ANNALES DE L'I. H. P., SECTION A

MARCIN MOSZYŃSKI

On classical intrinsically resonant formal perturbation theory

Annales de l'I. H. P., section A, tome 63, nº 2 (1995), p. 125-154

<http://www.numdam.org/item?id=AIHPA_1995__63_2_125_0>

© Gauthier-Villars, 1995, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section A » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Henri Poincaré,

Vol. 63, n° 2, 1995, p. 125-154. *Physique théorique*

On classical intrinsically resonant formal perturbation theory

by

Marcin MOSZYŃSKI [1]

Instytut Matematyki Stosowanej i Mechaniki, Uniwersytet Warszawski ul. Banacha 2, 02-097 Warszawa, Poland.

ABSTRACT. – We discuss conditions under which the Birkhoff normal form for perturbations of d-dimensional resonant harmonic oscillator is defined for all orders. In particular for $d = 2$ we provide simple and verifiable geometric conditions showing that this is the case on many open sets of the phase space $\mathbb{R}_+^2 \times \mathbb{T}^2$.

Key words: Classical mechanics, perturbation theory, resonant systems, normal form.

RÉSUMÉ. – On considère les conditions sur la perturbation d'un oscillateur harmonique résonant d-dimensionnel, sous lesquelles tous les termes de la forme normale de Birkhoff sont bien définis. Dans le cas particulier $d = 2$, on formule des conditions géométriques simples et vérifiables.

0. INTRODUCTION

One of the fundamental problems of perturbation theory is to find normal forms for perturbations of integrable hamiltonian systems. Various versions of this problem appear in the two most prominent fields of perturbation

[1] Research partially supported by the KBN grant 2/1167/91/01.

theory: the KAM theory (*see* for instance [2], [7]) and the Nekhoroshev theory (*see* for instance [3], [8], [13], [17]). Generally speaking, normal form is a canonical form of perturbed hamiltonian which, in some sense, depends weakly on the angle variables (for intrinsically resonant systems one often uses the normal form depending on the actions and on the resonant angles, *see* for instance [8], [18]). When it does not depend on angles at all, it is called the Birkhoff normal form. The perturbed system is usually non-integrable [16], so such a form may exist in a formal sense only. For instance, it can be viewed as a formal power series in the "small parameter" ε, the Birkhoff series, when we deal with a family $h_{(\varepsilon)}(A, \varphi) = h_0(A) + \varepsilon v(A, \varphi)$ of perturbations of h_0, and A, φ are the action-angle variables.

If we consider non-resonant systems, the Birkhoff series can be constructed under some additional assumptions. Technically, the most serious problem is the apperance of the so called "small denominators". One often puts a diophantine conditions on $\omega = \partial_A h_0$, which also makes it possible to obtain various types of estimates. The two algorithms used for the calculation of the terms of the Birkhoff series are the Poincaré – von Zeipel algorithm based on generating functions, and the Lie method (*see* for instance [2], [4], [5], [8], [11]). Both methods lead to the same normal form, as proved in [9].

In the resonant case these algorithms generally break down because "zero denominators" appear in place of "small denominators".

Intrinsically resonant systems are systems for which all actions are resonant. They are of great interest in classical hamiltonian mechanics and have been intensively studied. One of the problems considered is the construction of approximate prime integrals of motion for perturbations of such systems (*see* for instance [6], [10], [12], [15]). There are also some non-perturbative results (for instance [13], [14]).

In this paper we study a possibility of such a modification of the Lie method (in Deprit version; *see* [11]) which would allow us to construct the Birkhoff series for perturbations of resonant harmonic oscillator, the simplest but highly significant example of intrinsically resonant system. The way to deal with the "zero denominators" suggested here consists of performing a preliminary canonical transformation before actually calculating the normal form, so as to obtain "zero numerators" for all "zero denominators". We establish some general conditions guaranteing the existence of all the terms of the Birkhoff series for any number of degrees of freedom. These conditions are somewhat abstract, but we exhibit examples when they are satisfied (even globally and for any dimension). However in the two-dimensional case we can present much more precise geometric conditions. These conditions state that, essentially, a certain auxiliary one-dimensional hamiltonian system has its phase space (or a part of it) smoothly foliated by periodic orbits. They may be verified by examining the level sets of the auxiliary hamiltonian. The main point of the paper is that while the Birkhoff normal form, when and where it exists, may be very difficult to calculate, it is rather easy to check where it does exist (for $d = 2$).

All our results are formulated in the framework of smooth functions. We are interested in the formal aspect of the problem only, that is we do not study any bounds on terms of obtained series. Nevertheless, we use here some diophantine conditions which seem to be necessary to obtain existence and smoothness of the terms.

In Section 1, after some heuristic considerations, we formulate and prove Theorem 1.1 on the existence of the formal perturbation theory in the resonant case. This result is applied to the two-dimensional case in Section 2 and illustrate with two concrete examples. We also prove a result concerning the non-unique choice of the preliminary canonical transformation (Theorem 2.3). Some technical lemmas are collected in Appendix.

This paper was motivated by ideas of Jan Herczyński concerning the two-dimensional case, used in Section 2. The referee's comments on the previous version of the paper are gratefully acknowledged.

1. FORMAL PERTURBATION THEORY – MODIFICATION OF THE LIE METHOD

We consider the hamiltonian of d-dimensional oscillator, which in the usual p, q variables in $\mathbb{R}^{2d}$ has the form

$$\frac{1}{2} \sum_{j=1}^{d} (p_j^2 + \omega_j^2 q_j^2),$$

where $d \geq 2$, $\omega = (\omega_1 \dots \omega_d) \in \mathbb{R}_+^d$, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. This system is canonically integrable on the subset $\{(p, q) \in \mathbb{R}^{2d} : p_j^2 + q_j^2 \neq 0,\ j = 1 \dots d\}$ of the phase space in the standard action-angle variables $(A, \varphi) \in \mathbb{R}_+^d \times \mathbb{T}^d$ defined by

$$\sqrt{2 A_j} \cos \varphi_j = \sqrt{\omega_j}\, q_j, \qquad \sqrt{2 A_j} \sin \varphi_j = p_j / \sqrt{\omega_j}$$

for $j = 1 \dots d$. $\mathbb{T}$ is here the circle identified with the quotient group $\mathbb{R}/2\pi\mathbb{Z}$. In these variables the oscillator hamiltonian has the form $h_0(A, \varphi) = \omega A$

(here $xy = \sum_{j=1}^{d} x_j\, y_j$ for $x,\, y \in \mathbb{C}^d$). We shall be interested especially in the situation when ω is resonant, *i.e.* when there exists $\nu \in \mathbb{Z}^d$, $\nu \neq 0$, such that $\omega\nu = 0$. Suppose that v is a function on the phase space $\mathbb{R}^d_+ \times \mathbb{T}^d$. We consider a family of the hamiltonians $h_{(\varepsilon)} = h_0 + \varepsilon\, v$ for ε near 0. We shall try to find a family of canonical transformations $u_{(\varepsilon)}$ mapping $U \times \mathbb{T}^d$ onto $W \subset \mathbb{R}^d_+ \times \mathbb{T}^d$ and such that $h_{(\varepsilon)} \circ u_{(\varepsilon)} = k_{(\varepsilon)}$, where U is a certain open subset of $\mathbb{R}^d_+$, $k_{(\varepsilon)}$ are hamiltonians in the normal form on $U \times \mathbb{T}^d$ (that is they depend only on the "new" actions $A \in U$) and $k_{(0)} = h_0$. As is known, such $u_{(\varepsilon)}$ do not usually exist, so our goal will be to make this normalization only in the formal sense explained below. We denote $u_{(0)} = u_0$, but in contrast with the non-resonant case we cannot assume that u_0 is the identity transformation! We require that u_0 be a canonical diffeomorphism of $U \times \mathbb{T}^d$ onto W (and the condition $k_{(0)} = h_0$ implies that h_0 is u_0-invariant). When we write $u_{(\varepsilon)}$ in the form $u_{(\varepsilon)} = u_0 \circ \tilde{u}_{(\varepsilon)}$, with $\tilde{u}_{(\varepsilon)}$ being a canonical diffeomorphism of $U \times \mathbb{T}^d$ onto itself then $\tilde{u}_{(0)}$ will be the identity transformation. Thus, analogously as in the usual Lie method, we can assume that the family of canonical transformations $\tilde{u}_{(\varepsilon)}$ is a hamiltonian flow generated by some auxilliary time-dependent hamiltonian $\tilde{w}_{(\varepsilon)}$ and ε plays the role of time. Let $t_{(\varepsilon)}$, $\tilde{t}_{(\varepsilon)}$ be the operators acting on functions defined on W and on $U \times \mathbb{T}^d$, respectively, given by

$$\tilde{t}_{(\varepsilon)}\, f = f \circ \tilde{u}_{(\varepsilon)},$$
$$t_{(\varepsilon)}\, f = f \circ u_{(\varepsilon)} = f \circ u_0 \circ \tilde{u}_{(\varepsilon)} = \tilde{t}_{(\varepsilon)}\, t_0\, f,$$

where $t_0\, f = t_{(0)}\, f = f \circ u_0$. We then have

$$\frac{d}{d\varepsilon}\,(\tilde{t}_{(\varepsilon)}\, f) = \tilde{t}_{(\varepsilon)}\, \{ f,\, \tilde{w}_{(\varepsilon)} \}, \qquad \tilde{t}_{(0)}\, f = f,$$

where the Poisson bracket $\{ f,\, g \} = \sum_{j=1}^{d} \left(\frac{\partial f}{\partial \varphi_j} \frac{\partial g}{\partial A_j} - \frac{\partial g}{\partial \varphi_j} \frac{\partial f}{\partial A_j} \right)$ for differentiable functions f, g defined on an open subset of $\mathbb{R}^d_+ \times \mathbb{T}^d$. Additionally we let $w_{(\varepsilon)}$ be such that $\tilde{w}_{(\varepsilon)} = \frac{d}{d\varepsilon}\, w_{(\varepsilon)}$. If we assume now that $\tilde{t}_{(\varepsilon)}$, $t_{(\varepsilon)}$, $\tilde{w}_{(\varepsilon)}$, $k_{(\varepsilon)}$ have formal power series expansions in ε

$$\tilde{t}_{(\varepsilon)} = \sum_{n=0}^{\infty} \varepsilon^n\, \tilde{t}_n, \qquad t_{(\varepsilon)} = \sum_{n=0}^{\infty} \varepsilon^n\, t_n,$$

$$\tilde{w}_{(\varepsilon)} = \sum_{n=0}^{\infty} \varepsilon^n\, (n+1)\, w_{n+1}, \qquad k_{(\varepsilon)} = \sum_{n=0}^{\infty} \varepsilon^n\, k_n$$

(the last of which is the Birkhoff series), then $t_n = \tilde{t}_n\, t_0$, $k_0 = h_0$ and $\tilde{t}_n$ will satisfy the recurent equations

$$\tilde{t}_0\, f = f, \qquad \tilde{t}_n\, f = \frac{1}{n} \sum_{j=0}^{n-1} (n-j)\ \tilde{t}_j \{ f, w_{n-j} \} \tag{1}$$

for $n \geq 1$. The condition $t_{(\varepsilon)}\, h_{(\varepsilon)} = k_{(\varepsilon)}$ can be expressed in terms of t_n (and thus, indirectly, in terms of t_0 and w_n) as

$$t_0\, h_0 = h_0$$

and

$$t_n\, h_0 + t_{n-1}\, v = k_n,$$

for $n \geq 1$, which is called the homological equation of the perturbation theory. If we require that k_n depend on actions only then the above are the normalization conditions in the sense of formal series. It will be our main goal to satisfy them in some regions of the phase space for some perturbations v.

Assume now that W is an open subset of $\mathbf{R}_+^d \times \mathbf{T}^d$ and that v is a smooth (which we always take to mean C^∞) real-valued function on $\mathbf{R}_+^d \times \mathbf{T}^d$. For any open $U \subset \mathbf{R}_+^d$ we denote $W_U = U \times \mathbf{T}^d$. On $\mathbf{R}_+^d \times \mathbf{T}^d$ we consider the symplectic form $\sum_{j=1}^{d} dA_j \wedge d\varphi_j$. The preceding informal considerations lead us to the following definition.

DEFINITION 1.1. – *The formal perturbation theory for $h_0 + \varepsilon\, v$ is well-defined on W* if and only if there exist an open subset U of $\mathbf{R}_+^d$, a smooth canonical diffeomorphism u_0 of W_U onto W and the functions $k_n,\ w_n \in C^\infty(W_U)$, $n = 1, \ldots$, such that

(i) k_n depend only on actions $A \in U$;

(ii) $t_0 (h_0|_W) = h_0|_{W_U}$;

(iii) $t_n (h_0|_W) + t_{n-1} (v|_W) = k_n$

for $n = 1, \ldots$, where $t_n : C^\infty(W) \to C^\infty(W_U)$ are given by

$$t_0\, f = f \circ u_0, \quad t_n = \tilde{t}_n\, t_0 \tag{2}$$

and $\tilde{t}_n : C^\infty(W_U) \to C^\infty(W_U)$ are the operators obtained recursively by (1) from the functions w_n.

If these conditions are satisfied then u_0, $\{w_n\}_{n=1}^\infty$, $\{k_n\}_{n=1}^\infty$ will be called *the coefficients of perturbation theory for $h_0 + \varepsilon\, v$ on W*.

We use the following notation. Let $\mathbb{Z}_\omega = \{\nu \in \mathbb{Z}^d : \nu\omega = 0\}$; $|\nu| = \sum_{j=1}^{d} |\nu_j|$ for $\nu \in \mathbb{Z}^d$. For an open $U \subset \mathbb{R}_+^d$, $f \in C^\infty(W_U)$ and $\nu \in \mathbb{Z}^d$ let f_ν denote the ν-th Fourier coefficient in $\varphi \in \mathbb{T}^d$ of f. As follows from the Lemma in the Appendix, $f_\nu \in C^\infty(U)$. Let

$$(P_r f)(A, \varphi) = \sum_{\nu \in \mathbb{Z}_\omega} f_\nu(A) \exp(i\nu\varphi),$$

$$(P_{nr} f)(A, \varphi) = \sum_{\nu \in \mathbb{Z}^d \setminus \mathbb{Z}_\omega} f_\nu(A) \exp(i\nu\varphi),$$

$$(P_0 f)(A, \varphi) = f_0(A)$$

for $(A, \varphi) \in W_U$. Thus P_r, P_{nr}, P_0 are respectively operators of taking the *resonant part*, the *non-resonant part* and the *average over the angles* of the function. Note that they are linear operators from $C^\infty(W_U)$ into itself, as seen from the Lemma in the Appendix. Moreover, they are commuting projections such that $P_r + P_{nr} = I$ and $\operatorname{Ran} P_0 \subset \operatorname{Ran} P_r$. For an open $W \subset \mathbb{R}_+^d \times \mathbb{T}^d$ we define the restriction operator $\pi_W : C^\infty(\mathbb{R}_+^d \times \mathbb{T}^d) \to C^\infty(W)$

$$\pi_W f = f_{|W}.$$

We shall use also the following notation: if $k \in C^\infty(W_U)$ and $P_0 k = k$, then by $\boldsymbol{k}$ we denote the function from $C^\infty(U)$ such that $k(A, \varphi) = \boldsymbol{k}(A)$ for $(A, \varphi) \in W_U$.

The conditions (i), (ii), (iii) of Definition 1.1 can be rewritten now in the form

$$P_0 k_n = P_r k_n = k_n, \tag{3}$$

$$t_0 \pi_W h_0 = \pi_{W_U} h_0, \tag{4}$$

$$\tilde{t}_n t_0 \pi_W h_0 + \tilde{t}_{n-1} t_0 \pi_W v = k_n \tag{5}$$

for $n \geq 1$, where t_0 is given by (2). The condition (4) stating that h_0 is t_0-invariant is automatically satisfied in the non-resonant case by the trivial choice $t_0 = I$. As we shall see, this choice is usually wrong in the resonant case. The choice of appropriate u_0 (and thus also of t_0) is in fact the most important problem in the resonant perturbation theory and depends essentially on the perturbation v.

For our purposes it is crucial that (4) implies that P_r commutes with t_0.

LEMMA 1.1. – *Let U, W be open subsets of $\mathbb{R}_+^d$ and $\mathbb{R}_+^d \times \mathbb{T}^d$, respectively. If $u_0 : W_U \to W$ is a canonical C^∞-diffeomorphism satisfying* (4) *with*

t_0 given by (2), *then*

$$P_r\, t_0\, \pi_W = t_0\, \pi_W\, P_r$$

and

$$P_{nr}\, t_0\, \pi_W = t_0\, \pi_W\, P_{nr},$$

where the above are equalities of operators from $C^\infty(\mathbb{R}^d_+ \times \mathbb{T}^d)$ *to* $C^\infty(W_U)$.

We prove the above lemma in the Appendix. In our calculations we will use the following simple relations between the introduced operators and the Poisson bracket.

LEMMA 1.2. – *Let* U *be an open subset of* $\mathbb{R}^d_+$ *and* $f,\, f_1,\, f_2 \in C^\infty(W_U)$. *Then*

$$\begin{aligned}
P_r\,\{P_r\, f_1,\, P_{nr}\, f_2\} &= 0,\\
P_r\,\{P_r\, f_1,\, P_r\, f_2\} &= \{P_r\, f_1,\, P_r\, f_2\},\\
P_r\,\{f_1,\, f_2\} &= \{P_r\, f_1,\, P_r\, f_2\} + P_r\,\{P_{nr}\, f_1,\, P_{nr}\, f_2\},\\
P_{nr}\,\{\pi_{W_U}\, h_0,\, f\} &= P_{nr}\,\{\pi_{W_U}\, h_0,\, P_{nr}\, f\} = \{\pi_{W_U}\, h_0,\, P_{nr}\, f\},\\
P_r\,\{\pi_{W_U}\, h_0,\, f\} &= 0,\\
P_0\,\{P_0\, f_1,\, f_2\} &= 0.
\end{aligned}$$

We now look for the solutions of the homological equation (5) assuming that (4) holds for W with a certain U and u_0. We will do this recursively. To write down this recursion we employ (1) to have (5) in the form

$$\{\pi_{W_U}\, h_0,\, w_n\} + v_n = k_n, \tag{6}$$

where

$$v_n = \tilde{t}_{n-1}\, t_0\, \pi_W\, v + \sum_{j=1}^{n-1} \frac{n-j}{n}\, \tilde{t}_j\, \{\pi_{W_U}\, h_0,\, w_{n-j}\} \tag{7}$$

for $n > 1$, and $v_1 = t_0\, \pi_W\, v$. Thus v_n depends only on v, t_0 and on the functions w_j for $j = 1, \ldots n-1$. Observe that if (3) holds then from Lemma 1.2 it follows that (6) is equivalent to the two conditions:

$$P_r\, v_n = P_0\, v_n = k_n \tag{8}$$

and

$$\{\pi_{W_U}\, h_0,\, P_{nr}\, w_n\} = -P_{nr}\, v_n. \tag{9}$$

Thus if (6) has solutions then v_n must satisfy

$$P_r\, v_n = P_0\, v_n. \tag{10}$$

Note that the equation (6) determines only k_n and the non-resonant part of w_n and they are given by

$$(P_{nr}\, w_n)(A,\, \varphi) = \sum_{\nu \in \mathbb{Z}^d \setminus \mathbb{Z}_\omega} \frac{1}{i\, \omega\nu} (v_n)_\nu\, (A)\, \exp(i\, \nu\varphi), \tag{11}$$

$$k_n = P_0\, v_n. \tag{12}$$

It turns out, however, that the choice of the resonant part is tightly connected with the fulfilment of the necessary condition (10) on the next, $n+1$'th, level. To convince oneself of that let us rewrite more carefuly the formula (7). By (1) we have

$$\begin{aligned} v_{n+1} &= \tilde{t}_n\, t_0\, \pi_W\, v + \sum_{j=1}^{n} \frac{n+1-j}{n+1} \tilde{t}_j \{\pi_{W_U}\, h_0,\, w_{n+1-j}\} \\ &= \tilde{t}_n \left(t_0\, \pi_W\, v + \frac{1}{n+1} \{\pi_{W_U}\, h_0,\, w_1\} \right) \\ &\quad + \frac{n}{n+1} \tilde{t}_1 \{\pi_{W_U}\, h_0,\, w_n\} \\ &\quad + \sum_{j=2}^{n-1} \frac{n+1-j}{n+1} \tilde{t}_j \{\pi_{W_U}\, h_0,\, w_{n+1-j}\} \\ &= \sum_{j=0}^{n-1} \frac{n-j}{n} \tilde{t}_j \{ t_0\, \pi_W\, v + \frac{1}{n+1} \{\pi_{W_U}\, h_0,\, w_1\},\, w_{n-j}\} \\ &\quad + \frac{n}{n+1} \tilde{t}_1 \{\pi_{W_U}\, h_0,\, w_n\} \\ &\quad + \sum_{j=2}^{n-1} \frac{n+1-j}{n+1} \tilde{t}_j \{\pi_{W_U}\, h_0,\, w_{n+1-j}\} \\ &= \{t_0\, \pi_W\, v,\, w_n\} + \frac{1}{n+1} \{\{\pi_{W_U}\, h_0,\, w_1\},\, w_n\} \\ &\quad + \frac{n}{n+1} \{\{\pi_{\dot{W}_U}\, h_0,\, w_n\},\, w_1\} + \tilde{v}_n \end{aligned} \tag{13}$$

for $n > 1$ and

$$v_2 = \{t_0\, \pi_W\, v,\, w_1\} + \frac{1}{2} \{\{\pi_{W_U}\, h_0,\, w_1\},\, w_1\}, \tag{13'}$$

where

$$\tilde{v}_n = \sum_{j=1}^{n-1} \frac{n-j}{n} \tilde{t}_j \{ t_0 \, \pi_W \, v + \frac{1}{n+1} \{ \pi_{W_U} \, h_0, \, w_1 \}, \, w_{n-j} \} + \sum_{j=2}^{n-1} \frac{n+1-j}{n+1} \tilde{t}_j \{ \pi_{W_U} \, h_0, \, w_{n+1-j} \}, \tag{14}$$

(we use here and below the convention that $\sum_{j=k}^{l} x_j = 0$ if $l < k$). By Lemma 1.2 we obtain now

$$P_r \, v_{n+1} = \{ P_r \, t_0 \, \pi_W \, v, \, P_r \, w_n \} + \breve{v}_n, \tag{15}$$

with $\breve{v}_n$ given by

$$\begin{aligned} \breve{v}_n = P_r \, \tilde{v}_n + P_r \, \{ P_{nr} \, t_0 \, \pi_W \, v, \, P_{nr} \, w_n \} \\ + \frac{1}{n+1} P_r \, \{\{ \pi_{W_U} \, h_0, \, P_{nr} \, w_1 \}, \, P_{nr} \, w_n \} \\ + \frac{n}{n+1} P_r \, \{\{ \pi_{W_U} \, h_0, \, P_{nr} \, w_n \}, \, P_{nr} \, w_1 \} \end{aligned} \tag{16}$$

for $n > 1$ and

$$\begin{aligned} \breve{v}_1 = P_r \, \{ P_{nr} \, t_0 \, \pi_W \, v, \, P_{nr} \, w_1 \} \\ + \frac{1}{2} P_r \, \{\{ \pi_{W_U} \, h_0, \, P_{nr} \, w_1 \}, \, P_{nr} \, w_1 \}. \end{aligned} \tag{16'}$$

This form of $P_r \, v_{n+1}$ is convenient for our purpose, because $\breve{v}_n$ depends only on t_0, the functions w_j for $j = 1, \dots n-1$ and $P_{nr} \, w_n$, yet it doesn't depend on $P_r \, w_n$! Moreover

$$P_r \, \breve{v}_n = \breve{v}_n. \tag{17}$$

Let us now have a closer look at the term $P_r \, t_0 \, \pi_W \, v$ in (15). If (6) holds for $n = 1$ then by (8) and Lemma 1.1 we have $P_r \, t_0 \, \pi_W \, v = t_0 \, \pi_W \, P_r \, v = k_1$. This means in particular that the resonant part of v depends only on actions in coordinates given by u_0. Thus we introduce the following definition.

DEFINITION 1.2. – Let $\omega \in \mathbb{R}_+^d$, U, W–open subsets of $\mathbb{R}_+^d$ and $\mathbb{R}_+^d \times \mathbb{T}^d$, respectively. Let $g \in C^\infty (\mathbb{R}_+^d \times \mathbb{T}^d)$ and $k \in C^\infty (W_U)$ be with $P_0 \, k = k$.

Assume that u_0 is a canonical C^∞-diffeomorphism of W_U onto W satisfying (4), with t_0 given by (2). In this situation we say that *g on W is ω-integrable into k on W_U by u_0* if

$$t_0\, \pi_W\, g = k.$$

We say also that *W is a domain of integrability for ω, g* if and only if there exist U, u_0, k as above such that g on W is ω-integrable into k on W_U by u_0.

Roughly speaking, g on W is ω-integrable into k on W_U by u_0, if the change of variables given by u_0 transfers h_0 into h_0 and g into a function k depending only on actions. When it holds for $g = P_r\, v$ and $k = k_1$ then the condition (10) for $n+1$ can be written in the form

$$i\,(P_r\, w_n)_\nu\,(A)\,(\nabla_A\, \boldsymbol{k}_1\,(A))\,\nu = (\breve{v}_n)_\nu\,(A), \quad \nu \in \mathbb{Z}_\omega \backslash \{0\}. \tag{18}$$

So we can try to invert our considerations and use the formula (11), (12) and (18) to define recursively the solutions w_n, k_n of the homological equation (5). Obviously, to carry on the recursion we need some extra assumptions guaranteing that RHS of (11) is a smooth function if $v_n \in C^\infty(W_U)$ and that $P_r\, w_n$ may be obtained as a smooth function from (18) if $\breve{v}_n \in C^\infty(W_U)$. Thus it will be convenient to use the following diophantine conditions.

DEFINITION 1.3. – Let $\omega \in \mathbb{R}^d_+$, U –open subset of $\mathbb{R}^d$ and let $f \in C^\infty(U)$. We say that ω *is resonant-diophantine* if there exist $C, \gamma > 0$, such that for $\nu \in \mathbb{Z}^d \backslash \mathbb{Z}_\omega$

$$|\,\nu\omega\,| \geq C\,|\,\nu\,|^{-\gamma}.$$

We say that f is *ω-diophantine on U* if for any compact $K \subset U$ there exist C_K, $\gamma_K > 0$ such that

$$\inf_{A \in K}\; |(\nabla_A\, f\,(A))\,\nu\,| \geq C_K\,|\,\nu\,|^{-\gamma_K}$$

for any $\nu \in \mathbb{Z}_\omega \backslash \{0\}$.

Note that when ω is non-resonant (that is when $\mathbb{Z}_\omega = \{0\}$), then each $f \in C^\infty(U)$ is ω-diophantine, moreover in this case ω is resonant-diophantine if and only if it is diophantine.

We are ready now to formulate the main theorem.

THEOREM 1.1. – *Assume that ω is resonant-diophantine. Let $v \in C^\infty(\mathbb{R}^d_+ \times \mathbb{T}^d)$ be such that $P_r\, v$ on W is ω-integrable into k_1 on W_U by u_0 with $\boldsymbol{k}_1$ being ω-diophantine on U. Then the formal perturbation* *theory for* $h_0 + \varepsilon\, v$ *is well defined on* W *and as its coefficients one may take* u_0, $\{w_n\}_{n=1}^{\infty}$, $\{k_n\}_{n=1}^{\infty}$, *where* k_n *is given by* (12) *for* $n \geq 2$ *and* w_n *for* $n \geq 1$ *is recurently defined:* $P_{nr}\, w_n$ *by* (11) *and*

$$P_r\, w_n\,(A,\,\varphi) = \sum_{\nu \in \mathbb{Z}_\omega \setminus \{0\}} \frac{1}{i\,(\nabla_A\, \boldsymbol{k}_1\,(A))\,\nu}\,(\breve{v}_n)_\nu\,(A)\,\exp\,(i\,\nu\varphi) + (w_n)_0\,(A), \tag{19}$$

with $\breve{v}_n$ *given by* (16) *and* (16′), *and with* $(w_n)_0$ *an arbitrary function from* $C^\infty\,(U)$.

Proof. – We shall inductively prove that k_n, w_n are smooth (and thus that the recurent definition is correct). Obviously if $w_j \in C^\infty\,(W_U)$ for $j \leq n-1$ then $v_n \in C^\infty\,(W_U)$. Using the resonant diophantine condition on ω and the results described in the first part of the Appendix we can easily check that $P_{nr}\, w_n$, $k_n \in C^\infty\,(W_U)$. Thus by (16), (16′) $\breve{v}_n$ is smooth. Using once again the Lemma in the Appendix and the ω-diophantine condition on $\boldsymbol{k}_1$ we obtain that $P_r\, w_n$ and so w_n is smooth too. The fact that u_0, $\{w_n\}_{n=1}^{\infty}$, $\{k_n\}_{n=1}^{\infty}$ are coefficients of perturbation theory for $h_0 + \varepsilon\, v$ immediatelly follows from our previous calculations. □

Remarks. – (i) We use two diophantine conditions above. Though we did not prove any estimate of coefficients of obtained formal series, these conditions cannot be replaced by only algebraic nondegeneracy conditions (i.e. conditions of the type "$\alpha\nu \neq 0$" instead of "$|\alpha\nu| \geq C\,|\nu|^{-\gamma}$"). Without the diophantine conditions, smoothness of the coefficients would be in general impossible to obtain (*see* Appendix). However, the smoothness seems to be the weakest reasonable requirement. Adding some extra assumptions like analyticity of v and u_0 and choosing the appropriate norms (*see* for instance [8]) we can obtain some estimates of the coefficients from the described formulas. Yet, because the choice of the norms highly depends on applications planned, we shall not deal with such estimates.

(ii) The two diophantine conditions are very simple in the case $d = 2$. Each resonant $\omega \in \mathbb{R}_+^2$ is resonant-diophantine and, moreover, if g on W is ω-integrable into k_1 on $U \times \mathbb{T}^2$ by u_0, then $\boldsymbol{k}_1$ is ω-diophantine on U if and only if

$$(\nabla_{A,\,\varphi}\, g)\,(A,\,\varphi) \notin \operatorname{lin}\,\{(\omega,\,0)\} \tag{20}$$

for $(A,\,\varphi) \in W$.

In the general multidimensional case it is easy to see that ω is resonant-diophantine when for instance all ω_j are rational. Note also that if ω is such that $\{ \nu \in \mathbb{Z}_\omega : \nu_1 \dots \nu_d \text{ are relatively prime} \}$ is a finite set, then generically $\boldsymbol{k}_1$ is ω-diophantine on U for some "large" open U. (For example $\omega = (\sqrt{2}, 1, 1)$ satisfy the above condition on ω and it is resonant-diophantine, too). There is also a class of linear functions ω-diophantine on the whole $\mathbb{R}^d_+$ (*see* Example 1.1).

(iii) Note that (12) holds also for $n = 1$.

(iv) The coefficients of perturbation theory for $h_0 + \varepsilon v$ are not uniquely determined. The choice of u_0 is not unique, and after choosing u_0 there is still a freedom of the choice of $\{ P_0 w_n \}_{n=1}^\infty$. It should be interesting to see how the choice of u_0 is connected with $\{ k_n \}_{n=1}^\infty$. We shall take up the first problem partially in Section 2. An important open question is whether $\{ k_n \}_{n=1}^\infty$ depends on the second choice.

(v) If $W \subset \mathbb{R}^d_+ \times \mathbb{T}^d$ is a domain of integrability for ω, $P_r v$ then there exists a canonical transformation u_0 of W onto W_U which defines action-angle variables for the simplified perturbed hamiltonians $\tilde{h}_{(\varepsilon)} = h_0 + \varepsilon P_r v$, $\varepsilon \in \mathbb{R}$. In a certain sense the normal form for $h_{(\varepsilon)}$ is constructed in two steps: first we find the exact canonical action-angle variables for $\tilde{h}_{(\varepsilon)}$, then we can treat $P_{nr} v$ as a kind of non-resonant perturbation of $\tilde{h}_{(\varepsilon)}$ and find the normal form of $h_{(\varepsilon)} = \tilde{h}_{(\varepsilon)} + \varepsilon P_{nr} v$. Theorem 1.1 states that if the first step is possible with ω-diophantine $\boldsymbol{k}_1$, then the second step also is possible to all orders in ε.

(vi) For the non-resonant harmonic oscillator with diophantine ω we may put u_0 to be the identity and obtain the C^∞-version of the theorem on the global existence of perturbation theory.

In the following example we apply Theorem 1.1 to find the coefficients of perturbation theory for a class of perturbations of some resonant harmonic oscillators.

Example 1.1. – Let ω be resonant-diophantine, $s \in \mathbb{N}$, $\nu^{(1)}, \dots \nu^{(s)} \in \mathbb{Z}_\omega$, and let $w \in C^\infty(\mathbb{R}^d_+ \times \mathbb{T}^s)$ satisfy the conditions

(i) first order partial derivatives of w have the continuous extensions over $[0, +\infty)^d \times \mathbb{T}^s$;

(ii) $\lim\limits_{A' \to A} \dfrac{\partial w}{\partial \psi_j}(A', \psi) = 0$ for any $j = 1, \dots s$, $\psi \in \mathbb{T}^s$ and $A \in \mathbb{R}^d$ with $A_j = 0$, $A_k \geq 0$, $k = 1, \dots d$;

(iii) second order partial derivatives of w are bounded on $\mathbb{R}^d_+ \times \mathbb{T}^s$.

Let $w_0 \in C^\infty(\mathbb{R}^d_+ \times \mathbb{T}^d)$ be an auxiliary hamiltonian defined by

$$w_0(A, \varphi) = w(A, \nu^{(1)} \varphi, \dots, \nu^{(s)} \varphi).$$

For a fixed $\bar{t} \in \mathbb{R}$ define $u_0 : \mathbb{R}^d_+ \times \mathbb{T}^d \to \mathbb{R}^d_+ \times \mathbb{T}^d$ as the hamiltonian flow with the hamiltonian w_0 after the time $\bar{t}$. It may be proved that u_0 is a well-defined canonical C^∞-diffeomorphism of $W = \mathbb{R}^d_+ \times \mathbb{T}^d$ (the solutions of the Hamilton equation in $\mathbb{R}^d_+ \times \mathbb{T}^d$ with the hamiltonian w_0 exist for all times t) and $h_0 \circ u_0 = h_0$, that is (4) holds for $U = \mathbb{R}^d_+$ and t_0 given by (2). Let us consider an arbitrary $k_1 \in C^\infty(\mathbb{R}^d_+ \times \mathbb{T}^d)$ such that $P_r\, k_1 = k_1$ with $\boldsymbol{k}_1$ ω-diophantine on $\mathbb{R}^d_+$. Finally, let $v = v^{(1)} + v^{(2)}$, where $v^{(1)} = t_0^{-1}\, k_1 = k_1 \circ u_0^{-1}$ and $v^{(2)}$ is an arbitrary function from $C^\infty(\mathbb{R}^d_+ \times \mathbb{T}^d)$ with $P_r\, v^{(2)} = 0$. Then the assumptions of Theorem 1.1 are fulfilled.

It is easy to find many ω, w, k_1 as above. For instance, for some ω there exist $\omega' \in \mathbb{R}^d$ and C', $\gamma' > 0$ such that $|\omega' \nu| \geq C' |\nu|^{-\gamma'}$ for $\nu \in \mathbb{Z}_\omega \backslash \{0\}$ (e.g., we may take $\omega = (1, 1, 1)$, $\omega' = (1, \sqrt{2}, 0)$ for $d = 3$). Then $\boldsymbol{k}_1$ is ω-diophantine on $\mathbb{R}^d_+$ for $k_1(A, \varphi) = \omega' A$. The function w_0 may be of the form $f(A)\, g(\nu^{(1)} \varphi, \ldots, \nu^{(s)} \varphi)$, where $f \in C^\infty(\mathbb{R}^d_+)$, $g \in C^\infty(\mathbb{T}^s)$, f has bounded derivatives of order 0, 1, 2, derivatives of order 1 continuously extendable over $[0, +\infty)^d$ and $\lim_{A' \to A} f(A') = 0$ for $A \in [0, +\infty)^d \backslash (0, +\infty)^d$.

An important property of Example 1.1 is that the perturbation theory is globally defined, that is $W = \mathbb{R}^d_+ \times \mathbb{T}^d$. The canonical diffeomorphism u_0 is constructed as a hamiltonian flow with some hamiltonian w_0 after a fixed time, similarly as the formal transformation $\tilde{u}_{(\varepsilon)}$ is "generated by the flow" given by the formal hamiltonian $\tilde{w}_{(\varepsilon)}$. A weak point of this example is that v is defined indirectly. Therefore it is rather hard to compute the explicit form of v and u_0 and thus also of the coefficients w_n, k_n of perturbation theory.

2. TWO-DIMENSIONAL CASE

As follows from Remark (ii) after Theorem 1.1 both diophantine conditions are particularly simple when $d = 2$. The condition (20) for $g = P_r\, v$ is independent of the choice of u_0 and of the explicit form of $\boldsymbol{k}_1$, that is it can be verified *a priori* when only ω, v, and W are given. It simply states that the hamiltonian h_0 and the resonant part of v are functionally independent. We obtain

THEOREM 2.1. – *If $\omega \in \mathbb{R}^2_+$ is resonant, $v \in C^\infty(\mathbb{R}^2_+ \times \mathbb{T}^2)$, W is a domain of integrability for ω, $P_r\, v$, and $P_r\, v$ and h_0 are functionally independent on W then the formal perturbation theory for $h_0 + \varepsilon\, v$ is well-defined.*

The assumption most serious and difficult to check here is that W is a domain of integrability for ω, $P_r\, v$. We want now to replace it with a simpler and verifiable condition. To this end observe that by Definition 1.2 the above assumption implies that there exist common action-angle variables for two hamiltonians $-P_r\, v$ and h_0 on W. Moreover, by Lemma 1.2, $P_r\, v$ and h_0 are in involution. Since we assume that $d = 2$, the existence of action-angle variables follows from the Arnold-Liouville theorem (see [1]). However, we cannot simply apply this theorem for three reasons:

(a) the canonical change of variables which we need should preserve the form of h_0,

(b) u_0 should be defined on possibly "large" open sets, not just locally as in [1],

(c) we require that actions be positive.

Fortunately, the simple form of our two prime integrals allows us to follow the construction of Arnold quite faithfully and prove that certain well defined open sets W are in fact domains of integrability for ω, $P_r\, v$, provided that some verifiable geometric conditions hold.

We start with the definition of a class of new canonical coordinates B, ψ in which these conditions may be easier expressed. In these coordinates h_0 is a function of B_2 and $P_r\, v$ is a function of B, ψ_1 only. Thus any domain of integrability for ω, $P_r\, v$ will have the form $W = \mathcal{G} \times \mathbb{T}^1$ in B, ψ coordinates, with a certain $\mathcal{G} \subset \mathbb{R}^2 \times \mathbb{T}^1$.

Assume that $\omega \in \mathbb{R}^2_+$ is resonant and let $\omega^\perp$ be the (uniquely determined) element of $\mathbb{Z}_\omega$ with $\omega_1^\perp > 0$ and $\omega_1^\perp$, $\omega_2^\perp$ relatively prime. Let

$$\mathcal{A}_\omega = \{\, \alpha \in \mathbb{Z}^2 \,:\, \alpha_1\, \omega_2^\perp - \alpha_2\, \omega_1^\perp = +1 \text{ or } -1 \,\}.$$

Obviously $\mathcal{A}_\omega$ is non-empty. For any $\alpha \in \mathcal{A}_\omega$ let N_α be the 2×2 matrix

$$N_\alpha = \begin{pmatrix} \omega^\perp \\ \alpha \end{pmatrix}.$$

In what follows we identify a matrix with the appropriate linear transformation of $\mathbb{R}^d$ or $\mathbb{T}^d$. To define the new coordinates we consider the transformations $\mathcal{C}_\alpha$ and $\mathcal{C}_\alpha^{-1}$ in $\mathbb{R}^2_+ \times \mathbb{T}^2$

$$\begin{aligned}(A,\, \varphi) &= \mathcal{C}_\alpha\,(B,\, \psi) = (N_\alpha^T\, B,\; N_\alpha^{-1}\, \psi),\\ (B,\, \psi) &= \mathcal{C}_\alpha^{-1}\,(A,\, \varphi) = ((N_\alpha^{-1})^T\, A,\; N_\alpha\, \varphi)\end{aligned}$$

and denote $\mathbb{R}^2_{\omega,\alpha} = (N_\alpha^{-1})^T\, \mathbb{R}^2_+$. One can show that

$$\mathbb{R}^2_{\omega,\,\alpha} = \{\, (B_1,\, B_2) \in \mathbb{R}^2 \,:\, \alpha\omega\, B_2 > 0,\; c_\alpha^-\, B_2 < B_1 < c_\alpha^+\, B_2 \,\},$$

with

$$c_\alpha^- = -\frac{\alpha_1}{\omega_1^\perp}, \qquad c_\alpha^+ = -\frac{\alpha_2}{\omega_2^\perp} \tag{21}$$

and that $\mathcal{C}_\alpha^{-1}$ is a canonical diffeomorphism of $\mathbb{R}_+^2 \times \mathbb{T}^2$ onto $\mathbb{R}_{\omega,\alpha}^2 \times \mathbb{T}^2$. Moreover

$$(h_0 \circ \mathcal{C}_\alpha)(B, \psi) = \alpha\omega B_2 \tag{22}$$

and $(P_r f) \circ \mathcal{C}_\alpha$ does not depend on ψ_2 for any $f \in C^\infty(\mathbb{R}_+^2 \times \mathbb{T}^2)$. Thus, for f with $P_r f = f$ we introduce the notation:

$$(f \circ \mathcal{C}_\alpha)(B, \psi) = f^{[\alpha]}(B, \psi_1)$$

for $B \in \mathbb{R}_{\omega,\alpha}^2$ and $\psi \in \mathbb{T}^2$.

Remark. – In the context of perturbation theory (B_1, ψ_1) are called "slow" variables and (B_2, ψ_2) "fast" variables (independently of the choice of $\alpha \in \mathcal{A}_\omega$).

We can now formulate "geometric conditions" on $P_r v$ mentioned above in terms of B, ψ variables. Suppose that $\mathcal{G}$ is an open, connected subset of $\mathbb{R}_{\omega,\alpha}^2 \times \mathbb{T}^1$ for some $\alpha \in \mathcal{A}_\omega$ and that $f \in C^\infty(\mathcal{G})$. Denote

$$\mathcal{G}_{B_2} = \{ (B_1, \psi_1) \in \mathbb{R} \times \mathbb{T}^1 : (B_1, B_2, \psi_1) \in \mathcal{G} \}$$

and let $f_{B_2} \in C^\infty(\mathcal{G}_{B_2})$,

$$f_{B_2}(B_1, \psi_1) = f(B_1, B_2, \psi_1).$$

Since $\mathcal{G}$ is open and connected we can consider the interval

$$(a, b) = \{ B_2 \in \mathbb{R} : \mathcal{G}_{B_2} \neq \emptyset \}.$$

Denote also

$$\tilde{a}_{B_2} = \inf_{(B_1, \psi_1) \in \mathcal{G}_{B_2}} f(B_1, B_2, \psi_1), \qquad \tilde{b}_{B_2} = \sup_{(B_1, \psi_1) \in \mathcal{G}_{B_2}} f(B_1, B_2, \psi_1),$$

and

$$\tilde{\Gamma} = \{ (x, B_2) \in \mathbb{R}^2 : a < B_2 < b,\ \tilde{a}_{B_2} < x < \tilde{b}_{B_2} \}.$$

Obviously, $\mathcal{G}_{B_2}$ and $\tilde{\Gamma}$ are open subsets of $\mathbb{R} \times \mathbb{T}^1$ and $\mathbb{R}^2$, respectively.

DEFINITION 2.1. – *$\mathcal{G}$ is smoothly foliated into circles by f with $B_2 =$ const if and only if there exists a C^∞-function $\tilde{\eta} : \mathcal{G} \to \mathbb{T}^1$ such that for any $B_2 \in (a, b)$ the map*

$$\tilde{\Phi}_{B_2}(B_1, \psi_1) = (f(B_1, B_2, \psi_1),\ \tilde{\eta}(B_1, B_2, \psi_1))$$

is a diffeomorphism of $\mathcal{G}_{B_2}$ onto $(\tilde{a}_{B_2}, \tilde{b}_{B_2}) \times \mathbb{T}^1$.

In practice, it will be convenient to use the following lemma.

LEMMA 2.1. – *$\mathcal{G}$ is smoothly foliated into circles by f with B_2 =* const *if and only if the following three conditions are satisfied:*

(i) *for any $B_2 \in (a, b)$ the set $\mathcal{G}_{B_2}$ is connected and all level sets of f_{B_2} are homeomorphic to the circle*;

(ii) $(\nabla_{B_1, \psi_1} f)(B, \psi_1) \neq (0, 0)$ *for* $(B, \psi_1) \in \mathcal{G}$;

(iii) *there exists a C^∞-function $\chi : \tilde{\Gamma} \to \mathbb{R} \times \mathbb{T}^1$ such that $\chi(x, B_2) \in f_{B_2}^{-1}(\{x\})$ for $(x, B_2) \in \tilde{\Gamma}$.*

Proof. – If $\mathcal{G}$ is smoothly foliated into circles by f with B_2 = const, we can define $\chi(x, B_2) = \tilde{\Phi}_{B_2}^{-1}(x, 0)$ and then (i), (ii), and (iii) obviously hold. Thus, suppose that we have (i), (ii), and (iii). From (i) and (ii) the hamiltonian flow $S_{B_2, t}$ in $\mathcal{G}_{B_2}$ generated by f_{B_2} is defined for all times t and is a C^∞-function of $(B_1, B_2, \psi_1, t) \in \mathcal{G} \times \mathbb{R}$. Let $\tilde{t}_{B_2}(B_1, \psi_1)$ be the time needed for the initial data $\chi(f_{B_2}(B_1, \psi_1), B_2)$ to reach the data (B_1, ψ_1), which is defined up to the period $T_{B_2}(f_{B_2}(B_1, \psi_1))$. Using the implicit function theorem we obtain that

$$\tilde{\eta}(B_1, B_2, \psi_1) = \tilde{t}_{B_2}(B_1, \psi_1) \frac{2\pi}{T_{B_2}(f_{B_2}(B_1, \psi_1))}$$

is a C^∞-function with values in $\mathbb{T}^1$ satisfying the conditions of Definition 2.1. □

We may start now the main part of this section and prove the existence of domains of integrability for ω, $P_r\, v$, for open sets satisfying the "geometric conditions". We assume that an open, connected $\mathcal{G} \subset \mathbb{R}^2_{\omega, \alpha} \times \mathbb{T}^1$ is smoothly foliated into circles by $f = (P_r\, v)^{[\alpha]}_{|\mathcal{G}}$ with B_2 = const. We shall first carefully follow the construction of a canonical transformation integrating the system given by f_{B_2} on $\mathcal{G}_{B_2}$ for a fixed B_2 (see [1]). Then we shall "canonically extend" the constructed transformation to all the coordinates B_1, B_2, ψ_1, ψ_2 and after some simple modifications we shall reach our main goal, that is we shall obtain u_0, k_1, and U such that $P_r v$ on $W = \mathcal{C}_\alpha(\mathcal{G} \times \mathbb{T}^1)$ will be ω-integrable into k_1 on $U \times \mathbb{T}^2$ by u_0. Analogously to the proof of Lemma 2.1 we consider the smooth hamiltonian flow $S_{B_2, t}$ in $\mathcal{G}_{B_2}$ generated by f_{B_2}. For any $x \in (\tilde{a}_{B_2}, \tilde{b}_{B_2})$

$$M_{B_2, x} = f_{B_2}^{-1}(\{x\})$$

is a one-dimensional invariant torus for $S_{B_2, t}$. We choose the orientation on $M_{B_2, x}$ consistently with the direction of the flow $S_{B_2, t}$. Since $\tilde{\Gamma}$ is connected and $M_{B_2, x}$ is homeomorphic to $\mathbb{T}^1$, two different cases are possible:

libration: for any $(x, B_2) \in \tilde{\Gamma}$

$$\frac{1}{2\pi} \int_{M_{B_2, x}} d\psi_1 = 0, \tag{23a}$$

rotation: for any $(x, B_2) \in \tilde{\Gamma}$

$$\frac{1}{2\pi} \int_{M_{B_2, x}} d\psi_1 = \pm 1, \tag{23b}$$

where the above integrals means the integrals of the differencial form $d\psi_1$ over $M_{B_2, x}$. In the libration case "the circle" $M_{B_2, x}$ does not round the cylinder $(c_\alpha^- B_2, c_\alpha^+ B_2) \times \mathbb{T}^1$ and in the rotation case it rounds this cylinder once in the direction depending on the RHS of (23*b*). Note that in the libration case

$$\Pi(x, B_2) = \left| \int_{M_{B_2, x}} B_1 \, d\psi_1 \right| > 0$$

for any $(x, B_2) \in \tilde{\Gamma}$ because by Stoke's theorem $\Pi(x, B_2)$ is the area of the set bounded by $M_{B_2, x}$. Moreover, Π is a continuous function on $\tilde{\Gamma}$. Thus we can define

$$\sigma = \operatorname{sign}\left(\int_{M_{B_2, x}} B_1 \, d\psi_1 \right) \tag{24a}$$

in the libration case and

$$\sigma = \operatorname{sign}\left(\int_{M_{B_2, x}} d\psi_1 \right) \tag{24b}$$

in the rotation case, and in both cases σ is independent of $(x, B_2) \in \tilde{\Gamma}$. We also define

$$J_{B_2}(x) = \frac{\sigma}{2\pi} \int_{M_{B_2, x}} B_1 \, d\psi_1. \tag{25}$$

J is a C^∞-function of x, B_2 and

$$\frac{\partial J_{B_2}}{\partial x}(x) = \frac{1}{2\pi} T_{B_2}(x), \tag{26}$$

where $T_{B_2}(x)$ is the period of one round of $S_{B_2, t}$ on $M_{B_2, x}$. Let $t_{B_2}(B_1, \psi_1)$ be the time needed for the initial data $\tilde{\Phi}_{B_2}^{-1}(f_{B_2}(B_1, \psi_1), 0)$ to reach the data (B_1, ψ_1) which is defined up to $T_{B_2}(f_{B_2}(B_1, \psi_1))$. By the implicit function theorem η given by

$$\eta(B_1, B_2, \psi_1) = t_{B_2}(B_1, \psi_1) \frac{2\pi}{T_{B_2}(f_{B_2}(B_1, \psi_1))}$$

is a smooth $\mathbb{T}^1$-valued function. Consider now the interval

$$(a_{B_2}, b_{B_2}) = J_{B_2}((\tilde{a}_{B_2}, \tilde{b}_{B_2}))$$

(by (26) J_{B_2} is monotone) and the transformation $\Phi_{B_2} : \mathcal{G}_{B_2} \to (a_{B_2}, b_{B_2}) \times \mathbb{T}^1$,

$$\Phi_{B_2}(B_1, \psi_1) = (\boldsymbol{I}(B_1, B_2, \psi_1), \boldsymbol{\eta}(B_1, B_2, \psi_1)),$$

where

$$\boldsymbol{I}(B_1, B_2, \psi_1) = J_{B_2}(f_{B_2}(B_1, \psi_1)).$$

As seen from the construction, Φ_{B_2} is a canonical diffeomorphism onto $(a_{B_2}, b_{B_2}) \times \mathbb{T}^1$ smoothly depending on B_2 (see [1]). Moreover

$$\Gamma = \{ (I, B_2) \in \mathbb{R}^2 : a < B_2 < b, \; a_{B_2} < I < b_{B_2} \}$$

is an open, connected subset of $\mathbb{R}^2$, because it is diffeomorphic to $\tilde{\Gamma}$. Define now the functions $\boldsymbol{B}_1 : \Gamma \times \mathbb{T}^1 \to \mathbb{R}$, $\boldsymbol{\psi}_1 : \Gamma \times \mathbb{T}^1 \to \mathbb{T}^1$ by

$$(\boldsymbol{B}_1(I, B_2, \eta), \boldsymbol{\psi}_1(I, B_2, \eta)) = \Phi_{B_2}^{-1}(I, \eta)$$

for $(I, B_2) \in \Gamma$, $\eta \in \mathbb{T}^1$. From the canonicity of $\Phi_{B_2}^{-1}$ for any $B_2 \in (a, b)$ we have

$$\frac{\partial \boldsymbol{B}_1}{\partial I} \frac{\partial \boldsymbol{\psi}_1}{\partial \eta} - \frac{\partial \boldsymbol{B}_1}{\partial \eta} \frac{\partial \boldsymbol{\psi}_1}{\partial I} = 1. \tag{27}$$

Consider now the differential form ξ_{B_2} on $(a_{B_2}, b_{B_2}) \times \mathbb{T}^1$

$$\xi_{B_2} = \left(\frac{\partial \boldsymbol{B}_1}{\partial I} \frac{\partial \boldsymbol{\psi}_1}{\partial B_2} - \frac{\partial \boldsymbol{B}_1}{\partial B_2} \frac{\partial \boldsymbol{\psi}_1}{\partial I} \right) dI + \left(\frac{\partial \boldsymbol{B}_1}{\partial \eta} \frac{\partial \boldsymbol{\psi}_1}{\partial B_2} - \frac{\partial \boldsymbol{B}_1}{\partial B_2} \frac{\partial \boldsymbol{\psi}_1}{\partial \eta} \right) d\eta. \tag{28}$$

We have

$$d\xi_{B_2} = \frac{\partial}{\partial B_2} \left(\frac{\partial \boldsymbol{B}_1}{\partial I} \frac{\partial \boldsymbol{\psi}_1}{\partial \eta} - \frac{\partial \boldsymbol{B}_1}{\partial \eta} \frac{\partial \boldsymbol{\psi}_1}{\partial I} \right) dI \wedge d\eta$$

and thus by (27) $d\xi_{B_2} = 0$, that is ξ_{B_2} is a closed form. For a fixed $I \in (a_{B_2}, b_{B_2})$ let us define the curve $\gamma_I : [0, 2\pi] \to (a_{B_2}, b_{B_2}) \times \mathbb{T}^1$,

$\gamma_I(\eta) = (I, \eta)$. Integrating by parts we obtain

$$\begin{aligned}\int_{\gamma_I} \xi_{B_2} &= \int_0^{2\pi} \left(\frac{\partial \boldsymbol{B}_1}{\partial \eta} \frac{\partial \psi_1}{\partial B_2} - \frac{\partial \boldsymbol{B}_1}{\partial B_2} \frac{\partial \psi_1}{\partial \eta} \right) (I, B_2, \eta)\, d\eta \\ &= \boldsymbol{B}_1(I, B_2, 2\pi) \frac{\partial \psi_1}{\partial B_2}(I, B_2, 2\pi) \\ &\quad - \boldsymbol{B}_1(I, B_2, 0) \frac{\partial \psi_1}{\partial B_2}(I, B_2, 0) \\ &\quad - \int_0^{2\pi} \left(\boldsymbol{B}_1 \frac{\partial^2 \psi_1}{\partial B_2\, \partial \eta} + \frac{\partial \boldsymbol{B}_1}{\partial B_2} \frac{\partial \psi_1}{\partial \eta} \right) (I, B_2, \eta)\, d\eta \\ &= - \frac{\partial}{\partial B_2} \left(\int_0^{2\pi} \left(\boldsymbol{B}_1 \frac{\partial \psi_1}{\partial \eta} \right) (I, B_2, \eta)\, d\eta \right) \\ &= - \frac{\partial}{\partial B_2} \left(\int_{\Phi_{B_2}^{-1}(\gamma_I)} B_1\, d\psi_1 \right) \\ &= - \frac{\partial}{\partial B_2} \left(\int_{M_{B_2, J_{B_2}^{-1}(I)}} B_1\, d\psi_1 \right) \sigma \\ &= -\sigma \frac{\partial}{\partial B_2} \left(\frac{2\pi}{\sigma} J_{B_2}(J_{B_2}^{-1}(I)) \right) = -2\pi \frac{\partial I}{\partial B_2} = 0. \end{aligned} \quad (29)$$

Let now $\tilde{\boldsymbol{c}} \in C^\infty((a, b))$ be such that $\tilde{\boldsymbol{c}}(B_2) \in (a_{B_2}, b_{B_2})$ for $B_2 \in (a, b)$ (such a function may be easily constructed since Γ is an open set) and let $\boldsymbol{c} : (a, b) \to \mathbb{R} \times \mathbb{T}^1$, $\boldsymbol{c}(B_2) = (\tilde{\boldsymbol{c}}(B_2), 0)$. Define the function $\tilde{\psi}_2 : \Gamma \times \mathbb{T}^1 \to \mathbb{T}^1$ as follows:

$$\tilde{\psi}_2(I, B_2, \eta) = \int_{\boldsymbol{c}(B_2)}^{(I, \eta)} \xi_{B_2},$$

with integration over an arbitrary curve in $(a_{B_2}, b_{B_2}) \times \mathbb{T}^1$ joining $\boldsymbol{c}(B_2)$ and (I, η). Since we have (29) and ξ_{B_2} is a closed form, $\tilde{\psi}_2$ is a well-defined C^∞-function. From (28) we also have

$$\begin{aligned} \frac{\partial \tilde{\psi}_2}{\partial I} &= \frac{\partial \boldsymbol{B}_1}{\partial I} \frac{\partial \psi_1}{\partial B_2} - \frac{\partial \boldsymbol{B}_1}{\partial B_2} \frac{\partial \psi_1}{\partial I}, \\ \frac{\partial \tilde{\psi}_2}{\partial \eta} &= \frac{\partial \boldsymbol{B}_1}{\partial \eta} \frac{\partial \psi_1}{\partial B_2} - \frac{\partial \boldsymbol{B}_1}{\partial B_2} \frac{\partial \psi_1}{\partial \eta}. \end{aligned} \quad (30)$$

Thus we can define the transformation $\Theta : \Gamma \times \mathbb{T}^2 \to \mathcal{G} \times \mathbb{T}^1$

$$\Theta(I, B_2, \eta, \psi_2) = (\boldsymbol{B}_1(I, B_2, \eta), B_2, \psi_1(I, B_2, \eta), \psi_2 + \tilde{\psi}_2(I, B_2, \eta)).$$

It is canonical because $d\boldsymbol{B}_1 \wedge d\psi_1 + dB_2 \wedge d(\psi_2 + \tilde{\psi}_2) = dI \wedge d\eta + dB_2 \wedge d\psi_2$ by (27), (30) and hence it is also a local diffeomorphism. As we can see from the construction, Θ is in fact a global canonical diffeomorphism onto $\mathcal{G} \times \mathbb{T}^1$ and Θ^{-1} has the form:

$$\Theta^{-1}(B_1, B_2, \psi_1, \psi_2) = (\boldsymbol{I}(B_1, B_2, \psi_1), B_2, \boldsymbol{\eta}(B_1, B_2, \psi_1), \psi_2 - \tilde{\tilde{\psi}}_2(B_1, B_2, \psi_1)),$$

where $\tilde{\tilde{\psi}}_2(B_1, B_2, \psi_1) = \tilde{\psi}_2(\boldsymbol{I}(B_1, B_2, \psi_1), B_2, \boldsymbol{\eta}(B_1, B_2, \psi_1))$. Moreover,

$$[(P_r v) \circ \mathcal{C}_\alpha \circ \Theta](I, B_2, \eta, \psi_1) = J_{B_2}^{-1}(I) \tag{31}$$

and

$$[h_0 \circ \mathcal{C}_\alpha \circ \Theta](I, B_2, \eta, \psi_1) = \alpha\omega B_2 \tag{32}$$

for $(I, B_2, \eta, \psi_1) \in \Gamma \times \mathbb{T}^2$ and thus $P_r v$ and h_0 are functions depending on actions only in I, B_2, η, ψ_1 coordinates (by (26) $J_{B_2}^{-1}$ is a well-defined, smooth function of I, B_2).

Our goal is to find the domains of integrability for ω, $P_r v$, that is we must find such W and such new canonical coordinates on W that $P_r v$ and h_0 depend on the new actions only, h_0 has the same form as in the standard A, φ coordinates, and the new actions belong to $\mathbb{R}_+^2$. The first has been already satisfied. To satisfy the remaining conditions we must make one more simple canonical transformation ("simple" means here that it does not "mix" the angles with the actions). It will have the form $\mathcal{C}_{\alpha'} \circ \mathcal{S}$ for the appropriately choosen α' and $\mathcal{S}$. Let us study more closely the set Γ. Assume that $(I, B_2) \in \Gamma$. Then $I = J_{B_2}(x)$ for some $x = f_{B_2}(B_1, \psi_1)$, $\psi_1 \in \mathbb{T}^1$ and B_1 such that $(B_1, B_2) \in \mathbb{R}^2_{\omega, \alpha}$. Thus in the libration case by (24a) and (25) $I = \frac{1}{2\pi}\Pi(x, B_2)$, so we have

$$I \in (0, c_\alpha^+ B_2 - c_\alpha^- B_2). \tag{33a}$$

In the rotation case, from (24b), (25), and Stoke's theorem we obtain

$$\frac{-1}{2\pi}\int_{\gamma_-} B_1\, d\psi_1 + J_{B_2}(x) = \frac{1}{2\pi}\tilde{\Pi}(x, B_2),$$

where $\gamma_-(t) = (c_\alpha^- B_2, t) \in [c_\alpha^- B_2, c_\alpha^+ B_2] \times \mathbb{T}^1$, $t \in [0, 2\pi]$ and $\tilde{\Pi}(x, B_2)$ is the area of that part of the cylinder $[c_\alpha^- B_2, c_\alpha^+ B_2] \times \mathbb{T}^1$ which is bounded by the curve γ_- and $M_{B_2, x}$. Since $\frac{1}{2\pi}\int_{\gamma_-} B_1\, d\psi_1 = c_\alpha^- B_2$ we find

$$I \in (c_\alpha^- B_2, c_\alpha^+ B_2). \tag{33b}$$

Therefore in the rotation case $\Gamma \subset \mathbb{R}^2_{\omega,\alpha}$, so it is enough to take $\mathcal{S}$ the identity and $\alpha' = \alpha$. The libration case is more complicated. It may happen that Γ is not a subset of $\mathbb{R}^2_{\omega,\alpha'}$ for any $\alpha' \in \mathcal{A}_\omega$ and thus, usually, our goal may be reached only "locally" (that is for "small" Γ). For $s \in \{-1, 1\}$, $r \in \mathbb{R}$ denote

$$\Gamma_{s,r} = \{(sI + r, B_2) : (I, B_2) \in \Gamma\} \tag{34}$$

and suppose that Γ is so "small" that $\Gamma_{s,r} \subset \mathbb{R}^2_{\omega,\alpha'}$ for some $\alpha' \in \{\alpha + l\,\omega^\perp : l \in \mathbb{Z}\} \subset \mathcal{A}_\omega$, $s \in \{-1, 1\}$, $r \in \mathbb{R}$. We define the (canonical) transformation $\mathcal{S} : \Gamma \times \mathbb{T}^2 \to \Gamma_{s,r} \times \mathbb{T}^2$

$$\mathcal{S}(I, B_2, \eta, \psi_2) = (sI + r, B_2, s\,\eta, \psi_2).$$

To make the notation uniform in both cases, we put $r = 0$, $s = 1$, and $\alpha' = \alpha$ in the rotation case (so $\mathcal{S}$ is the identity). Thus we define

$$\begin{aligned} W &= \mathcal{C}_\alpha(\mathcal{G} \times \mathbb{T}^1), \\ U &= N^T_{\alpha'}\,\Gamma_{s,r}, \\ u_0 &= \mathcal{C}_\alpha\,\Theta\,\mathcal{S}^{-1}\,\mathcal{C}^{-1}_{\alpha'}\,_{|U\times\mathbb{T}^2}, \\ k_1(A, \varphi) &= J^{-1}_{(\rho\omega_1^\perp A_2 - \rho\omega_2^\perp A_1)}(s\,\rho\alpha'_2 A_1 - s\,\rho\alpha'_1 A_2 - sr), \end{aligned} \tag{35}$$

where $\rho = \det N_{\alpha'} = \det N_\alpha$ (equal to 1 or -1). The function k_1 is smooth and φ-independent, u_0 is a canonical C^∞-diffeomorphism of $U \times \mathbb{T}^2$ onto W. By (31), (35) we have $(P_r\,v) \circ u_0 = k_1$ and by (22), (32), (35) we have $(h_0 \circ u_0)(A, \varphi) = \omega A$ (since $\omega\alpha' = \omega\alpha + k\,\omega\omega^\perp = \omega\alpha$). Obviously U and W are open subsets of $\mathbb{R}^2_+$, $\mathbb{R}^2_+ \times \mathbb{T}^2$, respectively. Using (22) and Lemma 2.1 we can check that $P_r\,v$ and h_0 are functionally independent on W. Hence, the following theorem has been proved.

THEOREM 2.2. – *Assume that ω is resonant, $v \in C^\infty(\mathbb{R}^2_+ \times \mathbb{T}^2)$, $\alpha \in \mathcal{A}_\omega$ and an open connected $\mathcal{G} \subset \mathbb{R}^2_{\omega,\alpha} \times \mathbb{T}^1$ is smoothly foliated into circles by $f = (P_r\,v)^{[\alpha]}_{|\mathcal{G}}$ with $B_2 =$ const. If (23a) holds, assume also that*

$$\Gamma_{s,r} \subset \mathbb{R}^2_{\omega,\alpha'} \tag{36}$$

for some $\alpha' \in \{\alpha + k\,\omega^\perp : k \in \mathbb{Z}\}$, $s \in \{-1, 1\}$, $r \in \mathbb{R}$, and with $\Gamma_{s,r}$ defined by (34). Then $P_r\,v$ on W is ω-integrable into k_1 on $U \times \mathbb{T}^2$ by u_0 and the formal perturbation theory for $h_0 + \varepsilon\,v$ is well defined on W, where W, U, u_0, k_1 are given by (35).

Remarks. – (i) u_0 is not uniquely determined, due to a choice of $\alpha \in \mathcal{A}_\omega$ and also a choice of α', s, r. There is however no $1-1$ correspondence between parameters (α, α', s, r) and transformations

u_0. The non-uniqueness of u_0 is described in Theorem 2.3. Note that the necessity of choosing the additional (and somewhat inconvenient) parameters α', s, r to satisfy (36) in the libration case is caused by the requirement that actions be in $\mathbb{R}^2_+$. Without this restriction we would have a free choice of these parameters.

(ii) Under the remaining assumptions of the theorem, the additional assumption (36) in the libration case may obviously be "locally" satisfied in the following sense: $\mathcal{G}$ may be expressed in the form

$$\bigcup_{(I,\, B_2)\,\in\,\Gamma} \mathcal{G}_{I,\, B_2}$$

with such open, connected $\mathcal{G}_{I,\, B_2} \subset \mathcal{G}$, that (36) holds for some α', s, and r if we take $\mathcal{G}_{I,\, B_2}$ instead of $\mathcal{G}$. It is clear that the above decomposition of $\mathcal{G}$ may turn out to be, in fact, finite (for instance for bounded Γ).

(iii) If $\omega_1 = 1$ or $\omega_2 = 1$, then (36) is automatically satisfied for $r = 0$ and for appropriate choice of s and α'. It follows from (33a) that it is enough to take $s = -1$, $\alpha' = \alpha + \alpha_2\, \omega^{\perp}$ when $\omega_1 = 1$, and $s = 1$, $\alpha' = \alpha + \alpha_1\, \omega^{\perp}$ when $\omega_2 = 1$, because by (21) we obtain $c^+_{\alpha'}\, B_2 = 0$ or $c^-_{\alpha'}\, B_2 = 0$ then, respectively.

We will now exemplify the constructions of Theorem 2.1 in concrete cases.

Example 2.1. – Let $\omega = (1,\, 1)$ and let v expressed in the standard p, q coordinates be the potential $q_1^2\, q_2^2$. Then

$$v\,(A,\, \varphi) = 4\, A_1\, A_2\, \cos^2\, \varphi_1\, \cos^2\, \varphi_2$$

and

$$(P_r\, v)\,(A,\, \varphi) = \frac{1}{2}\, A_1\, A_2\, (\cos\, 2\,(\varphi_1 - \varphi_2) + 2).$$

We have $\omega^{\perp} = (1,\, -1)$ and we take $\alpha = (0,\, 1)$. Thus we find

$$\mathcal{C}_\alpha\,(B,\, \psi) = (B_1,\, B_2 - B_1,\, \psi_1 + \psi_2,\, \psi_2),$$
$$\mathcal{C}_\alpha^{-1}\,(A,\, \varphi) = (A_1,\, A_1 + A_2,\, \varphi_1 - \varphi_2,\, \varphi_2).$$

Moreover

$$c^-_\alpha\, B_2 = 0, \qquad c^+_\alpha\, B_2 = B_2,$$
$$\mathbb{R}^2_{\omega,\, \alpha} = \{\, (B_1,\, B_2)\, :\, B_2 > 0,\;\, B_1 \in\, (0,\, B_2)\,\},$$
$$(h_0 \circ \mathcal{C}_\alpha)\,(B,\, \psi) = B_2,$$
$$(P_r\, v)^{[\alpha]}\,(B,\, \psi_1) = \frac{1}{2}\, B_1\,(B_2 - B_1)\,(\cos\, 2\,\psi_1 + 2).$$

We look for open sets foliated into circles by $(P_r\, v)^{[\alpha]}$. Supposing $B_2 > 0$ to be fixed, we first find the separatrices of $(P_r\, v)^{[\alpha]}$, that is the level sets passing through the critical points on the cylinder $(0,\, B_2) \times \mathbb{T}^1$. We have four critical points: $C_j = \left(\frac{1}{2}\, B_2,\, j\, \frac{\pi}{2}\right)$ for $j = 0,\, 1,\, 2,\, 3$ with the appropriate values of $(P_r v)^{[\alpha]}$ equal to $\frac{1}{8} B_2^2\, ((-1)^j + 2)$. Thus we have two separatrices: one which reduces to two-point set $\{\, C_0,\, C_2 \,\}$, and another being the curve given by the equation

$$4\, B_1\, (B_2 - B_1)\, (\cos\, 2\, \psi_1 + 2) = B_2^2$$

in the cylinder. If we take both separatrices out, the cylinder will be divided into four connected subsets. Each subset is a sum of level sets which are homeomorphic to the circle and described by the equation

$$B_1\, (B_2 - B_1)\, (\cos\, 2\, \psi_1 + 2) = 2\, x,$$

where $x \in \left(0,\, \frac{1}{8}\, B_2^2\right) \cup \left(\frac{1}{8}\, B_2^2,\, \frac{3}{8}\, B_2^2\right)$ (*see* Figure 1). We consider the following subsets of $\mathbb{R}^2_{\omega,\, \alpha} \times \mathbb{T}^1$:

$$\mathcal{G}_j = \Big\{ (B_1,\, B_2,\, \psi_1) \in \mathbb{R}^2 \times \mathbb{T}^1 \; : \; B_2 > 0,\; \frac{j-1}{2} B_2 < B_1 < \frac{j}{2}\, B_2, \\ 0 < 4\, B_1\, (B_2 - B_1)\, (\cos\, 2\, \psi_1 + 2) < B_2^2 \Big\}$$

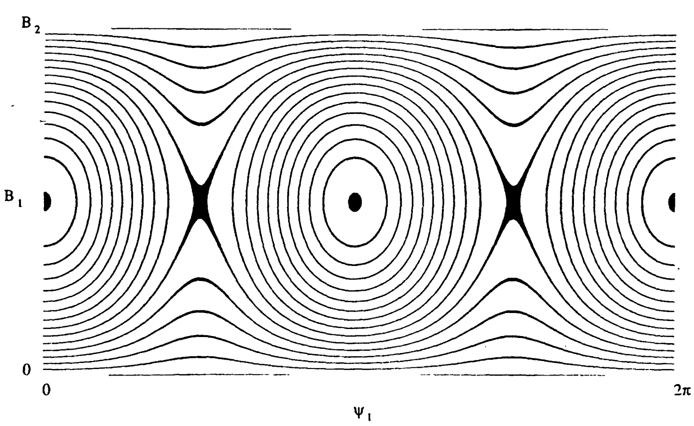

Fig. 1. – The level sets of $(P_r\, v)^{[\alpha]}_{B_2}$ for v of Example 2.1 and $B_2 = 1$.

for $j = 1, 2$ and

$$\mathcal{G}_j = \Big\{ (B_1, B_2, \psi_1) \in \mathbb{R}^2 \times \mathbb{T}^1 : B_2 > 0,\ 0 < B_1 < B_2,$$
$$\left(j - \frac{5}{2}\right)\pi < \psi_1 < \left(j - \frac{7}{2}\right)\pi,$$
$$B_2^2 < 4 B_1 (B_2 - B_1)(\cos 2\psi_1 + 2) < 3 B_2^2 \Big\}$$

for $j = 3, 4$, where we identify $\mathbb{T}^1$ with $\left(\frac{1}{2}\pi, \frac{5}{2}\pi\right]$. Using Lemma 2.1 we can easily check that for any j the set $\mathcal{G}_j$ is smoothly foliated into circles by $(P_r\, v)^{[\alpha]}_{|\mathcal{G}_j}$ with $B_2 = \text{const}$. Thus, using Theorem 2.2 and remark (iii) we obtain that the formal perturbation theory is well-defined on $W_j = \mathcal{C}_\alpha(\mathcal{G}_j \times \mathbb{T}^1)$ for $j = 1, \dots 4$. Moreover, the Lebesgue measure of $\mathbb{R}^2_+ \times \mathbb{T}^2 \setminus \bigcup_{j=1}^4 W_j$ equals zero.

The following example of "global purely rotational case" is obtained by a modification of the potential from Example 2.1.

Example 2.2. – Let us consider the same $\omega = (1, 1)$ and v of the form $q_1^2 q_2^2 + \lambda (q_1^2 + p_1^2)^2$ in p, q coordinates, with a constant $\lambda > \frac{3}{8}$. Taking as before $\alpha = (0, 1)$, we obtain

$$(P_r\, v)^{[\alpha]}(B, \psi_1) = \frac{1}{2} B_1 (B_2 - B_1)(\cos 2\psi_1 + 2) + 4\lambda B_1^2,$$

and so $(P_r\, v)^{[\alpha]}_{B_2}$ has no critical points on the cylinder $(0, B_2) \times \mathbb{T}^1$.

It turns out that the level sets of $(P_r\, v)^{[\alpha]}_{B_2}$ are all homeomorphic to the circle (*see* Figure 2) and also the remaining conditions from Lemma 2.1 may be easily checked. Thus the whole phase space $\mathbb{R}^2_+ \times \mathbb{T}^2$ is a domain of integrability for ω, $P_r\, v$, and the formal perturbation theory for $h_0 + \varepsilon\, v$ is well-defined on $\mathbb{R}^2_+ \times \mathbb{T}^2$. Moreover, it can be checked in this case that $U = \mathbb{R}^2_+$, which means that u_0 is a smooth canonical automorphism of $\mathbb{R}^2_+ \times \mathbb{T}^2$.

As we could see in the proof of Theorem 2.2 the choice of the canonical transformation u_0 is not unique. We shall now describe this non-uniqueness more precisely. Assume that (20) holds for $P_r\, v$ on W, that $P_r\, v$ on W is ω-integrable into $k^{(j)}$ on $U^{(j)} \times \mathbb{T}^2$ by $u_0^{(j)}$ for $j = 1, 2$ and consider $s : U^{(2)} \times \mathbb{T}^2 \to U^{(1)} \times \mathbb{T}^2$, $s = (u_0^{(1)})^{-1} \circ u_0^{(2)}$. Thus we have

$$h_0 \circ s = h_{0_{|U^{(2)} \times \mathbb{T}^2}}, \qquad k^{(1)} \circ s = k^{(2)} \tag{37}$$

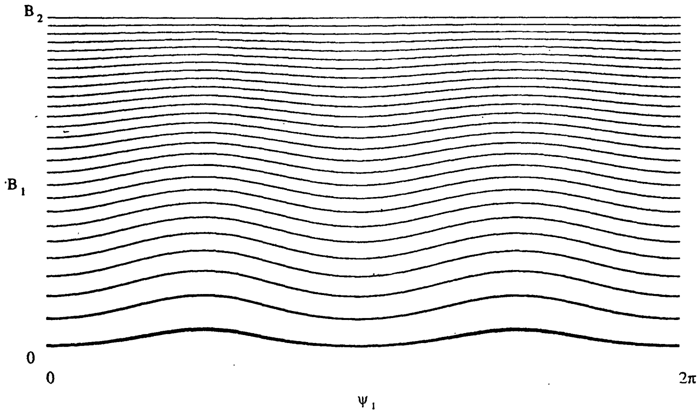

Fig. 2. – The level sets of $(P_r\ v)^{[\alpha]}_{B_2}$ for v of Example 2.2 and $B_2 = \lambda = 1$.

and

$$(\nabla_\varphi k^{(j)})(A, \varphi) = 0, \qquad (\nabla_A k^{(j)})(A, \varphi) \notin \mathrm{lin}(\{\omega\}) \tag{38}$$

for $j = 1, 2$. Let us fix $\alpha \in \mathcal{A}_\omega$ with $\alpha_1 \omega_2^\perp - \alpha_2 \omega_1^\perp = 1$.

THEOREM 2.3. – *If $U^{(1)}$, $U^{(2)}$ are open, connected, non-empty subsets of $\mathbb{R}^2_+$, $k^{(j)} \in C^\infty(U^{(j)} \times \mathbb{T}^2)$ satisfy* (38) *for $j = 1, 2$ and s is a canonical C^∞-diffeomorphism of $U^{(2)} \times \mathbb{T}^2$ onto $U^{(1)} \times \mathbb{T}^2$ satisfying* (37), *then s has a form*

$$s(A, \varphi) = (NA + c,\ (N^{-1})^T\ \varphi + g(A)), \tag{39}$$

where

(i) *N is the identity for ω non-resonant and*

$$N = N_0 + n \begin{pmatrix} \omega_1^\perp \omega_2^\perp & (\omega_1^\perp)^2 \\ (\omega_2^\perp)^2 & -\omega_1^\perp \omega_2^\perp \end{pmatrix} \tag{40}$$

for ω resonant, where $n \in \mathbb{Z}$ and N_0 is equal either to the identity or to

$$\begin{pmatrix} 1 - 2\alpha_2 \omega_1^\perp & 2\alpha_1 \omega_1^\perp \\ -2\alpha_2 \omega_2^\perp & 1 + 2\alpha_1 \omega_2^\perp \end{pmatrix}; \tag{41}$$

(ii) *g is a C^∞-function from $U^{(2)}$ into $\mathbb{T}^2$ satisfying*

$$((D_A g)(A))^T N = N^T (D_A g)(A) \tag{42}$$

for $A \in U^{(2)}$;

(iii) *$c \in \mathbb{R}^2$ which $c\omega = 0$.*

Proof. – Let $s(A, \varphi) = (\mathcal{A}(A, \varphi), \phi(A, \varphi))$. Observe that by (38) the map $F : U^{(1)} \to \mathbb{R}^2$, $F(A) = (\omega A, k^{(1)}(A))$ is a local diffeomorphism.

Thus, if $A_0 \in U^{(1)}$ then there exists an open neighbourhood V_0 of A_0 and a smooth function G defined in a neighbourhood of $F(A_0)$ such that for any $A \in V_0$ we have

$$A = G(F(A)) \tag{43}$$

(that is $G = (F_{|V_0})^{-1}$). Let now A_0', φ_0' be such that $\mathcal{A}(A_0', \varphi_0') = A_0$ and let $C_0 = \{\varphi \in \mathbb{T}^2 : \mathcal{A}(A_0', \varphi) = A_0\}$. The set C_0 is closed in $\mathbb{T}^2$ and non-empty ($\varphi_0' \in C_0$). Let $\tilde{C}_0 = \{\varphi \in \mathbb{T}^2 : \mathcal{A}(A_0', \varphi) \in V_0\}$. It is an open set in $\mathbb{T}^2$ and $C_0 \subset \tilde{C}_0$. Moreover $C_0 = \tilde{C}_0$ by (37) and (43) because we have for $\varphi \in \tilde{C}_0$

$$\begin{aligned}\mathcal{A}(A_0', \varphi) &= G(\omega\,\mathcal{A}(A_0', \varphi), \boldsymbol{k}^{(1)}(\mathcal{A}(A_0', \varphi))) = G(\omega\, A_0', \boldsymbol{k}^{(2)}(A_0'))\\ &= G(\omega\,\mathcal{A}(A_0', \varphi_0'), \boldsymbol{k}^{(1)}(\mathcal{A}(A_0', \varphi_0'))) = \mathcal{A}(A_0', \varphi_0') = A_0\end{aligned}$$

that is $\varphi \in C_0$. Thus, using the connecticity of $\mathbb{T}^2$, we obtain $C_0 = \mathbb{T}^2$ and hence $\mathcal{A}$ is φ independent (since A_0 and thus also A_0' was arbitrarily choosen), so the differencial of s has the matrix (4×4) of the form

$$(Ds)(A, \varphi) = \begin{pmatrix} N(A) & 0 \\ P(A, \varphi) & R(A, \varphi) \end{pmatrix},$$

where $N(A) = (D_A\,\mathcal{A})(A, \varphi)$, $P(A, \varphi) = (D_A\,\phi)(A, \varphi)$, $R(A, \varphi) = (D_\varphi\,\phi)(A, \varphi)$. Since s is canonical we have (*see* [7])

$$\begin{aligned}R(A, \varphi) &= ((N(A))^{-1})^T, (P(A, \varphi))^T\\ &= (N(A))^T\,P(A, \varphi)\,(N(A))^{-1}\end{aligned} \tag{44}$$

and thus R is φ independent. The map $\varphi \to (\mathcal{A}(A, \varphi), \phi(A, \varphi))$ is a diffeomorphism for any fixed A, but $\mathcal{A}$ is φ independent, hence $\phi(A, \cdot) : \mathbb{T}^2 \to \mathbb{T}^2$ is a diffeomorphism, too. From (44) we have

$$\phi(A, \varphi) = ((N(A))^{-1})^T\,\varphi + g(A), \tag{45}$$

where g is a smooth function from $U^{(2)}$ into $\mathbb{T}^2$ and thus, since $\phi(A, \cdot)$ is a diffeomorphism of the torus, $N(A)$ and $(N(A))^{-1}$ must have coefficients in $\mathbb{Z}$. In particular, it follows that N and R are constant matrices because $U^{(2)}$ is connected. Moreover, det $N = 1$ or -1. By (45) we also have

$$P(A, \varphi) = (D_A\,g)(A), \tag{46}$$

so P depends on A only. Finally, from (44), (45), (46) we obtain (39) with N being a certain matrix with coefficients in $\mathbb{Z}$ and $|\det N| = 1$, with $g-$ a C^∞-function from $U^{(2)}$ into $\mathbb{T}^2$ satisfying (42), and with a certain

$c \in \mathbb{R}^2$. From (37) we see that

$$\omega\,(N A + c) = \omega\, A \tag{47}$$

for $A \in U^{(2)}$. Since $U^{(2)}$ is open and non-empty (47) is equivalent to

$$N^T\,\omega = \omega \quad \text{and} \quad c\,\omega = 0. \tag{48}$$

For ω non-resonant it follows immediatelly that N is the identity. Suppose that ω is resonant. Let

$$N = \begin{pmatrix} n_1 & n_3 \\ n_2 & n_4 \end{pmatrix}.$$

Since $(-\omega_2^\perp, \omega_1^\perp) \in \operatorname{lin}\{\omega\}$, by (48) we obtain

$$-(n_1 - 1)\,\omega_2^\perp + n_2\,\omega_1^\perp = -n_3\,\omega_2^\perp + (n_4 - 1)\,\omega_1^\perp = 0$$

and thus there exist $m_1, m_2 \in \mathbb{Z}$ satisfying

$$\begin{aligned} n_1 - 1 &= -m_1\,\omega_1^\perp, \qquad & n_2 &= -m_1\,\omega_2^\perp, \\ n_3 &= -m_2\,\omega_1^\perp, \qquad & n_4 - 1 &= -m_2\,\omega_2^\perp. \end{aligned} \tag{49}$$

From (49) we conclude

$$\begin{aligned} \det N &= (1 - m_1\,\omega_1^\perp)\,(1 - m_2\,\omega_2^\perp) - m_1\,m_2\,\omega_2^\perp\,\omega_1^\perp \\ &= 1 - m_1\,\omega_1^\perp - m_2\,\omega_2^\perp. \end{aligned} \tag{50}$$

If $\det N = 1$, then from (85) we have $m_1\,\omega_1^\perp = -m_2\,\omega_2^\perp$ and thus $m_1 = -n\,\omega_2^\perp$, $m_2 = n\,\omega_1^\perp$ for some $n \in \mathbb{Z}$. This gives us (40) with $N_0 = I$. If $\det N = -1$, then (50) follows that $m_1\,\omega_1^\perp + m_2\,\omega_2^\perp = 2$ and thus using our assumption on α we obtain $\omega_2^\perp\,(2\,\alpha_1 + m_1) = -\,\omega_1^\perp\,(m_1 - 2\,\alpha_2)$. Therefore $m_1 = -n\,\omega_2^\perp + 2\,\alpha_2$, $m_2 = n\,\omega_1^\perp - 2\,\alpha_1$ for some $n \in \mathbb{Z}$ and from (49) we conclude (40) with N_0 of the form (41). □

Remark. – It is easy to check that the converse result also holds. Namely, if $P_r\,v$ on W is ω-integrable into $k^{(1)}$ on $U^{(1)} \times \mathbb{T}^2$ by $u_0^{(1)}$, s has the form as in the above theorem, and

$$U^{(2)} = \{\, N^{-1}\,A - N^{-1}\,c \in \mathbb{R}^2 \,:\, A \in U^{(1)} \,\} \subset \mathbb{R}^2_+,$$

then $P_r\,v$ on W is ω-integrable into $k^{(2)}$ on $U^{(2)} \times \mathbb{T}^2$ by $u_0^{(2)}$, where $k^{(2)} = k^{(1)} \circ s$ and $u_0^{(2)} = u_0^{(1)} \circ s$.

APPENDIX

The Fourier Expansion of Smooth Functions on $U \times \mathbb{T}^d$

We formulate here a technical lemma on the Fourier expansion of smooth functions on $U \times \mathbb{T}^d$, where U is an open subset of $\mathbb{R}^{d'}$. Suppose that $f \in C(U \times \mathbb{T}^d)$ and that the series

$$\sum_{\nu \in \mathbb{Z}^d} f_\nu (A) \exp(i \nu \varphi) \tag{1A}$$

is the Fourier expansion of f with $A \in U$ fixed.

LEMMA. – a) *$f \in C^\infty (U \times \mathbb{T}^d)$ if and only if $f_\nu \in C^\infty (U)$ for any $\nu \in \mathbb{Z}^d$ and for any compact $K \subset U$, $\alpha \in \mathbb{N}^{d'}$, and $n \in \mathbb{N}$*

$$\sup_{A \in K, \nu \in \mathbb{Z}^d} \left| \frac{\partial^{|\alpha|} f_\nu}{\partial A^\alpha} (A) \right| (|\nu| + 1)^n < +\infty.$$

b) *If $f \in C^\infty (U \times \mathbb{T}^d)$ then the series $(1A)$ is absolutely and uniformly convergent to f on any compact subset of $U \times \mathbb{T}^d$. Moreover, $\frac{\partial^{|\alpha|+|\beta|}}{\partial A^\alpha \partial \varphi^\beta} f$ has the expansion*

$$\sum_{\nu \in \mathbb{Z}^d} \frac{\partial^{|\alpha|} f_\nu}{\partial A^\alpha} (A) (i \nu)^\beta \exp(i \nu \varphi)$$

for any $\alpha \in \mathbb{N}^{d'}$, $\beta \in \mathbb{N}^d$.

c) *If $f \in C^\infty (U \times \mathbb{T}^d)$ and g_ν, for $\nu \in \mathbb{Z}^d$, are functions from $C^\infty (U)$ satisfying the condition: for any compact $K \subset U$, $\alpha \in \mathbb{N}^{d'}$ there exist constants $D_{\alpha, K}$, $\gamma_{\alpha, K} > 0$ such that for any $\nu \in \mathbb{Z}^d$*

$$\sup_{A \in K} \left| \frac{\partial^{|\alpha|} g_\nu}{\partial A^\alpha} (A) \right| \leq D_{\alpha, K} (|\nu| + 1)^{\gamma_{\alpha, K}},$$

then $\sum_{\nu \in \mathbb{Z}^d} f_\nu (A)\, g_\nu (A) \exp(i \nu \varphi)$ is the expansion of a function from $C^\infty (U \times \mathbb{T}^d)$.

Remark. – In particular the conditions of (c) of the above lemma are satisfied by any system of functions of the form

$$g_\nu (A) = \begin{cases} 0 & \nu \notin \Omega \\ (b(A)\nu)^{-1} & \nu \in \Omega \end{cases},$$

where $\Omega \subset \mathbb{Z}^d$ and $b : U \to \mathbb{R}^d$ is a C^∞-function satisfying: for any compact $K \subset U$ there exist such D_K, $\gamma_K > 0$ that for $\nu \in \Omega$

$$\inf_{A \in K} |b(A)\nu| \geq D_K |\nu|^{-\gamma_K}.$$

The Proof of Lemma 1.1

Let us denote $\sigma_\omega = \{ i\nu\omega \ : \ \nu \in \mathbb{Z}^d \}$. Let $g \in C^\infty (U \times \mathbb{T}^d)$. For $(A, \varphi) \in U \times \mathbb{T}^d$ we have

$$\{ g, \pi_{W_U} h_0 \} (A, \varphi) = \sum_{j=1}^{d} \frac{\partial g}{\partial \varphi_j} (A, \varphi) \omega_j = \sum_{\nu \in \mathbb{Z}^d} g_\nu (A) i \nu\omega \exp (i \nu\varphi).$$

Thus for $\lambda \in \mathbb{C}$

$$\begin{gathered} \{ g, \pi_{W_U} h_0 \} = \lambda g \\ \text{iff} \\ \lambda \in \sigma_\omega \quad \text{and} \quad g(A, \varphi) = \sum_{i\nu\omega=\lambda} g_\nu (A) \exp (i \nu\varphi), \end{gathered} \tag{2A}$$

unless $g = 0$. Let now $f \in C^\infty (\mathbb{R}^d_+ \times \mathbb{T}^d)$. Then, using (2$A$) with $U = \mathbb{R}^d_+$ and $g = f$, we can write

$$f = \sum_{\lambda \in \sigma_\omega} f_{[\lambda]},$$

where $f_{[\lambda]} \in C^\infty (\mathbb{R}^d_+ \times \mathbb{T}^d)$, $\{ f_{[\lambda]}, h_0 \} = \lambda f_{[\lambda]}$, and by the Lemma from the first part of Appendix the above series is uniformly convergent on any compact subset of $\mathbb{R}^d_+ \times \mathbb{T}^d$ for any ordering of σ_ω. From (4) and the canonicity of u_0 we have

$$\begin{aligned} \{ t_0 \pi_W f_{[\lambda]}, \pi_{W_U} h_0 \} &= \{ t_0 \pi_W f_{[\lambda]}, t_0 \pi_W h_0 \} \\ &= t_0 \pi_W \{ f_{[\lambda]}, h_0 \} = \lambda t_0 \pi_W f_{[\lambda]} \end{aligned}$$

for $\lambda \in \sigma_\omega$. Using now (2$A$) for U and $g = t_0 \pi_W f_{[\lambda]}$ we have $(t_0 \pi_W f_{[\lambda]})_\nu = 0$ for $i\nu\omega \neq \lambda$. Thus

$$P_r t_0 \pi_W f_{[\lambda]} = 0 \tag{3A}$$

for $\lambda \neq 0$ and $P_r t_0 \pi_W f_{[0]} = t_0 \pi_W f_{[0]}$, but also $P_r f = f_{[0]}$, so

$$P_r t_0 \pi_W f_{[0]} = t_0 \pi_W P_r f. \tag{4A}$$

Let us denote $\tilde{f} = \sum_{\lambda \in \sigma_\omega \setminus \{0\}} f_{[\lambda]}$ and let S_n be the partial sum sequence of this series. The sequence S_n is convergent to $\tilde{f}$ uniformly on any compact subset of $\mathbb{R}^d_+ \times \mathbb{T}^d$ and the same is true of $t_0 \pi_W S_n$ and $t_0 \pi_W \tilde{f}$. Thus, by (3A), for any $A \in U$ and $\nu \in \mathbb{Z}_\omega$

$$(t_0 \pi_W \tilde{f})_\nu (A) = \lim_{n \to +\infty} (t_0 \pi_W S_n)_\nu (A) = 0.$$

Hence, we have $P_r t_0 \pi_W \tilde{f} = 0$ so, from (4A)

$$P_r t_0 \pi_W f = P_r t_0 \pi_W f_{[0]} = t_0 \pi_W P_r f,$$

which completes the proof.

REFERENCES

[1] W. I. ARNOLD, *Mathematical Methods of Classical Mechanics*, Moscow, Nauka, 1974 (in Russian).

[2] G. BENETTIN, L. GALGANI, A. GIORGILLI and J.-M. STRELCYN, A proof of Kolmogorov's theorem on invariant tori using canonical transformations defined by the Lie method, *Il Nuovo Cimento*, Vol. **79 B**, No. 2, 1984, pp. 201-223.

[3] G. BENETTIN and G. GALLAVOTTI, Stability of motion near resonances in quasi integrable hamiltonian systems, *Journal of Statistical Physics*, Vol. **44**, 1986, pp. 293-338.

[4] M. DEGLI ESPOSTI, S. GRAFFI and J. HERCZYŃSKI, Quantization of the classical Lie algorithm in the Bargman representation, *Annals of Physics*, Vol. **209**, 1991, pp. 364-392.

[5] F. FASSÒ, Lie series method for vector fields and hamiltonian perturbation theory, *Journal of Applied Mathematics and Physics (ZAMP)*, Vol. **41**, 1990, pp. 843-864.

[6] J. FORD and G. H. LUNSFORD, Stochastic behaviour of resonant nearly linear oscillator systems in the limit of zero nonlinear coupling, *Physical Review*, Vol. **1 A**, 1970, pp. 59-70.

[7] G. GALAVOTTI, *The Elements of Mechanics*, New York, Springer, 1983.

[8] A. GIORGILLI and L. GALGANI, Rigorous estimates for the series expansions of hamiltonian perturbation theory, *Celestial Mechanics*, Vol. **37**, 1985, pp. 95-112.

[9] J. HERCZYŃSKI, A note on uniqueness of the normal form for quasi-integrable systems, *Meccanica*, Vol. **27**, 1992, pp. 281-284.

[10] E. F. JAGER and A. L. LICHTENBERG, Resonant modification and destruction of adiabatic invariants, *Annals of Physics*, Vol. **71**, 1972, pp. 319-356.

[11] P. V. KOSELEFF, Calcul formel pour les méthodes de Lie en mécanique hamiltonienne, *Thèse*, École Polytechnique, Paris, 1993.

[12] A. J. LICHTENBERG and M. A. LIEBERMAN, *Regular and Stochastic Motion*, Berlin, Springer, 1983.

[13] P. LOCHAK, Canonical perturbation theory via simultaneous approximation, *Russian Mathematical Surveys*, Vol. **47:6**, 1992, pp. 57-133.

[14] P. LOCHAK, Stability of hamiltonian systems over exponentially long times. The near-linear case, *Proceedings of the Cincinnati 1992 Conference on Hamiltonian Dynamical Systems*, IMA Publ.

[15] B. MCNAMARA and R. I. WHITEMAN, Invariants of nearly periodic hamiltonian systems, *Culham Laboratory Report, CLM P111*, 1966, Culham England.

[16] H. POINCARÉ, *Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. **1**, Paris, Gauthier-Villars, 1897.

[17] J. PÖSCHEL, Nekhoroshev estimates for quasiconvex hamiltonian systems, *Mathematishe Zeitschrift*, Vol. **213**, 1993, pp. 187-216.

[18] H. SCOTT DUMAS, A Nekhoroshev-like theory of classical particle channeling in perfect crystals, *Dynamics Reported, , Expositions in Dynamical Systems*, New Series, Vol. **2**, 1993, pp. 69-115.

(Manuscript received January 18, 1994;
Revised version received November 28, 1994.)