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Radiation conditions and resolvent estimates for relativistic Schrödinger operators

by

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ABSTRACT. – We establish the limiting absorption principle for relativistic Schrödinger operators with short-range potentials. We study the asymptotic behavior of the extended resolvents as the energy goes to ∞ . In order to distinguish the extended resolvents, we introduce radiation conditions.

Key words: Relativistic Schrödinger operators, resolvents estimates, radiation conditions.

RÉSUMÉ. – Nous prouvons un principe d'absorption limitée pour des opérateurs de Schrödinger relativiste avec des potentiels à courte portée. Nous étudions le comportement asymptotique des valeurs aux bords de la résolvante lorsque l'énergie tend vers ∞ . Nous introduisons des conditions du rayonnement pour distinguer des valeurs aux bords de la résolvante.

1. INTRODUCTION

The operator

$$(1.1) \quad H = \sqrt{-\Delta + 1} + V(x),$$

which we call a relativistic Schrödinger operator, occurs naturally when one tries to make relativistic corrections to the mathematical theory based on

the Schrödinger operator $-\Delta + V(x)$. There has been a substantial amount of literature for relativistic Schrödinger operators: spectral properties have been investigated by Daubechies [5], Dereziński [6], Gérard [7], Herbst [9] and Weder [25], [26]; decays of eigenfunctions have been discussed by Carmona, Masters and Simon [4], Helffer and Parisse [8] and Nardini [16]; the stability of relativistic matters has been studied extensively by many authors (*see* Lieb [13] and references therein).

The purpose of the present paper is to investigate various properties of the resolvent of the relativistic Schrödinger operator H with a short-range potential $V(x)$, in connection with the limiting absorption principle. Establishing the limiting absorption principle for the operator H , we discuss the asymptotic behavior of the extended resolvents $R^\pm(\lambda)$ of H as $\lambda \rightarrow \infty$. For a function f in some weighted L^2 -space, we show that both $u^+ := R^+(\lambda)f$ and $u^- := R^-(\lambda)f$ satisfy the same equation

$$(\sqrt{-\Delta + 1} + V(x) - \lambda)u = f$$

in distribution sense. In order to distinguish between these two solutions, we introduce an outgoing and an incoming radiation conditions for the operator (1.1).

There are some facts which motivate us to investigate these properties of the resolvent of the relativistic Schrödinger operator. We should recall that the limiting absorption principle is a basic tool in the study of scattering theory for the Schrödinger operator. In fact, the extended resolvents of the Schrödinger operator play crucial roles in time-independent scattering theory (*cf.* Agmon [1], Kuroda [12]) as well as in the inverse scattering problem (*cf.* Saitō [22], [23]).

We now introduce the notation which will be used in this paper. For $x \in \mathbf{R}^n$, $|x|$ denotes the Euclidean norm of x and

$$\langle x \rangle = \sqrt{1 + |x|^2}.$$

For $s \in \mathbf{R}$ and a positive integer m , we define the weighted Hilbert spaces L_s^2 and H_s^m by

$$L_s^2 = \{f : \langle x \rangle^s f \in L^2(\mathbf{R}^n)\},$$

and

$$H_s^m = \{f : \langle x \rangle^s \partial_x^\alpha f \in L^2(\mathbf{R}^n), |\alpha| \leq m\}.$$

When $s = 0$, we write $L^2 = L_0^2$ and $H^m = H_0^m$. The inner products and norms in L_s^2 and H_s^m are given by

$$\begin{cases} (f, g)_s = \int_{\mathbf{R}^n} \langle x \rangle^{2s} f(x) \overline{g(x)} dx, \\ \|f\|_s = [(f, f)_s]^{1/2}, \end{cases}$$

and

$$(1.2) \quad \begin{cases} (f, g)_{m, s} = \int_{\mathbf{R}^n} \langle x \rangle^{2s} \sum_{|\alpha| \leq m} \{ \partial_x^\alpha f(x) \overline{\partial_x^\alpha g(x)} \} dx \\ \|f\|_{m, s} = [(f, f)_{m, s}]^{1/2}, \end{cases}$$

respectively. By $\mathcal{S}(\mathbf{R}^n)$ we mean the set of all rapidly decreasing functions on \mathbf{R}^n . For a pair of Hilbert spaces X and Y , $\mathbf{B}(X, Y)$ denotes the Banach space of all bounded linear operators from X to Y , equipped with the operator norm

$$\|T\|_{(X, Y)} = \sup_{x \in X \setminus \{0\}} \|Tx\|_Y / \|x\|_X,$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms in X and Y . We set $\mathbf{B}(X) = \mathbf{B}(X, X)$. For $T \in \mathbf{B}(L_s^2, L_t^2)$, its operator norm will be denoted by $\|T\|_{(s, t)}$.

The plan of the paper is as follows. In Section 2 we establish the limiting absorption principle for the free relativistic Schrödinger operator $H_0 = \sqrt{-\Delta + 1}$. In Section 3 we study the asymptotic behavior of the extended resolvents of the operator H_0 . We discuss the radiation conditions for the operator H_0 in Section 4. In Section 5 we extend the results obtained in Sections 2-4 to the relativistic Schrödinger operator H .

2. THE LIMITING ABSORPTION PRINCIPLE FOR H_0

In this section we show the existence of the extended resolvents of the free relativistic Schrödinger operator H_0 . We start with the precise definition of the operator H_0 . Let H_0 denote the selfadjoint operator in L^2 given by

$$(2.1) \quad H_0 = \sqrt{-\Delta + 1} \quad \text{with domain } H^1.$$

It is known that H_0 restricted on $C_0^\infty(\mathbf{R}^n)$ is essentially selfadjoint, and that

$$\sigma(H_0) = \sigma_{ac}(H_0) = [1, \infty)$$

(cf. [15], [24, Theorem 1]), where $\sigma(H_0)$ and $\sigma_{ac}(H_0)$ are the spectrum and the absolutely continuous spectrum of H_0 respectively. Here we introduce some more notation. The resolvent of H_0 will be denoted by $R_0(z)$, *i.e.*

$$R_0(z) = (H_0 - z)^{-1} \quad (z \in \rho(H_0)),$$

where $\rho(H_0)$ is the resolvent set of H_0 . The upper and lower half-planes will be denoted by \mathbf{C}^+ and \mathbf{C}^- respectively, *i.e.*

$$\mathbf{C}^\pm = \{z \in \mathbf{C} \mid \pm \text{Im } z > 0\}.$$

The main result in this section is

THEOREM 2.1. – *Let H_0 be as in (2.1), and let $s > 1/2$. Then*

(i) *For any $\lambda > 1$, there exist the limits*

$$R_0^\pm(\lambda) = \lim_{\mu \downarrow 0} R_0(\lambda \pm i\mu) \quad \text{in } \mathbf{B}(L_s^2, H_{-s}^1).$$

(ii) *The operator-valued functions $R_0^\pm(z)$ defined by*

$$R_0^\pm(z) = \begin{cases} R_0(z) & \text{if } z \in \mathbf{C}^\pm \\ R_0^\pm(\lambda) & \text{if } z = \lambda > 1 \end{cases}$$

are $\mathbf{B}(L_s^2, H_{-s}^1)$ -valued continuous functions.

Remark. – In [2 Theorem 4.3 and Remark 4.5, b)], Arsu derived a result in a general setting which is similar to conclusion (i) of Theorem 2.1 and conclusion (i) of Theorem 5.5 in the present paper. It should be noted that the operator topology adopted in [2] is weaker than ours.

Before giving the proof of Theorem 2.1, we need some prerequisites. We first need to show a boundedness result on pseudodifferential operators in weighted L^2 Sobolev spaces.

DEFINITION. – *A C^∞ function $p(x, \xi)$ on $\mathbf{R}^n \times \mathbf{R}^n$ is said to be in the class $S_{0,0}^\mu$ ($\mu \in \mathbf{R}$) if for any pair α and β of multi-indices there exists a constant $C_{\alpha\beta} \geq 0$ such that*

$$\left| \left(\frac{\partial}{\partial \xi} \right)^\alpha \left(\frac{\partial}{\partial x} \right)^\beta p(x, \xi) \right| \leq C_{\alpha\beta} \langle \xi \rangle^\mu.$$

The class $S_{0,0}^\mu$ is a Fréchet space equipped with the seminorms

$$|p|_\ell^{(\mu)} = \max_{|\alpha|, |\beta| \leq \ell} \sup_{x, \xi} \left\{ \left| \left(\frac{\partial}{\partial \xi} \right)^\alpha \left(\frac{\partial}{\partial x} \right)^\beta p(x, \xi) \right| \langle \xi \rangle^{-\mu} \right\}$$

$$(\ell = 0, 1, 2, \dots).$$

LEMMA 2.2. – *Let $p(x, \xi)$ be in $S_{0,0}^{-m}$ for some integer $m \geq 0$ and let $s \geq 0$. Then there exist a nonnegative constant $C = C_{m,s}$ and a positive integer $\ell = \ell_{m,s}$ such that*

$$(2.2) \quad \|p(x, D)f\|_{m,s} \leq C |p|_\ell^{(-m)} \|f\|_s \quad (f \in \mathcal{S}(\mathbf{R}^n)).$$

Proof. – We first prove the lemma in the case where $m = 0$. Define the oscillatory integral

$$r(x, \xi) = \text{Os} - \int \int_{\mathbf{R}^{2n}} e^{-iy \cdot \eta} p(x, \xi + \eta) \langle x + y \rangle^{-s} (2\pi)^{-n} dy d\eta.$$

Differentiating under the oscillatory integral sign and integrating by parts, we deduce that for any pair of multi-indices α and β

$$(2.3) \quad \left| \left(\frac{\partial}{\partial \xi} \right)^\alpha \left(\frac{\partial}{\partial x} \right)^\beta r(x, \xi) \right| \leq C |p|_k^{(0)} \langle x \rangle^{-s},$$

where $C = C_{s\alpha\beta}$ is a nonnegative constant and $k = k_{s\alpha\beta}$ is a positive integer. In view of Kumano-go [11, Theorem 2.6, p. 74], we have

$$(2.4) \quad r(x, D) = p(x, D) \langle x \rangle^{-s}.$$

Combining (2.3) and (2.4), we see that $\langle x \rangle^s p(x, D) \langle x \rangle^{-s}$ is an operator with symbol belonging to $S_{0,0}^0$. Applying the Calderón-Vaillancourt theorem, we get (2.2) with $m = 0$.

We next prove the lemma in the case where m is a positive integer. By definition (see (1.2)), we see that

$$(2.5) \quad \|p(x, D) f\|_{m,s}^2 = \sum_{|\alpha| \leq m} \left\| \left(\frac{\partial}{\partial x} \right)^\alpha p(x, D) f \right\|_s^2.$$

Note that $(\partial/\partial x)^\alpha p(x, D)$, $|\alpha| \leq m$, can be regarded as an operator belonging to $S_{0,0}^0$, and that each seminorm of $S_{0,0}^0$ of the symbol of $(\partial/\partial x)^\alpha p(x, D)$ can be estimated by a constant multiplied by a seminorm of $S_{0,0}^{-m}$ of $p(x, \xi)$ (see Kumano-go [11, Chapter 2, § 2, Theorems 2.5 and 2.6]). Then applying the result obtained in the first part of the proof to $(\partial/\partial x)^\alpha p(x, D)$, we see that for any α with $|\alpha| \leq m$

$$(2.6) \quad \left\| \left(\frac{\partial}{\partial x} \right)^\alpha p(x, D) f \right\|_s \leq C |p|_\ell^{(-m)} \|f\|_s \quad (f \in \mathcal{S}(\mathbf{R}^n)),$$

where $C = C_{ms}$ and $\ell = \ell_{ms}$ are a nonnegative constant and a positive integer respectively. Combining (2.5) with (2.6) gives the lemma. \square

To prove Theorem 2.1, we first note that for $z \in \mathbf{C} \setminus [1, \infty)$

$$(2.7) \quad R_0(z) = \mathcal{F}^{-1} \left[\frac{1}{\langle \xi \rangle - z} \right] \mathcal{F} = \mathcal{F}^{-1} \left[\frac{\langle \xi \rangle + z}{|\xi|^2 - (z^2 - 1)} \right] \mathcal{F},$$

where \mathcal{F} is the Fourier transform:

$$[\mathcal{F} f](\xi) = \hat{f}(\xi) = \int_{\mathbf{R}^n} e^{-x \cdot \xi} f(x) dx.$$

We shall show a representation formula for $R_0(z)$ in terms of the resolvent of the free Schrödinger operator $-\Delta$. For $1 < a < b < \infty$, we define a subset $K = K_{ab}$ of the complex plane \mathbf{C} by

$$(2.8) \quad K = \left\{ z = \lambda + i\mu \in \mathbf{C} : a \leq \lambda \leq b, |\mu| \leq \frac{\sqrt{a^2 - 1}}{2} \right\}.$$

It is easy to see that

$$(2.9) \quad z \in K \Rightarrow 0 < \frac{3(a^2 - 1)}{4} \leq \operatorname{Re}(z^2 - 1) \leq b^2 - 1.$$

We choose a cutoff function $\gamma \in C_0^\infty(\mathbf{R}_\xi^n)$ so that

$$\gamma(\xi) = \begin{cases} 1 & \text{if } \frac{\sqrt{3(a^2 - 1)}}{4} \leq |\xi| \leq \frac{3\sqrt{b^2 - 1}}{2}, \\ 0 & \text{if } 0 \leq |\xi| \leq \frac{\sqrt{3(a^2 - 1)}}{6}, \quad |\xi| \geq 2\sqrt{b^2 - 1}. \end{cases}$$

We now introduce simple pseudodifferential operators: for each $z \in K$, define

$$(2.10) \quad \begin{cases} A(z)f = zI + \mathcal{F}^{-1}[\gamma(\xi)\langle \xi \rangle]\mathcal{F}f & (f \in \mathcal{S}(\mathbf{R}^n)), \\ B(z)f = \mathcal{F}^{-1}\left[\frac{(1 - \gamma(\xi))\langle \xi \rangle}{|\xi|^2 - (z^2 - 1)}\right]\mathcal{F}f & (f \in \mathcal{S}(\mathbf{R}^n)). \end{cases}$$

Note that

$$(2.11) \quad \begin{aligned} & ||\xi|^2 - (z^2 - 1)| \\ & \geq \begin{cases} \frac{9}{16}(a^2 - 1) & \text{if } z \in K \text{ and } \xi \in \operatorname{supp}[1 - \gamma], \\ \frac{1}{2}|\xi|^2 & \text{if } z \in K \text{ and } |\xi| \geq \frac{3\sqrt{b^2 - 1}}{2}. \end{cases} \end{aligned}$$

The representation formula for $R_0(z)$ follows immediately from (2.7) and (2.10):

LEMMA 2.3. – For $z \in K$ with $\operatorname{Im} z \neq 0$,

$$(2.12) \quad R_0(z)f = \Gamma_0(z^2 - 1)A(z)f + B(z)f \quad (f \in \mathcal{S}(\mathbf{R}^n)),$$

where $\Gamma_0(z)$ is the resolvent of $-\Delta$, i.e. $\Gamma_0(z) = (-\Delta - z)^{-1}$.

On the operator $A(z)$ we have

LEMMA 2.4. – Let $s \geq 0$. Then for each $z \in K$, $A(z)$ can be uniquely extended to a bounded operator on L_s^2 . Furthermore $A(z)$ is a $\mathbf{B}(L_s^2)$ -valued continuous function on K .

Proof. – Since $\gamma(\xi)\langle\xi\rangle$ belongs to $S_{0,0}^0$ it follows from Lemma 2.2 that $\gamma(D)\langle D\rangle$ can be extended to a bounded operator on L_s^2 . The fact that $A(z) = zI + \gamma(D)\langle D\rangle$ implies the lemma. \square

As for the operator $B(z)$ we have

LEMMA 2.5. – Let $s \geq 0$. Then for each $z \in K$, $B(z)$ can be uniquely extended to a bounded operator from L_s^2 to H_s^1 . Furthermore $B(z)$ is a $\mathbf{B}(L_s^2, H_s^1)$ -valued continuous function on K .

Proof. – Noting (2.11), we see that for any α there corresponds a constant $C_\alpha > 0$ such that

$$(2.13) \quad \left| \left(\frac{\partial}{\partial \xi} \right)^\alpha \left\{ \frac{(1 - \gamma(\xi))\langle \xi \rangle}{|\xi|^2 - (z^2 - 1)} \right\} \right| \leq C_\alpha \langle \xi \rangle^{-1-|\alpha|}$$

for all $z \in K$. Lemma 2.2 together with (2.13) implies the first conclusion of the lemma. Furthermore, noting the identity

$$\begin{aligned} & \frac{1}{|\xi|^2 - (z_1^2 - 1)} - \frac{1}{|\xi|^2 - (z_2^2 - 1)} \\ &= \frac{(z_1 - z_2)(z_1 + z_2)}{\{|\xi|^2 - (z_1^2 - 1)\}\{|\xi|^2 - (z_2^2 - 1)\}}, \end{aligned}$$

we have for any α

$$\begin{aligned} & \left| \left(\frac{\partial}{\partial \xi} \right)^\alpha \left\{ \frac{(1 - \gamma(\xi))\langle \xi \rangle}{|\xi|^2 - (z_1^2 - 1)} - \frac{(1 - \gamma(\xi))\langle \xi \rangle}{|\xi|^2 - (z_2^2 - 1)} \right\} \right| \\ & \leq C_\alpha |z_1 - z_2| \langle \xi \rangle^{-1-|\alpha|} \quad (z_1, z_2 \in K). \end{aligned}$$

Lemma 2.2 together with this inequality implies the second conclusion of the lemma. \square

Proof of Theorem 2.1. – Let $\lambda > 1$ be given. Choose a and b so that $1 < a < \lambda < b$. We define the set K by (2.8). According to Lemma 2.3, we have

$$(2.14) \quad R_0(\lambda \pm i\mu) = \Gamma_0((\lambda^2 - \mu^2 - 1) \pm i(2\lambda\mu))A(\lambda \pm i\mu) + B(\lambda \pm i\mu)$$

for $z = \lambda \pm i\mu \in K$, where $\mu > 0$. It follows from Agmon [1, Theorem 4.1] that

$$(2.15) \quad \lim_{\mu \downarrow 0} \Gamma_0((\lambda^2 - \mu^2 - 1) \pm i(2\lambda\mu)) = \Gamma_0^\pm(\lambda^2 - 1) \quad \text{in } \mathbf{B}(L_s^2, H_{-s}^2).$$

In view of Lemmas 2.4 and 2.5, it follows from (2.14) and (2.15) that conclusion (i) holds, and moreover, that $R_0^\pm(\lambda) = \Gamma_0^\pm(\lambda^2 - 1)A(\lambda) + B(\lambda)$.

To show conclusion (ii), we note that $\Gamma_0^\pm(z)$ defined by

$$\Gamma_0^\pm(z) = \begin{cases} \Gamma_0(z) & \text{if } z \in \mathbf{C}^\pm \\ \Gamma_0^\pm(\lambda) & \text{if } z = \lambda > 0 \end{cases}$$

are $\mathbf{B}(L_s^2, H_{-s}^2)$ -valued continuous functions (see the proof of Agmon [1, Theorem 4.1]). Now conclusion (ii) is apparent from this fact and Lemmas 2.3, 2.4 and 2.5. \square

In view of the proof of Theorem 2.1, we obviously have:

COROLLARY 2.6. – *Let K be as in (2.8). Then*

$$R_0^\pm(z) = \Gamma_0^\pm(z^2 - 1)A(z) + B(z) \quad (z \in K^\pm),$$

where $K^\pm = \{z \in K : \pm \text{Im } z \geq 0\}$.

Remark. – The proof of Theorem 2.1 is based upon the decomposition (2.12) in Lemma 2.3. Note that it is possible to prove Theorem 2.1 with the aid of the following decomposition

$$(2.16) \quad R_0(z) = (H_0 + z)\Gamma_0(z^2 - 1)$$

instead of (2.12). The decomposition (2.16) is an analogue of the decomposition of the resolvent of the free Dirac operator, the decomposition which was exploited in Balslev and Helffer [3]. The reason to use the decomposition (2.12) is that it is suitable to derive radiation conditions for H_0 . See Theorem 4.3 in Section 4.

3. ASYMPTOTIC BEHAVIOR OF THE RESOLVENT OF H_0

In this section we study in detail the asymptotic behavior of the extended resolvents $R_0^\pm(\lambda)$, of which existence has been established in the preceding section. Throughout this section, we regard $R_0^\pm(\lambda)$ as operators belonging to $\mathbf{B}(L_s^2, L_{-s}^2)$ rather than $\mathbf{B}(L_s^2, H_{-s}^1)$.

PROPOSITION 3.1. – *Let $s > 1/2$. Then*

$$\|R_0^\pm(\lambda)\|_{(s, -s)} = O(1) \quad (\lambda \rightarrow \infty).$$

Proposition 3.1 follows from Murata [14, Proposition 4.1], in which he discussed high energy resolvent estimates for a general class of first order pseudodifferential operators. However, we should remark that a proof along the lines of Pladdy, Saitō and Umeda [17, Theorem 2.3] is possible.

We now state the main theorem in this section.

THEOREM 3.2. – *Let $s > 1/2$. Then*

(i) $R_0^\pm(\lambda)$ converge strongly to 0 as $\lambda \rightarrow \infty$, i.e. for any $f \in L_s^2$

$$R_0^\pm(\lambda) f \rightarrow 0 \quad \text{in } L_{-s}^2 \quad \text{as } \lambda \rightarrow \infty.$$

(ii) $\liminf_{\lambda \rightarrow \infty} \|R_0^\pm(\lambda)\|_{(s, -s)} > 0$.

Here we make a remark. The asymptotic behavior of the extended resolvents $R_0^\pm(\lambda)$ demonstrated in Theorem 3.2 is exactly the same as that of the extended resolvents of Dirac operators (cf. [17], [18]), but quite different from that of extended resolvents of Schrödinger operators. In fact, the norm in $\mathbf{B}(L_s^2, L_{-s}^2)$ of the extended resolvents of Schrödinger operators goes to 0 with the rate $\lambda^{-1/2}$ as λ tends to ∞ . See Saitō [20], [21].

We shall give the proof of Theorem 3.2 with a series of lemmas.

LEMMA 3.3. – *Define*

$$\chi_0 = \{f \in \mathcal{S}(\mathbf{R}_x^n) : \mathcal{F} f \in C_0^\infty(\mathbf{R}_\xi^n)\}.$$

Then χ_0 is dense in L_s^2 for any $s \in \mathbf{R}$.

For the proof, see [18, Lemma 7.1].

LEMMA 3.4. – *For $z \in \mathbf{C}$, put*

$$R(\xi; z) = \frac{1}{\langle \xi \rangle - z}.$$

Then for any $L > 1$ and any α , there corresponds a constant $C_{\alpha L} > 0$ such that

(i) $|(\partial/\partial\xi)^\alpha R(\xi; z)| \leq C_{\alpha L}/|z|$ for all ξ and z satisfying $\langle \xi \rangle \leq L$ and $|z| \geq 2L$,

(ii) $|(\partial/\partial\xi)^\alpha [R(\xi; z_1) - R(\xi; z_2)]| \leq C_{\alpha L}|z_1 - z_2|$ for all z_1, z_2 and ξ satisfying $|z_1|, |z_2| \geq 2L$ and $\langle \xi \rangle \leq L$.

Proof. – We can prove property (i) by induction on the length of α . Property (i) holds for $|\alpha| = 0$, since $|\langle \xi \rangle - z| \geq |z|/2$ for $\langle \xi \rangle \leq L$, $|z| \geq 2L$. To show that property (i) holds for $|\alpha| > 0$, we differentiate the both sides of $(\langle \xi \rangle - z)R(\xi; z) = 1$. In order to show property (ii), we use the indensity $R(\xi; z_1) - R(\xi; z_2) = (z_1 - z_2)R(\xi; z_1)R(\xi; z_2)$ and property (i). We omit the details. \square

In the following lemma we regard $\mathcal{S}(\mathbf{R}^n)$ as a Fréchet space equipped with the semi-norms

$$|f|_{\ell, \mathcal{S}} = \sum_{|\alpha+\beta| \leq \ell} \sup_x \left\{ \left| x^\alpha \left(\frac{\partial}{\partial x} \right)^\beta f(x) \right| \right\} \quad (\ell = 0, 1, 2, \dots).$$

LEMMA 3.5. – Suppose that $f \in \mathcal{S}(\mathbf{R}^n)$, and that

$$(3.1) \quad \text{supp } [\hat{f}] \subset \{\xi \in \mathbf{R}^n : \langle \xi \rangle \leq L\}$$

for some $L > 1$. For each $z \in \mathbf{C}$ with $|z| \geq 2L$, define

$$v_z(x) = \mathcal{F}^{-1} [R(\xi; z)] \mathcal{F} f,$$

where $R(\xi; z)$ is the function introduced in Lemma 3.4. Then

(i) For each $\ell \geq 0$, there corresponds a constant C_ℓ such that $|v_z|_{\ell, \mathcal{S}} \leq C_\ell / |z|$.

(ii) For any $\lambda \in [2L, \infty)$, $v_{\lambda \pm i\mu} \rightarrow v_\lambda$ in $\mathcal{S}(\mathbf{R}^n)$ as $\mu \downarrow 0$.

Proof. – Differentiation under the integral sign and integration by parts give

$$(3.2) \quad x^\alpha \left(\frac{\partial}{\partial x} \right)^\beta v_z(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix \cdot \xi} \left(i \frac{\partial}{\partial \xi} \right)^\alpha \{R(\xi; z) (i\xi)^\beta \hat{f}(\xi)\} d\xi.$$

It follows from property (i) of Lemma 3.4 that

$$\begin{aligned} & \left| \text{the integrand on the RHS of (3.2)} \right| \\ & \leq C_{\alpha\beta} \frac{1}{|z|} |\hat{f}|_{|\alpha+\beta|+n+1, \mathcal{S}} \langle \xi \rangle^{-n-1}, \end{aligned}$$

which gives conclusions (i). Similarly, using property (ii) of Lemma 3.4, we have

$$\left| x^\alpha \left(\frac{\partial}{\partial x} \right)^\beta (v_{\lambda \pm i\mu}(x) - v_\lambda(x)) \right| \leq C_{\alpha\beta} \times \mu \quad (\mu > 0).$$

This inequality yields conclusion (ii). \square

Proof of Theorem 3.2 (i). – In view of Proposition 3.1 and Lemma 3.3, it is sufficient to show that for any $f \in \chi_0$

$$(3.3) \quad R_0^\pm(\lambda) f \rightarrow 0 \quad \text{in } L_{-s}^2 \quad (\lambda \rightarrow \infty).$$

Let $f \in \chi_0$ be given, and choose $L > 1$ so that (3.1) is valid. Define v_z in the same manner as in Lemma 3.5. Since $R_0(z) f = v_z$ for z with

$\text{Im } z \neq 0$ (recall (2.7)), we see by Theorem 2.1 (i) and Lemma 3.5 (ii) that $R_0^\pm(\lambda) f = v_\lambda$ for $\lambda \geq 2L$. By virtue of Lemma 3.5 (i), we obtain

$$\|R_0^\pm(\lambda) f\|_{-s} \leq \frac{C_s}{|\lambda|},$$

which implies (3.3). \square

We shall prove conclusion (ii) of Theorem 3.2 with making use of

LEMMA 3.6. – *There exists a sequence $\{h_j\}_{j=1}^\infty \subset \mathcal{S}(\mathbf{R}^n)$ such that*

- (i) $\sup_j \|h_j\|_s < +\infty$ for every $s \geq 0$,
- (ii) $\lim_{j \rightarrow \infty} (R_0^\pm(j+2)h_j, h_j)_0 \neq 0$.

Proof of Theorem 3.2 (ii). – Let $\{h_j\}$ be the sequence given in Lemma 3.6. For $s > 1/2$ we have

$$|(R_0^\pm(j+2)h_j, h_j)_0| \leq \|R_0^\pm(j+2)\|_{(s, -s)} \|h_j\|_s^2,$$

which, together with Lemma 3.6, implies

$$\liminf_{j \rightarrow \infty} \|R_0^\pm(j+2)\|_{(s, -s)} > 0.$$

This is equivalent to conclusion (ii) of Theorem 3.2. \square

There remains to give the proof of Lemma 3.6, in which we need

LEMMA 3.7. – *Let φ be a real-valued C^1 -function defined on the interval $[-1, 1]$. Then*

$$\lim_{\mu \downarrow 0} \int_{-1}^1 \frac{\varphi(\sigma)}{\sigma \mp i\mu} d\sigma = \pm i\pi\varphi(0) + \int_{-1}^1 \left\{ \int_0^1 \varphi'(\sigma\theta) d\theta \right\} d\sigma.$$

We omit the proof of Lemma 3.7, which is an exercise of residue calculus.

Proof of Lemma 3.6. (Following the idea of Yamada [27]) – Choose an even function φ so that

$$(3.4) \quad \text{supp } [\varphi] \subset (-1, 1)$$

and

$$(3.5) \quad \varphi(0) = 1.$$

Define $h_j \in \mathcal{S}(\mathbf{R}^n)$ by

$$(3.6) \quad \hat{h}_j(\xi) = |\xi|^{-(n-1)/2} \varphi(\langle \xi \rangle - j - 2) \quad (j = 1, 2, \dots).$$

It is easy to see that

$$\text{supp } [\hat{h}_j] \subset \{\xi \in \mathbf{R}^n : \sqrt{j(j+2)} \leq |\xi| \leq \sqrt{j^2 + 6j + 8}\}.$$

Let α be a given multi-index. Using the Plancherel theorem, we have

$$(3.7) \quad \|x^\alpha h_j\|_0^2 = (2\pi)^{-n} \int \left| \left(i \frac{\partial}{\partial \xi} \right)^\alpha \hat{h}_j(\xi) \right|^2 d\xi.$$

The integral on the right hand side is bounded by

$$(3.8) \quad C_{\alpha\varphi} \int_{\sqrt{j(j+2)} \leq |\xi| \leq \sqrt{j^2 + 6j + 8}} |\xi|^{-(n-1)} d\xi,$$

where $C_{\alpha\varphi}$ is a constant depending only on α and φ . A simple calculation shows that the integral in (3.8) is bounded by a constant independent of j . Combining this fact with (3.7), we obtain

$$(3.9) \quad \sup_j \|x^\alpha h_j\|_0 < \infty.$$

Since α is arbitrary, we can conclude from (3.9) that $\{h_j\}$ satisfies property (i) of the lemma.

We next show property (ii). Taking into account (3.6), and passing to the polar coordinate, we get

$$(R_0(z) h_j, h_j)_0 = (2\pi)^{-n} \omega_n \int_0^\infty \frac{1}{\sqrt{r^2 + 1} - z} \varphi(\sqrt{r^2 + 1} - j - 2)^2 dr,$$

where ω_n denotes the surface area of the unit sphere in \mathbf{R}^n . With a change of a variable we have

$$(3.10) \quad (R_0(z) h_j, h_j)_0 = (2\pi)^{-n} \omega_n \int_{-1}^1 \frac{1}{\sigma + j + 2 - z} \varphi(\sigma)^2 \times \frac{\sigma + j + 2}{\sqrt{(\sigma + j + 2)^2 - 1}} d\sigma,$$

where we have used (3.4). We now take $z = j + 2 \pm i\mu$ ($\mu > 0$) in (3.10) and take the limit of (3.10) as $\mu \downarrow 0$. Then, in view of Theorem 2.1 (i) and Lemma 3.7, we see that

$$(3.11) \quad \frac{(2\pi)^n}{\omega_n} (R_0^\pm(j+2) h_j, h_j)_0 = \pm i \pi \varphi(0)^2 \frac{j+2}{\sqrt{(j+2)^2 - 1}} + \int_{-1}^1 \left\{ \int_0^1 \psi'_j(\sigma\theta) d\theta \right\} d\sigma,$$

where

$$\psi_j(\sigma) = \varphi(\sigma)^2 \frac{\sigma + j + 2}{\sqrt{(\sigma + j + 2)^2 - 1}}.$$

It follows from the Lebesgue dominated convergence theorem that

$$(3.12) \quad \lim_{j \rightarrow \infty} \int_{-1}^1 \left\{ \int_0^1 \psi'_j(\sigma\theta) d\theta \right\} d\sigma = \int_{-1}^1 \left\{ \int_0^1 (\varphi^2)'(\sigma\theta) d\theta \right\} d\sigma.$$

Since φ^2 is an even function, one can easily see that the right hand side of (3.12) is equal to 0. It follows immediately from this fact, (3.11) and (3.5) that

$$\lim_{j \rightarrow \infty} \frac{(2\pi)^n}{\omega_n} (R_0^\pm(j+2)h_j, h_j)_0 = \pm i\pi,$$

which trivially implies property (ii) of the lemma. \square

4. RADIATION CONDITIONS FOR H_0

We first establish the uniqueness theorem for the equation $(\sqrt{-\Delta + 1} - \lambda)u = f$ whose solution satisfies the radiation conditions mentioned in the introduction. Here we introduce some notation. For tempered distributions $f \in \mathcal{S}'(\mathbf{R}_x^n)$ and $g \in \mathcal{S}'(\mathbf{R}_\xi^n)$, we define $(f, \varphi)_x$ and $(g, \psi)_\xi$ by

$$\begin{aligned} (f, \varphi)_x &= \langle f, \bar{\varphi} \rangle & (\varphi \in \mathcal{S}(\mathbf{R}_x^n)) \\ (g, \psi)_\xi &= (2\pi)^{-n} \langle g, \bar{\psi} \rangle & (\psi \in \mathcal{S}(\mathbf{R}_\xi^n)). \end{aligned}$$

Then by definition of Fourier transforms on $\mathcal{S}'(\mathbf{R}_x^n)$

$$(4.1) \quad (f, \varphi)_x = (\hat{f}, \hat{\varphi})_\xi \quad (f \in \mathcal{S}'(\mathbf{R}_x^n), \varphi \in \mathcal{S}(\mathbf{R}_x^n))$$

(cf. Kumano-go [11, p. 40]). For $x \in \mathbf{R}^n$ we write $\tilde{x}_k = x_k/|x|$ ($k = 1, \dots, n$).

THEOREM 4.1 (Uniqueness). - Let $1/2 < s < 1$. Suppose that $u \in L^2_{-s} \cap H^1_{loc}$ satisfies the equation

$$(4.2) \quad (\sqrt{-\Delta + 1} - \lambda)u = 0 \quad (\lambda > 1),$$

and, in addition, that u satisfies either of

$$(4.3\pm) \quad \left(\frac{\partial}{\partial x_k} \mp i\sqrt{\lambda^2 - 1} \tilde{x}_k \right) u \in L^2_{s-1} \quad (k = 1, \dots, n).$$

Then u vanishes identically.

Proof. – Let φ be a test function in $\mathcal{S}(\mathbf{R}_x^n)$. Then we have

$$\begin{aligned} ((-\Delta - (\lambda^2 - 1)) u, \varphi)_x &= ((\sqrt{|\xi|^2 + 1} - \lambda) \hat{u}, (\sqrt{|\xi|^2 + 1} + \lambda) \hat{\varphi})_\xi \\ &= ((\sqrt{-\Delta + 1} - \lambda) u, (\sqrt{-\Delta + 1} + \lambda) \varphi)_x \\ &= 0, \end{aligned}$$

where we have used (4.1) and (4.2). Thereby u satisfies the equation

$$(-\Delta - (\lambda^2 - 1)) u = 0,$$

which, together with the assumption that $u \in L^2_{-s} \cap H^1_{loc}$, implies that $u \in H^2_{loc}$. Since u satisfies either (4.3+) or (4.3–), we can apply Ikebe-Saitō [10, Theorem 1.5] with $L = -\Delta$ and $\kappa = \sqrt{\lambda^2 - 1}$, and conclude that u vanishes identically. \square

DEFINITION 4.2. – *The conditions (4.3+) and (4.3–) appeared in Theorem 4.1 are called the outgoing and the incoming radiation conditions respectively.*

We note that the radiation conditions introduced here are the same for Dirac operators (*cf.* Pladdy, Saitō and Umeda [19]).

For $\lambda > 1$ and $f \in L^2_s (s > 1/2)$, we set

$$u_0^\pm(\lambda, f) = R_0^\pm(\lambda) f.$$

We now state a theorem, which gives a characterization of $R_0^\pm(\lambda)$ by means of the equation $(\sqrt{-\Delta + 1} - \lambda) u = f$ with the radiation conditions.

THEOREM 4.3. – *Let $1/2 < s < 1$ and $\lambda > 1$. Then $u_0^\pm(\lambda, f)$ defined as above are in $L^2_{-s} \cap H^1_{loc}$ and satisfy the equation*

$$(\sqrt{-\Delta + 1} - \lambda) u = f.$$

Moreover, $u_0^+(\lambda, f)$ satisfies the outgoing radiation condition (4.3+), and $u_0^-(\lambda, f)$ satisfies the incoming radiation condition (4.3–).

Proof. – For simplicity, we give the proof only for $u_0^+(\lambda, f)$. For z with $\text{Im } z \neq 0$ and $f \in L^2_s$, we set

$$u_0(z, f) = R_0(z) f.$$

By Theorem 2.1 (i), we have

$$(4.4) \quad u_0(\lambda + i\mu, f) \rightarrow u_0^+(\lambda, f) \quad \text{in } H^1_{-s}$$

as $\mu \downarrow 0$. Since $u_0(z, f) \in \text{Dom}(H_0)$, we have

$$(4.5) \quad ((\sqrt{-\Delta + 1} - \lambda - i\mu) u_0(\lambda + i\mu, f), \varphi)_x = (f, \varphi)_x \quad (\mu > 0)$$

for $\varphi \in \mathcal{S}(\mathbf{R}_x^n)$. The left hand side of (4.5) is equal to

$$(4.6) \quad (u_0(\lambda + i\mu, f), (\sqrt{-\Delta + 1} - \lambda + i\mu)\varphi)_x.$$

Taking the limit as $\mu \downarrow 0$, we conclude from (4.4), (4.5) and (4.6) that

$$(\sqrt{-\Delta + 1} - \lambda)u_0^+(\lambda, f) = f \quad \text{in } \mathcal{S}'(\mathbf{R}^n).$$

Since $R_0^+(\lambda) \in \mathbf{B}(L_s^2, H_{-s}^1)$, it is evident that $u_0^+(\lambda, f) \in L_{-s}^2 \cap H_{\text{loc}}^1$. It remains to show that $u_0^+(\lambda, f)$ satisfies the outgoing radiation condition. According to Corollary 2.6, the function $u_0^+(\lambda, f)$ can be decomposed into the following sum:

$$u_0^+(\lambda, f) = w_1 + w_2, \quad w_1 = \Gamma_0^+(\lambda^2 - 1)A(\lambda)f, \quad w_2 = B(\lambda)f.$$

Since $A(\lambda)f \in L_s^2$ by Lemma 2.4, it follows from Ikebe-Saitō [10, Theorem 1.5] that

$$(\partial/\partial x_k - i\sqrt{\lambda^2 - 1}\tilde{x}_k)w_1 \in L_{s-1}^2, \quad (k = 1, \dots, n).$$

Since $B(\lambda) \in \mathbf{B}(L_s^2, H_s^1)$ by Lemma 2.5, it follows that

$$\left(\frac{\partial}{\partial x_k} - i\sqrt{\lambda^2 - 1}\tilde{x}_k\right)w_2 \in L_s^2 \subset L_{s-1}^2, \quad (k = 1, \dots, n).$$

Summing up, we have shown that $u_0^+(\lambda, f)$ satisfies the outgoing radiation condition. \square

5. THE EXTENDED RESOLVENTS OF H

The task of this section is to extend the results obtained in the previous sections to the operator $H = H_0 + V$. Unfortunately, we have to confine ourselves to the operator H with a small short-range potential for a technical reason (see Theorems 5.5, 5.6 and 5.7), although we anticipated that the results in the previous sections could hold for H with a general short-range potential, on which we would probably need to impose some differentiability conditions. We emphasize that we do not require differentiability of the potential.

ASSUMPTION 5.1. – $V(x)$ is a real-valued measurable function satisfying

$$|V(x)| \leq C\langle x \rangle^{-1-\varepsilon}$$

for some positive constants ε and C .

It is easy to see that under Assumption 5.1 the multiplication operator $V = V(x)\times$ is a bounded selfadjoint operator in L^2 . As a result,

$H = H_0 + V$ is a selfadjoint operator in L^2 with domain H^1 . The resolvent of H will be denoted by $R(z) : R(z) = (H - z)^{-1}$, $z \in \rho(H)$. The hypotheses of the main theorems in this section will be expressed in terms of the quantity

$$(5.1) \quad \omega(V) = \sup_x \langle x \rangle^{1+\varepsilon} |V(x)|.$$

Throughout this section we assume that

$$(5.2) \quad \frac{1}{2} < s < \min \left\{ 1, \frac{1+\varepsilon}{2} \right\}.$$

LEMMA 5.2. – For $a > 1$ set

$$J^\pm(a) = \{z \in \mathbf{C} \mid \operatorname{Re} z \geq a, 0 \leq \pm \operatorname{Im} z \leq 1\}.$$

Then

$$(5.3) \quad \rho^\pm(a) := \sup_{z \in J^\pm(a)} \|R_0^\pm(z)\|_{(s, -s)} < +\infty.$$

Proof. – According to Theorem 2.1 (ii), $R_0^\pm(z)$ are $\mathbf{B}(L_s^2, L_{-s}^2)$ -valued continuous functions. Therefore $\|R_0^\pm(z)\|_{(s, -s)}$ are locally bounded. This fact together with Murata [14, Proposition 4.1] implies the lemma. \square

LEMMA 5.3. – Let V obey Assumption 5.1. Then the multiplication operator V can be extended to a bounded operator from L_{-s}^2 to L_s^2 , and

$$(5.4) \quad \|V\|_{(-s, s)} \leq \omega(V).$$

Proof. – Let $f \in L_{-s}^2$. Then we have

$$\|Vf\|_s^2 \leq \int \langle x \rangle^{2s} |\omega(V) \langle x \rangle^{-1-\varepsilon} f(x)|^2 dx$$

by (5.1). Since $s - 1 - \varepsilon < -s$ by (5.2), it follows that $\|Vf\|_s \leq \omega(V) \|f\|_{-s}$. \square

Combining Lemma 5.3 with Lemma 5.2 gives

LEMMA 5.4. – Let V obey Assumption 5.1, and let $J^\pm(a)$ be as in Lemma 5.2. Then for $z \in J^\pm(a)$

$$(5.5) \quad \|VR_0^\pm(z)\|_{(s, s)} \leq \omega(V) \rho^\pm(a),$$

$$(5.6) \quad \|R_0^\pm(z)V\|_{(-s, -s)} \leq \omega(V) \rho^\pm(a).$$

In Theorems 5.5, 5.6 and 5.7 below, the number a is supposed to be close to 1. Note that by Theorem 3.2 $\rho^\pm(a)$ are bounded from below by a positive constant for all $a > 1$. We now establish the limiting absorption principle for H and derive the asymptotic behavior of the extended resolvents of H which is the same as that of $R_0^\pm(\lambda)$ (cf. Theorem 3.2).

THEOREM 5.5. – *Let V obey Assumption 5.1 and let s be as in (5.2). Let $a > 1$, and suppose $\omega(V)\rho^\pm(a) < 1$. Then*

(i) *For any $\lambda \geq a$, there exist the limits*

$$R^\pm(\lambda) = \lim_{\mu \downarrow 0} R(\lambda \pm i\mu) \quad \text{in } \mathbf{B}(L_s^2, H_{-s}^1).$$

(ii) *For $\lambda \geq a$, $I + VR_0^\pm(\lambda)$ are invertible in L_s^2 and*

$$R^\pm(\lambda) = R_0^\pm(\lambda)(I + VR_0^\pm(\lambda))^{-1}.$$

(iii) *For any $f \in L_s^2$, $R^\pm(\lambda)f \rightarrow 0$ in L_{-s}^2 as $\lambda \rightarrow \infty$.*

(iv) $\liminf_{\lambda \rightarrow \infty} \|R^\pm(\lambda)\|_{(s, -s)} > 0$.

Proof. – We give the proof only for $R^+(\lambda)$. We start with the equation

$$(5.7) \quad R(z)(I + VR_0(z)) = R_0(z) \quad \text{in } L^2 \quad (\text{Im } z \neq 0).$$

We need to regard $I + VR_0(z) (= I + VR_0^+(z))$ as an operator acting in L_s^2 . Since $\omega(V)\rho^+(a) < 1$ by assumption, we see by (5.5) that $I + VR_0^+(z)$ is invertible in L_s^2 for any $z \in J^+(a)$, and that

$$(5.8) \quad (I + VR_0^+(z))^{-1} = \sum_{\ell=0}^{\infty} (-VR_0^+(z))^\ell \quad \text{in } \mathbf{B}(L_s^2),$$

where the convergence is uniform in $J^+(a)$. In view of Lemma 5.3 and Theorem 2.1 (ii), $VR_0^+(z)$ is a $\mathbf{B}(L_s^2)$ -valued continuous function on $J^+(a)$. Hence so is $(I + VR_0^+(z))^{-1}$. By (5.7) we have

$$R(z) = R_0(z)(I + VR_0^+(z))^{-1},$$

which, together with Theorem 2.1, implies conclusions (i) and (ii).

To prove conclusion (iii), we use (5.8), conclusion (ii) and Lemmas 5.2 and 5.4. We then obtain

$$(5.9) \quad \limsup_{\lambda \rightarrow \infty} \|R^+(\lambda)f\|_{-s} \leq \rho^+(a) \left\{ \sum_{\ell=N+1}^{\infty} (\omega(V)\rho^+(a))^\ell \right\} \|f\|_s$$

for any $f \in L_s^2$ and any integer N , where we have used Theorem 3.2 (i). Since $\omega(V)\rho^+(a) < 1$, and since N is arbitrary, we conclude from (5.9) that

$$\lim_{\lambda \rightarrow \infty} \|R^+(\lambda)f\|_{-s} = 0.$$

There remains to prove conclusion (iv). Suppose, to get a contradiction, that

$$\liminf_{\lambda \rightarrow \infty} \|R^+(\lambda)\|_{(s, -s)} = 0.$$

Since, by (5.7),

$$R_0^+(\lambda) = R^+(\lambda)(I + VR_0^+(\lambda)) \quad \text{in } \mathbf{B}(L_s^2, L_{-s}^2),$$

Lemma 5.4 gives

$$\|R_0^+(\lambda)\|_{(s, -s)} \leq \|R^+(\lambda)\|_{(s, -s)}(1 + \omega(V)\rho^+(a)),$$

so that $\liminf_{\lambda \rightarrow \infty} \|R_0^+(\lambda)\|_{(s, -s)} = 0$. This contradicts conclusion (ii) of Theorem 3.2. \square

THEOREM 5.6 (Uniqueness). – *Let V obey Assumption 5.1 and let s be as in (5.2). Let $a > 1$ and $\omega(V)\rho^\pm(a) < 1$. Suppose that $u \in L_{-s}^2 \cap H_{\text{loc}}^1$ satisfies the equation*

$$(5.10) \quad (\sqrt{-\Delta + 1} + V(x) - \lambda)u = 0 \quad (\lambda \geq a),$$

and, in addition, that u satisfies either of

$$(5.11\pm) \quad \left(\frac{\partial}{\partial x_k} \mp i\sqrt{\lambda^2 - 1}\tilde{x}_k \right)u \in L_{s-1}^2 \quad (k = 1, \dots, n).$$

Then u vanishes identically.

Proof. – We show the uniqueness only for the solution of (5.10) satisfying the outgoing radiation condition (5.11+). Since $Vu \in L_s^2$, we see by Theorem 4.3 that

$$(\sqrt{-\Delta + 1} - \lambda)R_0^+(\lambda)Vu = Vu.$$

The right hand side equals

$$-(\sqrt{-\Delta + 1} - \lambda)u$$

by (5.10). Thus we obtain

$$(5.12) \quad (\sqrt{-\Delta + 1} - \lambda)(u + R_0^+(\lambda)Vu) = 0.$$

It follows from the hypotheses of the theorem and Theorem 4.3 that $u + R_0^+(\lambda)Vu$ belongs to $L_{-s}^2 \cap H_{\text{loc}}^1$ and $u + R_0^+(\lambda)Vu$ satisfies the outgoing radiation condition. Applying Theorem 4.1, we get

$$(5.13) \quad u + R_0^+(\lambda)Vu = 0.$$

Since

$$\|R_0^+(\lambda)V\|_{(-s,-s)} \leq \omega(V)\rho^+(a) < 1$$

by Lemma 5.4 and the hypothesis of the theorem, we see that $I + R_0^+(\lambda)V$ is invertible in L^2_{-s} , and therefore we conclude from (5.13) that $u = 0$. \square

Finally we state a theorem which, together with Theorem 5.6, gives a characterization of $R^\pm(\lambda)$ by means of the equation

$$(5.14) \quad (\sqrt{-\Delta + 1} + V(x) - \lambda)u = f$$

with the radiation conditions. To do so, we write

$$u^\pm(\lambda, f) = R^\pm(\lambda)f \quad (f \in L^2_s).$$

THEOREM 5.7. – *Let V obey Assumption 5.1 and let s be as in (5.2). Let $a > 1$ and $\omega(V)\rho^\pm(a) < 1$. Then $u^\pm(\lambda, f)$ with $\lambda > a$ are in $L^2_{-s} \cap H^1_{loc}$ and satisfy the equation (5.14). Moreover, $u^+(\lambda, f)$ satisfies the outgoing radiation condition (5.11+), and $u^-(\lambda, f)$ satisfies the incoming radiation condition (5.11–).*

Proof. – We give the proof only for $u^+(\lambda, f)$. By Theorem 5.5 (ii), we have

$$u^+(\lambda, f) = R_0^+(\lambda)(I + VR_0^+(\lambda))^{-1}f.$$

Applying Theorem 4.3 to the right hand side, we see that $u^+(\lambda, f) \in L^2_{-s} \cap H^1_{loc}$, and that $u^+(\lambda, f)$ satisfies the outgoing radiation condition. In order to show that $u^+(\lambda, f)$ satisfies the equation (5.14), we define

$$u(\lambda + i\mu, f) = R(\lambda + i\mu)f \quad (\mu > 0, f \in L^2_s).$$

Since $u(\lambda + i\mu, f) \in \text{Dom}(H)$, we see that for $\varphi \in \mathcal{S}(\mathbf{R}^n_x)$

$$(5.15) \quad ((\sqrt{-\Delta + 1} + V(x) - \lambda - i\mu)u(\lambda + i\mu, f), \varphi)_x = (f, \varphi)_x.$$

It follows from Theorem 5.5 (i) that $u(\lambda + i\mu, f) \rightarrow u^+(\lambda, f)$ in H^1_{-s} as $\mu \downarrow 0$. Now, taking the limit of (5.15) as $\mu \downarrow 0$, we conclude that $u^+(\lambda, f)$ satisfies the equation (5.14). \square

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