as (3) is satisfied.

First we note that f being compactly supported implies that $V_\omega(x)$ is a *non-deterministic* process in the sense of [18], *cf.* [14], [22]. Therefore the Lyapunov exponent $\gamma(E)$ of H_ω is strictly positive for (Lebesgue-) a.e. $E \in \mathbb{R}$, and, in particular, $\Sigma_{ac} = \emptyset$, a result known to hold in much more general situations [14]. It remains to be proved that $\Sigma_{sc} = \emptyset$ and that eigenfunctions decay exponentially at the rate of the Lyapunov exponent.

Since $\gamma(E) > 0$ for almost every E we can use the Osceledec-Ruelle theorem (e.g. [22, Sect. 11A]) to conclude that for a.e. E there exist non-trivial exponentially decaying (at Lyapunov rate) solutions at $+\infty$ as well as $-\infty$ of $-u'' + V_\omega u = Eu$ for a.e. ω. An application of Fubini yields

PROPOSITION 2. – *To a.e. $E \in \mathbb{R}$ there exist non-trivial solutions u_+ and u_- of $-u'' + V_\omega u = Eu$ such that u_+ is exponentially decaying at $+\infty$ and u_- is exponentially decaying at $-\infty$ at the rate of the Lyapunov exponent.*

This will serve as one of the basic ingredients in the proof of Theorem 1. The other ingredient is *spectral averaging*, a method which has turned out to be extremely useful in proofs of localization for various random models (e.g. [27], [20], [4]). Having the standard facts leading to Proposition 2 above and using the Kotani-Simon proof [20] of localization for the one-dimensional Anderson model as guidance, the main obstacle to overcome for the Poisson model was the lack of a result on spectral averaging. Such a result is given in Section 2, where a deterministic family of Schrödinger operators H_a involving a shift parameter a is studied.

In the proof of Theorem 1, which is completed in Section 3 this shift parameter will be found from the value of $Y_1 = X_1$, while all the other distances Y_n will be kept fixed (independent of Y_1). Since Y_1 has absolutely continuous distribution we can apply our result on spectral averaging from Section 2.

At the end of Section 3 we will comment on possible generalizations of Theorem 1. In particular, we state that our main result can be extended to "mixed Anderson-Poisson models" of the type

$$\tilde{V}_\omega(x) = \sum_n \lambda_n(\omega) f(x - X_n(\omega)).$$

We also note that after dropping the assumption $f \geq 0$ we at least get $\Sigma_{sc} \cap (0, \infty) = \emptyset$.

For convenience, we include an Appendix to collect suitable versions of some more or less standard results used in Section 2.

We have earlier announced our main result in [30].

2. SPECTRAL AVERAGING

Let $W_1 \in L^1_{loc}(\mathbb{R})$ vanish in $(-\infty, 0)$ and be such that the differential expression $-d^2/dx^2 + W_1$ is limit point at $+\infty$. Note that a quite general criterion for the latter to hold is given by $\int_0^x W_{1,-}(t)\,dt = \mathcal{O}(x^3)$ as $x \to \infty$ for the negative part $W_{1,-}$ of W_1 [9]. Also let $W_2 \in L^1_{loc}(\mathbb{R})$ vanish in $(0, \infty)$ and $-d^2/dx^2 + W_2$ be limit point at $-\infty$.

We define a family of potentials V_a, $a > 0$ on the real line by

$$V_a(x) := W_1(x - a) + W_2(x + a),$$

and denote the unique selfadjoint realization (in the sense of Sturm-Liouville theory) of $-d^2/dx^2 + V_a$ in $L^2(\mathbb{R})$ by H_a. (Under the slightly stronger assumption $W_i \in L^2_{loc}$, $i = 1, 2$, H_a coincides with the closure of $-d^2/dx^2 + V_a$ with domain $C_0^\infty(\mathbb{R})$.)

The operators H_a have spectral multiplicity one and their spectral type is determined by the Weyl-Titchmarsh spectral measures ρ_a, *i.e.* the trace measures corresponding to the standard 2×2-matrix valued spectral measures for H_a [3].

In the proof of the proposition below we will make use of Prüfer variables and their dependence on various parameters: For some potential V let u be the solution of $-u'' + Vu = Eu$ with $u(c) = \sin\theta$ and $u'(c) = \cos\theta$. Then the Prüfer amplitude and angle are

$$r_c(x, \theta, E, V) = (u(x)^2 + u'(x)^2)^{1/2}$$

and

$$\phi_c(x, \theta, E, V) = \arctan \frac{u}{u'} \quad \left(\text{resp.}\,\text{arccot}\,\frac{u'}{u}\right),$$

where ϕ_c is normalized such that $\phi_c(c, \theta, E, V) = \theta$ and $\phi_c(\cdot, \theta, E, V)$ is continuous.

The family H_a satisfies *spectral averaging* at positive energies:

PROPOSITION 3. – *For fixed $a_2 > a_1 > 0$ and arbitrary Borel sets $B \subset \mathbb{R}$ define*

$$\mu(B) := \int_{a_1}^{a_2} \rho_a(B)\, da.$$

Then the Borel measure μ is absolutely continuous on $(0, \infty)$.

Proof. – It is enough to show that μ is absolutely continuous in (E_0, E_1) for every fixed pair $E_1 > E_0 > 0$. This will in turn follow from the existence of a $C > 0$ such that

$$\int f(E)\, d\mu(E) \leq C \int f(E)\, dE \tag{6}$$

for every non-negative continuous f with compact support in (E_0, E_1). (So in fact we show local Lipshitz continuity of μ in $(0, \infty)$.)

In the proof of (6) we will make use of the fact that ρ_a is the weak limit of spectral measures to operators defined by $-d^2/dx^2 + V_a$ on finite intervals: Let $(\rho_a)_{-N,\alpha}^{N,\beta}$ be the Weyl-Titchmarsh spectral measure to the selfadjoint realization of $-d^2/dx^2 + V_a$ in $L^2(-N, N)$ corresponding to boundary conditions $f(-N)\cos\alpha - f'(-N)\sin\alpha = 0$, $f(N)\cos\beta - f'(N)\sin\beta = 0$. Define the averaged spectral measures $(\rho_a)_{-N}^{N}$ by

$$(\rho_a)_{-N}^{N}(B) = \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} (\rho_a)_{-N,\alpha}^{N,\beta}(B)\, d\alpha d\beta.$$

For every a we have that ρ_a is the weak limit of the $(\rho_a)_{-N}^{N}$:

$$\rho_a = w - \lim_{N\to\infty} (\rho_a)_{-N}^{N} = w - \lim_{N\to\infty} (\rho_a)_{-N-a}^{N+a}, \tag{7}$$

cf. [2, Ch. III].

The measures $(\rho_a)_{-N}^{N}$ are absolutely continuous with densities which can be expressed in terms of Prüfer amplitudes ([1], [2] for continuous V, [29, Appendix B] for more general V):

$$d(\rho_a)_{-N}^{N}(E) = \left(\frac{1}{\pi^2} \int_0^{\pi} r_0(N, \theta, E, V_a)^{-2}\, r_0(-N, \theta, E, V_a)^{-2}\, d\theta\right) dE.$$

Using (7) we can therefore write

$$\int f(E)\, d\mu(E) = \int_{a_1}^{a_2} \lim_{N\to\infty} \int f(E) \left(\frac{1}{\pi^2} \int_0^\pi r_0 (N+a,\, \theta,\, E,\, V_a)^{-2} \right.$$
$$\left. \times\, r_0(-N-a,\, \theta,\, E,\, V_a)^{-2}\, d\theta \right) dE\, da. \tag{8}$$

Solving $-u'' + V_a u = Eu$ first in $[0, a]$ (where $V_a = 0$) and then in $[a, N+a]$ shows

$$\begin{aligned} r_0(N+a,\, \theta,\, E,\, V_a) &= r_0(a,\, \theta,\, E,\, 0)\, r_a(N+a,\, \phi_0(a,\, \theta,\, E,\, 0),\, E,\, V_a) \\ &= r_0(a,\, \theta,\, E,\, 0)\, r_a(N,\, \phi_0(a,\, \theta,\, E,\, 0),\, E,\, W_1). \end{aligned}$$

Positivity of Prüfer amplitudes and continuous dependence of solutions of $-u'' = Eu$ on parameters shows $r_0(a,\, \theta,\, E,\, 0) \geq C > 0$ for all $a \in [a_1,\, a_2]$, $\theta \in [0,\, \pi]$ and $E \in [E_0,\, E_1]$. Together with a similar result for $r_0(-N-a,\, \theta,\, E,\, V_a)$ we get

$$\begin{aligned} &\int_{a_1}^{a_2} \int_0^\pi r_0(N+a,\, \theta,\, E,\, V_a)^{-2}\, r_0(-N-a,\, \theta,\, E,\, V_a)^{-2}\, d\theta\, da \\ &\leq C \int_{a_1}^{a_2} \int_0^\pi r_0(N,\, \phi_0(a,\, \theta,\, E,\, 0),\, E,\, W_1)^{-2} \\ &\quad \times r_0(-N,\, \phi_0(-a,\, \theta,\, E,\, 0),\, E,\, W_2)^{-2}\, d\theta\, da. \end{aligned} \tag{9}$$

In (9) we now substitute the new variables

$$\beta_1(a,\, \theta) = \phi_0(a,\, \theta,\, E,\, 0), \qquad \beta_2(a,\, \theta) = \phi_0(-a,\, \theta,\, E,\, 0).$$

We use

$$\begin{aligned} \frac{\partial \beta_1}{\partial a} &= \cos^2 \beta_1 + E \sin^2 \beta_1 \geq \min\{1,\, E_0\}, \\ \frac{\partial \beta_2}{\partial a} &= -\cos^2 \beta_2 - E \sin^2 \beta_2 \leq -\min\{1,\, E_0\} \end{aligned}$$

and (*see* Proposition 11 of the Appendix)

$$\begin{aligned} \frac{\partial \beta_1}{\partial \theta} &= r_0(a,\, \theta,\, E,\, 0)^{-2} \geq C > 0, \\ \frac{\partial \beta_2}{\partial \theta} &= r_0(-a,\, \theta,\, E,\, 0)^{-2} \geq C > 0 \end{aligned}$$

uniformly in $a \in [a_1, a_2]$, $\theta \in [0, \pi]$, $E \in [E_0, E_1]$. We get for the Jacobian

$$\det\left(\frac{\partial(\beta_1, \beta_2)}{\partial(a, \theta)}\right) \geq C > 0 \qquad (10)$$

uniformly in $E \in [E_0, E_1]$ and therefore the r.h.s. of (9) can be estimated by

$$C \int_{b_2}^{c_2} \int_{b_1}^{c_1} r_0(N, \beta_1, E, W_1)^{-2} r_0(-N, \beta_2, E, W_2)^{-2} d\beta_1 \, d\beta_2, \qquad (11)$$

where also $(\beta_1, \beta_2)([a_1, a_2] \times [0, \pi]) \subset [b_1, c_1] \times [b_2, c_2]$ uniformly in $E \in [E_0, E_1]$ was used.

The integral (11) factorizes in β_1 and β_2. The general identity

$$\int_0^\pi r_x(y, \theta, E, V)^{-2} \, d\theta = \pi$$

(*see* Corollary 12 of the Appendix) and the π-periodicity of $r_x(y, \theta, E, V)$ in θ now yield the uniform boundedness of (11) and therefore the r.h.s. of (9) in $E \in [E_0, E_1]$. This can be used to interchange the N-limit and the a-integral in (8) (dominated convergence) and to thereby get (6). #

Results on spectral averaging are related to absence of continuous spectrum for one-dimensional continuous and discrete Schrödinger operators by various versions of an argument known as Kotani's trick. So far, it has been applied to the case where the averaging parameter is (i) a boundary condition for a half-line Schrödinger operator, e.g. ([19] ,[15], [16]), or (ii) a coupling constant λ in a family $H_\lambda = H_0 + \lambda W$, the latter being basic to one of the proofs of localization for one-dimensional Anderson models [27], [30]. Equipped with Proposition 3 we can now apply these well known ideas to the model H_a from above $\Big($here $\sigma_c = \sigma_{sc} \cup \sigma_{ac}$ is the continuous spectrum and $W \in L^1_{loc, unif}[0, \infty)$ means $\sup_{x \geq 0} \int_x^{x+1} |W(t)| dt < \infty$ and $L^1_{loc, unif}(-\infty, 0]$ is defined analogously$\Big)$:

PROPOSITION 4. – *Let H_a be defined as above and assume $W_{1,-} \in L^1_{loc, unif}[0, \infty)$, $W_{2,-} \in L^1_{loc, unif}(-\infty, 0]$. Let I be an open subset of $(0, \infty)$. Suppose that for almost every $E \in I$ there exists a non-trivial solution $u_+ \in L^2(0, \infty)$ of $-u'' + W_1 u = Eu$ and a non-trivial solution $u_- \in L^2(-\infty, 0)$ of $-u'' + W_2 u = Eu$. Then*

$$\sigma_c(H_a) \cap I = \varnothing \quad for \;\; a.e. \;\; a > 0.$$

If the solutions u_+ and u_- are exponentially decaying at $+\infty$ and $-\infty$, respectively, then the eigenfunctions of H_a to eigenvalues $E \in I$ are exponentially decaying for a.e. $a > 0$.

Remarks. – In our application to the Poisson model in the next section we will have more information than used here, namely the existence of exponentially decaying solutions at $+\infty$ and $-\infty$ with all other solutions exponentially growing. This would actually allow a somewhat simpler proof of Proposition 4; compare the proof of Theorem 2.1 in [20]. We think, however, that it is helpful to realize that the proposition holds under the weakest natural assumption, namely square-integrability of the u_+ and u_-. In the proof we follow the line of argument used in [15, Lemma 1] for the discrete case.

We also expect that the uniform local integrability of $W_{i,-}$ can be replaced by merely assuming limit point behavior at $\pm\infty$. This is suggested by a corresponding result for the case of boundary condition variation given in [6], *see* Corollary 3.2 in connection with a remark at the end of the introduction there. But we do not yet know how to extend the method of proof used in [6] to the situation of Proposition 4.

Proof of Proposition 4. – Let $\mathcal{E}$ be the exceptional set of those $E \in I$ such that at either $+\infty$ or $-\infty$ there is no non-trivial L^2-solution. By assumption we have Lebesgue measure $|\mathcal{E}| = 0$ for this set. Therefore $\mu(\mathcal{E}) = 0$ with the measure μ from Proposition 3, independent of the choice of a_1 and a_2 there. This implies $\rho_a(\mathcal{E}) = 0$ for $a \in M$, where M is a full measure subset of $(0, \infty)$.

Now fix $a \in M$ and $s \in (1/2, 1]$. By $\rho_a(\mathcal{E}) = 0$ we have non-trivial solutions $u_+ \in L^2(0, \infty)$ of $-u'' + W_1 u = Eu$ and $u_- \in L^2(-\infty, 0)$ of $-u'' + W_2 u = Eu$ for ρ_a-a.e. $E \in I$. In addition, Shnol's theorem, *see* Proposition 9, gives the existence of non-trivial solutions $u \in L^2_{-s}(\mathbb{R}) := \{f : (1 + |x|^2)^{s/2} f \in L^2(\mathbb{R})\}$ of $-u'' + V_a u = Eu$ for ρ_a-a.e. $E \in I$.

Note that $u(. + a)$ is a solution of $-u'' + W_1 u = Eu$ on $(0, \infty)$ and $u(. - a)$ is a solution of $-u'' + W_2 u = Eu$ on $(-\infty, 0)$.

From Proposition 8 (and its analogue on half lines) we get $u'_+ \in L^2(0, \infty)$, $u'_- \in L^2(-\infty, 0)$ and $u' \in L^2_{-s}(\mathbb{R})$. Constancy of the Wronskian

$$C = u_+(x)\, u'(x + a) - u'_+(x)\, u(x + a)$$

yields

$$|C| \int_0^\infty (1 + |x|^2)^{-s/2}\, dx = \left| \int_0^\infty \left(u_+(x)\,(1 + |x|^2)^{-s/2}\, u'(x + a) - u'_+(x)\,(1 + |x|^2)^{-s/2}\, u(x + a)\right) dx \right| < \infty,$$

which can only hold with $C = 0$ since $s \leq 1$.

Thus we have shown that u is a multiple of u_+, *i.e.* square-integrable near $+\infty$ for ρ_a-a.e. $E \in I$. Using u_- in the same way we also get square-integrability of u at $-\infty$ for ρ_a-a.e. $E \in I$, and therefore $u \in L^2(\mathbb{R})$ for ρ_a-a.e. $E \in I$. This means that ρ_a-almost every $E \in I$ is an eigenvalue of H_a. The set of eigenvalues of H_a is countable, thus ρ_a is a point measure on I, *i.e.* $\sigma_c(H_a) \cap I = \emptyset$. Since we choose the eigenfunctions from the supply provided by the u_+ and u_-, the statement about exponential decay is obvious. #

3. PROOF OF THEOREM 1 AND COMMENTS

Combining the results of Section 2 with the general properties given in Section 1, the *proof of Theorem* 1 is now readily completed:

By Proposition 2 and Fubini's theorem we have that for almost every choice of the sequence $\ldots, Y_{-2}, Y_{-1}, Y_2, Y_2, \ldots$ there exist exponentially decaying solutions $u_\pm$ at $+\infty$ and $-\infty$, respectively, of $H_\omega u = Eu$ for almost every $E \in \mathbb{R}$. This does not depend on the value of Y_1 (since together with the potential we can shift the exponentially decaying solution).

Fix such a sequence $\ldots, Y_{-2}, Y_{-1}, Y_2, Y_3, \ldots$ and choose $\delta > 0$ with $\operatorname{supp} f \subset [-\delta, \delta]$. Defining

$$W_1(x) = f(x-\delta) + \sum_{k=2}^{\infty} f\left(x - \left(\delta + \sum_{i=2}^{k} Y_i\right)\right),$$

$$W_2(x) = \sum_{k=1}^{\infty} f\left(x + \left(\delta + \sum_{i=1}^{k} Y_{-i}\right)\right)$$

we are in the situation of Proposition 4. Thus for almost every $a > 0$ we have

$$\sigma_c(-d^2/dx^2 + W_1(x-a) + W_2(x+a)) \cap (0, \infty) = \emptyset$$

with exponentially decaying eigenfunctions. Since the operator $-d^2/dx^2 + W_1(x-a) + W_2(x+a)$ is unitary equivalent to H_ω for the choice $2a = Y_1 - 2\delta$ (by shifting the potential) this event has probability $\mathbb{P}(Y_1 > 2\delta) = \int_{2\delta}^{\infty} \alpha e^{-\alpha x}\, dx = e^{-2\alpha\delta}$. This holds for almost every $\ldots, Y_{-2}, Y_{-1}, Y_2, Y_3, \ldots$, thus by Fubini we finally get $\sigma_c(H_\omega) \cap (0, \infty) = \emptyset$ with probability $e^{-2\alpha\delta} > 0$. Since $\sigma(H_\omega) \subset [0, \infty)$ we get in particular that $\sigma_{sc}(H_\omega) = \emptyset$ with positive probability. This implies $\Sigma_{sc} = \emptyset$.

Noting that exponential decay of eigenfunctions at the Lyapunov rate also is a non-random property ([20, Theorem A1]) and that 0 is an eigenvalue of H_ω with zero probability [22, Thm. 2.12] completes the proof of Theorem 1. #

There are a number of generalizations, comments and open problems related to the above result, which we think are worth to be mentioned:

(i) It can be checked that all the basic properties of the Poisson process $\mu = \sum_n \delta_{X_n}$ needed in the proof of Theorem 1 are also satisfied for more general *compound Poisson processes* $\tilde{\mu} = \sum_n \lambda_n \delta_{X_n}$ discussed for example in [12]. Here the λ_n are non-negative i.i.d. random variables with $\mathbb{E}(\lambda_n^2) < \infty$, also independent from the Poisson points X_n. The associated metrically transitive potential is

$$\tilde{V}_\omega(x) = \int f(x-y)\, d\tilde{\mu}_\omega(y) = \sum_{n\in\mathbb{Z}} \lambda_n(\omega) f(x - X_n(\omega)). \tag{12}$$

By $\mathbb{E}(\lambda_n^2) < \infty$ one gets $\mathbb{E}(\tilde{V}_\omega(0)^2) < \infty$. We therefore have the following generalization of Theorem 1:

THEOREM 5. – *Let* $H_\omega = -d^2/dx^2 + \tilde{V}_\omega$ *with* $\tilde{V}_\omega$ *defined by* (12), f *as in* (1), (5) *and* $\tilde{\mu}$ *a compound Poisson process with* $\lambda_n \geq 0$ *i.i.d. and* $\mathbb{E}(\lambda_n^2) < \infty$. *Then the conclusion of Theorem* 1 *holds.*

It is enlightenning to compare our result for the "mixed Anderson-Poisson model" (12) with a result of Combes and Hislop [4] on localization for the multidimensional version of this model. Under stronger assumptions on the λ_n (bounded, absolutely continuous distribution) and suitable assumptions on f (in particular, *non*-compact support with suitable decay) they prove localization at low positive energies. They essentially use the randomness in the coupling constant in their proof (and can not treat the "pure" Poisson case), whereas we use the randomness of the Poisson points. So the Anderson-Poisson model is treated as an Anderson model in [4] and as a Poisson model here.

(ii) One can try to relax the assumptions on the single site potential f:

a) The proof of Proposition 3 actually does not need that f is non-negative as long as it is compactly supported, but note that spectral averaging only holds for positive energies. Proposition 4 can also be extended (note that the negative part of the potential may grow like $\ln|x|$ now [10], *i.e.* Proposition 4 does not apply directly). We can therefore prove

THEOREM 6. – *Let H_ω be defined as in Theorem* 1, *but with the weaker assumption that f is square-integrable, compactly supported and not identically zero. Then we have*

$$\Sigma_{ac} = \emptyset, \qquad \Sigma_{sc} \cap (0, \infty) = \emptyset$$

with eigenfunctions to positive energies decaying exponentially at the Lyapunov-exponent rate.

H_ω will now in general have negative spectrum. We expect that this spectrum is also pure point, but our method does not allow to prove this. Technically this is caused by the fact that $d\beta_1/da$ and $d\beta_2/da$ used in the proof of Proposition 3 will now have zeros. We note that a very similar problem arises in Minami's work on localization for Lévy noise potentials [21, p. 232].

b) If f is non-compatly supported, then generically H_ω will be deterministic in the sense of Kotani [14]. Nevertheless, it is known for many such f that $\sigma_{ac}(H_\omega) = \emptyset$ almost surely ([14], [11], [28], [29]). This leads to $\gamma(E) > 0$ for a.e. E and the validity of Proposition 2 again, leaving us with the need to extend Propositions 3 and 4 to a more general version of H_a, which includes interfering "tails" of W_1 and W_2. We do not see how to do that.

(iii) Our results in Theorems 1, 5 and 6 can, of course, not be extended to guarantee pure point spectrum for *every* ω. For example, if all the X_i are of equal distance (a zero probability event), then V_ω is periodic, *i.e.* H_ω absolutely continuous.

In addition, there are recent results which suggest that the spectrum of H_ω should be purely singular continuous for many values of ω. A result of del Rio, Makarov and Simon [5] and independently of Gordon [8] shows "genericity" of singular continuity for discrete Anderson models exhibiting localization. We expect similar ideas to apply to the Poisson model.

(iv) As a final comment we point out that it would be very useful to have a result on spectral averaging for models of the type

$$H_c = -\Delta + V(x) + W(x - c), \qquad c \in \mathbb{R}^d \tag{13}$$

in $L^2(\mathbb{R}^d)$, $d \geq 1$. Using this in $d = 1$, a proof of localization for the one-dimensional *random displacement model*

$$H_\omega = -d^2/dx^2 + \sum_{n \in \mathbb{Z}} f(x - n - \xi_n(\omega)) \tag{14}$$

with i.i.d. random variables ξ_n could be completed along the lines of the above proof. The multidimensional version of (14) has been studied by

Klopp [17]. For suitable f and ξ_n he proved localization at low energies in the semiclassical limit (*i.e.* under large coupling for f).

In $d > 1$ a result on spectral averaging for (13) would also be needed to treat the Poisson model, using in addition the ideas developed in [20] and [4] for the multidimensional continuum Anderson model.

Unfortunately, the model (13) does not seem to have any nice monotonicity properties in the parameter c, not even in $d = 1$. So far, monotonicity had a crucial part in all proofs of spectral averaging. In the proof of Proposition 3 above, it appears in the form of (10). Our inability to prove localization at negative energies in Theorem 6 is also caused by the breakdown of this monotonicity argument.

APPENDIX

In this Appendix we provide some technical results which were used in Section 2. All these results are more or less standard. We include proofs since some of them may not be known to hold under the given generality.

We start with a lemma providing infinitesimal formboundedness of $L^1_{loc,\,unif}$-potentials, but also showing that uniform constants can be chosen on general subintervals (a, b) of $\mathbb{R}$. Here $H^1(a, b) = \{f \in L^2(a, b)$: f absolutely continuous, $f' \in L^2(a, b)\}$ is the first order Sobolev space.

LEMMA 7. – *Let $W \in L^1_{loc,\,unif}(\mathbb{R})$, $\delta > 0$ and $\varepsilon > 0$. Then there is a constant C such that*

$$\int_a^b |f|^2 |W| dt \le \varepsilon \int_a^b |f'|^2 dt + C \int_a^b |f|^2\, dt \tag{15}$$

for every a and b with $-\infty \le a < b \le \infty$, $b - a \ge \delta$ and $f \in H^1(a, b)$.

Proof. – Let $I \subset (a, b)$ ($I \subset [a, b)$ or $I \subset (a, b]$ if a or b are finite) with $|I| = \delta$. There is $y \in I$ such that $|f(y)| = \min_{x \in I} |f(x)|$ (if a resp. b are finite, then $\lim_{x \to a} f(x)$ resp. $\lim_{x \to b} f(x)$ exist). For every choice of $\varepsilon' > 0$ and $x \in I$ we have

$$\begin{aligned} |f(x)|^2 &\le |f(y)|^2 + 2 \int_I |f f'|\, dt \\ &\le \frac{1}{\delta} \int_I |f|^2\, dt + \int_I \left(\frac{1}{\varepsilon'} |f|^2 + \varepsilon' |f'|^2 \right) dt, \end{aligned}$$

which implies that

$$\int_I |f|^2 |W|\, dt \leq \left\{ \left(\frac{1}{\delta} + \frac{1}{\varepsilon'} \right) \int_I |f|^2\, dt + \varepsilon' \int_I |f'|^2\, dt \right\} \int_I |W|\, dt$$
$$\leq \varepsilon' C_1(\delta) \int_I |f'|^2\, dt + C_2(\delta, \varepsilon') \int_I |f|^2\, dt.$$

From this we get (15) by covering (a, b) with intervals of length δ such that each $x \in (a, b)$ is covered by at most two intervals.

Next we study weighted L^2-solutions of $-u'' + Vu = zu$, using the notation $L_s^2 = \{f : k_s f \in L^2\}$, where $k_s(x) = (1 + |x|^2)^{s/2}$.

PROPOSITION 8. – *Let $V_+ \in L^1_{loc}(\mathbb{R})$ and $V_- \in L^1_{loc,\, unif}(\mathbb{R})$. If $z \in C$, $s \in \mathbb{R}$ and u is a solution of $-u'' + Vu = zu$ with $u \in L_s^2(\mathbb{R})$, then also $u' \in L_s^2(\mathbb{R})$.*

Proof. – From $(\overline{(k_s u)}(k_s u)')' = |(k_s u)'|^2 + \overline{(k_s u)}(k_s u)''$ and $(k_s u)'' = k_s'' u + 2k_s' u' + k_s(V - z) u$ we get

$$\begin{aligned} \operatorname{Re} \overline{(k_s u)(x)}\,(k_s u)'(x) &= \operatorname{Re} \overline{(k_s u)(0)}\,(k_s u)'(0) \\ &\quad + \int_0^x \{|(k_s u)'|^2 + \operatorname{Re} \overline{(k_s u)}\,(k_s'' u + 2k_s' u') \\ &\quad + |k_s u|^2 (V - \operatorname{Re} z)\}\, dt. \end{aligned} \tag{16}$$

Using $|k_s'| \leq |s|\, k_s$ one gets

$$\begin{aligned} |\overline{(k_s u)}\, k_s' u'| &\leq |s|\, |\overline{(k_s u)}\,(k_s u)'| + |s|\, |\overline{(k_s u)}\, k_s' u| \\ &\leq \frac{1}{4} |(k_s u)'|^2 + C(s)\, |k_s u|^2. \end{aligned}$$

From Lemma 7 we have

$$\int_0^x (V - \operatorname{Re} z)_- |k_s u|^2\, dt \leq \frac{1}{4} \int_0^x |(k_s u)'|^2\, dt + C \int_0^x |k_s u|^2\, dt$$

with C independent of x. If follows that the integral on the r.h.s. of (16) can be estimated by

$$\geq \int_0^x \left\{ \frac{1}{4} |(k_s u)'|^2 - \tilde{C}(s) |k_s u|^2 \right\} dt.$$

If we had $k_s u' \notin L^2(0, \infty)$, *i.e.* also $(k_s u)' \notin M^2(0, \infty)$, then this last expression would approach ∞ as $x \to +\infty$. But then (16) yields

$$(|(k_s u|^2)'(x) = 2\,\mathrm{Re}\,\overline{(k_s u)(x)}\,(k_s u)'(x) \to \infty \quad \text{as } x \to \infty,$$

leading to the contradiction $|k_s u|^2(x) \to \infty$. We have proved $u' \in L_s^2(0, \infty)$. In the same way we get $u' \in L_s^2(-\infty, 0)$. #

We are now going to provide a version of Shnol's theorem for operators H in $L^2(\mathbb{R})$ defined by $-d^2/dx^2 + V$, where $V_+ \in L^1_{loc}(\mathbb{R})$ and $V_- \in L^1_{loc,\,unif}(\mathbb{R})$. Note that by Lemma 8 V_- is infinitesimally formbounded with respect to $-d^2/dx^2 + V_+$, *i.e.* H can be defined by means of quadratic forms.

Let ρ be a spectral measure of H, *i.e.* a Borel measure ρ such that $\rho(\Delta) = 0$ if and only if $E(\Delta) = 0$, Δ being any Borel set and E the spectral resolution for H.

PROPOSITION 9. – *Let $h \in L^2(\mathbb{R})$ be fixed. Then for ρ-almost every $\lambda \in \mathbb{R}$ there is a non-trivial solution φ of $-\varphi'' + V\varphi = \lambda\varphi$ such that $h\varphi \in L^2(\mathbb{R})$.*

The classical example for which this result is well known (e.g. [25]) is given by the choice $h = k_{-s}$, $s > 1/2$. Below we provide a simple proof which is modeled after a method developed in [23]. We start with

LEMMA 10. – *For every $z \in \rho(H)$ we have that $h(H - z)^{-1}$ is a Hilbert-Schmidt operator.*

Proof. – From the formboundedness of V_- we get that $(-d^2/dx^2 - m)^{1/2}(H - m)^{-1/2}$ is a bounded operator for $m < \inf(\sigma(H), 0)$. A standard result (*cf.* [24]) says that $h(-d^2/dx^2 - m)^{-1/2}$ is Hilbert-Schmidt. Thus we get the Hilbert-Schmidt property for $h(H - m)^{-1/2}$, *i.e.* also for $h(H - m)^{-1}$. This extends to general $z \in \rho(H)$ by the resolvent equation. #

Proof of Proposition 9. – Let $U : L^2(\mathbb{R}) \to \oplus_j L^2(\mathbb{R}, d\rho_j)$ be an ordered spectral representation for H, *cf.* e.g. [31, Ch. 8]. Then $\rho = \rho_1$ is a spectral measure for H (e.g. Lemma 2b of [23]) and it remains to show the assertion with this ρ, since any two spectral measures for H are equivalent. By [31, Theorem 8.4] we have for every j that

$$(U_j f)(\lambda) = \int_{\mathbb{R}} \overline{v_j(x, \lambda)}\, f(x)\, dx \tag{17}$$

the ρ_j-almost every $\lambda \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$ with compact support. Here $v_j : \mathbb{R} \times \mathbb{R} \to C$ is $dx \times d\rho$-measurable and $\tau v_j(\cdot, \lambda) = \lambda v_j(\cdot, \lambda)$ with $v_j(\cdot, \lambda) \not\equiv 0$ for ρ_j-almost every λ. By Lemma 10 $(H - z)^{-1} h$ is the restriction of a Hilbert-Schmidt operator in $L^2(\mathbb{R})$, thus

$$(\lambda - z)^{-1} U_1 h = U_1 (H - z)^{-1} h \tag{18}$$

is the restriction of a Hilbert-Schmidt operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, d\rho)$ and thus has a kernel

$$G(\lambda, x) = L^2(\mathbb{R} \times \mathbb{R}, d\rho \times dx). \tag{19}$$

To a given $N \in \mathbb{N}$ let $f_n \in C_0^\infty(-N, N)$, $n = 1, 2, \ldots$, be given such that $\{f_n\}$ is dense in $L^2(-N, N)$. From (17) and (18) we get that for every n and ρ-almost every λ

$$(\lambda - z)^{-1} \int_{-N}^{N} \overline{v_1(x, \lambda)}\, h(x)\, f_n(x)\, dx = \int_{-N}^{N} G(\lambda, x)\, f_n(x)\, dx.$$

Since $\{f_n\}$ is countable and dense in $L^2(-N, N)$ this implies that for ρ-almost every λ

$$(\lambda - z)^{-1} \overline{v_1(x, \lambda)}\, h(x) = G(\lambda, x) \quad \text{for almost every } x \in [-N, N]. \tag{20}$$

N was arbitrary, therefore (20) in fact holds for almost every $x \in \mathbb{R}$. (19) and Fubini yield

$$(\lambda - z)^{-1} \overline{v_1(\cdot, \lambda)}\, h \in L^2(\mathbb{R}) \quad \text{for } \rho\text{-almost every } \lambda.$$

The assertion is proven with $\varphi = \overline{v_1(\cdot, \lambda)}$. #

We finally recall some properties of the Prüfer variables $r_c(x, \theta, E, V)$ and $\phi_c(x, \theta, E, V)$ used in the proof of Proposition 3. Since E and V are fixed here, we drop them from the notation.

PROPOSITION 11.

$$(\partial_c \phi_c)(x, \theta) = \frac{1}{r_c(x, \theta)^2}. \tag{21}$$

Proof. – Using $\partial_x \phi_c = \cos^2 \phi_c + (E - V) \sin^2 \phi_c$ and $\partial_x r_c = r_c (1 + V - E) \sin \phi_c \cos \phi_c$ we get

$$\begin{aligned} \partial_x \partial_\theta \phi_c(x, \theta) &= \partial_\theta \partial_x \phi_c(x, \theta) = \partial_\theta (\cos^2 \phi_c + (E - V) \sin^2 \phi_c) \\ &= 2 (\partial_\theta \phi_c)(E - V - 1) \sin \phi_c \cos \phi_c \\ &= -2 (\partial_\theta \phi_c) \frac{\partial_x r_c}{r_c}. \end{aligned}$$

$\partial_\theta \phi_c$ is found from this by the method of integrating factors.

Using $r_c(c, \theta^2)(\partial_\theta \phi_c)(c, \theta) = 1$ wet get (21). #

We note that under our assumption interchanging ∂_x and ∂_θ in the above proof is actually only justified for almost every pair (x, θ). This yields that (21) holds for every x and almost every θ, being sufficient for our applications.

COROLLARY 12. – *For any c, x and θ we have*

$$\frac{1}{\pi}\int_{\theta}^{\theta+\pi} r_c(x, \beta)^{-2}\, d\beta = 1.$$

Proof. – This follows by integrating (21) and using $\phi_c(x, \theta+\pi) - \phi_c(x, \theta) = \pi$. #

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