

ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 63, n° 2 (1995), p. 155-176

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Uniqueness of bounded observables

by

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ABSTRACT. – By an application of a new construction technique we construct a σ -orthomodular lattice with a strongly order-determining set of states and two bounded observables whose expectations are equal at each state. This answers negatively the uniqueness problem for bounded observables, formulated by S. Gudder (*Pacific J. Math.*, Vol. **19**, 1966, pp. 81-93).

RÉSUMÉ. – A l'aide d'une nouvelle technique, on construit un treillis σ -orthomodulaire admettant un ensemble fortement séparant d'états et deux observables bornées ayant la même espérance pour tous les états. C'est une réponse négative au problème d'unicité, présenté par S. Gudder (*Pacific J. Math.*, Vol. **19**, 1966, pp. 81-93).

1. INTRODUCTION

In the logico-algebraic approach to quantum mechanics (*see e.g.* [1], [7], [12]), the events of a quantum system are supposed to form a quantum *logic* (a σ -orthomodular poset). The states of the system correspond to σ -additive probability measures on a logic, called also *states*. Random variables are represented by *observables* (σ -homomorphisms of the Borel

¹ The author gratefully acknowledges the support of the EC grant PECO 3510PL922147 and of the grant no. 201/93/0953 of Grant Agency of Czech Republic.

σ -algebra into a logic). These notions generalize those of the classical probability theory and they allow to describe the most important example – the lattice of projections in a Hilbert space – as a special case.

It is natural to ask whether observables are uniquely determined by their expectations in all states. This holds for both Boolean σ -algebras and lattices of projections in a Hilbert space. For general logics, the question of uniqueness is reasonable only under an additional assumption that the state space is “sufficiently large” (R. Greechie has found examples of logics without states – they violate the uniqueness property trivially). The uniqueness problem, as formulated by S. Gudder in [5], remained open until now. Here we present an answer and we also add some other related new results in comparison to the quarter of a century’s history of the problem.

The new technique which enabled the progress is based on the ideas of R. Mayet and V. Rogalewicz and is treated in detail in another paper [10]. Here (in Lemma 4.5) we adopt only its very special case.

2. BASIC NOTIONS

Let us recall the basic notions we shall deal with in the sequel. In some places, the original terminology of [5] is replaced by more modern terms, especially those applied later by the same author in [6], [7], [8].

DEFINITION 2.1. – *A quantum logic (a σ -orthomodular poset) is a poset L with bounds $0, 1$ and with a unary operation $'$ (orthocomplementation) such that*

1. $a \leq b \Rightarrow b' \leq a'$,
2. $a'' = a$,
3. $a \vee a' = 1$,
4. *if $(a_i)_{i \in \mathbb{N}}$ is a sequence of mutually orthogonal elements in L (we define $b \perp c$ iff $b \leq c'$), then $\bigvee_{i \in \mathbb{N}} a_i$ exists in L ,*
5. $a \leq b \Rightarrow b = a \vee (a' \wedge b)$ (orthomodular law).

If, moreover, L is a lattice, it is called a lattice logic or a σ -orthomodular lattice.

A subset K of a logic L is called a *sublogic* of L if K is closed under orthocomplements and under countable orthogonal suprema. We sometimes index the logical operations with the respective logic, e.g. $\wedge_L, \vee_L, {}^L$, etc. Unless otherwise stated, L denotes a logic. For $a \in L$, we denote the intervals $a^\wedge = \{b \in L : b \leq a\}$ and $a^\vee = \{b \in L : b \geq a\}$. An element

$a \in L$ is called an *atom* if $a^\wedge = \{0, a\}$ and L is called *atomic* if, for each $b \in L \setminus \{0\}$, b^\wedge contains an atom.

Each logic can be viewed as the union of its maximal Boolean sub- σ -algebras, called *blocks*. Two elements $a, b \in L$ are called *compatible* (abbr. $a \leftrightarrow b$) if they are contained in a Boolean sub- σ -algebra of L . For $a \in L$, we use the notation $C(a) = \{b \in L : b \leftrightarrow a\}$ ($C(a)$ is the union of all blocks containing a).

DEFINITION 2.2. – A state on a logic L is a mapping $m : L \rightarrow [0, 1] \subset \mathbb{R}$ such that

1. $m(\mathbf{1}) = 1$,
2. if $(a_i)_{i \in \mathbb{N}}$ is a mutually orthogonal sequence in L , then $m\left(\bigvee_{i \in \mathbb{N}} a_i\right) =$

$$\sum_{i \in \mathbb{N}} m(a_i).$$

We denote by $S(L)$ the set of all states on L . A functional m on a subset of L is called *faithful* if $m(a) = 0$ implies $a = 0$.

There are logics admitting no states [4], so it is reasonable to assume the existence of a sufficiently large state space. We shall work with the following conditions (we refer to [13] for their detailed treatment):

DEFINITION 2.3. – A set S of states on L is called

unital if $\forall a \in L \setminus \{0\} \exists m \in S : m(a) = 1$,

separating if $\forall a, b \in L, a \neq b \exists m \in S : m(a) \neq m(b)$,

order-determining (OD) if $\forall a, b \in L, a \not\leq b \exists m \in S : m(a) > m(b)$

strongly order-determining (SOD) if $\forall a, b \in L, a \not\leq b \exists m \in S : m(a) = 1 > m(b)$.

Alternatively, taking $c = b'$, we may say that S is OD (resp. SOD) if $\forall a, c \in L, a \not\leq c \exists m \in S : m(a) + m(c) > 1$ (resp. $m(a) = 1, m(c) > 0$). If L is atomic, it suffices to verify the latter properties for a, c being atoms.

PROPOSITION 2.4. – Let L be a finite logic. A convex set S of states on L is SOD iff for each atom a of L there is a state $m_a \in S$ such that $m_a(a) = 1$ and $m_a \upharpoonright (L \setminus C(a))$ is faithful.

For an interval I of reals we denote by $\mathcal{B}(I)$ the Borel σ -algebra of subsets of I .

DEFINITION 2.5. – An observable on a logic L is a σ -homomorphism $x : \mathcal{B}(\mathbb{R}) \rightarrow L$, i.e. a mapping such that

1. $x(\emptyset) = 0$,
2. $x(R \setminus E) = x(E)'$,

3. $x(\bigcup_{i \in N} E_i) = \bigvee_{i \in N} x(E_i)$ whenever $E_i, i \in N$, are mutually disjoint.

An observable x is bounded if there are $\ell, u \in R$ such that $x[\ell, u] = \mathbf{1}$.²

A spectrum of x is the smallest closed set $C \subset R$ such that $x(R \setminus C) = \mathbf{0}$. The range $x(\mathcal{B}(R))$ of x is denoted by $\text{Range } x$. If m is a state on L and the value $E(x, m) = \int_R \lambda_m(x(d\lambda))$ exists, it is called the expectation of the observable x at the state m .

Two observables x, y are called *compatible* (resp. *totally noncompatible*) if $x(E) \leftrightarrow y(F)$ (resp. $x(E) \nleftrightarrow y(F)$) whenever $E, F \in \mathcal{B}(R)$, $x(E), y(F) \notin \{\mathbf{0}, \mathbf{1}\}$.

Remark 2.6. – Totally noncompatible observables x, y may be equal if they are constant, i.e. $x\{c\} = \mathbf{1}$ for some $c \in R$.

3. HISTORICAL OVERVIEW

It is natural to ask whether observables are uniquely determined by their expectations. To avoid trivial counterexamples (and also for reasonability of physical applications), one has to assume a large state space. The *uniqueness problem for bounded observables* appeared first in [5] in 1966. It can be formulated as follows:

Let x, y be bounded observables on a logic L admitting a SOD set of states. If the expectations $E(x, m), E(y, m)$ are equal at all states m on L , do x, y have to be the same?

Remark 3.1. – Originally, this problem was formulated in [5] for a different structure – without the orthomodular law, but with the *regularity* condition: If a, b, c are pairwise compatible, then $a \leftrightarrow b \vee c$. This difference is not important. First, the existence of a separating set of states entails the orthomodular law. Second, all examples concerning this problem were regular. In particular, all new examples in this paper are lattices and all lattice logics are regular [12]. In later publications ([7], [8], [6]) the author deals with σ -orthomodular posets. Sometimes he weakens the assumption – he requires the expectations to be equal only for a SOD subset of the state space.

We say that a logic L satisfies the *uniqueness property* if the equality of expectations at all states on L implies the equality of bounded observables.

² We sometimes do not close the arguments of observables into brackets.

Positive results

The first sufficient conditions for the uniqueness property were given by S. Gudder in [5]. Uniqueness holds for the most important examples of logics:

THEOREM 3.2 [5]. – *Boolean σ -algebras admitting unital sets of states satisfy the uniqueness property.*

THEOREM 3.3 [5]. – *Lattices of projections of Hilbert-spaces satisfy the uniqueness property.*

THEOREM 3.4 [5]. – *Let x, y be compatible bounded observables on a logic L admitting a unital separating set of states. If $E(x, m) = E(y, m)$ for all $m \in S(L)$, then $x = y$.*

THEOREM 3.5 [5]. – *Let x, y be bounded observables on a logic L admitting a SOD set of states. If the spectrum of x has at most one limit point and $E(x, m) = E(y, m)$ for all $m \in S(L)$, then $x = y$.*

P. Pták and V. Rogalewicz introduced in [14] and [15] another interesting condition:

A set S of states on a logic L is *regularly order-determining* (ROD) if

$$\forall \varepsilon > 0 \quad \forall a, c \in L, \\ a \not\leq c \quad \exists m \in S : \quad m(a) \geq 1 - \varepsilon, \quad m(c) \geq 1 - \varepsilon.$$

THEOREM 3.6 ([15], cf. also [14]). – *Logics admitting a ROD set of states satisfy the uniqueness property.*

Negative results

In [8], S. Gudder presents two different totally noncompatible observables x, y on a finite lattice logic L and an OD (but not unital) set of states $S \subsetneq S(L)$ such that $E(x, m) = E(y, m)$ for all $m \in S$. His example admits also a SOD (moreover: ROD) set of states in which, however, the expectations of x and y are not equal. C. Schindler has improved the latter result as follows:

THEOREM 3.7 [17]. – *There is a finite logic L admitting a unital OD set of states and two different observables x, y on L such that $E(x, m) = E(y, m)$ for all $m \in S(L)$.³*

³ As the original arguments of C. Schindler were not sufficient, the example required a computer verification. I am indebted to D. Foulis who kindly agreed with my making use of his program for this purpose.

The conditions on the state cannot be strengthened for finite logics – see Thm. 3.5.

V. Rogalewicz has constructed the following example:

THEOREM 3.8 [16]. – *There is a logic L with two different totally noncompatible bounded observables x, y and a SOD set of states $S \subsetneq S(L)$ such that $E(x, m) = E(y, m)$ for all $m \in S$.*

Remark 3.9. – The latter example is not a lattice. The expectations of x, y are not equal at all states from $S(L)$. Thus the uniqueness problem was not solved in its original form from [5].

Most of the results of this section will be generalized in the sequel.

4. BASIC TOOLS

In this section we summarize the results that enabled a progress in the investigations of the uniqueness property. These are mainly the pasting constructions developed in [11] and a special construction of lattice logics admitting SOD sets of states which is described in detail in [10].

DEFINITION 4.1 (see e.g. [7], [12]). – *Let $\mathcal{L} = \{L_\alpha : \alpha \in I\}$ be a collection of logics. We call the cartesian product $L = \prod_{\alpha \in I} L_\alpha$ a product of \mathcal{L} if it is endowed with the “pointwise” partial ordering and orthocomplementation, i.e. for all $a, b \in L$ we have $a \leq_L b$ (resp. $a = b'^L$) iff $a_\alpha \leq_{L_\alpha} b_\alpha$ (resp. $a_\alpha = b'^{L_\alpha}$) for all $\alpha \in I$.*

A subset I of a logic L is called an *orthoideal* if

1. $a < b \in I$ implies $a \in I$,
2. if $(a_i)_{i \in N} \subset I$ is an orthogonal sequence, then $\bigvee_{i \in N} a_i \in I$.

DEFINITION 4.2 [11]. – *Let \mathcal{L} be a collection of logics such that for each $P, Q \in \mathcal{L}$ the intersection $P \cap Q$ is a sublogic of both P and Q and, moreover, the orthocomplements and the partial orderings coincide on $P \cap Q$. Put $L = \bigcup_{P \in \mathcal{L}} P$ and define the binary relation \leq_L and the unary operation $'^L$ as follows:*

$a \leq_L b$ (resp. $a = b'^L$) iff $a \leq_P b$ (resp. $a = b'^P$) for some $P \in \mathcal{L}$.

The set L equipped with $\leq_L, '^L$ is called the pasting of the collection \mathcal{L} .

Sufficient conditions for a pasting to be a logic (resp. a lattice logic) are given in [11]. Here we shall apply only a special case. In order to obtain a lattice logic, we need the following generalization of [11], Prop. 4.3.

THEOREM 4.3. – *Let \mathcal{L} be a collection of logics satisfying the following conditions:*

(T1) $\forall P, Q \in \mathcal{L} : P \cap Q$ is of the form $\{a, a' \mid a \in I\}$, where I is an orthoideal in both P and Q .

(T2) $\exists W \in \mathcal{L} \forall P, Q \in \mathcal{L}, P \neq Q : P \cap Q \subset W$.

Then the pasting L of the collection \mathcal{L} is a logic. Suppose further that \mathcal{L} consists of lattice logics and satisfies the following properties:

(L3) $\forall P, Q \in \mathcal{L} : P \cap Q$ is closed under the lattice operations.

(L4) $\forall P \in \mathcal{L} \forall a \in P : a^{\vee P} \cap W$ has a least element.

Then L is a lattice logic.

Proof. – According to [11], Prop. 4.1 and 4.2, L is a logic; it remains to be proved that L is lattice. Let $a, b \in L$. If $a, b \in P$ for some $P \in \mathcal{L}$, then $a \vee_P b$ becomes also $a \vee_L b$. Suppose that a, b do not belong to a single logic of \mathcal{L} . We have $a \in P, b \in Q$ for some $P, Q \in \mathcal{L}, P \neq Q$. Let \bar{a} (resp. \bar{b}) be the least element of $a^{\vee P} \cap W$ (resp. $b^{\vee Q} \cap W$). Each upper bound c of a, b belongs to W , hence $c \geq \bar{a} \vee_W \bar{b}$ and $\bar{a} \vee_W \bar{b} = a \vee_L b$. The existence of $a \wedge_L b$ follows from de Morgan laws. \square

In the definition of the pasting we have supposed that the logics of the collection \mathcal{L} have already some common elements (at least $\mathbf{0}$ and $\mathbf{1}$). Alternatively, we can start with a collection of disjoint logics and form the pasting after identifying the elements of appropriate isomorphic sublogics. Let us formulate two special cases of this construction that will be of separate interest in the sequel:

THEOREM 4.4 [11], Thm. 6.1. – *Let K, L be logics. Suppose that a is an atom in K . Write $b = a^K$. Put $M = b^{\wedge K} \times L$. For all $c \in b^{\wedge K}$, let us identify $c \in K$ with $(c, \mathbf{0}_L) \in M$ and $c \vee_K a \in K$ with $(c, \mathbf{1}_L) \in M$. The pasting P of K and M is then a logic. If K, L are lattice logics, P is a lattice, too. We say that P originated by the substitution of the atom a in K with the logic L .*

After the above substitution, the interval $a^{\wedge P}$ becomes isomorphic to L , while the ordering inherited from K is preserved and extended canonically to P .

The distance $d(a, b)$ of two elements $a, b \in L$ is the minimal n for which there is a chain of blocks B_1, \dots, B_n such that $a \in B_1, b \in B_n$, and $B_i \cap B_{i+1} \not\supseteq \{\mathbf{0}, \mathbf{1}\}$.

LEMMA 4.5 [10]. – For each $k \in \mathbb{N}$, $k \geq 3$, there is a lattice logic M_k with a set of atoms $Z_k = \{x_1, \dots, x_k, y_1, \dots, y_k\}$ satisfying the following properties:

1. Elements of Z_k are mutually noncompatible. Moreover, $d(x_i, x_j) \geq 5$, $d(y_i, y_j) \geq 5$ for $i \neq j$, and $d(x_i, y_j) \geq 4$ for all i, j .
2. Each function $m : Z_k \rightarrow [0, 1]$ admits an extension to a state on M_k iff

$$(E) \quad \sum_{i \leq k} m(x_i) = \sum_{i \leq k} m(y_i).$$

3. Let $a \in M_k$ be an atom and let $m : Z_k \rightarrow [0, 1]$ be a function satisfying

$$\sum_{i \leq k} m(x_i) = \sum_{i \leq k} m(y_i) < k - 1$$

and

$$m(b) \begin{cases} = 1 & \text{for } b = a, \\ = 0 & \text{for } b \perp a, \\ \in (0, 1) & \text{otherwise.} \end{cases}$$

Then m admits an extension to a state μ on M_k such that $\mu(a) = 1$ and $\mu \upharpoonright (M_k \setminus C(a))$ is faithful. In particular, M_k admits a SOD set of states.

Proof (see [10] for more details and a generalization). – For $k = 3$ one may take the logic portrayed in Figure 1a. The construction for greater k is analogous, but the hexagon has to be replaced by a larger polygon – see Fig. 2, where p is an even number greater than $2k + 1$. (In order to simplify the arguments in the following proof, we do not seek here for the least possible value of p – e.g. $p = 2$ is sufficient for $k = 3$.) Let us verify the properties of the lemma. The proof of the equivalence 2 is divided into two parts corresponding to the two implications.

1. Trivial.

2'. Suppose that m is a state on M_k and verify (E). We demonstrate the proof for $k = 3$. Denote by A_3 the set of all atoms of M_3 . Summing the values of m over the atoms of the 11 blocks displayed in Figure 1b, we obtain

$$\sum_{a \in A_3} m(a) = 11 + \sum_{i \leq k} m(x_i).$$

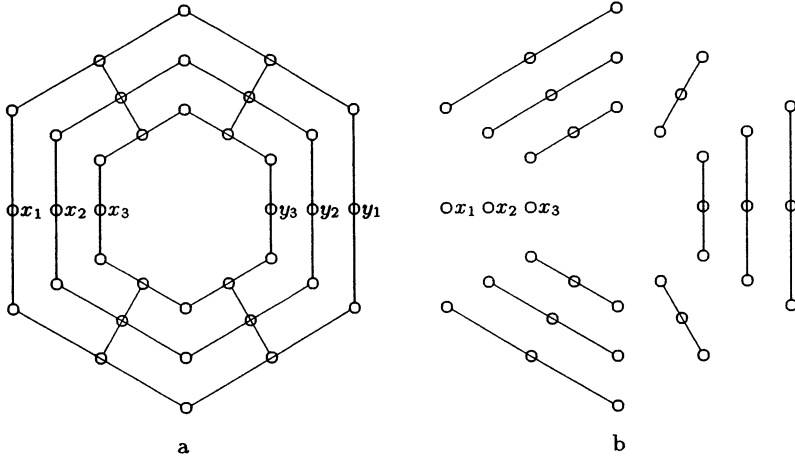


Fig. 1. – The lattice logic M_3 of Lemma 4.5.

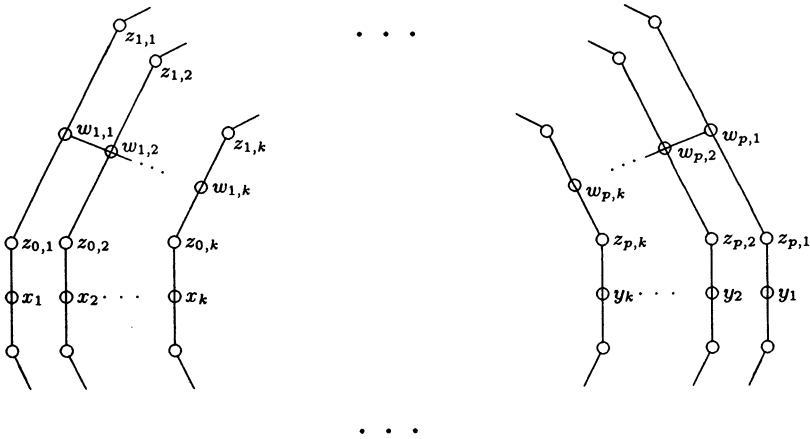


Fig. 2. – The lattice logic M_k of Lemma 4.5.

For the symmetric selection of blocks we obtain

$$\sum_{a \in A_3} m(a) = 11 + \sum_{i \leq k} m(y_i),$$

which yields (E). Analogous argument works for the logic of Figure 2 provided that p is even.

2''. Suppose that m satisfies (E) and find its extension $\mu \in S(M_k)$.

Step 1. – We choose values $\mu(z_{0,i})$, $i \leq k$, such that $\mu(z_{0,i}) > 0$ whenever $m(x_i) < 1$, and $\sum_{i \leq k} \mu(z_{0,i}) < k - 1$. This can be done e.g.

by putting $\mu(z_{0,i}) = (1 - m(x_i))/2$, $i \leq k$. We choose values $\mu(z_{p,i})$ analogously, with a further requirement $\sum_{i \leq k} \mu(z_{p,i}) = \sum_{i \leq k} \mu(z_{0,i})$. We

can put e.g. $\mu(z_{p,i}) = (1 - m(y_i))/2$. Notice that this selection satisfies the inequality

$$\sum_{i \leq k} |\mu(z_{p,i}) - \mu(z_{0,i})| \leq k/2.$$

Step 2. – For all even j , $0 < j < p$, we choose the values $\mu(z_{j,i}) \in (0, 1)$ such that

$$\sum_{i \leq k} \mu(z_{j,i}) = \sum_{i \leq k} \mu(z_{0,i}), \quad j = 2, 4, \dots, p-2,$$

$$(D) \quad \sum_{i \leq k} |\mu(z_{j,i}) - \mu(z_{j-2,i})| < 2, \quad j = 2, 4, \dots, p.$$

Unless $\mu(z_{0,i}) = \mu(z_{p,i}) \in \{0, 1\}$, we may define

$$\mu(z_{j,i}) = \frac{j}{p} \cdot \mu(z_{p,i}) + \left(1 - \frac{j}{p}\right) \cdot \mu(z_{0,i}),$$

where (D) is satisfied for $p > \frac{1}{2} \cdot \sum_{i \leq k} |\mu(z_{p,i}) - \mu(z_{0,i})|$.

Step 3. – For all odd j , $0 < j < p$, we choose the values $\mu(z_{j,i}) \in [0, 1)$ such that

$$\sum_{i \leq k} \mu(z_{j,i}) = k - 1 - \sum_{i \leq k} \mu(z_{0,i}),$$

$$\mu(z_{j,i}) \leq 1 - \max(\mu(z_{j-1,i}), \mu(z_{j+1,i})),$$

and $\mu(z_{j,i}) > 0$ whenever the right-hand side of the latter inequality is nonzero. Such a solution exists because

$$\begin{aligned} & \sum_{i \leq k} [1 - \max(\mu(z_{j-1,i}), \mu(z_{j+1,i}))] \\ &= k - \sum_{i \leq k} \mu(z_{j-1,i}) - \frac{1}{2} \cdot \sum_{i \leq k} |\mu(z_{j-1,i}) - \mu(z_{j+1,i})| \\ &> k - \sum_{i \leq k} \mu(z_{0,i}) - 1. \end{aligned}$$

Step 4. – For all $j = 1, \dots, p$, $i = 1, \dots, k$, we define

$$\mu(w_{j,i}) = 1 - \mu(z_{j-1,i}) - \mu(z_{j,i}).$$

As

$$\sum_{i \leq k} \mu(w_{j,i}) = k - (k - 1) = 1,$$

μ is well-defined on the atoms of the “upper half” of the diagram of Figure 2.

The “lower half” of this diagram is processed analogously.

3. In order to find the desired extension of m , we apply the same principles as in $2''$, with the necessary modifications concerning the elements of $C(a)$. Due to the symmetries of the logic M_k , we may suppose, without any loss of generality, that a is one of the atoms in the “upper left quadrant” of the diagram of Figure 2 and that its second index is k , i.e. $a \in \{x_k\} \cup \{z_{j,k} : j \leq p/2\} \cup \{w_{j,k} : j \leq p/2\}$.

A. If $a \in x_k$, the procedure of $2''$ works without any change.

B. Let $a = z_{j,k}$ for some even $j \leq p/2$. Steps 1 and 2 of $2''$ are modified so that $\mu(z_{j,k}) = 1$, $\mu(z_{j,i}) = c \mu(z_{0,i})$, $i \leq k - 1$, where the constant $c \leq 1$ is determined by the condition $\sum_{i \leq k} \mu(z_{j,i}) = \sum_{i \leq k} \mu(z_{0,i})$.

This requires $\sum_{i \leq k} \mu(z_{0,i}) > 1$ which is possible due to the inequality

$$\sum_{i \leq k} \mu(x_i) < k - 1. \text{ Notice that}$$

$$\sum_{i \leq k} |\mu(z_{j,i}) - \mu(z_{0,i})| = 2(1 - \mu(z_{0,k})) < 2,$$

$$\sum_{i \leq k} |\mu(z_{j,i}) - \mu(z_{p,i})| < 2 \sum_{i \leq k} \mu(z_{p,i}) < 2k.$$

Steps 3 and 4 are possible if $p \geq 2k$ (in this case $p - j \geq k$).

C. Let $a = z_{j,k}$ for some odd $j \leq p/2$. Steps 1 and 2 are modified so that $\mu(z_{j-1,k}) = \mu(z_{j+1,k}) = 0$, $\mu(z_{j-1,i}) = \mu(z_{j+1,i}) = 1 - c \cdot (\mu(z_{0,i}))$,

$i \leq k - 1$, where the constant $c \leq 1$ is determined by $\sum_{i \leq k} \mu(z_{j-1, i}) = \sum_{i \leq k} \mu(z_0, i)$. Notice that

$$\sum_{i \leq k} |\mu(z_{j-1, i}) - \mu(z_0, i)| = 2\mu(z_0, k) < 2,$$

$$\sum_{i \leq k} |\mu(z_{j+1, i}) - \mu(z_{j-1, i})| = 0,$$

$$\sum_{i \leq k} |\mu(z_{j+1, i}) - \mu(z_p, i)| < 2k.$$

Steps 3 and 4 are possible if $p \geq 2k + 2$ (in this case $p - (j + 1) \geq k$).

D. Let $a = w_{j, k}$ for some $j \leq p/2$. When defining the values $\mu(z_j, i)$ for even j , this requires the zero value on one of these atoms and a limited difference of the values of μ on the atoms nearest to $w_{j, k}$. This situation is analogous to that in case C and can be solved by the same technique. \square

5. NEW RESULTS

This section, besides giving generalizations of some results of section 3, serves as an introduction to the main counterexample – we shall see some difficulties and the techniques to overcome them. We present a sufficient condition for the uniqueness. We introduce logics with uniformly order-determining sets of states and show that they satisfy the uniqueness for totally noncompatible observables. We construct a finite lattice logic with the properties of Theorem 3.7.

THEOREM 5.1. – *Let x, y be bounded observables on a logic L admitting a SOD set of states S . Suppose that there exists a maximal real number r with the property $x[r, \infty) \neq y[r, \infty)$. Then $E(x, m) \neq E(y, m)$ for some $m \in S$.*

Proof. – Because of the maximality of r , $x[r + \varepsilon, \infty) = y[r + \varepsilon, \infty)$ for all $\varepsilon > 0$ and $x(r, \infty) = y(r, \infty)$. Thus $x\{r\} \neq y\{r\}$. Suppose, without any loss of generality, that $x\{r\} \not\leq y\{r\}$. There is a state $m \in S$ such that $m(x\{r\}) = 1 > m(y\{r\})$. We shall show that $E(x, m) > E(y, m)$. Evidently, $E(x, m) = r$. As $m(y(r, \infty)) = m(x(r, \infty)) = 0$, $E(y, m) \leq r$ and the equality holds only for $m(y\{r\}) = 1$ which is not the case. Thus $E(y, m) < r = E(x, m)$. \square

As a corollary of Thm. 5.1 we obtain Thm. 3.5.

We introduce a new condition on the state space: A set S of states on a logic L is called *uniformly-order-determining* (UOD) if there is an $\varepsilon > 0$ such that

$$(U_\varepsilon) \quad \forall a, b \in L, \quad a \not\leq b \quad \exists s \in S : s(a) > s(b) + \varepsilon.$$

Alternatively, S is UOD iff

$$\exists \varepsilon > 0 \quad \forall a, c \in L, \quad a \not\leq c \quad \exists s \in S : s(a) + s(c) > 1 + \varepsilon.$$

If S is ROD, condition (U_ε) is satisfied for all $\varepsilon < 1$. If L is finite and S is OD, it is also UOD. No implication holds between the properties UOD and SOD.

THEOREM 5.2. – *Let x, y be different totally noncompatible bounded observables on a logic L admitting a UOD set of states S . Then $E(x, m) \neq E(y, m)$ for some $m \in S$.*

Proof. – Suppose that S satisfies (U_ε) for $\varepsilon > 0$ and $E(x, m) = E(y, m)$ for all $m \in S$. According to [5], the spectra of x and y have the same minima and maxima, say ℓ and u . We take a $\delta \in (0, u - \ell)$ sufficiently small (namely, $\delta \leq (u - \ell) \cdot \varepsilon / (1 + \varepsilon)$) and we define $a = x(u - \delta, u)$, $c = y(\ell, \ell + \delta)$. As $a, c \notin \{0, 1\}$, $a \leftrightarrow c$ and there is a state $m \in S$ such that $m(a) + m(c) > 1 + \varepsilon$. We shall show that $E(x, m) > E(y, m)$. The expectations satisfy the following inequalities:

$$E(x, m) > (u - \delta) \cdot m(a) + \ell \cdot (1 - m(a)) = \ell + (u - \ell - \delta) \cdot m(a),$$

$$E(y, m) < (\ell + \delta) \cdot m(c) + u \cdot (1 - m(c)) = u - (u - \ell - \delta) \cdot m(c).$$

We deduce

$$\begin{aligned} E(x, m) - E(y, m) &> -(u - \ell) + (u - \ell - \delta) \cdot (m(a) + m(c)) \\ &> -(u - \ell) + (u - \ell - \delta) \cdot (1 + \varepsilon) \geq 0, \end{aligned}$$

a contradiction with the assumption $E(x, m) = E(y, m)$. \square

Let us look for a counterexample to the uniqueness among finite logics. The latter two theorems imply that such an example cannot admit a SOD set of states and the observables in question cannot be totally noncompatible, so Schindler's result (Thm. 3.7) is "the best possible". However, his example requires a complex (computer-aided) verification and therefore it remained unpublished. Instead of it, we present here

another example which, moreover, is a lattice. It demonstrates the use of Lemma 4.5.

THEOREM 5.3. – *There is a finite lattice logic L admitting a unital UOD set of states and there are two different observables x, y on L whose expectations are equal at all states.*

Proof. – Let K be the lattice logic portrayed in Figure 3 (see [3] or [9] for the arguments that it is a lattice logic; alternatively, the pasting techniques from [11] are applicable).

Take the lattice logic M_4 of Lemma 4.5 for $k = 4$. (As we do not require a SOD set of states, it is sufficient to choose $p = 4$, see Fig. 2 and 4). We identify the following pairs of atoms of M_4 and K :

- x_1 with u_1 , y_1 with v_1 ,
- x_2 with v_2 , y_2 with v_2 ,
- x_3 with u_2 , y_3 with v_2 ,
- x_4 with v_{-1} , y_4 with u_{-1} .

Thus M_4 is “deformed” to a lattice logic M by the identifications $x_2 = x_3$, $y_2 = y_3$. However, this has no effect on the properties mentioned in Lemma 4.5. We identify also the respective coatoms (=complements of atoms) and the least and the greatest elements of K and M . We take for L the pasting of K and M ; it is represented by Figure 4. To see that L is a lattice logic, one may apply [3] or [11], Thm. 5.6.

Each state m on L satisfies the equality

$$(R) \quad 2m(u_2) + m(u_1) + m(v_{-1}) = 2m(v_2) + m(v_1) + m(u_{-1}).$$

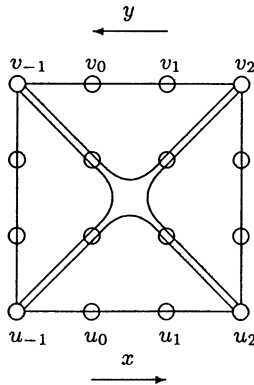


Fig. 3. – The astroid K of Theorem 5.3.

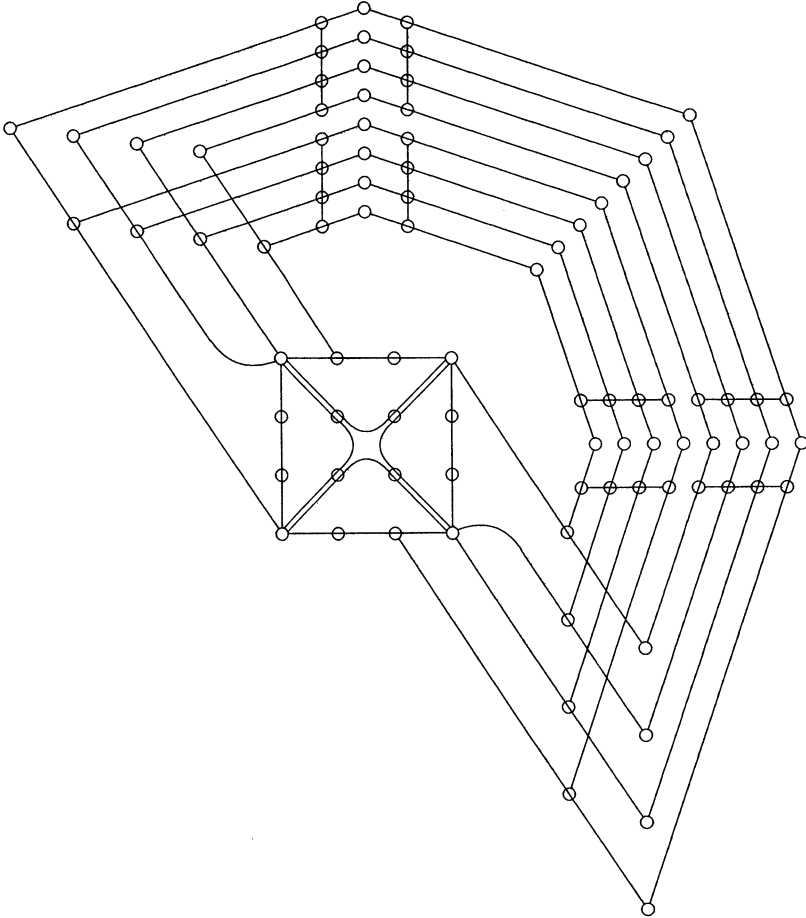


Fig. 4. – A lattice logic L admitting a unital OD set of states (Theorem 5.3).

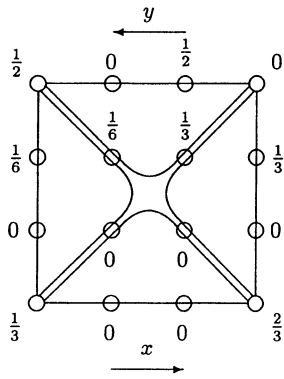
We define observables x, y on L with spectra $V = \{-1, 0, 1, 2\}$ by the rules $x \{i\} = u_i$, resp. $y \{i\} = v_i$ ($i \in V$). The expectations at a state m are

$$E(x, m) = 2m(u_2) + m(u_1) - m(u_{-1}),$$

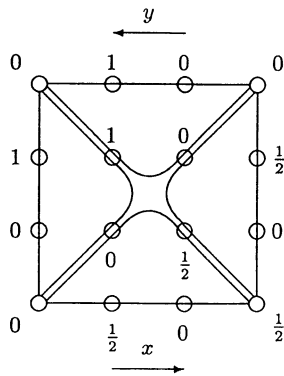
$$E(y, m) = 2m(v_2) + m(v_1) - m(v_{-1}).$$

According to (R), they are equal.

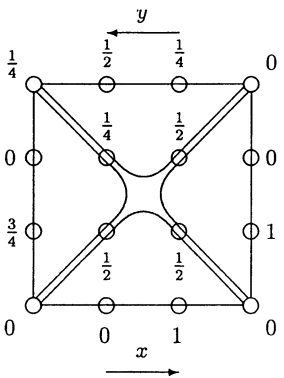
It remains to be proved that L admits a unital OD set of states. Recall that each state m on K which satisfies (R) admits an extension over L (Lemma 4.5). It is easy to find a unital set of states on L .



a



b



c

Fig. 5. – States on the astroid (Theorem 5.3).

Let us define an OD set of states on L . For nonorthogonal atoms $a, c \in L$ we have to find a state m satisfying $m(a) + m(c) > 1$. If $a = c$, we can even satisfy $m(a) + m(c) = 2$. If $a \in L \setminus K$, then we can find a state m such that $m(a) = 1$ and $m(c) > 0$. The only difficulty arises when $a, c \in K$. We shall distinguish several cases. It suffices to find a state on K satisfying (R) and $m(a) + m(c) > 1$; its extension to L exists. The most difficult case is that of $a = u_2$ (the cases of $a \in \{u_{-1}, v_2, v_{-1}\}$ are isomorphic to this one). For any choice of c , one of the states represented by Figure 5a, b or their isomorphic images solves the problem. The remaining choices of a and c are covered by some of the isomorphic copies of states from Figure 5b, c (Figure 5c is used for some choices with $a = u_1$). \square

6. MAIN COUNTEREXAMPLE

In order to answer the uniqueness problem for bounded observables, we first prove a lemma. It is a modification of Lemma 4.5 with "imprecise" action.

LEMMA 6.1. — *Let $n \in \mathbb{N}$. There is a finite lattice logic I_n with two sets of atoms $U_n = \{u_{1,n}, \dots, u_{n-1,n}\}$, $V_n = \{v_{1,n}, \dots, v_{n-1,n}\}$ satisfying the following properties:*

1. U_n is orthogonal, V_n is orthogonal, and $d(u_{i,n}, v_{j,n}) \geq 4$ for all $i, j < n$.
2. Each function $m : U_n \cup V_n \rightarrow [0, 1]$ admits an extension to a state on I_n iff it satisfies the inequalities

$$(I1) \quad \left| \sum_{i < n} im(u_{i,n}) - \sum_{i < n} im(v_{i,n}) \right| \leq 2,$$

$$(I2) \quad \sum_{i < n} m(u_{i,n}) \leq 1,$$

$$(I3) \quad \sum_{i < n} m(v_{i,n}) \leq 1.$$

3a. Let $a \in I_n \setminus (U_n \cup V_n)$ be an atom. Let $m : U_n \cup V_n \rightarrow [0, 1]$ be a function satisfying

$$(I1+) \quad \left| \sum_{i < n} im(u_{i,n}) - \sum_{i < n} im(v_{i,n}) \right| < 1,$$

$$(I2+) \quad \sum_{i < n} m(u_{i,n}) < 1,$$

$$(I3+) \quad \sum_{i < n} m(v_{i,n}) < 1,$$

and

$$m(b) \begin{cases} = 0 & \text{for } b \perp a, \\ \in (0, 1) & \text{otherwise.} \end{cases}$$

Then m admits an extension to a state μ on I_n such that $\mu(a) = 1$ and $\mu \upharpoonright (I_n \setminus C(a))$ is faithful.

3b. Let $a \in U_n$. Let $m : V_n \rightarrow (0, 1)$ be a function satisfying (I1+) and (I3+). Then m admits an extension to a state μ on I_n such that $\mu(a) = 1$ and $\mu \upharpoonright (I_n \setminus C(a))$ is faithful. (Analogously for the roles of U_n and V_n interchanged.)

As a consequence of 3a, 3b, I_n admits a SOD set of states.

Proof. – We apply Lemma 4.5 for $k = n + 1$ with a slight modification: We substitute the atom $x_1 \in M_k$ with the finite Boolean algebra with $n - 1$ atoms $u_{1,n}, \dots, u_{n-1,n}$. After this substitution, we identify x_i with $\bigvee_{j=i}^{n-1} u_{j,n}$ for all $i = 2, \dots, n - 1$. Analogously, we substitute the atom y_1 with the finite Boolean algebra with atoms $v_{1,n}, \dots, v_{n-1,n}$ and we identify y_i with $\bigvee_{j=i}^{n-1} v_{j,n}$ for all $i = 2, \dots, n - 1$. The result is a lattice

logic represented by Figure 6. The inequalities (I2), (I3) are equivalent to the obvious relations $m(x_1) \leq 1, m(y_1) \leq 1$.

We obtain $\sum_{i \leq k} m(x_i) = \sum_{i \leq k} m(y_i)$ without any other restriction on the values $m(x_i), m(y_i)$ (besides the obvious orderings $m(x_i) \geq m(x_{i+1}), m(y_i) \geq m(y_{i+1})$ for $i < n - 1$). As $\sum_{i < n} m(x_i) = \sum_{i < n} im(u_{i,n})$ and $\sum_{i < n} m(y_i) = \sum_{i < n} im(v_{i,n})$, (I1) follows from

$$\sum_{i < n} m(x_i) - \sum_{i < n} m(y_i) = m(y_n) + m(y_{n+1}) - m(x_n) - m(x_{n+1}).$$

The verification of 3a, b is routine. \square

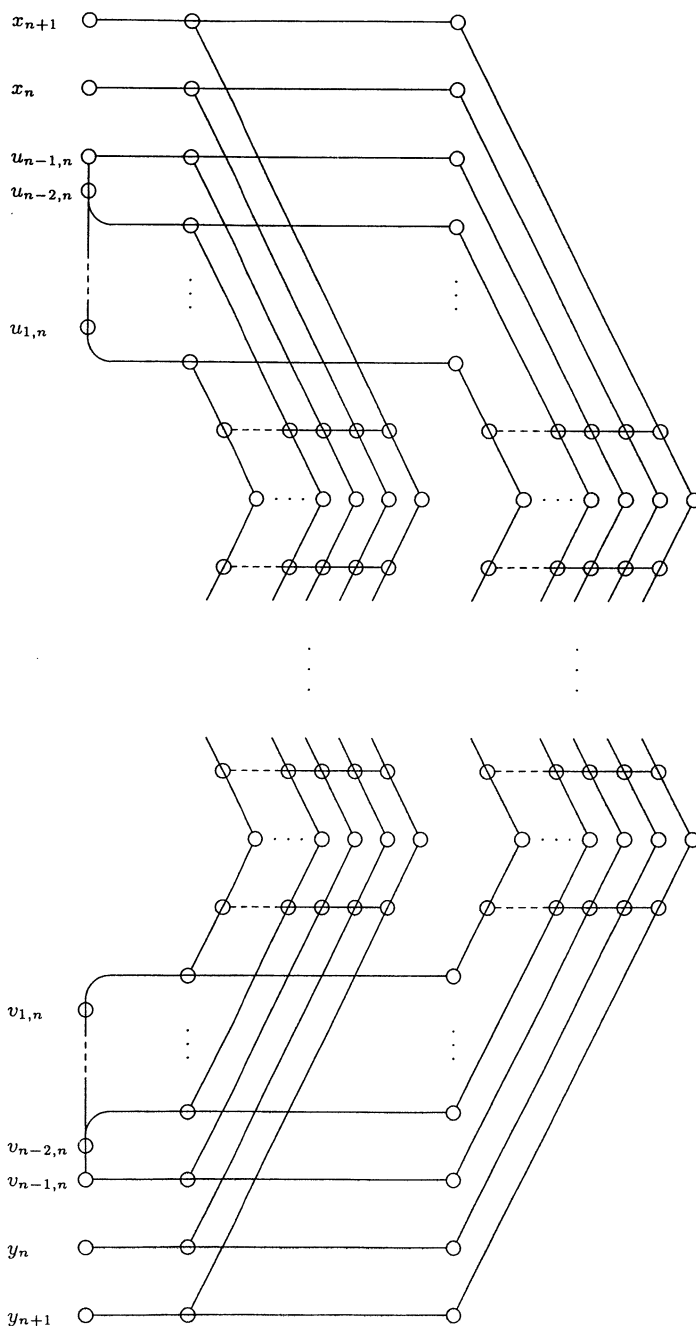


Fig. 6. – The lattice logic I_n of Lemma 6.1.

THEOREM 6.2. – *There is a lattice logic L and two different bounded observables x, y on L such that:*

1. L admits a SOD set of states.
2. The expectations $E(x, m), E(y, m)$ are equal for all states m on L .
3. The observables x, y are totally noncompatible.
4. Let t_x (resp. t_y) be a state on $\text{Range } x$ (resp. $\text{Range } y$). If $E(x, t_x) = E(y, t_y)$, then there is a state m on L such that $m|_{\text{Range } x} = t_x, m|_{\text{Range } y} = t_y$.

Proof. – Let us take two Boolean σ -algebras X, Y which are isomorphic to $\mathcal{B}((0, 1))$ under the isomorphisms x, y respectively. We may view x , resp. y , as an observable on X , resp. Y (with $\text{Range } x = X, \text{Range } y = Y$). We identify the least and the greatest elements of X, Y , and we form the pasting $W = X \cup Y$. It is a lattice logic. (It is the horizontal sum of X and Y , see [9]). The observables x, y may be viewed also as observables on W (and on any logic containing W as a sublogic).

Take an unbounded set $M \subset N$. For each $n \in M$, we take the lattice logic I_n of Lemma 6.1. In I_n , we substitute each $u_{i,n}, v_{i,n}$ ($i < n$) with a copy of $\mathcal{B}([0, 1])$ (which is isomorphic to $\mathcal{B}([q, r])$ for all $q, r \in R, q < r$). It is easy to verify that the result of the substitutions is a lattice logic.

We denote it by L_n . We denote $\bar{u}_n = \bigvee_{i=1}^{n-1} u_{i,n}$ and $\bar{v}_n = \bigvee_{i=1}^{n-1} v_{i,n}$. There is

an isomorphism e_n between $\bar{u}_n^\wedge \subset L_n$ and $x \left[\frac{1}{n}, 1 \right]^\wedge \subset X$ such that

$$e_n(u_{i,n}) = x \left[\frac{i}{n}, \frac{i+1}{n} \right) \quad (i < n).$$

Analogously, there is an isomorphism f_n between $\bar{v}_n^\wedge \subset L_n$ and $y \left[\frac{1}{n}, 1 \right)^\wedge \subset Y$ such that

$$f_n(v_{i,n}) = y \left[\frac{i}{n}, \frac{i+1}{n} \right) \quad (i < n).$$

We shall paste $W = X \cup Y$ with all $L_n, n \in M$. Before doing this, we identify the elements corresponding under the isomorphisms e_n, f_n ($n \in M$) and the respective complements. We shall describe this step in more detail: For each $n \in M$ and for each $a \in \bar{u}_n^\wedge, b \in \bar{v}_n^\wedge$, we identify the following pairs of elements:

- a with $e_n(a)$,
- a' with $e_n(a)'$,

b with $f_n(b)$,

b' with $f_n(b)'$.

For all $p, n \in M, p > n$, we have

$$L_p \cap L_n = x \left[\frac{1}{n}, 1 \right]^{\wedge} \cup \left(x \left[\frac{1}{n}, 1 \right]' \right)^{\vee} \cup y \left[\frac{1}{n}, 1 \right]^{\wedge} \\ \cup \left(y \left[\frac{1}{n}, 1 \right]' \right)^{\vee} \subset W.$$

Thus the pasting of $W, L_n (n \in M)$ is a lattice logic (Thm. 4.3). We denote it by L . We shall show that L satisfies the conditions of Theorem 6.2.

Let m be a state on L . For each $n \in M$, consider the “integral sums”

$$S_n = \sum_{i < n} \frac{i}{n} \cdot m \left(x \left[\frac{i}{n}, \frac{i+1}{n} \right] \right), \\ T_n = \sum_{i < n} \frac{i}{n} \cdot m \left(y \left[\frac{i}{n}, \frac{i+1}{n} \right] \right).$$

We have

$$E(x, m) - S_n \in \left[0, \frac{1}{n} \right), \\ E(y, m) - T_n \in \left[0, \frac{1}{n} \right),$$

and $S_n \rightarrow E(x, m), T_n \rightarrow E(y, m)$ for $n \rightarrow \infty$. Notice that

$$S_n = \frac{1}{n} \cdot \sum_{i < n} im(u_{i,n}), \\ T_n = \frac{1}{n} \cdot \sum_{i < n} im(v_{i,n}).$$

Lemma 6.1.2 implies that

$$|S_n - T_n| \leq \frac{2}{n}.$$

Thus $|S_n - T_n| \rightarrow 0$ for $n \rightarrow \infty$ and $E(x, m) = E(y, m)$. We have proved that the observables x, y have the same expectation at each state m .

The statement 4 is a consequence of Lemma 6.1; (I1+) follows from the equality $E(x, m) = E(y, m)$. It remains to be proved that L admits a SOD set of states. Let a, b be two nonorthogonal atoms of L . We have to find a state m satisfying $m(a) = 1, m(b) > 0$. If $a \in L \setminus W$, it suffices to take appropriate values of m on W (with $m(b) > 0$ if $b \in W$) and to apply Lemma 6.1.3a. If $a \in W$, we apply Lemma 6.1.3b analogously. \square

Theorem 6.2 answers negatively the original uniqueness problem from [5]. Surprisingly, the observables in question are totally noncompatible. Moreover, for each $a \in \text{Range } x, b \in \text{Range } y, a, b \notin \{0, 1\}$, we have $\text{Com}(a, b) = (a \vee b) \wedge (a' \vee b) \wedge (a \vee b') \wedge (a' \vee b') = 1$ ($\text{Com}(a, b)$ is the commutator of a, b ; see [2], [9] for more information). This means that the observables x, y are “logically independent”, but they are “stochastically equivalent”.

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(Manuscript received June 20, 1994;
Revised version received November 14, 1994)